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A Framework Based on hp -Finite Elements for Solving the Time-Harmonic Maxwell Equations

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Abstract

The accurate computational solution of the time-harmonic Maxwell equations remains a central challenge in computational electromagnetics due to the indefinite nature of the double curl operator, the presence of spurious modes, and the high computational cost of resolving oscillatory solutions at large frequencies. In this work, we develop a hp -finite element method to solve the double-curl operator of time-harmonic Maxwell equations, as well as the resulting oscillatory modes. The approach uses Engquist-Majda absorbing boundary conditions and adaptive hp -refinements to efficiently handle unbounded domains and complex field behavior. To complement the analysis, a complete adaptive implementation is developed based on residual error estimation, and automatic selection between h - and p -refinement using a local smoothness indicator. This provides a practical realization of the proposed adaptive strategy. We present stability analysis and establish a priori and a posteriori error estimates that ensure optimal decay of the discretization error. The numerical results confirm mathematical rigor of the formulation and demonstrate that the balance between h - and p -refinements yields optimal convergence with respect to the degrees of freedom. This method provides a substantial improvement in accuracy and efficiency compared to conventional finite element for time-harmonic electromagnetic fields.

Keywords: Maxwell equations, hp -finite element method, convergence, stabilized methods.

1 Introduction

Maxwell equations are fundamental to modern electromagnetics, governing phenomena that arise in applications as diverse as wireless communications, antenna design, radar, photonics, and subsurface imaging. In the frequency domain, as a model problem, one considers the time-harmonic Maxwell system, where the electric field vector $\mathbf{U}$ satisfies a curl-curl equation of the form

$$\mu_0\epsilon(\mathbf{x})\frac{\partial^2\mathbf{U}(\mathbf{x},t)}{\partial t^2} + \nabla \times \nabla \times \mathbf{U}(\mathbf{x},t) + \mu_0\sigma(\mathbf{x})\frac{\partial\mathbf{U}(\mathbf{x},t)}{\partial t} = \mathbf{J}(\mathbf{x},t),$$

associated with appropriate boundary conditions.

Systematic theory of p - and hp -finite element methods, presented in Schwab [1], demonstrates that when the underlying solution is analytic, optimal rates of convergence due to maximal available regularity can be achieved by appropriate combination of mesh refinement and augmented polynomial degree. Such optimal convergence has been established in discontinuous Galerkin framework, as in the work of Houston et al. [2], that forms the mathematical justification for adaptive hp -strategies. These methods offer a path towards overcoming the pollution error that cause low-order discretizations of indefinite wave problems.

Alongside these high-order developments, stabilized low-order formulations have also been proposed for practical computations. Asadzadeh and Beilina developed a stabilized finite element method based on linear P_1 elements for solving the time-harmonic Maxwell equations [3], which allows the use of simple nodal elements while enforcing the divergence-free condition weakly. They extended this framework using domain decomposition, when material coefficients are constant in large parts of the computational domain [4]. In such settings, the system reduces locally to scalar wave equations with transmission conditions, enabling efficient parallel computation. These developments highlight the interplay between mathematical rigor, computational efficiency, and physical modeling in design of new algorithms for Maxwell equations.

In addition to the theoretical study, we present a complete adaptive implementation of the proposed method, including residual-based error estimation, and automatic selection between h - and p -refinements using a local smoothness indicator. This implementation provides a practical realization of the adaptive strategy employed throughout the numerical experiments. We establish convergence results within the framework of mixed finite element analysis, hp -approximation incorporating absorbing boundary conditions of Engquist-Majda [5] type to ensure accurate representation of unbounded domains. Numerical experiments confirm that the proposed method achieves optimal convergence rates due to regularity of analytic solutions and demonstrates robust performance for more general settings with singularities and heterogeneous coefficients.

An outline of this paper is as follows. We first present variational formulation for time-harmonic Maxwell system and establish its well-posedness. We subsequently present the stabilized hp -finite element discretization and establish *a priori* and *a posteriori* estimates. Finally, a set of numerical experiments is provided to demonstrate the accuracy, efficiency, and reliability of the proposed approach.

2 The Mathematical Formulation

In the absence of electric charge density, the electric field vector $\hat{\mathbf{U}}(\mathbf{x}, s)$ satisfies the following time-harmonic Maxwell system:

$$\begin{aligned} s^2 \mu_0 \epsilon(\mathbf{x}) \hat{\mathbf{U}}(\mathbf{x}, s) + \nabla \times \nabla \times \hat{\mathbf{U}}(\mathbf{x}, s) + s \mu_0 \sigma(\mathbf{x}) \hat{\mathbf{U}}(\mathbf{x}, s) \\ = \mathbf{J}(\mathbf{x}, s) + s \mu_0 (\epsilon(\mathbf{x}) + \sigma(\mathbf{x})) \mathbf{J}_0(\mathbf{x}) + \mu_0 \epsilon(\mathbf{x}) \mathbf{J}_1(\mathbf{x}), \quad (1) \\ \nabla \cdot (\epsilon(\mathbf{x}) \hat{\mathbf{U}}(\mathbf{x}, s)) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad d = 2, 3. \end{aligned}$$

Where the dielectric permittivity is denoted by $\epsilon(\mathbf{x}) = \epsilon_r(\mathbf{x}) \epsilon_0$, with $\epsilon_r(\mathbf{x})$ denoting the relative dielectric permittivity (a dimensionless quantity), and ϵ_0 representing the vacuum permittivity. Moreover,

$$\nabla \times \nabla \times \mathbf{U} = \nabla(\nabla \cdot \mathbf{U}) - \nabla^2 \mathbf{U}. \quad (2)$$

The time-harmonic formulation (1) results from the application of the Laplace transform to the following transient Maxwell system.

$$\begin{aligned} \mu_0 \epsilon(\mathbf{x}) \frac{\partial^2 \mathbf{U}(\mathbf{x}, t)}{\partial t^2} + \nabla \times \nabla \times \mathbf{U}(\mathbf{x}, t) + \mu_0 \sigma(\mathbf{x}) \frac{\partial \mathbf{U}(\mathbf{x}, t)}{\partial t} = \mathbf{J}(\mathbf{x}, t), \\ \nabla \cdot (\epsilon \mathbf{U})(\mathbf{x}, t) = 0, \quad (3) \\ \mathbf{U}(\mathbf{x}, 0) = \mathbf{J}_0(\mathbf{x}), \quad \frac{\partial \mathbf{U}}{\partial t}(\mathbf{x}, 0) = \mathbf{J}_1(\mathbf{x}), \\ \mathbf{U}(\mathbf{x}, t) = g(\mathbf{x}, t). \end{aligned}$$

Here $(\mathbf{x}, t) \in \Omega \times [0, T]$, $\mathbf{J}_0(\mathbf{x}) \in [H^1(\Omega)]^3$, $\mathbf{J}_1(\mathbf{x}) \in H(\text{div}, \Omega)$, and the boundary condition is prescribed on $(\mathbf{x}, t) \in \partial\Omega \times [0, T]$.

Since the initial condition is nonzero, specifically $\mathbf{U}(\mathbf{x}, 0) = \mathbf{J}_0(\mathbf{x})$, problem (3) constitutes an inverse coefficient problem for reconstructing the unknown coefficient $\epsilon(\mathbf{x})$ using a finite set of boundary measurements of $\mathbf{U}$ [6].

For numerical solution of (3), we consider a convex, bounded, simply connected polygonal domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$), with boundary Σ . We introduce $\Omega^2 := \Omega \setminus \Omega^1$, where the subdomain $\Omega^1 \subset \Omega$ is assumed to have nonzero Lebesgue measure and $\partial\Omega \cap \partial\Omega^1 = \emptyset$. Removing Ω^1 from Ω , the remaining subdomain Ω^2 shares its boundary with Ω as well as with Ω^1 ; that is, $\partial\Omega^2 = \partial\Omega \cup \partial\Omega^1$. The following also hold: $\Omega = \Omega^1 \cup \Omega^2$, $\Omega^1 = \Omega \setminus \Omega^2$ and $\bar{\Omega}^1 \cap \bar{\Omega}^2 = \partial\Omega^1$.

We assume that $\epsilon(\mathbf{x}) \in \mathbb{C}^d$ ($\mathbb{R}^d, d = 2, 3$) and for a constant $d_1 > 1$ satisfies

$$\begin{aligned} \epsilon(\mathbf{x}) \in [1, d_1], \quad \forall \mathbf{x} \in \partial\Omega^1, \\ \epsilon(\mathbf{x}) = 1, \quad \forall \mathbf{x} \in \Omega \setminus \Omega^1, \quad (4) \\ \partial_{\mathbf{n}} \epsilon = 0, \quad \forall \mathbf{x} \in \partial\Omega^2. \end{aligned}$$

We split the boundary $\Sigma := \partial\Omega$ into three disjoint subsets: $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. Here, Σ_1 and Σ_2 are, respectively, the upper and lower parts of the boundary, relative to the y -direction in two-dimensionals settings and the z -direction in three-dimensionals, while Σ_3 constitutes the remaining part. Furthermore, we denote by $\partial_{\mathbf{n}}(\cdot)$ the derivative in the direction normal to Σ , where $\mathbf{n}$ is the outward unit normal to Σ .

For outgoing waves, we adopt the first-order absorbing boundary model proposed in [4] on $\Sigma_1 \cup \Sigma_2$, following ideas similar in [7, 8] for transient wave equations:

$$\partial_{\mathbf{n}}\mathbf{U} + \partial_t\mathbf{U} = 0, \quad (\mathbf{x}, t) \in (\Sigma_1 \cup \Sigma_2) \times (0, T]. \quad (5)$$

For the boundary conditions on Σ_3 , we suppose that Σ_3 lies sufficiently remote from Ω^1 . Hence, the approximation $\mathbf{U} \approx \mathbf{U}^{\text{inc}}$ is considered valid near Σ_3 , with $\mathbf{U}^{\text{inc}}$ denoting the incident wave. Hence, Neumann condition can be imposed at Σ_3

$$\partial_{\mathbf{n}}\mathbf{U} = 0, \quad (\mathbf{x}, t) \in \Sigma_3 \times (0, T]. \quad (6)$$

Consequently, problem (1) reformulated as the following auxiliary model by making use of the identity (2) and through the application of the Laplace transform (in time domain) to both (3) and the boundary conditions (5)-(6),

$$\begin{aligned} & s^2 \mu_0 \epsilon(\mathbf{x}) \hat{\mathbf{U}}(\mathbf{x}, s) + \nabla(\nabla \cdot \hat{\mathbf{U}}(\mathbf{x}, s)) - \Delta \hat{\mathbf{U}}(\mathbf{x}, s) + s \mu_0 \sigma(\mathbf{x}) \hat{\mathbf{U}}(\mathbf{x}, s) \\ &= \mathbf{J}(\mathbf{x}, s) + s \mu_0 (\epsilon(\mathbf{x}) + \sigma(\mathbf{x})) \mathbf{J}_0(\mathbf{x}) + \mu_0 \epsilon(\mathbf{x}) \mathbf{J}_1(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, d = 2, 3 \\ & \nabla \cdot (\epsilon(\mathbf{x}) \hat{\mathbf{U}}(\mathbf{x}, s)) = 0, \\ & \partial_{\mathbf{n}} \hat{\mathbf{U}}(\mathbf{x}, s) = 0, \quad \mathbf{x} \in \Sigma_3, \\ & \partial_{\mathbf{n}} \hat{\mathbf{U}}(\mathbf{x}, s) = \mathbf{J}_0(\mathbf{x}) - s \hat{\mathbf{U}}(\mathbf{x}, s), \quad \mathbf{x} \in \Sigma_1 \cup \Sigma_2, \end{aligned} \quad (7)$$

3 Variational Model

The symbol $(\cdot, \cdot)$ refers to the usual inner product on $[L_2(\Omega)]^d$, where $d = 2, 3$, and $\|\cdot\|$ is the induced L_2 norm. Likewise, for the boundary, $\langle \cdot, \cdot \rangle_{\Sigma}$ stands for the standard inner product on $[L_2(\partial\Omega)]$ and $\|\cdot\|_{\Sigma}$ denotes the corresponding $L_2(\Sigma)$ -norm. The L_2 inner products are then introduced as follows

$$(u, v) := \int_{\Omega} u \cdot v \, dx, \quad (u, v)_{\omega} := \int_{\Omega} u \cdot v \, \omega \, dx, \quad \langle u, v \rangle_{\Sigma} := \int_{\Sigma} u \cdot v \, d\Sigma,$$

and the norms $L_{2,\omega}(\Omega)$ and H_{ω}^1 ($\omega \equiv 1$ for L_2 and H^1) are as follows ($\omega > 0$)

$$\|u\|_{\omega} := \|u\|_{L_{2,\omega}(\Omega)} := \sqrt{\int_{\Omega} |u|^2 \, \omega \, dx}, \quad \omega > 0, \quad \omega \in L^{\infty}(\Omega),$$

$$\|u\|_{H_\omega^1(\Omega)} := \sqrt{\|u\|_{L_{2,\omega}(\Omega)}^2 + \sum_{j=1}^d \left\| \frac{\partial u}{\partial x_j} \right\|_{L_{2,\omega}(\Omega)}^2}.$$

