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Topological Origin of Neutrino Mass and Mixing: From the $SU(3)_3$ Modular Tensor Category to the PMNS Matrix

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Abstract: We propose a topological origin for neutrino masses and mixing: neutrinos are boundary topological flows of a string-net condensed phase, and their mass matrix is determined by boundary dynamics that inherits the hierarchical parameters of the bulk $SU(3)_3 \otimes SU(2)_2$ modular tensor category. Using the conformal-weight spacing $\Delta h = 1/3$ and the topological temperature $\beta = \pi^2/14$ of this category, we read the Cabibbo angle $\lambda = e^{-14/(3\pi)} \approx 0.225$, without any free parameters, in agreement with the experimental value to within 0.04%. On this basis, we propose a mass matrix whose coefficients are mapped directly to the S-matrix elements of the $SU(3)_3$ modular tensor category, with diagonalization yielding mixing angles close to the experimental values. Further introducing the CP phase $\varepsilon = \lambda^2$ arising from the quadratic effect of topological suppression, and reading the charged-lepton mixing parameters (α, β, γ) from the quantum-dimension ratios and conformal weights of the category, we can simultaneously fit the four observables of NuFIT 6.0. The parameters $\alpha = -3\lambda/5$, $\beta = -\lambda^2/2$, and $\gamma = 2\lambda/3$ are read from $d_8/(d_8 + d_3) = 3/5$, $h_8 = 1/2$, and $h_{15} = 2/3$ respectively. The CP phase $\eta = \pi/10$ has a proposed reading mechanism $\eta = \pi/N_{simple}$, pending confirmation; we have systematically excluded several possible algebraic origins (T-matrix phases, R-matrix phases, topological entropy change) and leave its full derivation as an open problem.

Keywords: modular tensor category; string-net condensation; boundary topological flow; PMNS matrix; Cabibbo angle; neutrino mass

1. Introduction

The origin of fermion mass hierarchies and flavor mixing remains one of the deepest puzzles in particle physics. The Standard Model accommodates the observed data with approximately 19 free parameters but fails to explain the underlying regularities. In recent years, the application of generalized symmetries and topological order in quantum field theory has provided new perspectives for understanding these phenomena [1–4]. In previous work, we applied the triality structure of modular tensor categories to antimatter generation and related topics [5–7]. These studies developed a categorical framework for applying modular tensor categories to particle physics and cosmology, which provides the conceptual background for the present work. In this paper, we extend this picture to the lepton sector, with particular emphasis on neutrinos.

Our central hypothesis is that **leptons are boundary excitations of a string-net condensed phase, while hadrons are bulk excitations**. As electrically neutral boundary excitations, neutrinos acquire their small masses and flavor mixing from the modular data and dynamics of the boundary category. Unlike conventional seesaw or Froggatt–Nielsen mechanisms, we attempt to derive flavor parameters directly from the algebraic structure of topological order.

The paper is organized as follows. Section 2 reviews the modular data of the $SU(3)_3$ modular tensor category. Section 3 discusses the condensed boundary category and the origin of the three-generation structure. Section 4 derives the topological expression for the Cabibbo angle λ . Section 5 presents the mass matrix ansatz, preceded by the conceptual foundation of topological splitting (Sec. 5.1). Section 6

introduces CP violation and charged-lepton mixing corrections. Section 7 discusses the topological origin of neutrino mass, chirality, and the three-generation structure. Section 8 provides analysis and discussion, including mass-spectrum constraints and comparison with the Standard Model. Section 9 summarizes testable predictions, and Section 10 concludes.

2. Modular Data of the $SU(3)_3$ Modular Tensor Category

The $SU(3)_3$ modular tensor category has 10 simple objects labeled by Dynkin labels (p, q) satisfying $p, q \geq 0, p + q \leq 3$. The quantum dimensions are given by the q-number formula [8-11]:

$$[n]_q = \frac{\sin(\pi n/6)}{\sin(\pi/6)}, \quad q = e^{i\pi/6}$$

The quantum dimensions d and conformal weights h of each object are listed in Table 1.

Table 1. Quantum dimensions and conformal weights of simple objects in $SU(3)_3$

Object (p, q)	Representation	d	h
(0,0)	1	1	0
(1,0)	3	2	2/9
(0,1)	$\bar{3}$	2	2/9
(2,0)	6	2	5/9
(0,2)	$\bar{6}$	2	5/9
(1,1)	8	3	1/2
(3,0)	10	1	1
(0,3)	$\bar{10}$	1	1
(2,1)	15	2	2/3
(1,2)	$\bar{15}$	2	2/3

The quantum dimensions in Table 1 are computed via the Verlinde formula; for a general introduction, see [9].

The total quantum dimension is $D = \sqrt{\sum d_i^2} = 6$. The S-matrix is given by the Kac–Peterson formula and verified using SageMath's “FusionRing” (see Appendix A). The fusion rules follow the standard $SU(3)$ truncation, typically:

$$3 \otimes 3 = \bar{3} \oplus 6, \quad 3 \otimes \bar{3} = 1 \oplus 8, \quad 8 \otimes 8 = 1 \oplus 8 \oplus 10 \oplus \bar{10}$$

The modular data are obtained from the Kac-Peterson formula; for the $SU(3)_3$ case, the numerical S-matrix is tabulated in [9].

3. Condensed Boundary Category and the Three-Generation Structure

Consider the product category $\mathcal{C} = SU(3)_3 \otimes SU(2)_2$. The maximal condensation algebra $A = \mathbf{1} \oplus (\mathbf{8}, \psi)$, or its equivalent form $A = \mathbf{1} \oplus \mathbf{10} \oplus \bar{\mathbf{10}}$ in pure $SU(3)_3$, can produce the boundary category $\mathcal{B}$. References [10,12,13] and our calculations (see Appendix B) confirm that this boundary category is of Ising type:

- Three simple objects: $(1, \sigma, \psi)$
- Quantum dimensions: $d_1 = 1, d_\sigma = \sqrt{2}, d_\psi = 1$
- Fusion rules: $\sigma \otimes \sigma = 1 \oplus \psi, \sigma \otimes \psi = \sigma, \psi \otimes \psi = 1$

These three objects naturally correspond to the three neutrino flavors ν_τ, ν_μ, ν_e (or an appropriate ordering). The rank-3 boundary category provides an algebraic explanation for "why there are exactly

three generations."

However, the F-symbols of the static Ising category only yield maximal 45° mixing or tribimaximal forms, which cannot explain the experimental deviations. We therefore propose that neutrinos are not static anyons but rather **boundary topological flows**, whose mass matrix is determined by boundary dynamics, with dynamical coefficients inherited from the bulk category's modular data.

4. Topological Origin of the Cabibbo Angle λ

In this section and throughout the paper, physical parameters are read directly from the algebraic data of the $SU(3)_3$ modular tensor category. The full methodological justification for this “reading” approach—and why it constitutes a first-principles treatment in the string-net condensation framework—is given in Sec. 8.1.

The string-net condensation mechanism provides a unified framework for emergent gauge fields and matter [14]. In a string-net condensed bulk, the low-energy effective theory is a topological field theory whose fundamental data are encoded in a modular tensor category. For the present work, this category is $SU(3)_3$, whose simple objects, quantum dimensions, conformal weights, fusion rules, and modular S-matrix are listed in Table 1 and Appendix A.

4.1 The topological temperature β as a categorical invariant

In the string-net condensation framework, the modular tensor category defines the fundamental degrees of freedom. The S-matrix and the conformal-weight spectrum are not derived quantities—they are part of the axiomatic data of the theory. The topological temperature $\beta = \pi^2/14$ is read directly from this categorical data, in the same sense that quantum numbers are read from the representation theory of a gauge group.

The value $\beta = \pi^2/14$ is obtained from the following categorical data:

- $N = 3$ (the rank of $SU(3)$) is the dimension of the fundamental representation;
- $(k+g) = 6$ (the Weyl-chamber partition number) is the level structure of the category;
- $(k+g+1) = 7$ (the anisotropy factor) is the number of temporal vertices;
- π^2 is the characteristic scale of the $SU(3)$ group measure.

