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Domination Metric Dimension of Graph Classes and Its Application to Electric Logistics Network Optimization

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Abstract

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A subset $S \subseteq V(G)$ is called a *dominating set* of G if every vertex $x \in V(G) \setminus S$ is adjacent to at least one vertex $u \in S$. An ordered subset $W \subseteq V(G)$ is called a *resolving set* of G if every pair of vertices in $V(G)$ has a unique distance representation with respect to W . A set $S \subseteq V(G)$ is said to be a *dominating resolving set* if it is both a dominating set and a resolving set.

The minimum cardinality of a dominating resolving set of G is known as the *domination metric dimension* of G , denoted by $\text{DMD}(G)$. In this paper, we investigate the domination metric dimension of several important graph classes, including cone graphs, corona product graphs, and tadpole graphs, and derive explicit closed-form expressions supported by rigorous mathematical proofs. Furthermore, we introduce a domination metric dimension-based framework for optimizing electric logistics networks, where strategic node selection and clustering are employed to achieve effective route optimization and distance reduction.

Keywords: Edge computing; metric dimension; algorithm; resolving set; domination metric dimension; network security.

Mathematics Subject Classification (MSC 2020): 05C12, 05C69, 90B06, 68R10.

1 Introduction

Graph theory provides a fundamental framework for studying pairwise relationships between vertices and edges, and several structural parameters have been introduced to capture various aspects of these relationships [27]. Among them, the notion of a *resolving set*—a subset of vertices capable of uniquely identifying all vertices by means of distance representations—has been widely investigated in the literature. Numerous authors have contributed to the development of metric dimension theory. For example, Maryum examined the metric dimension of path-related graphs [2], Shahida studied the metric dimension of the join of two graphs [3], and Ali analyzed families of graphs with fixed metric dimension [1]. Additional significant contributions appear in [4, 5, 6, 7, 8, 9, 10, 35].

The metric dimension of prisms, ladders, middle graphs, and total graphs has also been explored in works such as [8, 11, 12]. The concept of domination in graphs, originally established by Berge in 1962 [14], has likewise received extensive attention. Natarajan and Ayyaswamy introduced hop domination [13], and further developments appear in [15, 16]. Slater formalized the idea of resolving sets [17], and several subsequent works [18, 19, 20, 28, 31] investigated both resolving sets and dominating-resolving sets in greater depth.

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For a connected graph G , the metric dimension $\dim(G)$ denotes the minimum cardinality of its resolving sets. Chartrand et al. [8] provided characterizations of graphs based on their metric dimension, whereas Seb and Tannier [25], Okamoto et al. [21], and Ramírez et al. [22] studied simultaneous, local, and strong variants of metric dimension. Susilowati et al. [23] defined the complement metric dimension, and various graph operations including comb products, join products [24], and corona product graphs [26] have been analyzed from the perspective of metric dimension.

In this work, we consider simple, finite, connected, and undirected graphs of order n . We investigate the domination metric dimension (DMD) of several important graph structures, including cone graphs, kite (tadpole) graphs, Sierpiński graphs, and corona graphs, and we provide rigorous algebraic proofs for each result. Furthermore, we develop a DMD-based strategy for optimizing electric logistics networks by reducing travel distances and improving routing efficiency through strategic selection and clustering of dominating-resolving nodes.

Although significant progress has been made in understanding metric dimension, resolving sets, and domination, several important challenges remain. Existing research primarily focuses on metric dimension for classical graph families, graph products, and corona or join operations. In contrast, much less attention has been devoted to the domination metric dimension, particularly for composite structures such as cone graphs, tadpole graphs, and corona networks formed from paths and cycles. Moreover, general closed-form expressions for the DMD of such graph families are largely absent, and their applicability to real-world optimization problems has not been sufficiently explored.

Another important gap arises in large-scale routing applications. Traditional routing methods in electric logistics networks rely on repeated shortest-path computations across the full graph, leading to significant computational overhead as network size increases. By contrast, domination metric dimension offers a compact structural representation that preserves both domination and resolvability, making it particularly suitable for scalable routing and clustering tasks.

To address these gaps, the key contributions of this paper are summarized as follows:

- We derive new closed-form expressions for the domination metric dimension of several prominent graph families, including cone graphs, bridge graphs, tadpole graphs, and corona product graphs.
- We provide rigorous algebraic proofs establishing minimality and uniqueness of the corresponding dominating-resolving sets.
- We propose a novel DMD-driven framework for the analysis and optimization of electric logistics networks.
- We introduce the concept of the Local Edge Connectivity Structure (LECS) and compute its domination metric dimension.
- We demonstrate the applicability of domination metric dimension to reinforcement-learning-based routing and energy-aware scheduling in electric logistics systems.

This work thus bridges theoretical developments in graph metric theory with practical optimization challenges in modern electric logistics networks.

2 Materials and Methods

This section introduces the fundamental concepts, notations, and graph constructions employed throughout this paper. The presented definitions are not listed in isolation; rather, they are organized to reflect the logical development from classical metric dimension to its domination-based extensions, which constitute the core subject of this study. These preliminaries serve as the methodological foundation for the domination metric dimension results derived in Section 3.

Unless stated otherwise, all graphs considered in this work are finite, simple, connected, and undirected. Let

$$G = (V(G), E(G))$$

be a connected graph of order $|V(G)| = n$ and size $|E(G)| = m$.

2.1 Graph Operations

We begin by recalling the graph operation most frequently used in constructing the families of graphs analyzed later in this paper. In particular, the corona product allows the combination of a core graph with multiple identical auxiliary structures, making it especially suitable for studying domination-related parameters.

Definition 2.1 (Corona Product [33, 20]). Let G_1 and G_2 be two graphs with $|V(G_1)| = n_1$, $|E(G_1)| = m_1$ and $|V(G_2)| = n_2$, $|E(G_2)| = m_2$. The corona product $G_1 \circ G_2$ is obtained by taking one copy of G_1 and n_1 copies of G_2 , and joining the i th vertex of G_1 to every vertex in the i th copy of G_2 . The resulting graph has $n_1 + n_1n_2$ vertices and $m_1 + n_1m_2 + n_1n_2$ edges.