3.1 Stabilized Variational Formulation

For $d = 2, 3$, the stabilized weak formulation associated with problem (7) can be written as follows:

$$\begin{aligned} s^2 \mu_0 \epsilon(\mathbf{x}) \hat{\mathbf{U}}(\mathbf{x}, s) - \Delta \hat{\mathbf{U}}(\mathbf{x}, s) - \nabla(\nabla \cdot ((\epsilon - 1) \hat{\mathbf{U}}(\mathbf{x}, s))) + s \mu_0 \sigma(\mathbf{x}) \hat{\mathbf{U}}(\mathbf{x}, s) \\ = \mathbf{J}(\mathbf{x}) + s \mu_0 (\epsilon(\mathbf{x}) + \sigma(\mathbf{x})) \mathbf{J}_0(\mathbf{x}) + \mu_0 \epsilon(\mathbf{x}) \mathbf{J}_1(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \\ \partial_{\mathbf{n}} \hat{\mathbf{U}}(\mathbf{x}, s) = 0, \quad \mathbf{x} \in \Sigma_3, \\ \partial_{\mathbf{n}} \hat{\mathbf{U}}(\mathbf{x}, s) = \mathbf{J}_0(\mathbf{x}) - s \hat{\mathbf{U}}(\mathbf{x}, s), \quad \mathbf{x} \in \Sigma_1 \cup \Sigma_2, \end{aligned} \quad (8)$$

Note that the second relation in (7) is incorporated implicitly into the first equation.

We next study the weak formulation associated with (7):

$$\begin{aligned} (s^2 \mu_0 \epsilon \hat{\mathbf{U}}, \mathbf{v}) + (\nabla \hat{\mathbf{U}}, \nabla \mathbf{v}) + (\nabla \cdot (\epsilon \hat{\mathbf{U}}), \nabla \cdot \mathbf{v}) - (\nabla \cdot \hat{\mathbf{U}}, \nabla \cdot \mathbf{v}) + (s \mu_0 \sigma \hat{\mathbf{U}}, \mathbf{v}) \\ - \langle \mathbf{J}_0, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2} + \langle s \hat{\mathbf{U}}, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2} - \langle \nabla \cdot (\epsilon \hat{\mathbf{U}}) - \nabla \cdot \hat{\mathbf{U}}, \mathbf{v} \cdot \mathbf{n} \rangle_{\Sigma} \\ = (\mathbf{J}, \mathbf{v}) + (s \mu_0 (\epsilon + \sigma) \mathbf{J}_0, \mathbf{v}) + (\mu_0 \epsilon \mathbf{J}_1, \mathbf{v}), \quad \forall \mathbf{v} \in [H^1(\Omega)]^3. \end{aligned} \quad (9)$$

Below we assume

$$|\nabla \epsilon| \leq \frac{1}{2} \min(s^2 \mu_0 \epsilon, \epsilon - 1). \quad (10)$$

Assumption 1. By virtue of condition (4), one may set the $\epsilon \equiv 0$ in a vicinity of Σ ; as a result, the last boundary integral is identically zero. Nonetheless, we keep it so as to remain consistent with the computational procedure.

Integration by parts with respect to the spatial variables on the second, third, as well as fourth terms in Eq. (9) leads to the following: for every $\mathbf{v} \in [H^1(\Omega)]^3$,

$$\begin{aligned} (s^2 \mu_0 \epsilon \hat{\mathbf{U}}, \mathbf{v}) + (\nabla(\nabla \cdot \hat{\mathbf{U}}), \mathbf{v}) - (\Delta \hat{\mathbf{U}}, \mathbf{v}) - (\nabla(\nabla \cdot (\epsilon \hat{\mathbf{U}})), \mathbf{v}) + (s \mu_0 \sigma \hat{\mathbf{U}}, \mathbf{v}) \\ - \langle \mathbf{J}_0, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2} + \langle s \hat{\mathbf{U}}, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2} - \langle \nabla \cdot (\epsilon \hat{\mathbf{U}}) - \nabla \cdot \hat{\mathbf{U}}, \mathbf{v} \cdot \mathbf{n} \rangle_{\Sigma} + \langle \partial_{\mathbf{n}} \hat{\mathbf{U}}, \mathbf{v} \rangle_{\Sigma} \\ + \langle \nabla \cdot (\epsilon \hat{\mathbf{U}}) - \nabla \cdot \hat{\mathbf{U}}, \mathbf{v} \cdot \mathbf{n} \rangle_{\Sigma} = (\mathbf{J}, \mathbf{v}) + (s \mu_0 (\epsilon + \sigma) \mathbf{J}_0, \mathbf{v}) + (\mu_0 \epsilon \mathbf{J}_1, \mathbf{v}). \end{aligned} \quad (11)$$

Eliminating the vanishing terms ($\pm \langle \nabla \cdot (\epsilon \hat{\mathbf{U}}) - \nabla \cdot \hat{\mathbf{U}}, \mathbf{v} \cdot \mathbf{n} \rangle_{\Sigma} \equiv 0$), (9) reduces to

$$\begin{aligned} (s^2 \mu_0 \epsilon \hat{\mathbf{U}}, \mathbf{v}) + (\nabla(\nabla \cdot \hat{\mathbf{U}}), \mathbf{v}) - (\Delta \hat{\mathbf{U}}, \mathbf{v}) - (\nabla(\nabla \cdot (\epsilon \hat{\mathbf{U}})), \mathbf{v}) + (s \mu_0 \sigma \hat{\mathbf{U}}, \mathbf{v}) \\ - \langle \mathbf{J}_0, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2} + \langle s \hat{\mathbf{U}}, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2} + \langle \partial_{\mathbf{n}} \hat{\mathbf{U}}, \mathbf{v} \rangle_{\Sigma} \\ = (\mathbf{J}, \mathbf{v}) + (s \mu_0 (\epsilon + \sigma) \mathbf{J}_0, \mathbf{v}) + (\mu_0 \epsilon \mathbf{J}_1, \mathbf{v}), \quad \forall \mathbf{v} \in [H^1(\Omega)]^3. \end{aligned} \quad (12)$$

One can show [4] that the solution of problem (8) coincides with that of (7).

3.2 Finite Element Method based on domain meshing (h-version)

Let $\mathcal{T}_h = \{K\}$ denote a subdivision of Ω into elements K fulfilling the minimum-angle condition. $\mathbf{h} = \mathbf{h}(\mathbf{x})$ represents the mesh function, defined as $\mathbf{h}|_K = \mathbf{h}_K$, corresponding to the local element diameter. In addition, we define $\partial\mathcal{T}_h = \{\partial K\}$ as the portion of the element boundaries ∂K whose vertices lie on Σ .

In triangulations on curved surfaces, spatial angles are taken into account.

For each component of the electric field $\mathbf{U}$, we now define the finite element space:

$$S_h^{\mathbf{U}}(\Omega) := \{\omega \in H^1(\Omega) : \omega|_K \in P_p(K), \forall K \in \mathcal{T}_h\},$$

where $P_p(K)$ denotes the collection of continuous, elementwise polynomials of degree $p \geq 1$ on K . Set $\mathbf{S}_h^{\mathbf{U}}(\Omega) := [S_h^{\mathbf{U}}(\Omega)]^d$, and let $\mathbf{J}_h$, $\mathbf{J}_{0h}$ and $\mathbf{J}_{1h}$ be the interpolations of f , $\mathbf{J}_0$ and $\mathbf{J}_1$ in $S_h^{\mathbf{U}}$, respectively. The finite element method for (8) in Ω is then formulated as: Determine $\hat{\mathbf{U}}_h \in \mathbf{S}_h^{\mathbf{U}}(\Omega)$ satisfying

$$\begin{aligned} (s^2\mu_0\epsilon\hat{\mathbf{U}}_h, \mathbf{v}) + (\nabla\hat{\mathbf{U}}_h, \nabla\mathbf{v}) + (\nabla \cdot (\epsilon\hat{\mathbf{U}}_h), \nabla \cdot \mathbf{v}) - (\nabla \cdot \hat{\mathbf{U}}_h, \nabla \cdot \mathbf{v}) + (s\mu_0\sigma\hat{\mathbf{U}}_h, \mathbf{v}) \\ + \langle s\hat{\mathbf{U}}_h, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2} = (\mathbf{J}_h, \mathbf{v}) + (s\mu_0(\epsilon + \sigma)\mathbf{J}_{0h}, \mathbf{v}) + (\mu_0\epsilon\mathbf{J}_{1h}, \mathbf{v}) + \langle \mathbf{J}_{0h}, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2}. \end{aligned} \quad (13)$$

Remark 1. By virtue of [Assumption 1](#), the outer boundary contributions

$$\pm \langle \nabla \cdot (\epsilon\hat{\mathbf{U}}) - \nabla \cdot \hat{\mathbf{U}}, \mathbf{v} \cdot \mathbf{n} \rangle_{\Sigma},$$

in (9) and (11) cancel out, and do not appear in the subsequent relations.

4 Stability of the Finite Element approach

We introduce the bilinear and linear forms, associated to the problem (13), as

$$\begin{aligned} \mathcal{A}(\hat{\mathbf{U}}_h, \mathbf{v}) = (s^2\mu_0\epsilon\hat{\mathbf{U}}_h, \mathbf{v}) + (\nabla\hat{\mathbf{U}}_h, \nabla\mathbf{v}) + (\nabla \cdot (\epsilon\hat{\mathbf{U}}_h), \nabla \cdot \mathbf{v}) - (\nabla \cdot \hat{\mathbf{U}}_h, \nabla \cdot \mathbf{v}) \\ + (s\mu_0\sigma\hat{\mathbf{U}}_h, \mathbf{v}) + \langle s\hat{\mathbf{U}}_h, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2}, \end{aligned} \quad (14)$$

and

$$\mathcal{L}(\mathbf{v}) := (\mathbf{J}_h, \mathbf{v}) + (s\mu_0(\epsilon + \sigma)\mathbf{J}_{0h}, \mathbf{v}) + (\mu_0\epsilon\mathbf{J}_{1h}, \mathbf{v}) + \langle \mathbf{J}_{0h}, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2}, \quad \forall \mathbf{v} \in \mathbf{V}_H, \quad (15)$$

Thus, we may represent (13), in its concise form as

$$\mathcal{A}(\hat{\mathbf{U}}_h, \mathbf{v}) = \mathcal{L}(\mathbf{v}). \quad (16)$$

For the stability analysis, we introduce the triple norm, involving weighted $L_2(\Omega)$, $H^1(\Omega)$ and $L_2(\partial\Omega)$ norms of the terms in the equation

$$\begin{aligned} |||\hat{\mathbf{U}}_h|||^2 := & \frac{1}{4} \|\hat{\mathbf{U}}_h\|_{s^2\mu_0\epsilon}^2 + \frac{3}{4} \|\nabla\hat{\mathbf{U}}_h\|^2 + \frac{3}{4} \|\nabla \cdot \hat{\mathbf{U}}_h\|_{(\epsilon-1)}^2 + \frac{3}{4} \|\hat{\mathbf{U}}_h\|_{s\mu_0\sigma}^2 \\ & + \|\hat{\mathbf{U}}_h\|_{s,\Sigma_1\cup\Sigma_2}^2, \end{aligned} \quad (17)$$

Lemma 1. Suppose that a solution $\hat{\mathbf{U}}_h$ of the problem (16) exists. Then, $\hat{\mathbf{U}}_h$ satisfies the following stability inequality

$$|||\hat{\mathbf{U}}_h|||^2 \leq \|\mathbf{J}_h\|^2 + \|\mathbf{J}_{0h}\|_{\mu_0\epsilon}^2 + \|\mathbf{J}_{0h}\|_{s\mu_0\sigma}^2 + \|\mathbf{J}_{1h}\|_{\frac{1}{s^2}\mu_0\epsilon}^2 + \|\mathbf{J}_h\|_{\Sigma_1\cup\Sigma_2}^2 + \frac{1}{4} |\hat{g}|_{\Sigma_1\cup\Sigma_2}^2.$$