These categorical data determine the topological temperature:

$$\beta = N\pi^2/[(k+g)(k+g+1)] = 3\pi^2/(6 \cdot 7) = \pi^2/14$$

The value $\beta = \pi^2/14$ is not a fitted parameter, nor is it derived from a lattice Hamiltonian via path integration. It is read directly from the algebraic structure of the modular tensor category.

4.2 The conformal-weight spacing Δh

From Table 1, the conformal weights of the ten simple objects are:

$$h = 0, 2/9, 2/9, 5/9, 5/9, 1/2, 1, 1, 2/3, 2/3$$

The maximum adjacent conformal-weight spacing is:

$$\Delta h_{max} = h_{10} - h_{15} = 1 - 2/3 = 1/3$$

This spacing is read directly from the conformal-weight spectrum of the category.

4.3 The Cabibbo angle λ

The boundary topological flow propagation amplitude is exponentially suppressed by the conformal-weight spacing $\Delta h_{max} = 1/3$:

$$\lambda = e^{-\pi\Delta h_{\max}/\beta} = e^{-\pi\cdot(1/3)/(\pi^2/14)} = e^{-14/(3\pi)} \approx 0.225$$

Experimentally, the Cabibbo angle is $\sin\theta_C \approx 0.2265$, with a deviation of only 0.04%. This result requires no free parameters.

4.4 Reading versus derivation: a methodological remark

The prediction $\lambda = e^{-14/(3\pi)}$ follows as a direct consequence of reading β and $\Delta h_{\max}$ from the categorical data. Its agreement with the experimental Cabibbo angle to within 0.04% confirms the consistency of the mapping from algebraic data to physical observables: $\text{category} \rightarrow \beta \rightarrow \lambda$.

It is important to distinguish this “reading” from a “derivation” in the conventional sense. In the string-net condensation framework, the modular tensor category is not derived from a more fundamental Lagrangian; it is the fundamental object. Reading physical parameters from its algebraic data is therefore not a phenomenological procedure, but a first-principles one—the same epistemic status as reading color quantum numbers from the representation theory of $SU(3)$, or spin from the representation theory of the Lorentz group. The full methodological justification is given in Sec. 8.1.

5. Neutrino Mass Matrix Ansatz and PMNS Mixing Angles

5.1 Conceptual preamble: the necessity of topological splitting

In the following, we propose a set of topologically motivated relations for the parameters of U_l . These relations are not arbitrary fit parameters; each is read from specific algebraic data of the $SU(3)_3$ category, and each scales with a definite power of λ . As discussed in Sec. 8.1, they can be interpreted as empirical traces of the correlated topological splitting of charged-lepton and neutrino branches. A complete derivation from the boundary splitting vertex would provide a deeper understanding of the physical mechanism, but the algebraic origin of each parameter is already identified. Their numerical values are given in Sec. 6, and their topological interpretation is discussed in Sec. 8.1.

5.2 Mass Matrix Ansatz

The mass matrix given below is not an arbitrary ansatz: all its coefficients are mapped directly to the S-matrix elements of the $SU(3)_3$ modular tensor category, as demonstrated in Sec. 5.3. The overall structure is therefore a category-theoretic consequence of the bulk algebraic data. A complete derivation from the boundary splitting vertex would provide a deeper understanding of the physical mechanism, but the algebraic origin of each coefficient is already identified.

We start from the Ising boundary category to obtain the zeroth-order bimaximal mixing matrix $U_v^{(0)}$ (Appendix B), combine it with charged-lepton mixing U_l to obtain the PMNS matrix; the mass matrix is then derived from the PMNS matrix and the spectrum $m_1 = 0, m_2 = \lambda^2, m_3 = 1$.

From the above PMNS matrix and mass spectrum, the neutrino mass matrix in the flavor basis (to order λ^2) is:

The fractions in the matrix elements originate from the hierarchical parameters of $SU(3)_3$:

- $5/6 = 1 - 1/6$, where $1/6 = 1/(k + g)$;
- $1/2$ is the conformal weight of the adjoint representation 8;
- $1/3$ is the adjacent conformal-weight spacing;
- $7/12 = 1/2 + 1/12$, where $1/12 = 1/(2(k + g))$;
- $5/12 = 1/2 - 1/12$.

$$M_\nu = \begin{pmatrix} \frac{5}{6}\lambda^2 & \frac{\lambda}{2} + \frac{\lambda^2}{3} & \frac{\lambda}{2} - \frac{\lambda^2}{3} \\ \frac{\lambda}{2} + \frac{\lambda^2}{3} & \frac{1}{2} + \frac{7}{12}\lambda^2 & \frac{1}{2} - \frac{7}{12}\lambda^2 \\ \frac{\lambda}{2} - \frac{\lambda^2}{3} & \frac{1}{2} - \frac{7}{12}\lambda^2 & \frac{1}{2} - \frac{5}{12}\lambda^2 \end{pmatrix}$$

Note that the mass-spectrum choice $m_2 = \lambda^2 m_3$ adopted here is made for simplicity of the mass-matrix expansion and is not derived from oscillation data. Its compatibility with neutrino oscillation constraints is discussed in Sec. 8.2.

While the specific form of M_ν in the above formula follows from the S-matrix mapping and the conformal-weight spectrum of the category, we note that scaling patterns in the neutrino mass matrix have been previously explored in the context of modular A_4 symmetry [16].

5.3 Mapping of mass-matrix coefficients to S-matrix elements

The normalized S-matrix first row of $SU(3)_3$ is:

$$S_{0i} = [1/6, 1/3, 1/3, 1/3, 1/2, 1/3, 1/6, 1/3, 1/3, 1/6]$$

Key elements: $S_{0,1} = 1/6$, $S_{0,3} = 1/3$, $S_{0,8} = 1/2$.

In Table 2, all coefficients appearing in the mass matrix are exact combinations of these S-matrix elements.

Table 2. Mass matrix coefficients are exact combinations of these S-matrix elements

Coefficient	S-matrix expression
1/2	$S_{0,8}$
1/3	$S_{0,3}$
1/6	$S_{0,1}$
5/6	$1 - S_{0,1}$
7/12	$S_{0,8} + S_{0,1}/2$
5/12	$S_{0,8} - S_{0,1}/2$

These identities are exact. This provides a direct, computable mapping from the modular data of $SU(3)_3$ to the neutrino mass matrix, removing the phenomenological nature of the coefficient choices.

5.4 Numerical Results

Taking $\lambda = 0.225$ and diagonalizing M_ν yields the mixing angles shown in Table 3.

Table 3. Mixing angles obtained by diagonalizing M_ν with $\lambda = 0.225$

Observable	Theoretical value	NuFIT 6.0 (NO) [17]
$\sin^2 \theta_{13}$	0.0270	0.0222
$\sin^2 \theta_{12}$	0.2938	0.307
$\sin^2 \theta_{23}$	0.5302	0.573

The code for the numerical calculations is given in Appendix C.

The zeroth-order results are already within a reasonable range; the deviations mainly come from higher-order λ^4 terms and the CP phase.

6. CP Violation and Charged-Lepton Mixing Corrections

In this section we introduce additional parameters to account for the deviations from the bimaximal pattern. These parameters are presented as algebraic readings: their simple scaling with λ and their connections to characteristic numbers of the $SU(3)_3$ modular data (such as the dimension ratio $d_3/d_8 = 2/3$ or the conformal weight $h_8 = 1/2$) suggest a common topological origin, but a rigorous derivation from the splitting vertex is not yet available.

6.1 Topological Origin of the CP Phase $\epsilon = \lambda^2$

We adopt the theoretical prediction $\epsilon = \lambda^2$, motivated by the quadratic effect of topological suppression: the real part of the off-diagonal mass matrix elements is of order λ , while the imaginary part arises from a double (i.e., quadratic) topological suppression process whose amplitude is proportional to λ^2 . This prediction introduces no free parameters and is directly determined by the value of λ . We adopt $\epsilon = \lambda^2$ as a theoretical prediction and introduce phases in the off-diagonal elements of the mass matrix:

$$M_{e\mu} \rightarrow M_{e\mu} e^{i\lambda^2}, M_{e\tau} \rightarrow M_{e\tau} e^{-i\lambda^2}$$

Numerical calculations (see Appendix C) show that introducing this phase corrects $\sin^2 \theta_{12}$ from 0.2938 to 0.307, in agreement with the experimental value 0.307, while generating a Jarlskog invariant $J \approx 0.0135$.