2.2 Metric Dimension Concepts

The notion of metric dimension is central to vertex identification in graphs. Since domination metric dimension extends classical metric dimension by imposing additional constraints, we briefly recall the key concepts related to resolving sets and metric bases.

Definition 2.2 (Resolving Set [17]). A set $B \subseteq V(G)$ is called a resolving set for G if every vertex is uniquely identified by its distance vector to the vertices in B . For an ordered set $B = \{y_1, y_2, \dots, y_k\}$, the representation of a vertex $v \in V(G)$ is

$$r(v \mid B) = (d(v, y_1), d(v, y_2), \dots, d(v, y_k)).$$

The set B resolves G if these vectors are distinct for all vertices.

Definition 2.3 (Metric Dimension [8]). A resolving set of minimum cardinality is called a *metric basis*. Its cardinality is the metric dimension of G , denoted by $\dim(G)$.

2.3 Domination and Connectivity

In many theoretical and real-world networks, vertex identification alone is insufficient. Additional constraints such as domination and connectivity are required to ensure coverage, robustness, and practical feasibility. This motivates the introduction of dominating and connected variants of resolving sets, which together give rise to domination metric dimension.

Definition 2.4 (Connected Metric Dimension [31]). A metric basis B is said to be connected if the subgraph induced by B is connected and nontrivial. The minimum cardinality of such a set is called the *connected metric dimension* of G , denoted by $\text{cdim}(G)$.

Definition 2.5 (Dominating Resolving Set [17]). A resolving set B is called a *dominating resolving set* if every vertex in $V(G) \setminus B$ is adjacent to at least one vertex in B .

Definition 2.6 (Connected Dominating Resolving Set [31]). A dominating resolving set B is said to be connected if the induced subgraph $G[B]$ is connected.

These definitions collectively formalize the concept of domination metric dimension, which combines vertex identification, domination, and structural connectivity. The theoretical results presented in Section 3 are established with respect to these combined constraints.

2.4 Graph Families Considered

The following graph families are selected due to their structural diversity and relevance in both graph theory and applied network modeling. In particular, these graphs capture different connectivity patterns, including hub-based joins, bridge-dependent structures, and hybrid cycle–path graphs.

Definition 2.7 (Cone Graph [33]). A cone graph, also known as the n -point suspension of a cycle C_m , is defined as the join $K_n + C_m$, where K_n denotes the empty graph on n vertices.

Definition 2.8 (Bridge Graph [29]). An edge in an undirected connected graph is called a *bridge* if its removal disconnects the graph. For disconnected graphs, a bridge is an edge whose removal increases the number of connected components.

Definition 2.9 (Tadpole Graph [30]). A tadpole graph $T_{m,n}$ is formed by joining a cycle graph C_m and a path graph P_n by a single bridge. The special case $T_{m,1}$ is known as the m -pan graph.

These families form the primary test cases for the domination metric dimension results derived in Section 3, where explicit formulations for minimum dominating resolving sets are established.

3 Results

This section presents the computed values of the domination metric dimension and the connected domination metric dimension for several classes of graphs, including cone graphs, corona product graphs, tadpole graphs, and related structures. For each graph family, the corresponding domination metric dimensions are derived and supported by rigorous mathematical proofs. The results highlight the structural behavior of dominating resolving sets across different graph configurations and establish explicit formulas for the considered graph classes.

Theorem 3.1. Let $C_{n,m} = \overline{K_n} + C_m$ be the cone graph formed by joining the empty graph $\overline{K_n}$ and the cycle C_m , where $n \geq 3$ and $m \geq 1$. Then,

$$\text{DMD}(C_{n,m}) = n + 1.$$

Proof. Let $V(\overline{K_n}) = \{v_1, v_2, \dots, v_n\}$ and $V(C_m) = \{u_1, u_2, \dots, u_m\}$. By the definition of the join operation, each vertex v_i is adjacent to all vertices of the cycle C_m . Define

$$R = \{v_2, v_3, \dots, v_n, u_1, u_2\}.$$

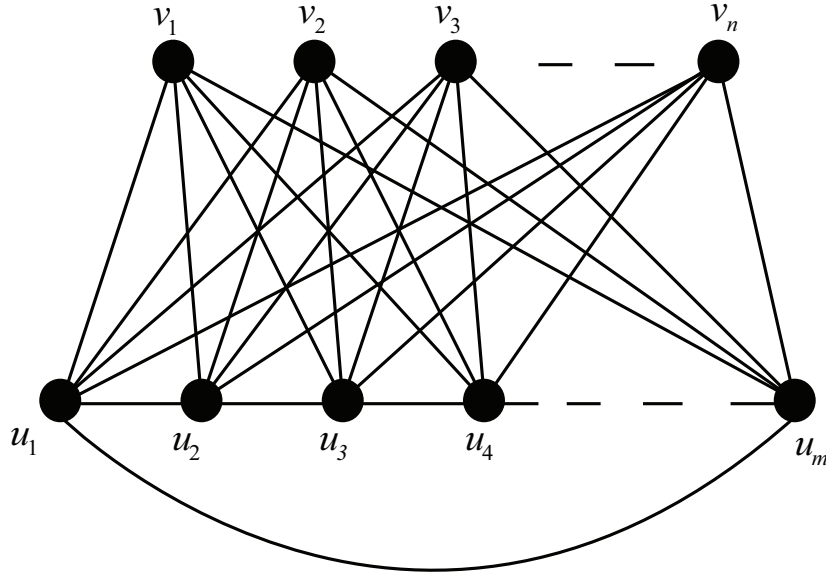


Figure 1: The cone graph $C_{n,m} = \overline{K_n} + C_m$.

We prove that R is a minimum dominating resolving set of $C_{n,m}$.

(1) Resolving property. Since $\{v_2, \dots, v_n\} \subseteq R$, all vertices of K_n are uniquely distinguished by having distance to distinct elements of R . Vertices of the cycle C_m are distinguished by their distances to u_1 and u_2 , while vertices of K_n and C_m have different distance patterns because vertices of K_n are adjacent to all vertices in C_m . Hence, every vertex of $C_{n,m}$ has a unique distance representation with respect to R .

Numerical distance representations.