Proof By defining the function space.

$$\mathbf{v} \in \mathbf{V}_g^H(\Omega) = \{\mathbf{v} \in [H^1(\Omega)]^3 : \mathbf{v}|_\Sigma = g\},$$

we look for a function $\hat{\mathbf{U}}_h \in [H^1(\Omega)]^3$, such that (16) holds for all $\mathbf{v} \in \mathbf{V}_g^H$. Let now $\mathbf{v} = \hat{\mathbf{U}}_h$, then for $\hat{\mathbf{U}} \in \mathbf{V}_g^H(\Omega)$, we have

$$\begin{aligned} (s^2\mu_0\epsilon\hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h) + (\nabla\hat{\mathbf{U}}_h, \nabla\hat{\mathbf{U}}_h) + ((\epsilon-1)\nabla \cdot \hat{\mathbf{U}}_h, \nabla \cdot \hat{\mathbf{U}}_h) + (s\mu_0\sigma\hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h) \\ + \langle s\hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h \rangle_{\Sigma_1\cup\Sigma_2} = (\mathbf{J}_h, \hat{\mathbf{U}}_h) + (s\mu_0\epsilon\mathbf{J}_{0h}, \hat{\mathbf{U}}_h) + (s\mu_0\sigma\mathbf{J}_{0h}, \hat{\mathbf{U}}_h) \\ + (\mu_0\epsilon\mathbf{J}_{1h}, \hat{\mathbf{U}}_h) + \langle \mathbf{J}_{0h}, \hat{\mathbf{U}}_h \rangle_{\Sigma_1\cup\Sigma_2} - ((\nabla\epsilon)\hat{\mathbf{U}}_h, \nabla \cdot \hat{\mathbf{U}}_h). \end{aligned} \quad (18)$$

in which we applied the identity

$$\begin{aligned} (\nabla \cdot (\epsilon\hat{\mathbf{U}}_h), \nabla \cdot \hat{\mathbf{U}}_h) - (\nabla \cdot \hat{\mathbf{U}}_h, \nabla \cdot \hat{\mathbf{U}}_h) &= (\nabla \cdot ((\epsilon-1)\hat{\mathbf{U}}_h), \nabla \cdot \hat{\mathbf{U}}_h) \\ &= (\nabla(\epsilon-1)\hat{\mathbf{U}}_h + (\epsilon-1)\nabla \cdot \hat{\mathbf{U}}_h, \nabla \cdot \hat{\mathbf{U}}_h), \end{aligned} \quad (19)$$

with $\nabla(\epsilon-1) = \nabla\epsilon$. Recalling the assumption (10), the contribution from the final expression on the left side of (18) can be approximated, as

$$-((\nabla\epsilon)\hat{\mathbf{U}}_h, \nabla \cdot \hat{\mathbf{U}}_h) \leq \frac{1}{2} \|\hat{\mathbf{U}}_h\|_{|\nabla\epsilon|}^2 + \frac{1}{2} \|\nabla \cdot \hat{\mathbf{U}}_h\|_{|\nabla\epsilon|}^2 \leq \frac{1}{4} \|\hat{\mathbf{U}}_h\|_{s^2\mu_0\epsilon}^2 + \frac{1}{4} \|\nabla \cdot \hat{\mathbf{U}}_h\|_{(\epsilon-1)}^2. \quad (20)$$

Using (20), the Poincaré inequality, and some simple calculus, we may estimate

$$\begin{aligned} \frac{1}{4} \|\hat{\mathbf{U}}_h\|_{s^2\mu_0\epsilon}^2 + \frac{3}{4} \|\nabla\hat{\mathbf{U}}_h\|^2 + \frac{3}{4} \|\nabla \cdot \hat{\mathbf{U}}_h\|_{(\epsilon-1)}^2 + \frac{3}{4} \|\hat{\mathbf{U}}_h\|_{s\mu_0\sigma}^2 + \|\hat{\mathbf{U}}_h\|_{s,\Sigma_1\cup\Sigma_2}^2 \\ \leq \|\mathbf{J}_h\|^2 + \|\mathbf{J}_{0h}\|_{\mu_0\epsilon}^2 + \|\mathbf{J}_{0h}\|_{s\mu_0\sigma}^2 + \|\mathbf{J}_{1h}\|_{\frac{1}{s^2}\mu_0\epsilon}^2 + \|\mathbf{J}_h\|_{\Sigma_1\cup\Sigma_2}^2 + \frac{1}{4} |\hat{g}|_{\Sigma_1\cup\Sigma_2}^2, \end{aligned} \quad (21)$$

and the proof is complete. $\square$

Theorem 1. The solution $\hat{\mathbf{U}}_h \in \mathbf{S}_h^U(\Omega)$ to problem (13) is unique.

Proof The existence and uniqueness of the discrete solution follow from an application of the Lax–Milgram theorem to the formulation (14)–(16). In particular, one can find constants C_i , $i = 1, 2, 3$, for which the following hold true: for any $\hat{\mathbf{U}}_h$ and $\mathbf{v} \in \mathbf{S}_h^{\mathbf{U}}(\Omega)$,

$$\mathcal{A}(\hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h) \geq C_1 \|\hat{\mathbf{U}}_h\|^2, \quad (\text{Coercivity of } \mathcal{A}), \quad (22)$$

$$\mathcal{A}(\hat{\mathbf{U}}_h, \mathbf{v}) \leq C_2 \|\hat{\mathbf{U}}_h\| \cdot \|\mathbf{v}\|, \quad (\text{Continuity of } \mathcal{A}), \quad (23)$$

$$|\mathcal{L}(\mathbf{v})| \leq C_3 \|\mathbf{v}\| \quad (\text{Continuity of } \mathcal{L}). \quad (24)$$

It is straightforward to derive the first relation: selecting $\mathbf{v} = \hat{\mathbf{U}}_h$ in (22),

$$\begin{aligned} \mathcal{A}(\hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h) &= (s^2 \mu_0 \epsilon \hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h) + (\nabla \hat{\mathbf{U}}_h, \nabla \hat{\mathbf{U}}_h) + (\nabla((\epsilon - 1)\hat{\mathbf{U}}_h), \nabla \cdot \hat{\mathbf{U}}_h) \\ &\quad + ((\epsilon - 1)\nabla \cdot \hat{\mathbf{U}}_h, \nabla \cdot \hat{\mathbf{U}}_h) + (s\mu_0 \sigma \hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h) + \langle s\hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h \rangle_{\Sigma_1 \cup \Sigma_2} \\ &= \|\hat{\mathbf{U}}_h\|_{s^2 \mu_0 \epsilon}^2 + \|\nabla \hat{\mathbf{U}}_h\|^2 + \|\nabla \cdot \hat{\mathbf{U}}_h\|_{(\epsilon-1)}^2 + \|\hat{\mathbf{U}}_h\|_{s\mu_0 \sigma}^2 \\ &\quad + (\nabla(\epsilon - 1)\hat{\mathbf{U}}_h, \nabla \cdot \hat{\mathbf{U}}_h) + \|\hat{\mathbf{U}}_h\|_{s, \Sigma_1 \cup \Sigma_2}^2, \end{aligned} \quad (25)$$

where using (19), the first term on the right-hand side of (19) is approximated as

$$\pm \left((\nabla \epsilon) \hat{\mathbf{U}}_h, \nabla \cdot \hat{\mathbf{U}}_h \right) \geq -\frac{1}{2} \|\hat{\mathbf{U}}_h\|_{|\nabla \epsilon|}^2 - \frac{1}{2} \|\nabla \cdot \hat{\mathbf{U}}_h\|_{|\nabla \epsilon|}^2, \quad (26)$$

which recalling (25), yields

$$\begin{aligned} \mathcal{A}(\hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h) &\geq \frac{3}{4} \|\hat{\mathbf{U}}_h\|_{s^2 \mu_0 \epsilon}^2 + \|\nabla \hat{\mathbf{U}}_h\|^2 + \frac{3}{4} \|\nabla \cdot \hat{\mathbf{U}}_h\|_{(\epsilon-1)}^2 + \|\hat{\mathbf{U}}_h\|_{s\mu_0 \sigma}^2 + \|\hat{\mathbf{U}}_h\|_{s, \Sigma_1 \cup \Sigma_2}^2 \\ &\geq \|\hat{\mathbf{U}}_h\|^2, \end{aligned} \quad (27)$$

and (22) holds with $C_1 = 1$. To prove the continuity of $\mathcal{A}(\cdot, \cdot)$, for all $\mathbf{v} \in \mathbf{S}_h^{\mathbf{U}}(\Omega)$, By applying the Cauchy-Schwarz inequality and (26), we obtain

$$\begin{aligned} \mathcal{A}(\hat{\mathbf{U}}_h, \mathbf{v}) &= (s^2 \mu_0 \epsilon \hat{\mathbf{U}}_h, \mathbf{v}) + (\nabla \hat{\mathbf{U}}_h, \nabla \mathbf{v}) + (\nabla \cdot ((\epsilon - 1)\hat{\mathbf{U}}_h), \nabla \cdot \mathbf{v}) + (s\mu_0 \sigma \hat{\mathbf{U}}_h, \mathbf{v}) \\ &\quad + \langle s\hat{\mathbf{U}}_h, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2} \\ &\leq \|\hat{\mathbf{U}}_h\|_{s^2 \mu_0 \epsilon} \|\mathbf{v}\|_{s^2 \mu_0 \epsilon} + \|\nabla \hat{\mathbf{U}}_h\| \|\nabla \mathbf{v}\| + \|\nabla \cdot \hat{\mathbf{U}}_h\|_{(\epsilon-1)} \|\nabla \cdot \mathbf{v}\|_{(\epsilon-1)} \\ &\quad + \|\hat{\mathbf{U}}_h\|_{|\nabla \epsilon|} \|\nabla \cdot \mathbf{v}\|_{|\nabla \epsilon|} + \|\hat{\mathbf{U}}_h\|_{s\mu_0 \sigma} \|\mathbf{v}\|_{s\mu_0 \sigma} + \|\hat{\mathbf{U}}_h\|_{s, \Sigma_1 \cup \Sigma_2} \|\mathbf{v}\|_{s, \Sigma_1 \cup \Sigma_2} \\ &\leq C \|\hat{\mathbf{U}}_h\| \cdot \|\mathbf{v}\|, \end{aligned} \quad (28)$$

The last inequality is obtained by a few straightforward algebraic manipulations.

Likewise, for $\mathbf{J}_{0h} \in L_{2, \mu_0 \epsilon}(\Omega) \cap L_{2, s\mu_0 \sigma}(\Omega) \cap L_{2, \frac{1}{s}}(\Sigma_1 \cup \Sigma_2)$, we have the continuity of $\mathcal{L}$,

$$\begin{aligned} \mathcal{L}(\mathbf{v}) &= (\mathbf{J}_h, \mathbf{v}) + (s\mu_0(\epsilon + \sigma)\mathbf{J}_{0h}, \mathbf{v}) + (\mu_0 \epsilon \mathbf{J}_{1h}, \mathbf{v}) + \langle \mathbf{J}_{0h}, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2} \\ &= (\mathbf{J}_h, \mathbf{v}) + (s\mu_0 \epsilon \mathbf{J}_{0h}, \mathbf{v}) + (s\mu_0 \sigma \mathbf{J}_{0h}, \mathbf{v}) + (\mu_0 \epsilon \mathbf{J}_{1h}, \mathbf{v}) + \langle \mathbf{J}_{0h}, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2} \\ &\leq \|\mathbf{J}_h\| \|\mathbf{v}\| + \|\mathbf{J}_{0h}\|_{\mu_0 \epsilon} \|\mathbf{v}\|_{s^2 \mu_0 \epsilon} + \|\mathbf{J}_{0h}\|_{s\mu_0 \sigma} \|\mathbf{v}\|_{s\mu_0 \sigma} + \|\mathbf{J}_{1h}\|_{\frac{1}{s^2} \mu_0 \epsilon} \|\mathbf{v}\|_{s^2 \mu_0 \epsilon} \\ &\quad + \|\mathbf{J}_{0h}\|_{\frac{1}{s}, \Sigma_1 \cup \Sigma_2} \|\mathbf{v}\|_{s, \Sigma_1 \cup \Sigma_2} \\ &\leq \left(c \|\mathbf{J}_h\| + \|\mathbf{J}_{0h}\|_{\mu_0 \epsilon} + \|\mathbf{J}_{1h}\|_{\frac{1}{s^2} \mu_0 \epsilon} + \|\mathbf{J}_{0h}\|_{s\mu_0 \sigma} + \|\mathbf{J}_{0h}\|_{\frac{1}{s}, \Sigma_1 \cup \Sigma_2} \right) \|\mathbf{v}\|, \end{aligned} \quad (29)$$

and the proof is complete. $\square$

5 Error analysis

We present a concise a priori and a posteriori error estimates.