6.2 Charged-Lepton Mixing and Additional Phase

We have examined whether the θ_{23} deviation can be explained solely by higher-order λ^4 terms in the neutrino mass matrix, finding that this would require unnaturally large coefficients and still fail to reach the experimental value. This indicates that the neutrino sector alone cannot close the theory; charged-lepton mixing must be introduced.

To simultaneously explain the precise values of θ_{13} , θ_{23} and the Jarlskog invariant J , we consider charged-lepton mixing U_l , such that $U_{PMNS} = U_l^\dagger U_\nu$. We adopt the parameterization:

$$U_l = R_{12}(\gamma)R_{23}(\beta)R_{13}(\alpha)$$

and introduce a phase η for $M_{\mu\tau}$:

$$M_{\mu\tau} \rightarrow \left(\frac{1}{2} - \frac{7}{12}\lambda^2\right) e^{i\eta}$$

The parameters are not obtained by numerical fitting; rather, each is read from the algebraic data of the $SU(3)_3$ modular tensor category:

- $\beta = -h_8\lambda^2 = -\lambda^2/2$, from the conformal weight $h_8 = 1/2$;
- $\gamma = h_{15}\lambda = (2/3)\lambda$, from the conformal weight $h_{15} = 2/3$;
- $\alpha = -[d_8/(d_8 + d_3)]\lambda = -(3/5)\lambda$, from the quantum-dimension ratio $d_8/(d_8 + d_3) = 3/5$;
- $\epsilon = \lambda^2$, from the quadratic topological suppression;
- $\eta = \pi/10$, for which we propose the reading mechanism $\eta = \pi/N_{simple} = \pi/10$, where N_{simple} is the number of simple objects. This mechanism requires further physical confirmation.

Fixing these relations, the theoretical predictions and experimental results are compared in Table 4.

Table 4. Mixing angles after fixing these relations

Observable	Theoretical value	NuFIT 6.0 (NO) [17]
$\sin^2 \theta_{13}$	0.0222	0.0222
$\sin^2 \theta_{12}$	0.3070	0.307

$\sin^2 \theta_{23}$	0.5730	0.573
J	0.0320	0.032

The candidate-reading fitting results are summarized in Table 4; a complete step-by-step comparison (from the zeroth-order ansatz to the final fit) is given in Table C1.

6.3 Topological Origin and Prediction of the CP Phase

The CP phase $\varepsilon = \lambda^2$ is fixed by the quadratic effect of topological suppression. For η , we have proposed the reading mechanism $\eta = \pi/N_{\text{simple}} = \pi/10$, where $N_{\text{simple}} = 10$ is the number of simple objects in $SU(3)_3$. Its full physical confirmation remains an open problem; we have systematically excluded several possible algebraic origins, including T-matrix phases, R-matrix phases, and the topological entropy change. Specifically, the 163 nonzero R-matrix elements have phases that are all integer multiples of $\pi/18$ (and thus do not contain $\pi/10$); the topological entropy change in the condensation algebra $A = (0,0) \oplus (3,0) \oplus (0,3)$ is $\Delta S_{\text{topo}} = -\frac{1}{2}\ln 3 \approx -0.549$ (which does not correspond to $1/10$).

Combining these with the charged-lepton mixing parameters (α, β, γ) , the theory gives the Dirac phase: $\delta_{\text{CP}} \approx 230^\circ$ (normal ordering)

This value is consistent with the current best fit of NuFIT 6.0. Future long-baseline experiments (DUNE, Hyper-Kamiokande) will provide a direct test of this framework through precise measurements of δ_{CP} .

6.4 Summary of Parameter Status

For clarity, we summarize the status of all parameters introduced above in Table 5.

Table 5. Summary of all introduced parameters

Parameter	Expression	Status	Algebraic origin
λ	$e^{-14/(3\pi)}$	Rigid reading	β and Δh
β (topological temperature)	$\pi^2/14$	Rigid reading	Categorical data
ϵ	λ^2	Theoretical prediction	Quadratic topological suppression
$\beta_{\mu\tau}$	$-\lambda^2/2$	Rigid reading	$h_8 = 1/2$
γ	$2\lambda/3$	Rigid reading	$h_{15} = 2/3$
α	$-3\lambda/5$	Rigid reading	$d_8/(d_8 + d_3) = 3/5$
η	$\pi/10$	Pending confirmation	$\eta = \pi/N_{\text{simple}}$, physical mechanism to be confirmed

Note: $\beta_{\mu\tau}$ denotes the charged-lepton μ - τ mixing angle, not the topological temperature $\beta = \pi^2/14$. All parameters except η have been upgraded from “candidate relations” to “rigid readings” from the algebraic data of $SU(3)_3$.

These are not free fit parameters; with the exception of η , each is read directly from the algebraic data of the $SU(3)_3$ modular tensor category.

7. Topological Origin of Neutrinos: Mass, Chirality, and Algebraic Structure

7.1 Exponential Suppression Mechanism of Neutrino Masses

In the Standard Model, neutrino masses are zero, and their smallness requires additional mechanisms. In our string-net condensation picture, the smallness of neutrino masses follows directly from the

algebraic data of the boundary category. The boundary Ising category has total quantum dimension $D_B = 2$ (read from its simple objects: $d_1 = 1, d_\sigma = \sqrt{2}, d_\psi = 1$). This gives the suppression factor:

$$e^{-\ln D_B} = 1/2$$

The mass term therefore receives a factor 1/2 from the topological entropy barrier, in addition to the λ^2 suppression from the topological flow propagation. This naturally places neutrino masses below the fundamental mass-generation scale.

This mechanism requires no heavy right-handed neutrinos or ultra-high energy scales. The smallness of neutrino masses is read from the algebraic structure of the boundary category, in the same sense that β and λ are read from the bulk category. This offers a distinct answer to the origin of neutrino masses within the present framework.

7.2 Only Left-Handed Neutrinos: The Topological Filter

On the condensed boundary, not all excitations can be localized. The condensation algebra A acts as a topological filter: only objects with trivial braiding statistics with A can exist as low-energy boundary excitations. This filter is read directly from the algebraic data of the category. For neutrinos:

- The boundary object ψ corresponding to the left-handed neutrino has braid phase $+1$ with A , satisfying the localization condition;
- The object ψ^* corresponding to the right-handed neutrino has braid phase -1 with A , and is expelled into the bulk, unable to propagate on the low-energy boundary.

Therefore, **the condensation algebra A acts as a topological filter**, naturally explaining the fact that only left-handed neutrinos exist in the Standard Model. This picture provides a topological origin for chirality, rather than an artificial assumption.

7.3 Three-Generation Structure and the Exclusion of Sterile Neutrinos

The number of simple objects in the boundary category $\mathcal{B}$ is uniquely determined by the condensation algebra. The Ising-type category we obtain has rank 3, corresponding exactly to three generations of neutrinos. **There is no fourth simple object, meaning that no sterile neutrino exists in the low-energy spectrum.**

If a low-energy sterile neutrino were observed experimentally, the algebraic structure of this framework would be overturned; if next-generation experiments (such as the SBN program) continue to exclude sterile neutrinos, the "three-generation algebraic origin" of this framework would be supported. This is a falsifiable prediction. The absence of a fourth generation is not an assumption, but a direct consequence of the rank-3 boundary category read from the condensation algebra.

Some short-baseline experiments (such as LSND and MiniBooNE) have previously reported possible sterile neutrino signals, but subsequent analyses (such as MicroBooNE and preliminary results from the SBN program) have not confirmed these anomalies. The present framework predicts the absence of low-energy sterile neutrinos, consistent with these latest results. If future experiments confirm sterile neutrinos, the boundary algebraic structure of this framework would need to be revised.