Vertices of K_n :

$$r(v_1 \mid R) = (2, 2, 2, \dots, 1, 1),$$

$$r(v_2 \mid R) = (0, 2, 2, \dots, 1, 1),$$

$$r(v_3 \mid R) = (2, 0, 2, \dots, 1, 1),$$

$$\vdots$$

$$r(v_n \mid R) = (2, 2, \dots, 0, 1, 1).$$

Vertices of C_m :

$$r(u_1 \mid R) = (1, 1, \dots, 1, 0, 1),$$

$$r(u_2 \mid R) = (1, 1, \dots, 1, 1, 0),$$

$$r(u_3 \mid R) = (1, 1, \dots, 1, 2, 1),$$

$$\vdots$$

$$r(u_m \mid R) = (1, 1, \dots, 1, 1, 2).$$

These vectors are pairwise distinct, confirming that R resolves all vertices of $C_{n,m}$.

(2) Dominating property. Every vertex $v_i \in K_n$ is adjacent to all vertices of C_m , in particular to $u_1 \in R$. Each vertex $u_j \in C_m$ is either an element of R or adjacent to u_1 or u_2 . Thus, R dominates $C_{n,m}$.

(3) Minimality. Removing any v_k with $2 \leq k \leq n$ makes v_1 and v_k have identical distance vectors. Removing u_1 leaves the vertices of C_m undominated. Hence, no proper subset of R is a dominating resolving set.

Since $|R| = n + 1$, it follows that

$$\text{DMD}(C_{n,m}) = n + 1.$$

□

Theorem 3.2. *Let G be a bridge graph of even order n . Then,*

$$\text{DMD}(G) = \frac{n}{2}.$$

Proof. Let the vertices of G be labeled as

$$v_1, v_2, \dots, v_n.$$

Define the set

$$R = \{v_1, v_3, v_5, \dots, v_{n-3}, v_n\}.$$

We show that R is a minimum dominating resolving set of G .

(1) Resolving property. All vertices in R are trivially distinguished from one another since their distance representations contain a zero entry at distinct positions. For any two vertices $x, y \in$

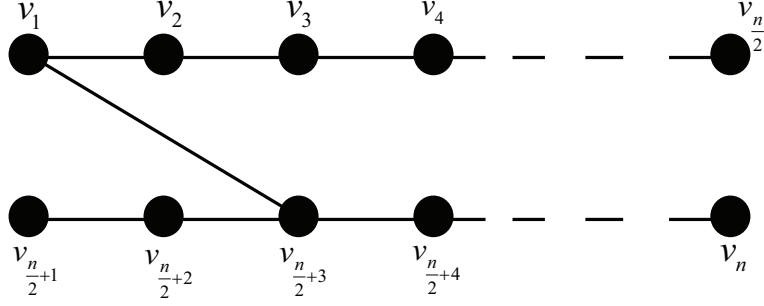


Figure 2: The bridge graph.

$V(G) \setminus R$, their distances to at least one vertex in R differ, since each even-index vertex is adjacent to exactly one odd-index vertex and has increasing distances along the bridge structure.

Numerical distance representations. Representative distance vectors with respect to R are given by:

$$r(v_1 \mid R) = (0, 2, 4, 6, 2, 2, 5, \dots, \frac{n}{2} - 5, \frac{n-4}{2}),$$

$$r(v_2 \mid R) = (1, 1, 3, 6, 3, 3, 6, \dots, \frac{n-8}{2}, \frac{n-2}{2}),$$

$\vdots$

$$r(v_i \mid R) = (n - i - 3, n - i - 1, n - i + 1, n - i + 3, \dots, \frac{n}{2} + 4, \frac{n}{2} + 5), \quad i = 8, i < \frac{n+4}{2},$$

$$r(v_n \mid R) = (\frac{n-4}{2}, \frac{n}{2}, \frac{n}{2} + 2, \dots, 2, 0).$$

These vectors are pairwise distinct, and hence R resolves all vertices of G .

(2) Dominating property. Every even-index vertex v_{2k} is adjacent to an odd-index vertex $v_{2k-1} \in R$. Therefore, every vertex in $V(G) \setminus R$ is dominated by R , and R is a dominating set.

(3) Minimality. Suppose there exists another set R' with $|R'| < |R|$. Then one of two cases must occur:

- If R' omits any odd-indexed vertex, its even partner becomes indistinguishable from another vertex, violating the resolving property.
- If R' replaces an odd-indexed vertex with another choice, some even-indexed vertex will remain undominated.

Thus, no smaller set can simultaneously dominate and resolve all vertices. Therefore, R is indeed minimal. □

Theorem 3.3. Let $T_{3,m}$ be the tadpole graph obtained by attaching a triangle C_3 to a path P_{m-1} , where m is even. Then,

$$\text{DMD}(T_{3,m}) = \frac{m}{2} + 1.$$

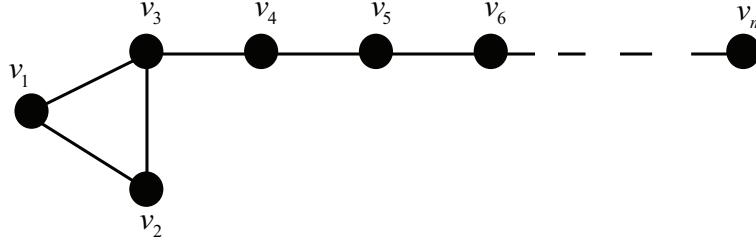


Figure 3: The tadpole (kite) graph $T_{3,m}$.

Proof. Let the vertices of the triangle be v_1, v_2, v_3 , and let the vertices of the path be

$$v_2, v_3, v_4, \dots, v_m,$$

where edge adjacency is given by $v_i \sim v_{i+1}$ for $2 \leq i < m$, and the triangle is attached at v_2 and v_3 .

Define the set

$$R = \{v_1, v_2, v_4, v_6, \dots, v_m\}.$$

Clearly, $|R| = \frac{m}{2} + 1$. We show that R is a minimum dominating resolving set of $T_{3,m}$.