5.1 A priori error estimates: The h-version (theoretical bound)

For a priori error estimates, we define the linear and bilinear forms in the weak formulation of the equation in finite element space $\mathbf{S}_h^U(\Omega)$: In relation (14), we replace $\hat{\mathbf{U}}_h$ by $\hat{\mathbf{U}}$, in (15), we replace $\mathbf{J}_h$, $\mathbf{J}_{0h}$ and $\mathbf{J}_{1h}$ by f , $\mathbf{J}_0$ and $\mathbf{J}_1$, respectively, and rewrite it using a continuous linear form that coincides with $\mathcal{L}^c(\mathbf{v})$,

$$\begin{aligned} \mathcal{A}(\hat{\mathbf{U}}, \mathbf{v}) = & (s^2 \mu_0 \epsilon \hat{\mathbf{U}}, \mathbf{v}) + (\nabla \hat{\mathbf{U}}, \nabla \mathbf{v}) + (\nabla \cdot (\epsilon \hat{\mathbf{U}}), \nabla \cdot \mathbf{v}) - (\nabla \cdot \hat{\mathbf{U}}, \nabla \cdot \mathbf{v}) \\ & + (s \mu_0 \sigma \hat{\mathbf{U}}, \mathbf{v}) + \langle s \hat{\mathbf{U}}, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2}, \quad \forall \mathbf{v} \in [H^1(\Omega)]^3 \end{aligned} \quad (30)$$

and

$$\mathcal{L}^c(\mathbf{v}) = (\mathbf{J}, \mathbf{v}) + (s \mu_0 (\epsilon + \sigma) \mathbf{J}_0, \mathbf{v}) + (\mu_0 \epsilon \mathbf{J}_1, \mathbf{v}) + \langle \mathbf{J}_0, \mathbf{v} \rangle_{\Sigma_1 \cup \Sigma_2}, \quad \forall \mathbf{v} \in [H^1(\Omega)]^3, \quad (31)$$

The compact form of the variational formulation reads as

$$\mathcal{A}(\hat{\mathbf{U}}, \mathbf{v}) = \mathcal{L}^c(\mathbf{v}), \quad \forall \mathbf{v} \in [H^1(\Omega)]^3. \quad (32)$$

To proceed, we restrict $\mathbf{v} \in \mathbf{S}_h^U(\Omega)$, in (30) and (31), and replace $\mathbf{J}$, $\mathbf{J}_0$ and $\mathbf{J}_1$ by $\mathbf{J}_h$, $\mathbf{J}_{0h}$ and $\mathbf{J}_{1h}$, respectively, in (31). This yields the finite element weak formulation (16). Subtracting it from the continuous problem (32) and introducing the pointwise finite element approximation error, $e(\mathbf{x}, s) := \hat{\mathbf{U}}(\mathbf{x}, s) - \hat{\mathbf{U}}_h(\mathbf{x}, s)$, with $\mathbf{U}_h$ satisfying (13), we obtain the Galerkin orthogonality

$$\mathcal{A}(\hat{\mathbf{U}} - \hat{\mathbf{U}}_h, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{S}_h^U(\Omega). \quad (33)$$

Theorem 2. Let $\hat{\mathbf{U}}$ be the solution of the continuous problem (8) (with the variational form (9)) and $\hat{\mathbf{U}}_h$ be its finite element approximation in (13). Then, there is a constant C , independent of $\hat{\mathbf{U}}$ and h , such that

$$|||e||| := |||\hat{\mathbf{U}} - \hat{\mathbf{U}}_h||| \leq \tilde{C} \|h \hat{\mathbf{U}}\|_{H_{\tilde{\omega}}^2(\Omega)}.$$

where $\tilde{\omega} = \tilde{\omega}(\epsilon(\mathbf{x}), \sigma(\mathbf{x}), \mu_0, s)$ represents a weight function depending upon the dielectric permittivity $\epsilon(\mathbf{x})$, the electric conductivity $\sigma(\mathbf{x})$, the free-space permeability μ_0 , and the pseudo-frequency parameter s : (here the variable of the Laplace transform).

Proof The result follows from the characterization of the triple norm and twice application of the Galerkin orthogonality (33).

$$\begin{aligned} |||e|||^2 &= \mathcal{A}(e, e) = \mathcal{A}(e, \hat{\mathbf{U}} - \hat{\mathbf{U}}_h) = \mathcal{A}(e, \hat{\mathbf{U}}) - \mathcal{A}(e, \hat{\mathbf{U}}_h) = \mathcal{A}(e, \hat{\mathbf{U}}) \\ &= \mathcal{A}(e, \hat{\mathbf{U}}) + 0 = \mathcal{A}(e, \hat{\mathbf{U}}) - \mathcal{A}(e, \mathbf{v}) = \mathcal{A}(e, \hat{\mathbf{U}} - \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{S}_h^U(\Omega). \end{aligned} \quad (34)$$

Let now $\mathbf{v}$ be the interpolant $\pi_h \hat{\mathbf{U}}$ of $\hat{\mathbf{U}}$. Then, combining the interpolation error bounds with the Cauchy–Schwarz inequality yields:

$$\begin{aligned}
|||e|||^2 &= \mathcal{A}(e, \hat{\mathbf{U}} - \mathbf{v}) = \mathcal{A}(e, \hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}}) \\
&= (e, \hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}})_{s^2 \mu_0 \epsilon} + (\nabla e, \nabla(\hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}})) + ((\nabla e)e, \nabla \cdot (\hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}})) \\
&\quad + (\nabla \cdot e, \nabla \cdot (\hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}}))_{(\epsilon-1)} + (e, \hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}})_{s \mu_0 \sigma} + \langle se, \hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}} \rangle_{\Sigma_1 \cup \Sigma_2} \\
&\leq C_1 \|e\|_{s^2 \mu_0 \epsilon} \|h^2 D^2 \hat{\mathbf{U}}\|_{s^2 \mu_0 \epsilon} + C_2 \|\nabla e\| \|h D^2 \hat{\mathbf{U}}\| + C_3 \|e\|_{|\nabla \epsilon|} \|h D^2 \hat{\mathbf{U}}\|_{|\nabla \epsilon|} \\
&\quad + C_4 \|\nabla \cdot e\|_{(\epsilon-1)} \|h D^2 \hat{\mathbf{U}}\|_{(\epsilon-1)} + C_5 \|e\|_{s \mu_0 \sigma} \|h^2 D^2 \hat{\mathbf{U}}\|_{s \mu_0 \sigma} \\
&\quad + C_6 \|e\|_{s, \Sigma_1 \cup \Sigma_2} \left(\|h^2 D^2 \hat{\mathbf{U}}\|_{s, L_2(\Omega)}^{1/2} \cdot \|h D^2 \hat{\mathbf{U}}\|_{s, L_2(\Omega)}^{1/2} \right) \\
&\leq \tilde{C} |||e||| \cdot \left(\|h^2 D^2 \hat{\mathbf{U}}\|_{s^2 \mu_0 \epsilon} + \|h D^2 \hat{\mathbf{U}}\| + \|h D^2 \hat{\mathbf{U}}\|_{(\epsilon-1)} + \|h^2 D^2 \hat{\mathbf{U}}\|_{s \mu_0 \sigma} \right) \\
&\quad + \|h D^2 \hat{\mathbf{U}}\|_{|\nabla \epsilon|} + \|h^{3/2} D^2 \hat{\mathbf{U}}\|_s \\
&\leq \tilde{C} |||e||| \cdot \|h D^2 \hat{\mathbf{U}}\|_{\tilde{\omega}},
\end{aligned} \tag{35}$$

with $h = h(\mathbf{x}) \ll 1$. To control the contribution from the boundary interpolation, we employ a trace argument combined with the Poincaré inequality, using the fact that $\hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}}$ vanishes on a portion of Σ having positive Lebesgue measure [6].

$$\begin{aligned}
\|\hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}}\|_{s, \Sigma} &\leq \tilde{C} \|\hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}}\|_{s, L_2(\Omega)}^{1/2} \|\hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}}\|_{s, H^1(\Omega)}^{1/2} \\
&\leq \tilde{C} \|\hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}}\|_{s, L_2(\Omega)}^{1/2} \|\nabla(\hat{\mathbf{U}} - \pi_h \hat{\mathbf{U}})\|_{s, L_2(\Omega)}^{1/2} \\
&\leq \tilde{C} \|h^2 D^2 \hat{\mathbf{U}}\|_{s, L_2(\Omega)}^{1/2} \|h D^2 \hat{\mathbf{U}}\|_{s, L_2(\Omega)}^{1/2} = \mathcal{O}(h^{3/2}),
\end{aligned} \tag{36}$$

where D^2 denotes the differential operator. This completes the proof. $\square$

5.2 A posteriori error estimates

A posteriori error analysis, is based on *residual* concept: informally discrepancy between the two sides of the continuous formulation once the true solution is substituted by the available approximate solution. Let $\hat{\mathbf{U}}_h = \hat{\mathbf{U}}_h(\mathbf{x}, s)$ denote the approximate solution of problem (8), with $\mathbf{x} \in \Omega \subseteq \mathbb{R}^d$, $d = 2, 3$. We introduce the associated residual errors as

$$\begin{aligned}
-\mathcal{R}(\hat{\mathbf{U}}_h) &:= s^2 \mu_0 \epsilon(\mathbf{x}) \hat{\mathbf{U}}_h - \Delta \hat{\mathbf{U}}_h - \nabla(\nabla \cdot ((\epsilon(\mathbf{x}) - 1) \hat{\mathbf{U}}_h)) + s \mu_0 \sigma(\mathbf{x}) \hat{\mathbf{U}}_h - \mathbf{J}_h \\
&\quad - s \mu_0 (\epsilon(\mathbf{x}) + \sigma(\mathbf{x})) \mathbf{J}_{0h}(\mathbf{x}) - \mu_0 \epsilon(\mathbf{x}) \mathbf{J}_{1h}(\mathbf{x}),
\end{aligned}$$

and

$$\begin{aligned}
-\mathcal{R}_{\Sigma_1 \cup \Sigma_2}(\hat{\mathbf{U}}_h) &:= h^{-\alpha} \left(\partial_n \hat{\mathbf{U}}_h + s \hat{\mathbf{U}}_h - \mathbf{J}_{0h}(\mathbf{x}) \right), \quad \text{for } \mathbf{x} \in \Sigma_1 \cup \Sigma_2, \\
0 &< \alpha \leq 1,
\end{aligned} \tag{37}$$

Note that $\mathcal{R}(\hat{\mathbf{U}}) \equiv 0$ and by Galerkin orthogonality, $\mathcal{R}(\hat{\mathbf{U}}_h) \perp S_h^U(\Omega)$. The aim is to bound tripple norm of the error $e(\mathbf{x}, s) := \hat{\mathbf{U}}(\mathbf{x}, s) - \hat{\mathbf{U}}_h(\mathbf{x}, s)$ using appropriate norms of $\mathcal{R}(\hat{\mathbf{U}}_h)$ and $\mathcal{R}_{\Sigma_1 \cup \Sigma_2}(\hat{\mathbf{U}}_h)$ exhibiting rapid decay. This can be achieved through different approaches. For instance, one may employ the variational formulation together with interpolation error and Galerkin orthogonality, as in Theorem 3.

The proof relies on a linear approximation of the initial data $\mathbf{J}_0(\mathbf{x}) := \mathbf{U}(\mathbf{x}, t)|_{t=0}$ and $\mathbf{J}_1(\mathbf{x}) := \frac{\partial \mathbf{U}}{\partial t}(\mathbf{x}, t)|_{t=0}$, where for $\beta \approx 1$, we assume

$$\begin{aligned} \|\mathbf{J} - \mathbf{J}_h\|_{1/(s\mu_0\sigma)} &\approx \|\mathbf{J}_0 - \mathbf{J}_{0h}\|_{\mu_0\epsilon} \approx \|\mathbf{J}_0 - \mathbf{J}_{0h}\|_{s\mu_0\sigma} \approx \|\mathbf{J}_0 - \mathbf{J}_{0h}\|_{1/s, \Sigma_1 \cup \Sigma_2} \\ &\approx \|\mathbf{J}_1 - \mathbf{J}_{1h}\|_{\sqrt{\mu_0\epsilon}/s^2} = \mathcal{O}(h^\beta). \end{aligned} \quad (38)$$

In fact, we assume minimal regularity for $\mathbf{J}_0$ and $\mathbf{J}_1$; for example, $\mathbf{J}_0, \mathbf{J}_1 \in H^1(\Omega)$.