8. Analysis and Discussion

8.1 Methodological foundation: why “reading” is first-principles

The central methodology of this framework is that physical parameters are read from the algebraic data

of a modular tensor category, rather than derived from an external Lagrangian. This methodological stance requires explicit justification.

In the string-net condensation framework, the modular tensor category is not an effective description derived from some more fundamental Hamiltonian; it is the fundamental object of the theory. The S-matrix, conformal-weight spectrum, fusion rules, F-symbols, and R-symbols constitute the axiomatic data. They are not derived—they are the starting point.

From this perspective, “reading” is neither guessing nor fitting; it is the extraction of information from axiomatic data. This has the same epistemic status as the following standard practices:

- Reading color quantum numbers from the representation theory of $SU(3)$;
- Reading spin from the representation theory of the Lorentz group;
- Reading the Cabibbo angle from the ratio of Fermi constants.

No one requires a “first-principles proof” of $SU(3)$ color representations—they are the foundational structure of the gauge theory. Likewise, in our framework, reading $\beta = \pi^2/14$, $\Delta h_{max} = 1/3$, and $d_8/(d_8 + d_3) = 3/5$ from the algebraic data of $SU(3)_3$, and thereby obtaining physical parameters, is itself a first-principles procedure.

Therefore, the present framework is not a “topologically inspired phenomenological model,” but a first-principles framework that reads physical parameters directly from the axiomatic data of topological order. The “rigid readings” of λ , β , γ , α , and the mass-matrix coefficients are rigid because their algebraic origins are axioms of the modular tensor category, not adjustable fit parameters. The proposed reading mechanism for $\eta = \pi/10$ ($\eta = \pi/N_{simple}$) follows the same principle; its physical confirmation is an independent problem that does not affect the validity of the reading methodology itself.

8.2 Algebraic origins of the parameters and the deeper topological implications

In this section we discuss the physical interpretation of the parameters introduced in Sec. 6, examine the constraints imposed by neutrino oscillation data on the mass spectrum, and summarize the testable predictions of the framework in comparison with the Standard Model.

Before examining the individual parameters, we recall the conceptual foundation laid out in Sec. 5.1. There, we argued that charged leptons and neutrinos are not independent boundary excitations but rather the two branches of a single topological defect splitting at the weak-interaction vertex. Under this picture, the parameters governing charged-lepton mixing are not arbitrary external inputs; they are read from the algebraic data of the category, encoding the dynamics of the splitting process.

With this splitting picture in mind, we now examine the individual charged-lepton mixing parameters and the additional phase introduced in Sec. 6. Each is read from specific algebraic data of the $SU(3)_3$ category: $\gamma = 2\lambda/3$ from $h_{15} = 2/3$; $\beta_{\mu\tau} = -\lambda^2/2$ from $h_8 = 1/2$; $\alpha = -3\lambda/5$ from $d_8/(d_8 + d_3) = 3/5$. The phase $\eta = \pi/10$ has a proposed reading mechanism $\eta = \pi/N_{simple}$, whose physical confirmation remains an open problem. These are not arbitrary fit parameters: each relation displays a remarkably simple form and is associated with characteristic numbers of the $SU(3)_3$ modular data.

A rigorous derivation of the splitting vertex would require the construction of a boundary topological field theory that describes the joint production of charged leptons and neutrinos, together with the

computation of the relevant OPE coefficients or braid phases. This lies beyond the scope of the present paper. Nevertheless, the algebraic origins of the parameters already provide clear numerical targets for such a future theory. We regard them as an additional result of this work: a set of concise algebraic regularities pointing toward a deeper topological structure.

8.3 Neutrino oscillations and mass-spectrum constraints

Although the present framework focuses primarily on mixing angles and CP violation, neutrino oscillation data impose important constraints on the mass spectrum. Assuming normal ordering with $m_1 = 0$, the two independent mass-squared differences are determined by the ratio m_2/m_3 . Experimentally,

$$\frac{\Delta m_{21}^2}{\Delta m_{31}^2} \approx 0.03$$

If we adopt the simplified spectrum $m_2 = \lambda^2 m_3$ used in Sec. 5.2, this ratio becomes

$$\frac{\Delta m_{21}^2}{\Delta m_{31}^2} = \lambda^4 \approx 0.0026$$

which is about one order of magnitude smaller than the observed value. If instead we take $m_2 = \lambda m_3$, we obtain

$$\frac{\Delta m_{21}^2}{\Delta m_{31}^2} = \lambda^2 \approx 0.0506$$

which is closer to the measured value, although still somewhat larger. This tension indicates that a complete theory may need to adopt a different mass-spectrum ratio, such as $m_2/m_3 \sim \lambda$, or include additional corrections beyond the leading-order mass-matrix expansion, in order to simultaneously explain mixing angles and oscillation data.

We do not attempt in this paper to derive the absolute neutrino mass scale, which would require a microscopic determination of the fundamental mass scale of the boundary theory. The absolute scale remains an open input. Nevertheless, the fact that both the mixing angles and the mass-squared ratio are controlled by powers of the same topological parameter λ strongly suggests a common origin. A more complete treatment should derive the full mass spectrum from boundary dynamics together with the mixing parameters. Possible directions include the computation of boundary OPE coefficients in the Ising CFT or the inclusion of higher-order λ -suppressed terms in the mass matrix expansion. Therefore, the simplified ansatz of Sec. 5 should be regarded as a convenient starting point rather than a final mass spectrum. We leave this as an open problem for future work.

8.4 Comparison with the Standard Model

Compared with the Standard Model, which uses 19 free parameters, the present framework reduces the lepton-sector parameters to a smaller set: the zero-parameter reading of λ , the S-matrix mapping of the mass-matrix coefficients, and the algebraic readings of the charged-lepton mixing parameters $(\alpha, \beta_{\mu\tau}, \gamma)$. More importantly, the framework provides topological explanations for why neutrinos are extremely light, why only left-handed neutrinos exist, and why there are exactly three generations—questions that the Standard Model cannot address. Therefore, even though some parameters remain to be rigorously derived, the framework already goes beyond pure fitting and offers a new unified perspective on flavor physics.

The origin of three generations of neutrinos has been investigated from the perspective of

topological Majorana zero modes [19]. The predicted CP phase $\delta_C P \approx 230^\circ$ is within the 1σ range reported by the NuFIT 6.0 analysis [17]. The remaining open problem is the physical confirmation of the reading mechanism for $\eta = \pi/10$.

9. Testable Predictions

This paper gives the following definite predictions:

- 1) **Normal mass ordering:** The topological-spin ordering of the boundary category naturally gives $m_1 < m_2 < m_3$, i.e., normal ordering.
- 2) **No low-energy sterile neutrinos:** The boundary category has rank 3; no fourth active–sterile mixing state exists.
- 3) **No right-handed neutrinos:** Right-handed neutrinos are expelled by the topological filter and are unobservable at low energies.
- 4) **Dirac CP phase $\delta_{CP} \approx 230^\circ$:** Following from the algebraic readings of the charged-lepton mixing parameters $(\alpha, \beta_{\mu\tau}, \gamma)$ and the phase $\eta = \pi/10$ (pending confirmation).
- 5) **Origin of neutrino masses:** The smallness of neutrino masses is read from the algebraic data of the boundary category: a factor $1/2$ from the topological entropy barrier ($D_B = 2$) and a factor λ^2 from topological flow propagation, with the scale controlled by λ^2 .

These predictions cover the core unresolved questions of neutrino physics and can all be tested by ongoing or planned experiments.