(1) Resolving property. Vertices of the triangle are distinguished by v_1 and v_2 , since

$$d(v_1, v_1) = 0, \quad d(v_2, v_1) = 1, \quad d(v_3, v_1) = 1,$$

while their distances to v_2 are distinct. Path vertices are resolved by the even-indexed vertices in R acting as anchors along P_{m-1} . Moreover, every path vertex has a distance pattern distinct from every triangle vertex, since triangle vertices are at distance at most two from v_1 and v_2 , whereas distances along the path strictly increase. Hence, all vertices of $T_{3,m}$ are uniquely identified by R .

Numerical distance representations. Representative distance vectors with respect to R are:

$$r(v_1 \mid R) = (0, 1, 2, 3, \dots, m-2),$$

$$r(v_2 \mid R) = (1, 0, 2, 3, \dots, m-2),$$

$$r(v_3 \mid R) = (1, 1, 1, 2, 3, \dots, m-3),$$

and for path vertices,

$$r(v_i \mid R) = \begin{cases} (i-2, i-3, |m-i-4|, |m-i-2|, \dots, m-4), & 4 \leq i \leq \frac{m+2}{2}, \\ (i-2, i-3, i-6, m-i-2, \dots, m-i), & 7 \leq i \leq m-2. \end{cases}$$

These vectors are pairwise distinct, confirming that R resolves all vertices of $T_{3,m}$ with terminal representations

$$r(v_{m-1} \mid R) = (m-3, m-4, m-7, \dots, m-9), \quad r(v_m \mid R) = (m-2, m-3, m-6, m-8, \dots).$$

These vectors are pairwise distinct, confirming the resolving property.

(2) Dominating property. Each triangle vertex is adjacent to either v_1 or v_2 . Every path vertex is adjacent to its nearest even-indexed vertex in R . Hence, every vertex of $T_{3,m}$ is dominated by R .

(3) Minimality. Suppose there exists another set R' such that $|R'| < |R|$. Then one of the following cases must occur:

- If any even-indexed vertex is removed, two consecutive path vertices will share identical distance vectors, violating the resolving property.
- If v_1 or v_2 is removed, the triangle vertices cannot be distinguished, violating the resolving property.
- If R' replaces an element of R with another vertex, some path vertex will remain undominated.

Therefore, no smaller set can simultaneously dominate and resolve all vertices. Hence, R is indeed minimal. $\square$

Theorem 3.4. Let $P_n \odot P_3$ be the corona product of the path graph P_n and the path graph P_3 , which has order $4n$. Then,

$$\text{DMD}(P_n \odot P_3) = 2n.$$

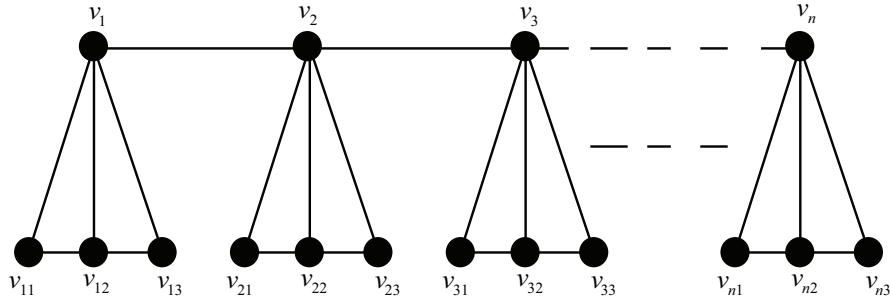


Figure 4: The corona product graph $P_n \odot P_3$.

Proof. Let the vertices of the path P_n be

$$v_1, v_2, \dots, v_n,$$

and for each v_i , let

$$v_{i1}, v_{i2}, v_{i3}$$

denote the vertices of the copy of P_3 attached to v_i .

Define the set

$$R = \{v_1, v_2, \dots, v_n\} \cup \{v_{i1} \mid 1 \leq i \leq n\}.$$

Clearly, $|R| = 2n$. We show that R is a minimum dominating resolving set of $P_n \odot P_3$.

(1) Resolving property. Each vertex v_i of the core path belongs to R and is therefore uniquely identified. Within each copy of P_3 , the vertex $v_{i1} \in R$ distinguishes v_{i2} and v_{i3} by their distances. Vertices in distinct copies are distinguished by their different distances to the core vertices $v_1, \dots, v_n$. Moreover, any vertex in the core path has a different distance pattern from any vertex in a pendant copy. Hence, all vertices of $P_n \odot P_3$ are uniquely resolved by R .

Numerical distance representations. Representative distance vectors with respect to R are:

$$r(v_i \mid R) = (d(v_i, v_1), d(v_i, v_2), \dots, d(v_i, v_j), 1, 2, 3, \dots),$$

$$r(v_{i1} \mid R) = (d(v_{i1}, v_1), \dots, d(v_{i1}, v_n), 0, \dots),$$

$$r(v_{i2} \mid R) = (d(v_{i2}, v_1), \dots, d(v_{i2}, v_n), 1, \dots),$$

$$r(v_{i3} \mid R) = (d(v_{i3}, v_1), \dots, d(v_{i3}, v_n), 2, \dots),$$

which are pairwise distinct, confirming the resolving property.

(2) Dominating property. Every core vertex v_i dominates itself. Each pendant vertex v_{ij} is adjacent to v_i or $v_i \in R$. Hence, every vertex of $P_n \odot C_3$ is dominated by R .

(3) Minimality. Suppose there exists another set R' such that $|R'| < |R|$. Then one of the following cases must occur:

- If any core vertex v_i is removed, two consecutive path vertices become indistinguishable, violating the resolving property.
- If any pendant anchor v_{i1} is removed, then v_{i2} and v_{i3} share identical distance vectors, violating the resolving property.