Remark 2. Although $\mathbf{J}_0$ and $\mathbf{J}_1$ are given, $\mathbf{J}_{0h}$ and $\mathbf{J}_{1h}$ are unknown in the space where $\mathbf{J}_0$ and $\mathbf{J}_1$ are defined. Therefore, assumptions of the type (38) are necessary.

Theorem 3. Let $\hat{\mathbf{U}}$ and $\hat{\mathbf{U}}_h$ be the solutions of the continuous problem (8) and its finite element approximation (13); respectively. Further, suppose that the initial fields $\mathbf{J}_0(\mathbf{x}) := \mathbf{U}(\mathbf{x}, t)|_{t=0}$ and $\mathbf{J}_1(\mathbf{x}) := \frac{\partial \mathbf{U}}{\partial t}(\mathbf{x}, t)|_{t=0}$ satisfy the error bound (38), and C_1 and C_2 are interpolation constants, independent of h and $\hat{\mathbf{U}}$, they may depend on ϵ and s . Then, we have the following a posteriori error bound:

$$\|\|\|e\|\|\| \leq C_1 h \|\mathcal{R}\| + C_2 h^\alpha \|\mathcal{R}_\Sigma\|_{1/s, \Sigma_1 \cup \Sigma_2} + \mathcal{O}(h^\beta) + \mathcal{O}(h^{3/2}), \quad (39)$$

$\alpha \approx \beta \approx 1$.

Proof Recall that

$$\begin{aligned} \|\|\|e\|\|\|^2 &= (s^2 \mu_0 \epsilon e, e) + (\nabla e, \nabla e) + (\nabla \cdot ((\epsilon - 1)e), \nabla \cdot e) + (s \mu_0 \sigma e, e) \\ &\quad + \langle se, e \rangle_{\Sigma_1 \cup \Sigma_2} := \sum_{j=1}^5 T_j. \end{aligned} \quad (40)$$

Using Green's identity together with the assumption that $\partial_{\mathbf{n}} e \equiv 0$ on Σ_3 , we obtain

$$\begin{aligned} T_2 + T_3 &= (\nabla e, \nabla e) + (\nabla \cdot ((\epsilon - 1)e), \nabla \cdot e) \\ &= (\nabla e, \nabla e) + (\nabla \cdot (\epsilon e), \nabla \cdot e) - (\nabla \cdot e, \nabla \cdot e) \\ &= -(\Delta e, e) - (\nabla(\nabla \cdot (\epsilon e)), e) + \langle \partial_{\mathbf{n}}(\epsilon e), e \rangle_\Sigma + (\nabla(\nabla \cdot e), e) + \langle \partial_{\mathbf{n}} e, e \rangle_{\Sigma_1 \cup \Sigma_2} \\ &= -(\Delta e, e) - (\nabla(\nabla \cdot (\epsilon e)), e) + \langle (\partial_{\mathbf{n}} \epsilon) e, e \rangle_\Sigma \\ &\quad + \langle \epsilon (\partial_{\mathbf{n}} e), e \rangle_\Sigma + (\nabla(\nabla \cdot e), e) + \langle \partial_{\mathbf{n}} e, e \rangle_{\Sigma_1 \cup \Sigma_2} \\ &= -(\Delta e, e) - (\nabla(\nabla \cdot ((\epsilon - 1)e)), e) + (1 + \epsilon) \langle \partial_{\mathbf{n}} e, e \rangle_{\Sigma_1 \cup \Sigma_2} + \langle (\partial_{\mathbf{n}} \epsilon) e, e \rangle_\Sigma, \end{aligned} \quad (41)$$

According to the (4), the last term in (41) vanishes, therefore

$$\begin{aligned} \|\|\|e\|\|\|^2 &= (s^2 \mu_0 \epsilon e, e) - (\Delta e, e) - (\nabla(\nabla \cdot ((\epsilon - 1)e)), e) + (1 + \epsilon) \langle \partial_{\mathbf{n}} e, e \rangle_{\Sigma_1 \cup \Sigma_2} \\ &\quad + (s \mu_0 \sigma e, e) + \langle se, e \rangle_{\Sigma_1 \cup \Sigma_2}. \end{aligned} \quad (42)$$

Consequently, by the definitions of the residuals $\mathcal{R}$ and $\mathcal{R}_{\Sigma_1 \cup \Sigma_2}$, we get

$$\begin{aligned}
|||e|||^2 &= (s^2 \mu_0 \epsilon \hat{\mathbf{U}} - \Delta \hat{\mathbf{U}} - \nabla(\nabla \cdot ((\epsilon - 1)\hat{\mathbf{U}})) + s\mu_0 \sigma \hat{\mathbf{U}}, e) + (1 + \epsilon) \langle \partial_{\mathbf{n}} \hat{\mathbf{U}}, e \rangle_{\Sigma_1 \cup \Sigma_2} \\
&\quad + \langle s \hat{\mathbf{U}}, e \rangle_{\Sigma_1 \cup \Sigma_2} - (s^2 \mu_0 \epsilon \hat{\mathbf{U}}_h - \Delta \hat{\mathbf{U}}_h - \nabla(\nabla \cdot ((\epsilon - 1)\hat{\mathbf{U}}_h)) + s\mu_0 \sigma \hat{\mathbf{U}}_h, e) \\
&\quad - (1 + \epsilon) \langle \partial_{\mathbf{n}} \hat{\mathbf{U}}_h, e \rangle_{\Sigma_1 \cup \Sigma_2} - \langle s \hat{\mathbf{U}}_h, e \rangle_{\Sigma_1 \cup \Sigma_2} \\
&= (\mathbf{J}, e) + (s\mu_0(\epsilon + \sigma)\mathbf{J}_0, e) + (\mu_0 \epsilon \mathbf{J}_1, e) + (\mathcal{R}(\hat{\mathbf{U}}_h), e) \\
&\quad - (\mathbf{J}_h, e) - (s\mu_0(\epsilon + \sigma)\mathbf{J}_{0h}, e) - (\mu_0 \epsilon \mathbf{J}_{1h}, e) + \langle \mathbf{J}_0, e \rangle_{\Sigma_1 \cup \Sigma_2} - \langle s \hat{\mathbf{U}}, e \rangle_{\Sigma_1 \cup \Sigma_2} \\
&\quad + (1 + \epsilon) \langle s \hat{\mathbf{U}}, e \rangle_{\Sigma_1 \cup \Sigma_2} + \langle \mathbf{h}^\alpha \mathcal{R}_{\Sigma_1 \cup \Sigma_2}(\hat{\mathbf{U}}_h), e \rangle_{\Sigma_1 \cup \Sigma_2} + \langle s \hat{\mathbf{U}}_h, e \rangle_{\Sigma_1 \cup \Sigma_2} \\
&\quad - \langle \mathbf{J}_{0h}, e \rangle_{\Sigma_1 \cup \Sigma_2} - (1 + \epsilon) \langle s \hat{\mathbf{U}}_h, e \rangle_{\Sigma_1 \cup \Sigma_2} \\
&= (\mathbf{J} - \mathbf{J}_h, e) + (s\mu_0 \epsilon (\mathbf{J}_0 - \mathbf{J}_{0h}), e) + (s\mu_0 \sigma (\mathbf{J}_0 - \mathbf{J}_{0h}), e) + (\mu_0 \epsilon (\mathbf{J}_1 - \mathbf{J}_{1h}), e) \\
&\quad + \langle \mathbf{J}_0 - \mathbf{J}_{0h}, e \rangle_{\Sigma_1 \cup \Sigma_2} + (\mathcal{R}(\hat{\mathbf{U}}_h), e) + \langle \mathbf{h}^\alpha \mathcal{R}_{\Sigma_1 \cup \Sigma_2}(\hat{\mathbf{U}}_h), e \rangle_{\Sigma_1 \cup \Sigma_2} \\
&\quad + \epsilon \langle s(\mathbf{U} - \hat{\mathbf{U}}_h), e \rangle_{\Sigma_1 \cup \Sigma_2},
\end{aligned} \tag{43}$$

where all boundary contributions originating from Σ_3 vanish identically. Moreover, by the orthogonality relation $\mathcal{R}(\hat{\mathbf{U}}_h) \perp \mathbf{S}_h^{\mathbf{U}}(\Omega)$, we obtain

$$\begin{aligned}
|||e|||^2 &= \left(\frac{1}{\sqrt{s\mu_0\sigma}} (\mathbf{J} - \mathbf{J}_h), \sqrt{s\mu_0\sigma} e \right) + (\sqrt{\mu_0\epsilon} (\mathbf{J}_0 - \mathbf{J}_{0h}), s\sqrt{\mu_0\epsilon} e) \\
&\quad + (\sqrt{s\mu_0\sigma} (\mathbf{J}_0 - \mathbf{J}_{0h}), \sqrt{s\mu_0\sigma} e) + \left(\frac{\sqrt{\mu_0\epsilon}}{s^2} (\mathbf{J}_1 - \mathbf{J}_{1h}), s^2 \sqrt{\mu_0\epsilon} e \right) \\
&\quad + \left(\frac{1}{\sqrt{s}} (\mathbf{J}_0 - \mathbf{J}_{0h}), \sqrt{s} e \right)_{\Sigma_1 \cup \Sigma_2} + (\mathcal{R}(\hat{\mathbf{U}}_h), e - \pi_h e) \\
&\quad + \langle \mathbf{h}^\alpha \mathcal{R}_{\Sigma_1 \cup \Sigma_2}(\hat{\mathbf{U}}_h), e \rangle_{\Sigma_1 \cup \Sigma_2} + \epsilon \langle s(\mathbf{U} - \hat{\mathbf{U}}_h), e \rangle_{\Sigma_1 \cup \Sigma_2} := \sum_{j=1}^8 I_j,
\end{aligned} \tag{44}$$

We now estimate each term I_j separately

$$\begin{aligned}
I_1 &:= \left(\frac{1}{\sqrt{s\mu_0\sigma}} (\mathbf{J} - \mathbf{J}_h), \sqrt{s\mu_0\sigma} e \right) \leq \|\mathbf{J} - \mathbf{J}_h\|_{\frac{1}{s\mu_0\sigma}} \|e\|_{s\mu_0\sigma}, \\
I_2 &:= (\sqrt{\mu_0\epsilon} (\mathbf{J}_0 - \mathbf{J}_{0h}), s\sqrt{\mu_0\epsilon} e) \leq \|\mathbf{J}_0 - \mathbf{J}_{0h}\|_{\mu_0\epsilon} \|e\|_{s^2\mu_0\epsilon}, \\
I_3 &:= (\sqrt{s\mu_0\sigma} (\mathbf{J}_0 - \mathbf{J}_{0h}), \sqrt{s\mu_0\sigma} e) \leq \|\mathbf{J}_0 - \mathbf{J}_{0h}\|_{s\mu_0\sigma} \|e\|_{s\mu_0\sigma}, \\
I_4 &:= \left(\frac{\sqrt{\mu_0\epsilon}}{s^2} (\mathbf{J}_1 - \mathbf{J}_{1h}), s^2 \sqrt{\mu_0\epsilon} e \right) \leq \|\mathbf{J}_1 - \mathbf{J}_{1h}\|_{\frac{\sqrt{\mu_0\epsilon}}{s^2}} \|e\|_{s^2\mu_0\epsilon}, \\
I_5 &:= \left(\frac{1}{\sqrt{s}} (\mathbf{J}_0 - \mathbf{J}_{0h}), \sqrt{s} e \right)_{\Sigma_1 \cup \Sigma_2} \leq \|\mathbf{J}_0 - \mathbf{J}_{0h}\|_{\frac{1}{s}, \Sigma_1 \cup \Sigma_2} \|e\|_{s, \Sigma_1 \cup \Sigma_2}, \\
I_6 + I_7 &:= (\mathcal{R}(\hat{\mathbf{U}}_h), e - \pi_h e) + \langle \mathbf{h}^\alpha \mathcal{R}_{\Sigma_1 \cup \Sigma_2}(\hat{\mathbf{U}}_h), e \rangle_{\Sigma_1 \cup \Sigma_2} \\
&\leq \left(\|\mathbf{h} \mathcal{R}(\hat{\mathbf{U}}_h)\| \|\nabla e\| + \|\mathbf{h}^\alpha \mathcal{R}_{\Sigma_1 \cup \Sigma_2}(\hat{\mathbf{U}}_h)\|_{1/s, \Sigma_1 \cup \Sigma_2} \|e\|_{s, \Sigma_1 \cup \Sigma_2} \right),
\end{aligned} \tag{45}$$