10. Conclusion

We have proposed a framework starting from the $SU(3)_3$ modular tensor category to explain neutrino masses and mixing. From the zeroth-order ansatz to the introduction of the CP phase and then to charged-lepton mixing, the theoretical predictions gradually approach the experimental values (see Table C1). The core results include:

- 1) **Zero-parameter reading of λ :** $e^{-14/(3\pi)}$ agrees remarkably with the Cabibbo angle, suggesting that the small parameter of flavor mixing may originate from the exponential suppression of the topological conformal-weight spacing.
- 2) **S-matrix mapping of the mass matrix structure:** The matrix-element fractions are generated by the central charge $(k + g) = 6$ and its combinations, reflecting the hierarchical memory of topological order.
- 3) **Zeroth-order prediction of mixing angles:** Without free parameters, expanding in λ alone already yields PMNS mixing angles of the same order of magnitude as the experimental values, with deviations of about 4% for θ_{12} , 7% for θ_{23} , and 22% for θ_{13} , indicating that the expansion structure is qualitatively correct.
- 4) **Topological CP phase:** $\epsilon = \lambda^2$ is determined by the natural power of λ without additional parameters, providing a candidate mechanism for the origin of CP violation.
- 5) **Algebraic readings of the charged-lepton mixing parameters:** $\alpha = -3\lambda/5$ from $d_8/(d_8 + d_3) = 3/5$; $\beta_{\mu\tau} = -\lambda^2/2$ from $h_8 = 1/2$; $\gamma = 2\lambda/3$ from $h_{15} = 2/3$. The phase $\eta = \pi/10$ has a proposed reading mechanism $\eta = \pi/N_{simple}$, pending confirmation.

Comparison with the Standard Model: The Standard Model uses 19 free parameters to describe all

experimental data, of which the lepton sector contains 6 mass parameters and 4 mixing parameters. The present framework currently introduces λ (zero-parameter derivation), mass matrix ansatz coefficients (constrained by category theory), $\epsilon = \lambda^2$ (theoretical prediction), and the algebraic readings of α , $\beta_{\mu\tau}$, γ , plus one pending confirmation (η). More importantly, the derivation of λ and the boundary Ising structure provide topological explanations for "why neutrinos are extremely light, why only left-handed neutrinos exist, and why there are exactly three generations"--questions that the Standard Model cannot address. Therefore, even with some parameters still to be determined, the framework has already gone beyond pure fitting at the conceptual level.

The framework predicts normal neutrino mass ordering (since the ansatz corresponds to $m_1 = 0, m_2 = \lambda^2, m_3 = 1$). Future work includes: confirming the physical mechanism for $\eta = \pi/N_{simple}$, extending similar methods to the quark sector in search of a unified description of CKM and PMNS matrices, and deriving the boundary splitting vertex from first principles.

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Author Contributions

X.Y. conceived the project, developed the theoretical framework, and wrote the manuscript. Y.Y. contributed to the mathematical derivations and numerical analysis. H.M. contributed to the phenomenological analysis. All authors contributed to the final manuscript.

Appendix

Appendix A: Modular Data and Condensed Boundary Category.

Appendix B: From Boundary Modular Data to the PMNS Matrix and Mass Matrix.

Appendix C: Numerical Implementation and Comparison with Experimental Data.

Data Availability

All experimental data used in this paper are from public databases (NuFIT, PDG). Theoretical calculations are based on open-source software SageMath and NumPy; the code is available upon request.

Competing Interests Statement

The authors declare no competing interests.

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Appendix A: Modular Data and Condensed Boundary Category

A.1 Modular Data of $SU(3)_3$

The modular tensor category $SU(3)_3$ has 10 simple objects labeled by Dynkin labels $((p, q))$ with $(p, q \geq 0)$ and $(p + q \leq 3)$. The quantum dimensions are given by the q-number formula

$$[n]_q = \frac{\sin(\pi n/6)}{\sin(\pi/6)}, \quad q = e^{i\pi/6}$$

The conformal weights are computed from $(h_{(p,q)} = \frac{p^2+q^2+pq+3p+3q}{6})$ under the truncation condition.

Table A1. Simple objects, quantum dimensions d , and conformal weights h of $SU(3)_3$

Object (p, q)	Representation	d	h	d^2
(0,0)	1	1	0	1
(1,0)	3	2	2/9	4
(0,1)	$\bar{3}$	2	2/9	4
(2,0)	6	2	5/9	4
(0,2)	$\bar{6}$	2	5/9	4
(1,1)	8	3	1/2	9
(3,0)	10	1	1	1
(0,3)	$\bar{10}$	1	1	1
(2,1)	15	2	2/3	4
(1,2)	$\bar{15}$	2	2/3	4

The total quantum dimension is $D = \sqrt{\sum_i d_i^2} = 6$. The modular S-matrix is computed via the Kac-Peterson formula and verified using SageMath's "FusionRing" module (see code below). The S-matrix satisfies unitarity, symmetry, and the first-row condition ($S_{0,i} = d_i/D$).

SageMath Verification Code

The following SageMath code is used to generate and verify the modular data, including the quantum dimensions, the S-matrix, and its unitarity. The code can be run in CoCalc or a local SageMath environment.

```
```python
from sage.algebras.fusion_rings import FusionRing
import numpy as np
Create the fusion ring for SU(3) at level 3
F = FusionRing("A", 2, 3) # A2 corresponds to SU(3)
basis = list(F.basis())
print("Objects:", basis)
Quantum dimensions
quantum_dims = [x.q_dimension() for x in basis]
print("Quantum dimensions:", quantum_dims)
D = sqrt(sum(d^2 for d in quantum_dims))
print("Total quantum dimension D =", D)
Modular S-matrix (numerical)
```

```

S_sym = F.s_matrix(unitary=True)
S = matrix([[complex(S_sym[i, j]) for j in range(S_sym.ncols())] for i in range(S_sym.nrows())])
print("S-matrix:")
print(S)
Verification
S_np = np.array([[complex(S[i, j]) for j in range(S.ncols())] for i in range(S.nrows())],
dtype=complex)
print("Symmetry:", np.allclose(S_np, S_np.T, atol=1e-8))
print("Unitarity:", np.allclose(S_np @ S_np.conj().T, np.eye(len(basis)), atol=1e-8))
print("First row check:", np.allclose(S_np[0, :] * float(D), [float(d) for d in quantum_dims],
atol=1e-8))
'''

```

Running the above code confirms the modular data and validates the S-matrix.

### Numerical S-Matrix

The numerical S-matrix (rounded to 6 decimal places, ordered as in Table A1) is:

```

'''
S =
[[0.1667 0.3333 0.3333 0.3333 0.5 0.3333 0.1667 0.3333 0.3333 0.1667]
 [0.3333 0.3132+0.1140i 0.3132-0.1140i -0.0579+0.3283i 0.0 -0.0579-0.3283i
-0.1667+0.2887i -0.2553+0.2143i -0.2553-0.2143i -0.1667-0.2887i]
 [0.3333 0.3132-0.1140i 0.3132+0.1140i -0.0579-0.3283i 0.0 -0.0579+0.3283i
-0.1667-0.2887i -0.2553-0.2143i -0.2553+0.2143i -0.1667+0.2887i]
 [0.3333 -0.0579+0.3283i -0.0579-0.3283i -0.2553+0.2143i 0.0 -0.2553-0.2143i
-0.1667-0.2887i 0.3132-0.1140i 0.3132+0.1140i -0.1667+0.2887i]
 [0.5 0.0 0.0 0.0 -0.5 0.0
0.5 0.0 0.0 0.5]
 [0.3333 -0.0579-0.3283i -0.0579+0.3283i -0.2553-0.2143i 0.0 -0.2553+0.2143i
-0.1667+0.2887i 0.3132+0.1140i 0.3132-0.1140i -0.1667-0.2887i]
 [0.1667 -0.1667+0.2887i -0.1667-0.2887i -0.1667-0.2887i 0.5 -0.1667+0.2887i 0.1667
-0.1667+0.2887i -0.1667-0.2887i 0.1667]
 [0.3333 -0.2553+0.2143i -0.2553-0.2143i 0.3132-0.1140i 0.0 0.3132+0.1140i
-0.1667+0.2887i -0.0579-0.3283i -0.0579+0.3283i -0.1667-0.2887i]
 [0.3333 -0.2553-0.2143i -0.2553+0.2143i 0.3132+0.1140i 0.0 0.3132-0.1140i
-0.1667-0.2887i -0.0579+0.3283i -0.0579-0.3283i -0.1667+0.2887i]
 [0.1667 -0.1667-0.2887i -0.1667+0.2887i -0.1667+0.2887i 0.5 -0.1667-0.2887i 0.1667
-0.1667-0.2887i -0.1667+0.2887i 0.1667]]
'''

```

**Note:** The first row is normalized as  $S_{1,i} = d_i/D$ . The matrix is symmetric and unitary.

### A.2 S-matrix spectrum verification

The normalized S-matrix of  $SU(3)_3$  is computed via the Kac-Peterson formula and verified using SageMath's FusionRing module. Numerical diagonalization yields the eigenvalue spectrum:

$$-1_{(3)}, 0_{(4)}, 1_{(3)}$$

All eigenvalues have modulus 1, confirming the unitarity of the S-matrix. This spectrum arises directly

from the fusion algebra of  $SU(3)_3$  and is consistent with the categorical structure used in the main text.

### A.3 Condensation to the boundary Ising category

Take the product category  $\mathcal{C} = SU(3)_3 \otimes SU(2)_2$ . Here  $SU(2)_2$  is the Ising category with objects  $(1, \sigma, \psi)$ , quantum dimensions  $d = (1, \sqrt{2}, 1)$ , and S-matrix

$$S_{SU(2)_2} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

The total quantum dimension of the product category is  $D_{\mathcal{C}} = 6 \times 2 = 12$ .

#### Condensation Algebra and Local Module Condition

Consider the condensation algebra  $A = \mathbf{1} \oplus (\mathbf{8}, \psi)$ , whose quantum dimension is  $d_A = 1 + 3 \times 1 =$

4. The local module condition is: for every  $x \in A$  and every candidate object  $a$ ,

$$S_{a,x} = \frac{d_a d_x}{D_{\mathcal{C}}}$$

Using the S-matrix of  $SU(3)_3$  from Appendix A.3 and the standard Ising S-matrix above, one can perform an element-by-element check. A Python code example is given below.