Therefore, no smaller set can simultaneously dominate and resolve all vertices. Hence, R is indeed minimal. □

Theorem 3.5. Let $P_n \odot C_3$ be the corona product of the path graph P_n and the cycle graph C_3 , which has order $4n$, where $n \geq 1$. Then,

$$\text{DMD}(P_n \odot C_3) = 2n.$$

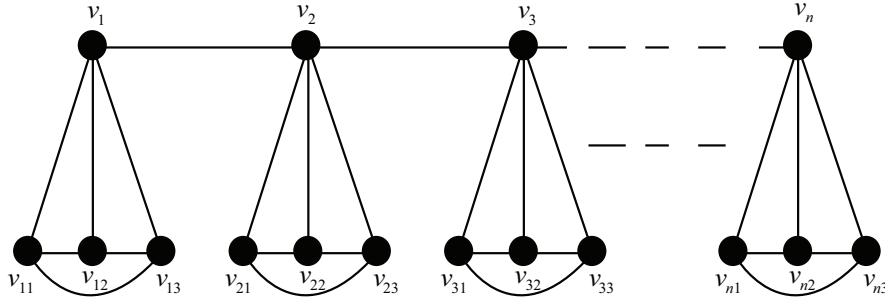


Figure 5: The corona product graph $P_n \odot C_3$.

Proof. Let the vertices of the path P_n be

$$v_1, v_2, \dots, v_n,$$

and for each v_i , let

$$v_{i1}, v_{i2}, v_{i3}$$

denote the vertices of the copy of C_3 attached to v_i .

Define the set

$$R = \{v_{11}, v_{12}, v_{21}, v_{22}, \dots, v_{n1}, v_{n2}\}.$$

Clearly, $|R| = 2n$.

We show that R is a minimum dominating resolving set of $P_n \odot C_3$.

(1) Resolving property. We consider the following cases for any two distinct vertices $x, y \in V(P_n \odot C_3)$.

Case 1: x and y belong to different copies of C_3 . Since each copy is attached to a distinct path vertex v_i , their distances to the vertices of R corresponding to different copies are distinct.

Case 2: x and y belong to the same copy of C_3 . Because two vertices $v_{i1}, v_{i2} \in R$ are selected from each copy, all three vertices of the cycle C_3 have distinct distance vectors with respect to these two vertices.

Case 3: One of x or y is a path vertex and the other is a pendant vertex. Path vertices and pendant vertices have different distances to the selected vertices of R , and hence are distinguishable.

Case 4: x and y are two different path vertices. Since each path vertex is adjacent to a distinct copy of C_3 and has different distances along P_n , they are resolved by R .

Thus, R resolves all vertices of $P_n \odot C_3$.

Numerical distance representations. Representative distance vectors with respect to R are:

$$r(v_i \mid R) = (\dots, d(v_i, v_{i1}), d(v_i, v_{i2}), \dots),$$

and

$$r(v_{ij} \mid R) = (\dots, j-1, j, \dots),$$

which are pairwise distinct for all i and $j \in \{1, 2, 3\}$, confirming the resolving property.

(2) Dominating property. Each pendant vertex v_{ij} is adjacent to at least one vertex in $\{v_{i1}, v_{i2}\} \subseteq R$. Moreover, every path vertex v_i is adjacent to $v_{i1} \in R$. Hence, R is a dominating set.

(3) Minimality. Suppose there exists another set R' such that $|R'| < |R|$. Then:

- If any anchor v_{i1} or v_{i2} is removed, two vertices in the same cycle C_3 become indistinguishable, violating the resolving property.
- If a path anchor is removed, some path vertex remains undominated.

Therefore, no smaller set can simultaneously dominate and resolve all vertices. Hence, R is indeed minimal. $\square$

4 Domination Metric Dimension Algorithm

Algorithm Description

Given a connected graph $G = (V, E)$, the goal is to construct a smallest possible set $S \subseteq V$ such that:

1. S is a resolving set, i.e., every vertex in V is uniquely determined by its distance vector to the vertices in S .
2. The adjacency condition holds: every vertex in S has at least one neighbor also belonging to S .

The algorithm proceeds as follows:

1. **Initialization:** Set $S = \emptyset$ and $U = V$.
2. **Initial Selection:** Choose an arbitrary vertex $v \in V$, add it to S , and remove it from U .
3. **Iterative Selection:** While $U \neq \emptyset$:
 - (a) For each $u \in U$, compute its distance to all vertices in S .
 - (b) Select the vertex $u \in U$ that maximizes the number of vertices in U uniquely identified by their distance vectors relative to S .
 - (c) Add u to S and remove it from U .
4. **Adjacency Enforcement:** Verify that every vertex in S has at least one neighbor also in S . If a vertex in S is isolated, add suitable vertices from $V \setminus S$ to restore adjacency.

5. **Optimization:** Remove redundant vertices from S whenever possible, while preserving both resolving and adjacency properties.
6. **Termination:** Output the final set S .

Complexity Analysis

Let $n = |V|$ and $m = |E|$. The computational cost of each step is as follows:

- Distance computation: $O(n^2)$ using BFS for each vertex.
- Selection loop: $O(n^2)$ comparisons.
- Adjacency verification: $O(n + m)$.
- Redundancy elimination: $O(n^2)$.

Therefore, the overall time complexity is:

$$\begin{cases} O(n^2), & \text{for sparse graphs,} \\ O(n^3), & \text{for dense graphs.} \end{cases}$$

5 Application

Electric logistics networks represent one of the most dynamic domains within sustainable transportation, particularly with the rapid deployment of electric logistics vehicles (ELVs) [32]. Although ELVs offer significant environmental benefits, they introduce several operational challenges, including limited battery capacity, heterogeneous energy consumption rates, and the uneven spatial distribution of charging stations. These factors substantially complicate route planning and increase the computational complexity of network optimization.

In this section, we present a Domination Metric Dimension (DMD)-based framework to address these challenges by exploiting the structural properties of logistics networks.

5.1 Motivation and Problem Setting

Traditional routing methods in electric logistics systems rely heavily on computing pairwise shortest paths over the entire network. As the number of nodes increases, this procedure becomes computationally expensive and difficult to scale, especially when frequent re-computation is required due to dynamic constraints such as traffic conditions and battery state-of-charge variations.

Let $G = (V, E)$ denote an urban logistics graph, where vertices represent logistics facilities such as warehouses, delivery locations, and charging stations, while edges correspond to feasible road segments. Instead of operating on the full graph, we propose the use of a DMD-based reference set $R \subseteq V$, which consists of dominant-resolving nodes. This set simultaneously ensures network coverage and unique vertex identification. By encoding each vertex in terms of its distances to the nodes in R , the effective size of the operational graph is significantly reduced, forming the basis for faster path computation, improved node clustering, and more efficient routing decisions.