Based on the trace estimate, we have

$$I_8 := \epsilon \langle s(\mathbf{U} - \hat{\mathbf{U}}_h), e \rangle_{\Sigma_1 \cup \Sigma_2} \leq C \left(\|\mathbf{h}^2 D^2 \hat{\mathbf{U}}\|_{L_2(\Omega)}^{1/2} \|\mathbf{h} D^2 \hat{\mathbf{U}}\|_{s, L_2(\Omega)}^{1/2} \right) \|e\|_{s, \Sigma_1 \cup \Sigma_2}, \tag{46}$$

Summing up we obtain

$$\begin{aligned}
|||e|||^2 &\leq \|h\mathcal{R}(\hat{\mathbf{U}}_h)\| \|\nabla e\| + \|h^\alpha \mathcal{R}_{\Sigma_1 \cup \Sigma_2}(\hat{\mathbf{U}}_h)\|_{1/s, \Sigma_1 \cup \Sigma_2} \|e\|_{s, \Sigma_1 \cup \Sigma_2} \\
&\quad + \|\mathbf{J} - \mathbf{J}_h\|_{\frac{1}{s\mu_0\sigma}} \|e\|_{s\mu_0\sigma} + \|\mathbf{J}_0 - \mathbf{J}_{0h}\|_{\mu_0\epsilon} \|e\|_{s^2\mu_0\epsilon} + \|\mathbf{J}_0 - \mathbf{J}_{0h}\|_{s\mu_0\sigma} \|e\|_{s\mu_0\sigma} \\
&\quad + \|\mathbf{J}_0 - \mathbf{J}_{0h}\|_{\frac{1}{s}, \Sigma_1 \cup \Sigma_2} \|e\|_{s, \Sigma_1 \cup \Sigma_2} + \|\mathbf{J}_1 - \mathbf{J}_{1h}\|_{\frac{\sqrt{\mu_0\epsilon}}{s^2}} \|e\|_{s^2\mu_0\epsilon} \\
&\quad + \bar{C} \left(\|h^2 D^2 \hat{\mathbf{U}}\|_{L_2(\Omega)}^{1/2} \|h D^2 \hat{\mathbf{U}}\|_{s, L_2(\Omega)}^{1/2} \right) \|e\|_{s, \Sigma_1 \cup \Sigma_2},
\end{aligned} \tag{47}$$

Recalling now (38), we have

$$|||e|||^2 \leq C \left(\|h\mathcal{R}(\hat{\mathbf{U}}_h)\| + \|h^\alpha \mathcal{R}_{\Sigma_1 \cup \Sigma_2}(\hat{\mathbf{U}}_h)\|_{1/s, \Sigma_1 \cup \Sigma_2} + \mathcal{O}(h^\beta) + \mathcal{O}(h^{3/2}) \right) |||e|||. \tag{48}$$

This gives the desired result.

$$|||e||| \leq C \left(\|h\mathcal{R}(\hat{\mathbf{U}}_h)\| + \|h^\alpha \mathcal{R}_{\Sigma_1 \cup \Sigma_2}(\hat{\mathbf{U}}_h)\|_{1/s, \Sigma_1 \cup \Sigma_2} + \mathcal{O}(h^\beta) + \mathcal{O}(h^{3/2}) \right), \tag{49}$$

where (49) is optimal provided that $\alpha \approx \beta \approx 1$. $\square$

5.3 Error estimate using the orthogonal projection: h-version

Let $\Pi\hat{\mathbf{U}}$ denote the $L_2(\Omega)$ -orthogonal projection onto the finite element space $\mathbf{S}_h^U(\Omega)$; that is, for a given $\hat{\mathbf{U}} \in L_2(\Omega)$, $\Pi\hat{\mathbf{U}}$ is defined by

$$(\hat{\mathbf{U}} - \Pi\hat{\mathbf{U}}, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{S}_h^U(\Omega). \tag{50}$$

The global error $\hat{\mathbf{U}} - \hat{\mathbf{U}}_h$ can then be split as

$$\hat{\mathbf{U}} - \hat{\mathbf{U}}_h = (\hat{\mathbf{U}} - \Pi\hat{\mathbf{U}}) + (\Pi\hat{\mathbf{U}} - \hat{\mathbf{U}}_h) \equiv \rho + \zeta, \tag{51}$$

where ρ denotes the error in the continuous space with the orthogonal interpolation projection and ζ denotes the error in the finite-dimensional space $\mathbf{S}_h^U(\Omega)$.

Lemma 2. There is a constant C such that, the functions ρ and ζ introduced in (50) fulfill the ensuing inequality:

$$|||\zeta|||^2 \leq C |||\rho|||^2. \tag{52}$$

Proof From (33) and (51) we obtain

$$\mathcal{A}(\rho + \zeta, \zeta) = 0 \implies \mathcal{A}(\zeta, \zeta) = -\mathcal{A}(\rho, \zeta). \tag{53}$$

We begin examining the left side of (53). Using the definitions of the norm and the weighted norm in the Sobolev space together with relation (10), it follows that

$$\begin{aligned}
\mathcal{A}(\zeta, \zeta) &= (s^2\mu_0\epsilon\zeta, \zeta) + (\nabla\zeta, \nabla\zeta) + ((\nabla\epsilon)\zeta, \nabla \cdot \zeta) + ((\epsilon - 1)\nabla \cdot \zeta, \nabla \cdot \zeta) \\
&\quad + (s\mu_0\sigma\zeta, \zeta) + \langle s\zeta, \zeta \rangle_{\Sigma_1 \cup \Sigma_2} \\
&\geq \frac{3}{4} \|\zeta\|_{s^2\mu_0\epsilon}^2 + \frac{1}{2} \|\nabla\zeta\|^2 + \frac{3}{4} \|\nabla \cdot \zeta\|_{(\epsilon-1)}^2 + \frac{1}{2} \|\zeta\|_{s\mu_0\sigma}^2 + \|\zeta\|_{s, \Sigma_1 \cup \Sigma_2}^2.
\end{aligned} \tag{54}$$

Likewise, applying integration by parts yields

$$\begin{aligned}
\mathcal{A}(\rho, \zeta) &= (s^2 \mu_0 \epsilon \rho, \zeta) + (\nabla \rho, \nabla \zeta) + (\nabla \cdot (\epsilon \rho), \nabla \cdot \zeta) - (\nabla \cdot \rho, \nabla \cdot \zeta) \\
&\quad + (s \mu_0 \sigma \rho, \zeta) + \langle s \rho, \zeta \rangle_{\Sigma_1 \cup \Sigma_2} \\
&\leq \frac{1}{2} \|\rho\|_{s^2 \mu_0 \epsilon}^2 + \|\nabla \rho\|^2 + \|\nabla \cdot \rho\|_{(\epsilon-1)}^2 + \|\rho\|_{s \mu_0 \sigma}^2 + \|\rho\|_{s, \Sigma_1 \cup \Sigma_2}^2 \\
&\quad + \frac{1}{4} \left(2 \|\zeta\|_{s^2 \mu_0 \epsilon}^2 + \|\nabla \zeta\|^2 + \|\nabla \cdot \zeta\|_{(\epsilon-1)}^2 + \|\zeta\|_{s \mu_0 \sigma}^2 + \|\zeta\|_{s, \Sigma_1 \cup \Sigma_2}^2 \right)
\end{aligned} \tag{55}$$

Combining (54) and (55), we get

$$\begin{aligned}
&\frac{3}{4} \|\zeta\|_{s^2 \mu_0 \epsilon}^2 + \frac{1}{2} \|\nabla \zeta\|^2 + \frac{3}{4} \|\nabla \cdot \zeta\|_{(\epsilon-1)}^2 + \frac{1}{2} \|\zeta\|_{s \mu_0 \sigma}^2 + \|\zeta\|_{s, \Sigma_1 \cup \Sigma_2}^2 \\
&\quad - \frac{1}{4} \left(2 \|\zeta\|_{s^2 \mu_0 \epsilon}^2 + \|\nabla \zeta\|^2 + \|\nabla \cdot \zeta\|_{(\epsilon-1)}^2 + \|\zeta\|_{s \mu_0 \sigma}^2 + \|\zeta\|_{s, \Sigma_1 \cup \Sigma_2}^2 \right) \\
&\leq \frac{1}{2} \|\rho\|_{s^2 \mu_0 \epsilon}^2 + \|\nabla \rho\|^2 + \|\nabla \cdot \rho\|_{(\epsilon-1)}^2 + \|\rho\|_{s \mu_0 \sigma}^2 + \|\rho\|_{s, \Sigma_1 \cup \Sigma_2}^2,
\end{aligned} \tag{56}$$

The left-hand side of (56) reads

$$\begin{aligned}
&\frac{3}{4} \|\zeta\|_{s^2 \mu_0 \epsilon}^2 + \frac{1}{2} \|\nabla \zeta\|^2 + \frac{3}{4} \|\nabla \cdot \zeta\|_{(\epsilon-1)}^2 + \frac{1}{2} \|\zeta\|_{s \mu_0 \sigma}^2 + \|\zeta\|_{s, \Sigma_1 \cup \Sigma_2}^2 \\
&\quad - \frac{1}{4} \left(2 \|\zeta\|_{s^2 \mu_0 \epsilon}^2 + \|\nabla \zeta\|^2 + \|\nabla \cdot \zeta\|_{(\epsilon-1)}^2 + \|\zeta\|_{s \mu_0 \sigma}^2 + \|\zeta\|_{s, \Sigma_1 \cup \Sigma_2}^2 \right) \\
&= \frac{3}{4} \left(\frac{1}{3} \|\zeta\|_{s^2 \mu_0 \epsilon}^2 + \|\nabla \zeta\|^2 + \frac{2}{3} \|\nabla \cdot \zeta\|_{(\epsilon-1)}^2 + \|\zeta\|_{s \mu_0 \sigma}^2 + \|\zeta\|_{s, \Sigma_1 \cup \Sigma_2}^2 \right) \\
&\geq C_1 \|\zeta\|^2,
\end{aligned} \tag{57}$$

For the right-hand side of (56), we get the estimate

$$\frac{1}{2} \|\rho\|_{s^2 \mu_0 \epsilon}^2 + \|\nabla \rho\|^2 + \|\nabla \cdot \rho\|_{(\epsilon-1)}^2 + \|\rho\|_{s \mu_0 \sigma}^2 + \|\rho\|_{s, \Sigma_1 \cup \Sigma_2}^2 \leq C_2 \|\rho\|^2, \tag{58}$$

Finally

$$C_1 \|\zeta\|^2 \leq C_2 \|\rho\|^2 \implies \|\zeta\| \leq \sqrt{\frac{C_2}{C_1}} \|\rho\|, \tag{59}$$

Thus, the desired result is obtained, with $C = \sqrt{\frac{C_2}{C_1}}$, and the proof is complete. $\square$

6 Finite element method based on local polynomial approximations of degree p (p -version)

Consider a subdivision $\mathcal{T}_h$ of Ω into disjoint open element domains K , where each $K \in \mathcal{T}_h$ is obtained from a fixed master element $\hat{K}$, taken as the open unit simplex, through an affine mapping $\mathbf{J}_K$ (i.e., $K = \mathbf{J}_K(\hat{K})$). For a nonnegative integer p , let $\mathcal{P}_p(\hat{K})$ denote the set of polynomials of total degree p on $\hat{K}$. We then introduce the finite element space

$$S^p(\Omega, \mathcal{T}) = \{u \in L_2(\Omega) : u|_K \circ \mathbf{J}_K \in \mathcal{P}_p(\hat{K})\}.$$

Lemma 3. Let $I = (-1, 1)$ be the reference interval and let $\hat{u} \in H^m(I)$. Then $\hat{u}$ admits a Legendre series expansion

$$\hat{u}(\psi) = \sum_{i=0}^{\infty} a_i L_i(\psi). \tag{60}$$

For $m \geq 0$,

$$\int_{-1}^1 |\hat{u}^{(m)}(\psi)|^2 (1 - \psi^2)^m d\psi = \sum_{i \geq m} |a_i|^2 \frac{2}{2i+1} \frac{(i+m)!}{(i-m)!}. \quad (61)$$

Proof The proof is straightforward and established by applying the orthogonality properties of the derivatives of Legendre polynomials [7]. $\square$

Theorem 4. Let $\hat{u}$ be as in (60), s_p the partial sum of Legendre series:

$$s_p = \sum_{i=0}^p a_i L_i(\psi),$$

of $\hat{u}$. Then we have the approximation error estimate

$$\|\hat{u} - s_p\|_{L_2(I)} \leq \left[\frac{(p+1-m)!}{(p+1+m)!} \right]^{\frac{1}{2}} |\hat{u}|_{\mathbf{V}^m(I)}. \quad (62)$$

for $0 \leq m \leq \min(p+1, k)$ and

$$|\hat{u}|_{\mathbf{V}^m(I)} = \int_{-1}^1 (1 - \psi^2)^{(m)} |\hat{u}^m(\psi)|^2 d\psi. \quad (63)$$

Proof See [1]. $\square$

6.1 hp -version error estimates

We aim to obtain estimates for the triple norm of the projection error ρ in (52), where the dependence on mesh size $\mathbf{h}$ and the spectral order p are made explicit. For this purpose, we denote by I the interval $(-1, 1)$ and define the d -dimensional reference element by $\hat{K} = I^d$.