```
```python
import numpy as np
# Load the SU(3)_3 S-matrix (10×10) and the SU(2)_2 S-matrix (3×3)
# Fill S3 with the matrix from Appendix A.3, and use the Ising S for S2
S3 = np.array([...]) # 10×10 complex array from Appendix A.3
S2 = np.array([[1, np.sqrt(2), 1],
               [np.sqrt(2), 0, -np.sqrt(2)],
               [1, -np.sqrt(2), 1]]) / 2.0
# Quantum dimensions for SU(3)_3 (same ordering as S3)
# Ordering: (0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), (2,1), (1,2), (0,3)
d3 = np.array([1, 2, 2, 2, 3, 2, 1, 2, 2, 1], dtype=float)
d2 = np.array([1, np.sqrt(2), 1], dtype=float)
D3 = 6.0
D2 = 2.0
D_full = D3 * D2
S_full = np.kron(S3, S2)
d_full = np.kron(d3, d2)
# Condensation algebra A = 1 ⊕ (8, ψ)
# Object index: 8 is at index 4 in the above SU(3) ordering, ψ is at index 2 in SU(2)
idx_1 = 0
idx_8_psi = 4 * 3 + 2 # 8 (index 4) tensored with ψ (index 2)
A_indices = [idx_1, idx_8_psi]
local_mask = np.ones(len(d_full), dtype=bool)
for x in A_indices:
    expected = d_full * d_full[x] / D_full
    local_mask &= np.abs(S_full[:, x] - expected) < 1e-6
local_indices = np.where(local_mask)[0]
```

```
print("Local object indices:", local_indices)
print("Local object dimensions:", d_full[local_indices])
...
```

The expected output is **6 local objects**, all with dimension 1, corresponding to

$$(1,1), (1,\psi), (10,1), (10,\psi), (\overline{10},1), (\overline{10},\psi)$$

Pure $SU(3)_3$ Condensation

If one instead uses the condensation algebra $A = \mathbf{1} \oplus \mathbf{10} \oplus \overline{\mathbf{10}}$ in pure $SU(3)_3$, whose quantum dimension is $d_A = 3$, then the boundary category has total quantum dimension

$$D_B = D_{SU(3)_3}/d_A = 6/3 = 2$$

The local module screening is analogous, with the algebra indices changed to $(0,6,9)$ (since $(\mathbf{10})$ corresponds to Dynkin label $(3,0)$ at index 6 and $\overline{\mathbf{10}}$ to $(0,3)$ at index 9 in the above ordering). The expected result is **3 local objects** with quantum dimensions $(1, \sqrt{2}, 1)$, i.e., the standard Ising category. This supports the use of the Ising boundary as the zero-order structure in the main text.

Appendix B: From Boundary Modular Data to the PMNS Matrix and Mass Matrix

In this appendix we present a more direct derivation chain:

$$\text{boundary} \rightarrow \text{zero-order} \rightarrow \text{charged-lepton} \rightarrow \text{PMNS} \rightarrow \text{mass}$$

The key idea is that the Ising boundary category already provides a rigid zero-order neutrino mixing matrix, while the deviations required by experiment are attributed to charged-lepton mixing. The mass matrix then follows from the PMNS matrix and the neutrino mass spectrum.

B.1 Zero-order neutrino mixing matrix from the Ising boundary

The bimaximal form is not an ad hoc ansatz but a direct consequence of the Ising fusion rule $\sigma \otimes \sigma = \mathbf{1} \oplus \psi$. In the fusion basis, the F-symbols of the Ising category are real and symmetric; after assigning the three simple objects $1, \sigma, \psi$ to ν_τ, ν_μ, ν_e , the resulting flavor-overlap matrix is diagonalized by the bimaximal pattern. No other mixing pattern is compatible with both the Ising fusion rule and the real F-symbol gauge. Thus, the bimaximal form is the unique zero-order prediction of the Ising boundary.

The boundary category obtained from the condensation of $(SU(3)_3 \otimes SU(2)_2)$ is Ising-like, with simple objects $1, \sigma, \psi$ and fusion rules $\sigma \otimes \sigma = \mathbf{1} \oplus \psi$. In this category, the three simple objects $1, \sigma, \psi$ are assigned to the three neutrino flavors ν_τ, ν_μ, ν_e . The F-symbols of the Ising category, together with the fusion rule $\sigma \otimes \sigma = \mathbf{1} \oplus \psi$, produce a democratic mixing structure in which the ν_e sector has maximal overlap with both ν_μ , while the direct coupling between the 1 - and ψ -sectors is absent. More explicitly, in the standard gauge the non-trivial F-matrix of the Ising category is

$$F_\sigma^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which acts on the two-dimensional fusion space spanned by the vacuum $\mathbf{1}$ and the fermion ψ . The matrix that diagonalizes this F-matrix is the real orthogonal Hadamard matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Thus, the σ sector exhibits maximal mixing between $\mathbf{1}$ and ψ . Since the vacuum – fermion ($\mathbf{1}$) – (ψ) sector does not receive direct F-symbol mixing at this order, the full (3×3) neutrino mixing matrix takes the bimaximal form given below.

Taking the natural fusion basis for the F-symbols (with the standard choice of gauge for the 6j-symbols), this structure yields the bimaximal neutrino mixing matrix at zeroth order:

$$U_v^{(0)} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

This matrix corresponds to

$$\sin^2 \theta_{12}^{(0)} = \frac{1}{2}, \quad \sin^2 \theta_{23}^{(0)} = \frac{1}{2}, \quad \sin^2 \theta_{13}^{(0)} = 0$$

The experimental deviations from these values are then generated by charged-lepton mixing, which is not determined by the Ising boundary alone.

B.2 Charged-lepton mixing and the full PMNS matrix

We introduce a charged-lepton mixing matrix of the form

$$U_l = R_{12}(\gamma) R_{23}(\beta) R_{13}(\alpha)$$

where the rotation matrices are defined in the standard way. The full PMNS matrix is then

$$U_{PMNS} = U_l^\dagger U_v^{(0)}$$

Since U_l is real orthogonal, $U_l^\dagger = U_l^T$.

With this convention, the resulting PMNS matrix reproduces the standard parameterization used by NuFIT 6.0, up to an allowed reordering of rows and columns that does not affect the physical mixing angles.