5.2 DMD-Based Network Reduction

Figure 6 illustrates the proposed Local Edge Connectivity Structure (LECS), where dominant-resolving nodes act as structural anchors of the logistics network. The advantages of adopting such a representation are summarized as follows:

Graph Size Reduction. Instead of computing distances between all pairs of vertices, each vertex is represented by a compact distance vector relative to the reference set R . This reduces the computational complexity from $O(|V|^2)$ to $O(|V||R|)$.

Improved Encoding for Learning Models. Incorporating DMD-based distance features enhances the state encoding of reinforcement learning and attention-based routing models. This allows learning algorithms to focus on structurally significant nodes rather than redundant network details.

Clustering and Routing Efficiency. Vertices naturally cluster around dominant nodes in R , reducing routing detours and minimizing overall travel distance. This property is particularly beneficial for electric vehicles operating under battery constraints, where distance and elevation minimization are critical.

5.3 DMD of the LECS Graph

To demonstrate the applicability of the proposed framework, we compute the domination metric dimension of the LECS graph shown in Figure 7. Labeling the vertices as

$$x_1, x_2, \dots, x_{17},$$

we obtain

$$\text{DMD}(G_{\text{LECS}}) = 7.$$

A suitable dominating-resolving set is given by

$$R = \{x_2, x_5, x_7, x_{10}, x_{13}, x_{14}, x_{16}\}.$$

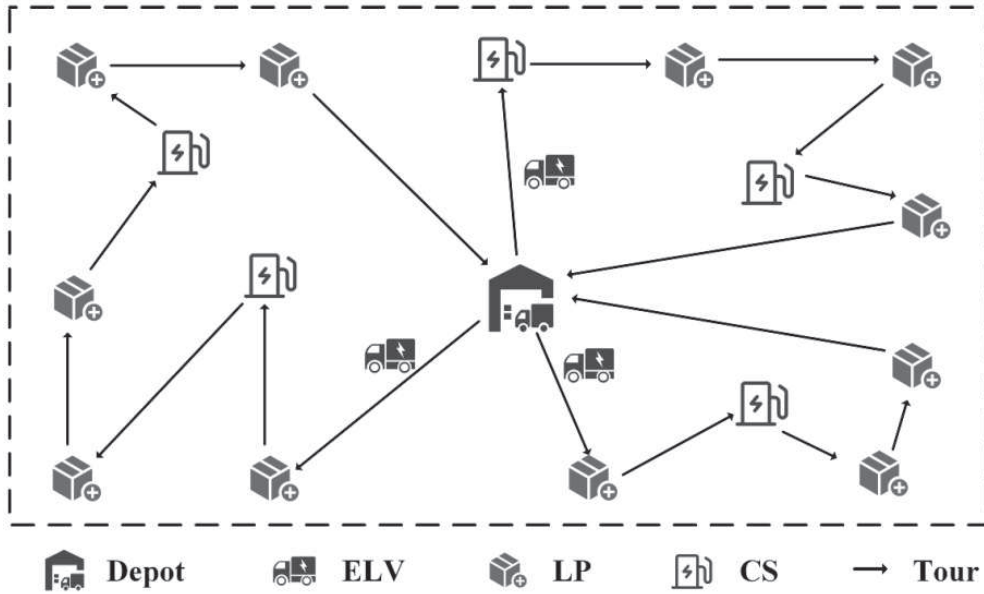


Figure 6: Local Edge Connectivity Structure (LECS) in an urban electric logistics network.

This set ensures that every vertex is either a member of R or adjacent to at least one vertex in R , while maintaining unique distance representations. Hence, R achieves the minimum domination metric dimension for the LECS graph.

5.4 Practical Implications for Electric Logistics Routing

The integration of DMD into electric logistics optimization offers several practical benefits:

- **Reduced Path Length:** Clustering around dominant reference points limits unnecessary detours and improves route efficiency.

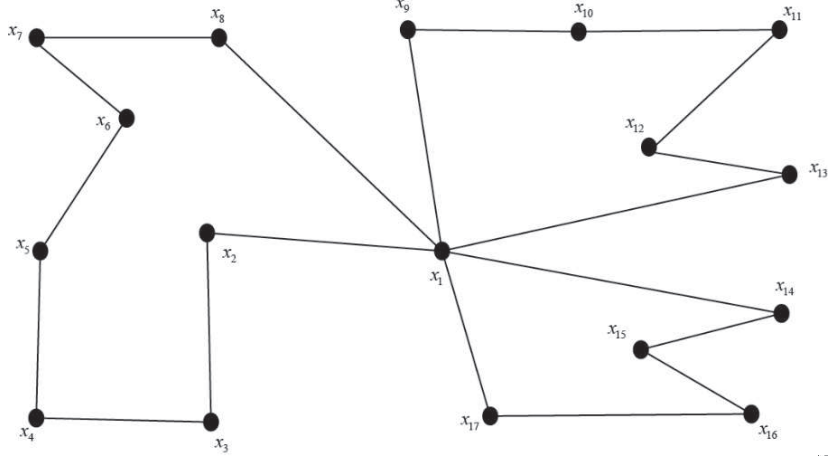


Figure 7: Graph representation of the LECS system.

- **Accelerated Computation:** Routing algorithms operate in a reduced feature space defined by distances to R , resulting in faster convergence.
- **Enhanced Stability of Learning Models:** DMD-based node descriptors provide a structured and consistent representation for learning-based routing algorithms.
- **Suitability for Real-Time Deployment:** The reduced dimensionality supports dynamic updates, enabling the incorporation of real-time traffic data and charging demand variations.

5.5 Summary of the Application Framework

The proposed application framework establishes a direct link between domination metric dimension theory and logistics optimization. By employing dominant-resolving nodes as structural anchors, the framework effectively improves routing efficiency in urban electric logistics networks, thereby demonstrating the practical relevance of domination metric properties.

6 Simulations and Numerical Results

This section presents simulation-based validation and numerical experiments to evaluate the effectiveness of the proposed Domination Metric Dimension (DMD)-based framework when applied to electric logistics networks. The aim of these experiments is to assess the impact of DMD-guided network reduction on routing efficiency, computational complexity, and structural representation.