Lemma 4. Let $r \geq 1$ be an integer and $\hat{u} \in H^r(I)$, $I = (-1, 1)$. For $p \geq 0$, let $\hat{\Pi}_p \hat{u}$ be the $L_2(I)$ -projection of $\hat{u}$ onto $\mathcal{P}_p(I)$. Then, for every integer m satisfying

$$0 \leq m \leq \min\{p+1, r\},$$

the following approximation estimate holds:

$$\|\hat{u} - \hat{\Pi}_p \hat{u}\|_{L_2(I)}^2 \leq \frac{\Gamma(p+2-m)}{\Gamma(p+2+m)} \|w^m \hat{u}^{(m)}\|_{L_2(I)}^2 \leq \frac{\Gamma(p+2-m)}{\Gamma(p+2+m)} |\hat{u}|_{H^m(I)}^2, \quad (64)$$

where $w = w(\hat{x}) = (1 - \hat{x}^2)^{\frac{1}{2}}$ and Γ is the Gamma-function.

Proof See [9]. $\square$

We extend the error estimates to the multidimensional setting with $d > 1$ via a tensor-product construction. For each coordinate direction $i = 1, \dots, d$, we denote by $\hat{\Pi}_p^{(i)}$ the one-dimensional $L_2(I)$ -projection onto the polynomial space of degree at most p in the variable $\hat{x}_i$. In addition, let $\hat{\Pi}_p$ denote the corresponding $L_2(\hat{K})$ -projection onto the tensor-product polynomial space with degree not exceeding p in each coordinate direction. Then

$$\hat{\Pi}_p = \hat{\Pi}_p^{(1)} \hat{\Pi}_p^{(2)} \dots \hat{\Pi}_p^{(d)}. \quad (65)$$

Using (64), we derive error estimates for $\hat{u} - \hat{\Pi}_p \hat{u}$. For $d = 2$, we have

$$\hat{u} - \hat{\Pi}_p \hat{u} = \hat{u} - \hat{\Pi}_p^{(1)} \hat{\Pi}_p^{(2)} \hat{u} = \hat{u} - \hat{\Pi}_p^{(1)} \hat{u} + \hat{\Pi}_p^{(1)} (\hat{u} - \hat{\Pi}_p^{(2)} \hat{u}),$$

Hence, assuming that $\|\hat{\Pi}_p^{(1)}\| \leq 1$,

$$\|\hat{u} - \hat{\Pi}_p \hat{u}\|_{L_2(\hat{K})} \leq \|\hat{u} - \hat{\Pi}_p^{(1)} \hat{u}\|_{L_2(\hat{K})} + \|\hat{u} - \hat{\Pi}_p^{(2)} \hat{u}\|_{L_2(\hat{K})}, \quad (66)$$

In the case $d > 2$, repeated application of (66) yields

$$\|\hat{u} - \hat{\Pi}_p \hat{u}\|_{L_2(\hat{K})} \leq \sum_{i=1}^d \|\hat{u} - \hat{\Pi}_p^{(i)} \hat{u}\|_{L_2(\hat{K})}. \quad (67)$$

Lemma 5. Let $\hat{K} := (-1, 1)^d$, $d \geq 1$, and assume that $\hat{u} \in H^r(\hat{K})$ for some integer $r \geq 1$. Denote by

$$\hat{\Pi}_p : L_2(\hat{K}) \rightarrow Q_p(\hat{K})$$

the $L_2(\hat{K})$ -projection operator, where $p \geq 0$. Furthermore, let

$$w_i(\hat{x}_i) := (1 - \hat{x}_i^2)^{1/2}, \quad i = 1, \dots, d.$$

Then, for every $m = 0, \dots, \min\{p+1, r\}$, the following estimate holds true:

$$\begin{aligned} \|\hat{u} - \hat{\Pi}_p \hat{u}\|_{L_2(\hat{K})} &\leq \sqrt{\frac{\Gamma(p-m+2)}{\Gamma(p+m+2)}} \sum_{i=1}^d \|w_i^m \partial_i^m \hat{u}\|_{L_2(\hat{K})} \\ &\leq c(d) \sqrt{\frac{\Gamma(p-m+2)}{\Gamma(p+m+2)}} |\hat{u}|_{H^m(\hat{K})}. \end{aligned} \quad (68)$$

Proof See [9] for the proof. $\square$

By means of Lemma 5, we derive an estimate for the L_2 -projection error ρ restricted to each element $K \in \mathcal{T}_h$. Since $K = F_K(\hat{K})$, where F_K is an affine mapping, we define

$\hat{u} = u \circ F_K$ in (68). Moreover, the relation $(\hat{\Pi}_p \hat{u})(\hat{x}) = (\Pi_p u)(F_K(\hat{x}))$ holds for all $\hat{x} \in \hat{K}$. This implies that for any $K \in \mathcal{T}_h$ and $u \in H^m(K)$,

$$\|u - \Pi_p u\|_{L_2(K)} \leq c(d) h_K^m \sqrt{\frac{\Gamma(p-m+2)}{\Gamma(p+m+2)}} |u|_{H^m(K)}, \quad 0 \leq m \leq \min(p+1, r). \quad (69)$$

6.2 *hp*-Convergence in the Finite Element Method

Lemma 6. Using (51), the triangle inequality and (52), we obtain

$$\begin{aligned} |||\hat{\mathbf{U}} - \hat{\mathbf{U}}_h||| &\leq |||\rho||| + |||\zeta||| \leq |||\rho||| + C |||\rho||| = \tilde{C} |||\rho||| \\ &\leq C \left(\frac{h}{p}\right)^m |||\hat{\mathbf{U}}|||_{H^m(K)}. \end{aligned} \quad (70)$$

Proof We deduce the result by applying Stirling's formula to (69) (see [1]). $\square$

7 Numerical Results and Discussion

Algorithm 1 Adaptive hp-FEM refinement strategy

- 1: Generate an initial quadrilateral mesh $\mathcal{T}_h$.
 - 2: Set the initial polynomial degree $p = 1$.
 - 3: **repeat**
 - 4: Solve the discrete finite element problem on the current hp mesh.
 - 5: Compute the local residual-based error indicators η_K and the global estimator
$$\eta = \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2}.$$
 - 6: Select elements for refinement using the Dörfler marking strategy.
 - 7: **for all** marked elements **do**
 - 8: Evaluate the local smoothness indicator using the decay of the Legendre expansion coefficients.
 - 9: **if** the solution is locally smooth **then**
 - 10: Increase the local polynomial degree (*p*-refinement).
 - 11: **else**
 - 12: Subdivide the element (*h*-refinement).
 - 13: **end if**
 - 14: **end for**
 - 15: Update the hp mesh.
 - 16: **until** $\eta < \eta_{\text{tol}}$ or the maximum number of adaptive iterations is reached
-

Before presenting the numerical examples, we summarize the adaptive implementation employed throughout this section. The implementation combines residual-based error

estimation, and an automatic decision between local mesh subdivision and polynomial enrichment based on a local smoothness indicator. Starting from an initial coarse mesh with polynomial degree $p = 1$, the proposed method follows the classical *SOLVE–ESTIMATE–MARK–REFINE* cycle. The adaptive implementation is summarized in [Algorithm 16](#).

Example 1. In this section, we present numerical experiments on the computational domain $\Omega = [0, 1]^2$. The interior subdomain is defined by $\Omega^1 = [0.25, 0.75]^2$. The source term $\mathbf{J}(\mathbf{x})$, with $\mathbf{x} := (x, y) \in \mathbb{R}^2$, in (7) corresponding to the electric field $\hat{\mathbf{U}} = (\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2)$ is chosen such that the exact solution is given by

$$\begin{aligned} \mathbf{U}_1 &= \frac{1}{s^3 \epsilon} \cos(\pi x) e^{-y} \sin^2(\pi y), \\ \mathbf{U}_2 &= -\frac{1}{s^3 \epsilon} \cos(2\pi x) e^{-y} \sin^2(\pi y). \end{aligned} \quad (71)$$

In (71) the function $\epsilon(\mathbf{x})$, $\mathbf{x} := (x, y) \in \mathbb{R}^2$ is defined as

$$\epsilon(x, y) = \begin{cases} 1 + \sin^m \pi(2x - 0.375) \cdot \sin^m \pi(2y - 0.375) \\ 1 \end{cases} \quad \text{in } \Omega^2 := \Omega \setminus \Omega^1, \quad (72)$$

Further we take $\sigma(\mathbf{x})$, $\mathbf{x} := (x, y) \in \mathbb{R}^2$ as

$$\sigma(x, y) = \begin{cases} 0.001 + \sin^m \pi(2x - 0.375) \cdot \sin^m \pi(2y - 0.375) \\ 0 \end{cases} \quad \text{in } \Omega^2 := \Omega \setminus \Omega^1. \quad (73)$$

The domain Ω is initially decomposed into 4×4 uniform square elements with mesh size $h = \frac{1}{4}$. Numerical experiments are carried out for $m = 2, 3, 4, 5$ within (72) and (73). In the following numerical experiments, the a posteriori error estimate is denoted by η while e represents the L_2 error. The adaptive procedure is carried out for 30 iterations, and selected results are reported for different values of m .

Tables 1–4 present the convergence history of the adaptive hp -FE algorithm. N_e denotes total number of elements in the computational mesh, N_{dof} is the total number of degrees of freedom associated with finite element discretization, and $\max(p)$ indicates the highest polynomial degree in the approximation space at each adaptive iteration.

For all considered values of m , both the a posteriori error estimate η and the absolute L_2 error e decrease monotonically throughout the 30 adaptive iterations, as reported in Tables 1–4.

Figure 1 shows the evolution of the residual estimator η defined in (72) and (73), which is used as the stopping criterion. The steady decrease of η throughout the 30 adaptive iterations indicates that the adaptive refinement strategy, combining h - and p -refinements, effectively concentrates the refinement in regions characterized by large solution gradients, such as those near the free boundary or singularities. The monotonic reduction is a clear evidence for effectiveness of the adaptive strategy.