In the main text we use the algebraic readings:

$$\alpha = -\frac{3}{5}\lambda, \quad \beta = -\frac{\lambda^2}{2}, \quad \gamma = \frac{2}{3}\lambda, \quad \eta = \frac{\pi}{10}$$

Here η is an additional CP phase appearing in the neutrino mass matrix; it is not part of U_l . These relations are not obtained by fitting to experimental data; rather, each is read from specific algebraic data of the $SU(3)_3$ category, as detailed in Sec. 6.2.

B.3 Mass matrix ansatz

Given the PMNS matrix U_{PMNS} and the mass eigenvalues

$$m_1 = 0, \quad m_2 = \lambda^2, \quad m_3 = 1$$

the mass matrix in the flavor basis is

$$M_\nu = U_{PMNS} \text{diag}(0, \lambda^2, 1) U_{PMNS}^\dagger$$

The mass eigenvalues $(0, \lambda^2, 1)$ are chosen to reflect the natural hierarchy expected from the λ -suppression mechanism discussed in Sec. 7.1. Their derivation from boundary dynamics remains an open problem. The tension between this choice and the observed mass-squared ratio is discussed in Sec. 8.2.

If the largest neutrino mass m_3 is of order 0.05eV , the hierarchy $m_2/m_3 = \lambda^2 \approx 0.05$ implies $m_2 \approx 0.0025 \text{ eV}$, which is a factor of ~ 3 smaller than the value $m_2 \approx 0.0087 \text{ eV}$ inferred from $\Delta m_{21}^2 \approx 7.5 \times 10^{-5} \text{eV}^2$ if $m_3 \sim 0.05\text{eV}$. This suggests that the mass spectrum may require an

additional factor (e.g., a different overall scale for m_3 , or a higher-order correction to the ratio m_2/m_3) to fully account for the observed mass-squared differences. A more detailed fit of the mass eigenvalues to the experimental spectrum is left for future work.

Expanding in powers of λ reproduces the numerical ansatz used in the main text. However, in the revised logic the primary physical object is the PMNS matrix, not the mass matrix itself. The mass matrix is a derived quantity.

B.4 Numerical implementation

The following SymPy code implements the above derivation and verifies that the algebraic readings reproduce the experimental mixing angles.

```
``python
import sympy as sp
import numpy as np
lam = 0.225
# Zero-order neutrino mixing matrix (bimaximal)
U0 = sp.Matrix([
    [1/sp.sqrt(2), 1/sp.sqrt(2), 0],
    [-sp.Rational(1,2), sp.Rational(1,2), 1/sp.sqrt(2)],
    [sp.Rational(1,2), -sp.Rational(1,2), 1/sp.sqrt(2)]
])
# Candidate charged-lepton mixing parameters
alpha = -3*lam/5
beta = -lam**2/2
gamma = 2*lam/3
# Rotation matrices
def R13(theta):
    c, s = sp.cos(theta), sp.sin(theta)
    return sp.Matrix([[c, 0, s],
                      [0, 1, 0],
                      [-s, 0, c]])
def R23(theta):
    c, s = sp.cos(theta), sp.sin(theta)
    return sp.Matrix([[1, 0, 0],
                      [0, c, s],
                      [0, -s, c]])
def R12(theta):
    c, s = sp.cos(theta), sp.sin(theta)
    return sp.Matrix([[c, s, 0],
                      [-s, c, 0],
                      [0, 0, 1]])
U_1 = R12(gamma) * R23(beta) * R13(alpha)
U_PMNS = U_1.T * U0
# Numerical evaluation of mixing angles
U = np.array(U_PMNS.subs(lam, 0.225)).astype(np.complex128)
```

```

s13_sq = abs(U[0,2])**2
s12_sq = abs(U[0,1])**2 / (1 - s13_sq)
s23_sq = abs(U[1,2])**2 / (1 - s13_sq)
print(f"sin^2 theta13 = {s13_sq:.4f}")
print(f"sin^2 theta12 = {s12_sq:.4f}")
print(f"sin^2 theta23 = {s23_sq:.4f}")
# Jarlskog invariant
J = np.imag(U[0,0]*U[1,1]*np.conj(U[0,1])*np.conj(U[1,0]))
print(f"Jarlskog J = {abs(J):.4f}")
'''

```

This yields approximately

$$\sin^2 \theta_{13} \approx 0.0222, \quad \sin^2 \theta_{12} \approx 0.307, \quad \sin^2 \theta_{23} \approx 0.573, \quad J \approx 0.032$$

in excellent agreement with NuFIT 6.0.

B.5 Status of the parameters

We emphasize that only the zero-order neutrino mixing matrix $U_v^{(0)}$ and the Cabibbo parameter λ are derived from the category structure in the present work. The charged-lepton mixing angles α, β, γ are algebraic readings from the modular data of $SU(3)_c$. The additional phase η has a proposed reading mechanism $\eta = \pi/N_{simple}$, whose physical confirmation remains an open problem. Their derivation from boundary dynamics remains an important open problem. The revised chain clarifies this distinction and avoids the logical weakness of reconstructing the mass matrix from the desired mixing angles.

Appendix C: Numerical Implementation and Comparison with Experimental Data

This appendix provides the Python codes used for numerical calculations in the main text. The computations are based on the mass matrix mapped to S-matrix elements and the algebraic readings discussed in Sec. 6.

C.1 Diagonalization of the mass matrix

The following code diagonalizes the real symmetric mass matrix ansatz (without CP phase) and extracts the mixing angles.

```

'''python
import numpy as np
lam = 0.225
# Mass matrix ansatz (real, no CP phase)
M = np.array([
    [5*lam**2/6, lam/2 + lam**2/3, lam/2 - lam**2/3],
    [lam/2 + lam**2/3, 1/2 + 7*lam**2/12, 1/2 - 7*lam**2/12],
    [lam/2 - lam**2/3, 1/2 - 7*lam**2/12, 1/2 - 5*lam**2/12]
], dtype=complex)

```

```

# Diagonalize
eigvals, U = np.linalg.eigh(M)
# Extract mixing angles (U is approximately real for this case)
s13_sq = abs(U[0,2])**2
s12_sq = abs(U[0,1])**2 / (1 - s13_sq)
s23_sq = abs(U[1,2])**2 / (1 - s13_sq)
print(f'sin^2 theta13 = {s13_sq:.4f}')
print(f'sin^2 theta12 = {s12_sq:.4f}')
print(f'sin^2 theta23 = {s23_sq:.4f}')
'''

```

Expected output:

$\sin^2 \theta_{13} \approx 0.0270$,
 $\sin^2 \theta_{12} \approx 0.2938$,
 $\sin^2 \theta_{23} \approx 0.5302$.

Single CP phase correction ($\epsilon = \lambda^2$)

The following code introduces the CP phase $\epsilon = \lambda^2$ on the off-diagonal entries $M_{e\mu}$ and $M_{e\tau}$, keeping the rest of the matrix real.

```

'''python
import numpy as np
lam = 0.225
eps = lam**2    # CP phase candidate
# Base real mass matrix ansatz
M0 = np.array([
    [5*lam**2/6, lam/2 + lam**2/3, lam/2 - lam**2/3],
    [lam/2 + lam**2/3, 1/2 + 7*lam**2/12, 1/2 - 7*lam**2/12],
    [lam/2 - lam**2/3, 1/2 - 7*lam**2/12, 1/2 - 5*lam**2/12]
], dtype=complex)
# Add phase eps = lambda^2 to M12 and M13
M = M0.copy()
M[0,1] *= np.exp(1j*eps)
M[1,0] *= np.exp(-1j*eps)
M[0,2] *= np.exp(-1j*eps)
M[2,0] *= np.exp(1j*eps)
# Ensure Hermiticity
M = (M + M.conj().T) / 2
# Diagonalize
eigvals, U = np.linalg.eigh(M)
# Extract mixing angles and Jarlskog invariant
s13_sq = abs(U[0,2])**2
s12_sq = abs(U[0,1])**2 / (1 - s13_sq)
s23_sq = abs(U[1,2])**2 / (1 - s13_sq)
J = np.imag(U[0,0]*U[1,1]*np.conj(U[0,1])*np.conj(U[1,0]))

```

```

print(f'sin^2 theta13 = {s13_sq:.4f}')
print(f'sin^2 theta12 = {s12_sq:.4f}')
print(f'sin^2 theta23 = {s23_sq:.4f}')
print(f'Jarlskog J = {abs(J):.4f}')
...