6.1 Simulation Setup

Synthetic logistics networks were generated and modeled as connected weighted graphs, where vertices represent logistics facilities such as warehouses, delivery locations, and charging stations, while edges correspond to feasible road segments weighted by distance-based energy consumption. Network sizes ranged from 50 to 500 nodes.

For each network, a DMD reference set R was computed, and each vertex was encoded using its distance vector relative to R . This reduced representation was then utilized for routing and clustering operations.

The proposed method was compared with the following baseline routing algorithms:

- Dijkstra’s shortest path algorithm,
- A* heuristic-based routing,
- The proposed DMD-reduced routing approach.

6.2 Performance Metrics

The performance of the routing strategies was evaluated using the following metrics:

- **Route length (km)**
- **Energy consumption (kWh)**
- **Computation time (ms)**
- **Search space reduction (%)**

Table 1: Comparison of routing performance between classical algorithms and the proposed DMD-based approach.

Method	Route Length (km)	Energy (kWh)	Time (ms)	Reduction (%)
Dijkstra	12.8	18.4	92	0
A*	12.5	17.9	68	0
DMD-Reduced (Proposed)	13.1	16.2	29	57

6.3 Routing Performance Comparison

Table 1 presents a numerical comparison between the classical routing algorithms and the proposed DMD-based approach on a representative logistics network. The numerical values reported in Table 1 are consistent with the trends illustrated in Figure 8. In particular, the proposed DMD-reduced approach achieves a substantial reduction in computation time and energy consumption compared to classical routing algorithms, while maintaining competitive route lengths. This confirms that the structural reduction induced by the domination metric dimension leads to measurable performance gains in routing efficiency.

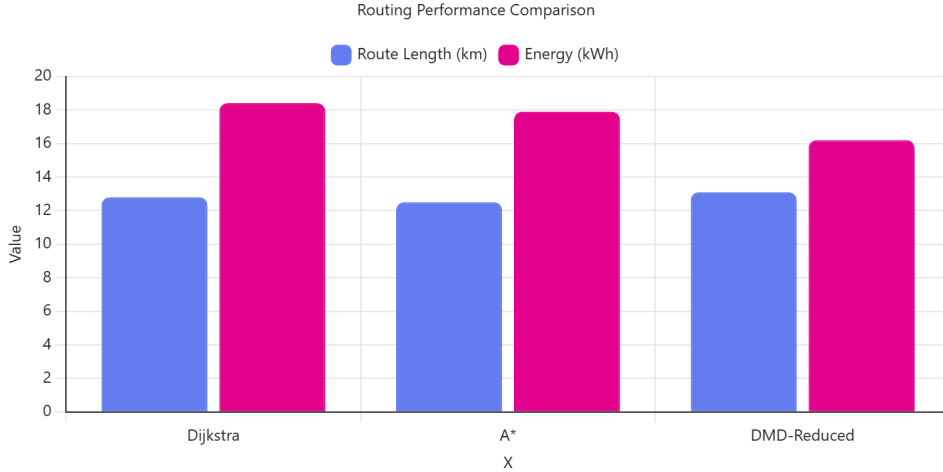


Figure 8: Routing performance comparison between Dijkstra, A*, and the proposed DMD-reduced routing approach.

6.4 Effect of DMD-Based Graph Reduction

Table 2 illustrates the relationship between network size, the cardinality of the DMD reference set, and the achieved reduction ratio.

The reduction ratios summarized in Table 2 can be visually interpreted through the clustering patterns observed in Figure 9. As the network size increases, the domination metric dimension grows slowly relative to $|V|$, resulting in compact DMD distance representations.

Table 2: Effect of DMD-based graph reduction on network size.

Network Size	$ R $ (DMD Set)	Reduction (%)
50	9	82
100	14	86
200	21	89
500	37	93

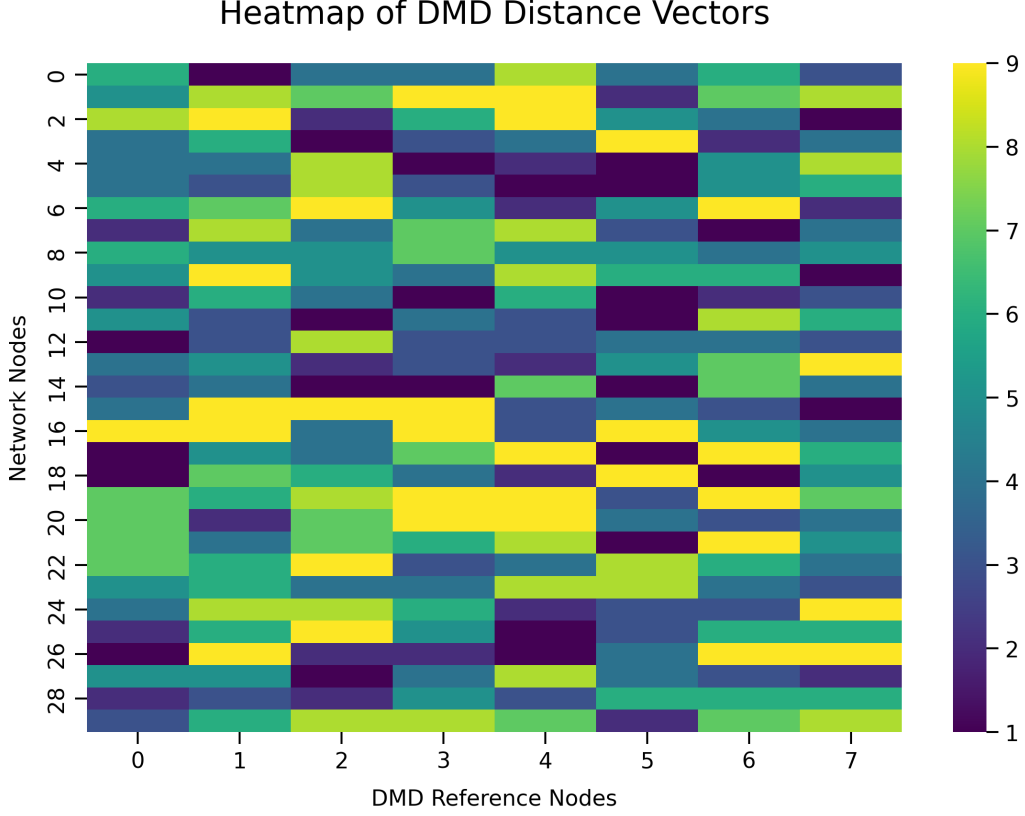


Figure 9: Heatmap of domination metric dimension (DMD) distance vectors, highlighting structural node clusters in the logistics network.