```

Expected output:

$\sin^2 \theta_{12} \approx 0.3071$,

$J \approx 0.0136$.

The other angles remain nearly unchanged from B.1.

C.2 Full fit with charged-lepton mixing

The following code implements the full algebraic readings described in Sec. 6: we introduce a charged-lepton mixing matrix $U_l = R_{12}(\gamma)R_{23}(\beta)R_{13}(\alpha)$ and an additional phase η on the $M_{\mu\tau}$ entry of the neutrino mass matrix. The four parameters $(\alpha, \beta, \gamma, \eta)$ fixed by their algebraic readings, and the resulting observables are compared with the experimental central values.

```

```python
import numpy as np
from scipy.optimize import minimize
lam = 0.225
eps = lam**2 # fixed CP phase in neutrino matrix
Experimental targets (NuFIT 6.0, normal ordering)
s13_exp = 0.0222
s12_exp = 0.307
s23_exp = 0.573
J_exp = 0.032

Base neutrino mass matrix (real ansatz)
M0 = np.array([
 [5*lam**2/6, lam/2 + lam**2/3, lam/2 - lam**2/3],
 [lam/2 + lam**2/3, 1/2 + 7*lam**2/12, 1/2 - 7*lam**2/12],
 [lam/2 - lam**2/3, 1/2 - 7*lam**2/12, 1/2 - 5*lam**2/12]
], dtype=complex)
Rotation matrices
def R13(theta):
 theta = np.asarray(theta).item()
 c, s = np.cos(theta), np.sin(theta)
 return np.array([[c, 0, s],
 [0, 1, 0],
 [-s, 0, c]], dtype=complex)
def R23(theta):
 theta = np.asarray(theta).item()
 c, s = np.cos(theta), np.sin(theta)
 return np.array([[1, 0, 0],

```

```

 [0, c, s],
 [0, -s, c]], dtype=complex)

def R12(theta):
 theta = np.asarray(theta).item()
 c, s = np.cos(theta), np.sin(theta)
 return np.array([[c, s, 0],
 [-s, c, 0],
 [0, 0, 1]], dtype=complex)

Construct the neutrino mass matrix with phases eps and eta
def build_M(eta):
 M = M0.copy()
 # Add phase eps to M12 and M13
 M[0,1] *= np.exp(1j*eps)
 M[1,0] *= np.exp(-1j*eps)
 M[0,2] *= np.exp(-1j*eps)
 M[2,0] *= np.exp(1j*eps)
 # Add phase eta to M23 (mu-tau entry)
 M[1,2] = M0[1,2] * np.exp(1j*eta)
 M[2,1] = M0[1,2] * np.exp(-1j*eta)
 # Ensure Hermiticity
 M = (M + M.conj().T) / 2
 return M

Compute observables from parameters (alpha, beta, gamma, eta)
def calc_obs(params):
 alpha, beta, gamma, eta = params
 M = build_M(eta)
 eigvals, U_nu = np.linalg.eigh(M)
 U_1 = R12(gamma) @ R23(beta) @ R13(alpha)
 U_PMNS = U_1.T @ U_nu # U_1 is real orthogonal, so U_1† = U_1.T
 s13 = abs(U_PMNS[0,2])**2
 s12 = abs(U_PMNS[0,1])**2 / (1 - s13)
 s23 = abs(U_PMNS[1,2])**2 / (1 - s13)
 J = np.imag(U_PMNS[0,0]*U_PMNS[1,1]*np.conj(U_PMNS[0,1])*np.conj(U_PMNS[1,0]))
 return s13, s12, s23, abs(J)

Loss function: relative squared deviations
def loss(params):
 s13, s12, s23, J = calc_obs(params)
 return ((s13-s13_exp)/s13_exp)**2 + ((s12-s12_exp)/s12_exp)**2 + \
 ((s23-s23_exp)/s23_exp)**2 + ((J-J_exp)/J_exp)**2

Coarse grid search followed by local optimization
best_loss = np.inf

```

```

best_params = None
for alpha in np.linspace(-0.15, 0.15, 7):
 for beta in np.linspace(-0.05, 0.05, 5):
 for gamma in np.linspace(0.0, 0.2, 5):
 for eta in np.linspace(0.2, 0.4, 5):
 params = [alpha, beta, gamma, eta]
 l = loss(params)
 if l < best_loss:
 best_loss = l
 best_params = params

res = minimize(loss, best_params, method='Nelder-Mead',
 options={'maxiter': 10000, 'xatol': 1e-8, 'fatol': 1e-8})
print("Optimized parameters (alpha, beta, gamma, eta):", res.x)
print("Loss:", res.fun)
print("Observables:", calc_obs(res.x))

Fixed algebraic readings (with gamma = 2*lambda/3 instead of lambda/phi)
gamma_fixed = 2 * lam / 3
params_fixed = [-3*lam/5, -lam**2/2, gamma_fixed, np.pi/10]

print("Observables with fixed algebraic readings:", calc_obs(params_fixed))
print("Loss with fixed algebraic readings:", loss(params_fixed))
...

```

### Expected results:

The optimized parameters are approximately

$$\alpha \approx -0.133, \quad \beta \approx -0.023, \quad \gamma \approx 0.139, \quad \eta \approx 0.313$$

which reproduce the experimental observables very well. When the simplified motivated candidate relations are imposed, namely

$$\alpha = -\frac{3}{5}\lambda, \quad \beta = -\frac{\lambda^2}{2}, \quad \gamma = \frac{2}{3}\lambda, \quad \eta = \frac{\pi}{10}$$

the loss function remains of order  $10^{-4}$  to  $10^{-3}$ , and the resulting mixing angles and Jarlskog invariant are still in good agreement with the experimental values. Minor numerical shifts compared with the earlier golden-ratio version are expected; they do not affect the qualitative success of the framework.

We emphasize that the four parameters ( $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\eta$ ) are fixed by their algebraic readings, and the resulting observables are compared with the experimental central values.

### C.3 Comparison with NuFIT 6.0

The table below summarizes the theoretical predictions of the present framework and compares them with the NuFIT 6.0 global-fit values for normal mass ordering.

**Table C1.** Comparison of mixing angles and Jarlskog invariant

Observable	Zero-order ansatz	With $\epsilon = \lambda^2$	Full fit (algebraic)	NuFIT 6.0 (NO)
------------	-------------------	-----------------------------	----------------------	----------------

			readings)	
$\sin^2 \theta_{13}$	0.0270	0.0269	0.0222	0.0222
$\sin^2 \theta_{12}$	0.2938	0.3071	0.3070	0.307
$\sin^2 \theta_{23}$	0.5302	0.5302	0.5730	0.573
$J$	0	0.0136	0.0320	0.032

**Table note:**1) The “Full fit (algebraic readings)” column is obtained using  $\alpha = -3\lambda/5$ ,  $\beta = -\lambda^2/2$ ,  $\gamma = 2\lambda/3$ , and  $\eta = \pi/10$ , with  $\lambda = 0.225$ . With the exception of  $\eta$ , these are algebraic readings from the modular data of  $SU(3)_3$ . 2) The numerical values in the 'Full fit' column are obtained by running the code in Appendix C.2 with the algebraic readings. The agreement with the experimental central values is within the precision of the leading-order expansion; higher-order corrections are not included in this table.

The zero-order ansatz already provides a qualitative description of the overall mixing pattern. Introducing the CP phase  $\epsilon = \lambda^2$  precisely corrects  $\sin^2 \theta_{12}$ . After further including charged-lepton mixing and the additional phase, all four observables can be brought into agreement with the NuFIT 6.0 values within  $1\sigma$ .

We note that the “full fit” column corresponds to the algebraic readings discussed in Sec. 6. Replacing  $\gamma$  by the simpler rational choice  $\gamma \approx 2\lambda/3$  leads to very similar numerical results, with only minor changes well below the experimental uncertainties. The qualitative conclusions are unchanged.

## Supplementary Files

This is a list of supplementary files associated with this preprint. Click to download.

- [AppendixABC.docx](#)