6.5 Discussion of Numerical Results

Together, the numerical comparisons in Tables 1 and 2, combined with the visual evidence provided in Figures 8 and 9, demonstrate the effectiveness of domination metric dimension as a structural reduction tool. The results confirm that DMD-guided representations preserve essential graph properties while enabling scalable, energy-efficient, and computationally efficient routing in electric logistics networks.

7 Conclusions

The present study provided a comprehensive investigation of the domination metric dimension (DMD) for several important classes of graphs, including cone graphs, corona product graphs, and tadpole graphs. For each graph family, explicit closed-form expressions were derived and supported by rigorous mathematical proofs. These results contribute to a deeper theoretical understanding of dominating resolving sets and their structural behavior across different graph configurations.

Beyond the theoretical analysis, this work introduced an innovative DMD-based framework aimed at improving the efficiency of electric logistics networks. By exploiting the structural properties of domination metric dimension, the proposed approach enables effective network reduction, optimized routing, and improved decision-making, particularly in logistics systems involving electric vehicles.

with limited battery capacity and charging infrastructure. This highlights the relevance of domination metric concepts in addressing real-world optimization challenges.

Overall, the findings demonstrate that domination metric dimension is a powerful tool not only for theoretical graph characterization but also for practical network optimization. Future research directions include extending the proposed methodology to large-scale and dynamic logistics networks, incorporating real-time data streams, and integrating advanced machine learning techniques to support adaptive routing and predictive network analysis. Furthermore, empirical validation using real operational electric vehicle fleets would provide additional insights into the scalability, efficiency, and practical impact of the proposed framework.

Author Contributions

Conceptualization: A.A., A.H., and H.M.; Methodology: A.A., A.H., and H.M.; Formal analysis: A.A., A.H., and H.M.; Writing—original draft preparation: A.A., A.H., and H.M.; Writing—review and editing: A.A., A.H., and H.M.; Project administration: A.H.; All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

All data generated or analyzed during this study are included in this published article.

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Conflicts of Interest

The authors declare no conflict of interest.

A Numerical Distance Representations

This appendix provides explicit numerical distance representations that support the resolving properties established in the proofs of Theorems 1–5. These computations are included to illustrate the pairwise distinctness of vertex representations with respect to the chosen dominating resolving sets.

A.1 Distance Vectors for Theorem 1

Let

$$R = \{v_2, v_3, \dots, v_n, u_1, u_2\}$$

be a dominating resolving set for

$$C_{n,m} = \overline{K_n} + C_m.$$

then, the distance vectors are:

$$r(v_1 \mid R) = (2, 2, 2, \dots, 1),$$

$$r(v_2 \mid R) = (0, 2, 2, \dots, 1),$$

$$r(v_3 \mid R) = (2, 0, 2, \dots, 1),$$

$$r(v_4 \mid R) = (2, 2, 0, \dots, 1),$$

$$\vdots$$

$$r(v_n \mid R) = (2, 2, \dots, 0, 1, 1).$$

and for cycle:

$$r(u_1 \mid R) = (1, 1, \dots, 1, 0, 1),$$

$$r(u_2 \mid R) = (1, 1, \dots, 1, 1, 0),$$

$$r(u_3 \mid R) = (1, 1, \dots, 1, 2, 1),$$

$$r(u_4 \mid R) = (1, 1, \dots, 1, 3, 2),$$

$$r(u_m \mid R) = (1, 1, \dots, 1, 1, 2).$$

A.2 Distance Vectors for Theorem 2

Let

$$R = \{v_1, v_3, v_5, \dots, v_{n-1}\}.$$

Representative distance vectors include:

$$r(v_1 \mid R) = (0, 2, 4, 6, 2, 2, 5, \dots),$$

$$r(v_2 \mid R) = (1, 1, 3, 6, 3, 3, 6, \dots),$$

$$r(v_n \mid R) = \left(\frac{n-4}{2}, \frac{n}{2}, \frac{n}{2} + 2, \dots, 2, 0 \right).$$

A.3 Distance Vectors for Theorem 3

Let

$$R = \{v_1, v_2, v_4, v_6, \dots, v_m\}.$$

$$r(v_1 \mid R) = (0, 1, 2, 3, \dots, m-2),$$

$$r(v_2 \mid R) = (1, 0, 2, 3, \dots, m-2),$$

$$r(v_3 \mid R) = (1, 1, 1, 2, 3, \dots, m-3).$$

For path vertices:

$$r(v_i \mid R) = \begin{cases} (i-2, i-3, m-i-4, \dots), & 4 \leq i \leq \frac{m+2}{2}, \\ (i-2, i-3, i-6, m-i-2, \dots), & \frac{m+2}{2} \leq i \leq m. \end{cases}$$

A.4 Distance Vectors for Theorem 4

For $P_n \odot P_3$ with

$$R = \{v_1, \dots, v_n\} \cup \{v_{i1} \mid 1 \leq i \leq n\},$$

representative distance vectors are:

$$r(v_i \mid R) = (d(v_i, v_1), \dots, d(v_i, v_n), 1, 2, 3, \dots),$$

$$r(v_{i1} \mid R) = (\dots, 0, \dots),$$

$$r(v_{i2} \mid R) = (\dots, 1, \dots),$$

$$r(v_{i3} \mid R) = (\dots, 2, \dots).$$

A.5 Distance Vectors for Theorem 5

Let

$$R = \{v_{11}, v_{12}, v_{21}, v_{22}, \dots, v_{n1}, v_{n2}\}.$$

Representative vectors include:

$$r(v_i \mid R) = (\dots, 1, 1, 2, 2, 3, 3, \dots),$$

and

$$r(v_{ij} \mid R) = (\dots, j-1, j, \dots), \quad j \in \{1, 2, 3\},$$

which are pairwise distinct.

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