<i>Iteration</i>	<i>N_e(Elements)</i>	<i>N_{dof}(DoFs)</i>	<i>max(p)</i>	<i>η(ErrorEst.)</i>	<i>e(L₂Error)</i>
1	16	50	1	1.105e-03	6.236e-06
5	198	548	2	1.599e-04	1.066e-06
10	368	1070	2	1.110e-04	8.031e-07
15	588	1700	4	8.691e-05	6.996e-07
20	858	2632	4	7.182e-05	5.940e-07
25	1178	3652	4	6.018e-05	5.707e-07
30	1548	4926	4	5.157e-05	5.043e-07

Table 1: Adaptive hp -FEM convergence for $m = 2$ in (72) and (73): a posteriori error estimate η and absolute L_2 error e at selected iterations

<i>Iteration</i>	<i>N_e(Elements)</i>	<i>N_{dof}(DoFs)</i>	<i>max(p)</i>	<i>η(ErrorEst.)</i>	<i>e(L₂Error)</i>
1	16	50	1	1.146e-03	6.552e-06
5	210	506	2	1.515e-04	8.224e-07
10	380	950	2	1.112e-04	6.834e-07
15	600	1688	3	8.443e-05	5.770e-07
20	870	2504	4	6.910e-05	5.172e-07
25	1190	3446	4	5.922e-05	5.481e-07
30	1560	4562	6	5.149e-05	4.888e-07

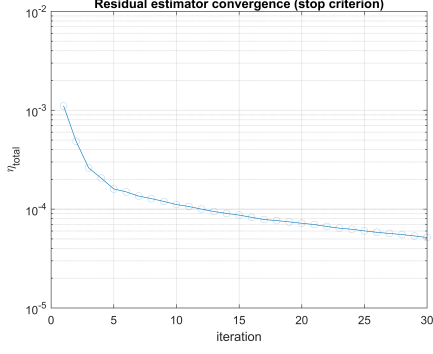
Table 2: Adaptive hp -FEM convergence for $m = 3$ in (72) and (73): a posteriori error estimate η and absolute L_2 error e at selected iterations

<i>Iteration</i>	<i>N_e(Elements)</i>	<i>N_{dof}(DoFs)</i>	<i>max(p)</i>	<i>η(ErrorEst.)</i>	<i>e(L₂Error)</i>
1	16	50	1	1.175e-03	6.757e-06
5	256	634	2	1.455e-04	9.985e-07
10	462	1280	3	9.698e-05	6.639e-07
15	702	2000	4	7.858e-05	6.080e-07
20	992	2868	4	6.512e-05	5.762e-07
25	1332	3928	4	5.561e-05	5.210e-07
30	1722	5100	4	4.878e-05	4.802e-07

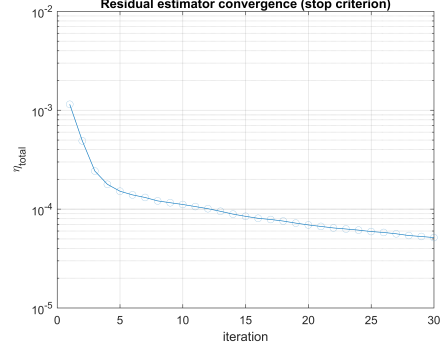
Table 3: Adaptive hp -FEM convergence for $m = 4$ in (72) and (73): a posteriori error estimate η and absolute L_2 error e at selected iterations

<i>Iteration</i>	<i>N_e(Elements)</i>	<i>N_{dof}(DoFs)</i>	<i>max(p)</i>	<i>η(ErrorEst.)</i>	<i>e(L₂Error)</i>
1	16	50	1	1.195e-03	6.884e-06
5	240	600	2	1.503e-04	9.693e-07
10	441	1194	3	1.026e-04	6.698e-07
15	676	1936	4	8.098e-05	6.170e-07
20	961	2832	6	6.679e-05	6.150e-07
25	1296	4218	6	5.636e-05	4.930e-07
30	1681	5958	6	4.995e-05	4.666e-07

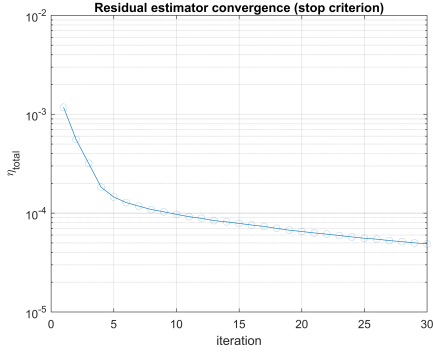
Table 4: Adaptive hp -FEM convergence for $m = 5$ in (72) and (73): a posteriori error estimate η and absolute L_2 error e at selected iterations



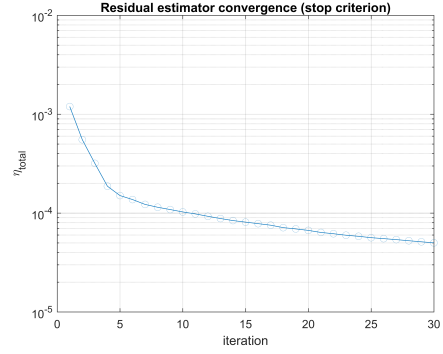
(a) $m=2$



(b) $m=3$



(c) $m=4$

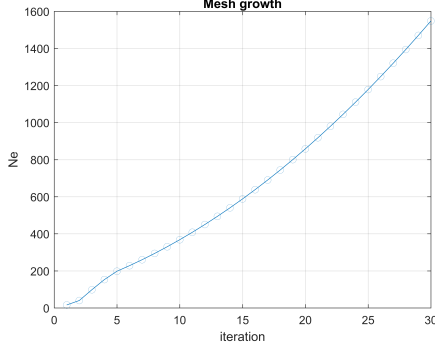


(d) $m=5$

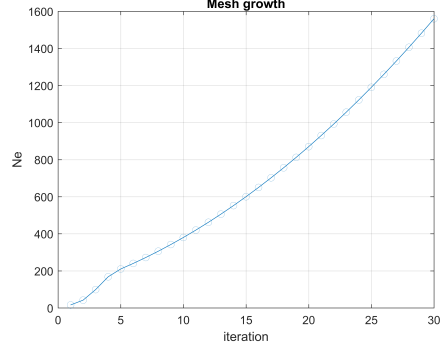
Fig. 1: History of the a posteriori residual error estimator η over adaptive iterations for $m = 2, 3, 4, 5$ in (72) and (73).

Figure 2 presents the evolution of the number of mesh elements, N_e , over the adaptive iterations. Starting from an initial coarse mesh consisting of 16 elements, the mesh is refined progressively as the adaptive procedure advances. The gradual increase in N_e indicates that refinement is applied selectively rather than uniformly across the computational domain, thereby focusing computational effort in regions where additional resolution is needed. This behavior is consistent with the expected characteristics of an efficient hp -adaptive refinement strategy.

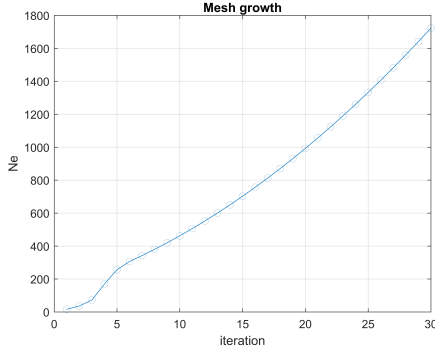
Figure 3 illustrates the evolution of the polynomial degree in the adaptive procedure, demonstrating the characteristic behavior of an hp -adaptive refinement strategy. While h -refinement is performed in regions of low regularity, p -refinement is applied in smooth regions. As shown in panel (3a) of Fig. 3 for $m = 2$, the polynomial degree increases in a stepwise manner, from $p = 1$ to $p = 2$ at iteration 3 and subsequently



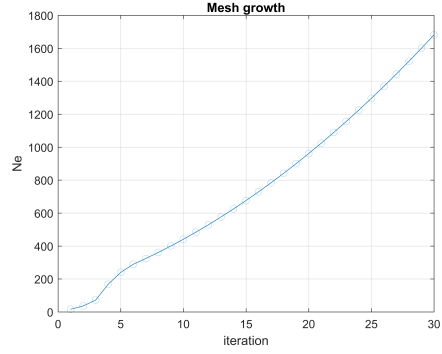
(a) $m=2$



(b) $m=3$



(c) $m=4$



(d) $m=5$

Fig. 2: Mesh growth dynamics for $m = 2, 3, 4, 5$ in (72) and (73): number of elements N_e vs. iteration count. The gradual increase from a coarse initial mesh (16 elements) to a final refined mesh indicates controlled adaptive refinement.

to $p = 3$ at iteration 12. A characteristic behavior of effective hp -adaptive refinement, allowing the approximation space to be enriched only where appropriate while avoiding unnecessary refinement at earlier stages. The subsequent stabilization of the maximum polynomial degree suggests that the prescribed error tolerance is achieved without further increase of the approximation order.

Figure 4 illustrates the exponential convergence behavior of the adaptive hp -FEM characterized by the estimate

$$e \leq C \exp\left(-b N_{\text{dof}}^{1/3}\right),$$

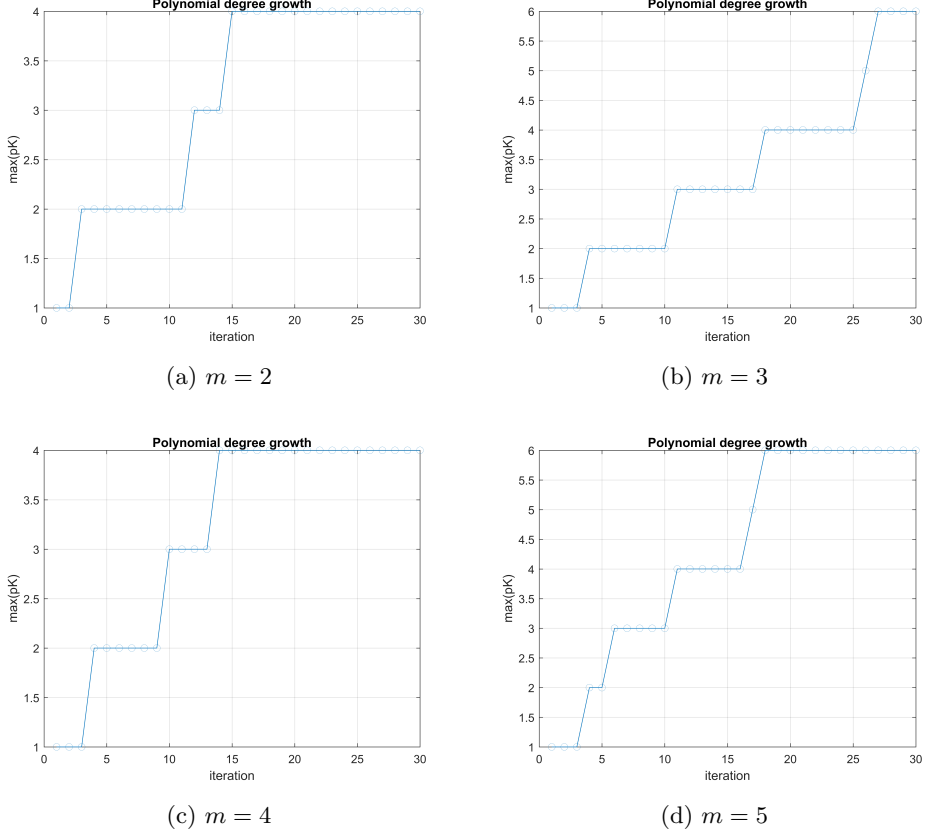
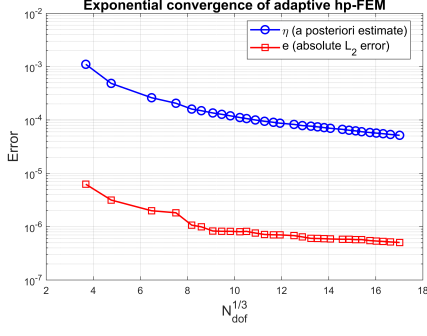


Fig. 3: Evolution of the maximum polynomial order $\max(p)$ during the adaptive process for $m = 2, 3, 4, 5$ in (72) and (73). The stepwise increase from $p = 1$ to $p = 3$ for $m = 2$ and $m = 4$ (up to $p = 5$ and $p = 6$ for $m = 5$ and $m = 3$, respectively) demonstrates the activation of p -refinement alongside h -refinement.

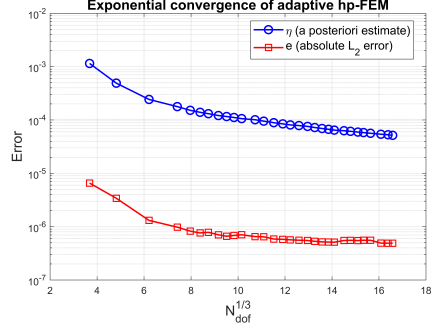
where $C > 0$ and $b > 0$ are constants independent of the discretization. Consequently, an approximately straight line on a semi-logarithmic plot with respect to $N_{\text{dof}}^{1/3}$ provides numerical evidence of exponential convergence.

The nearly linear decay of both the residual-based error estimator η and the absolute L_2 error is consistent with the exponential convergence expected for adaptive hp finite elements. This behavior reflects the combined effect of localized h -refinement and selective p -enrichment, which enables the approximation error to decrease rapidly while maintaining a moderate growth in the number of degrees of freedom.

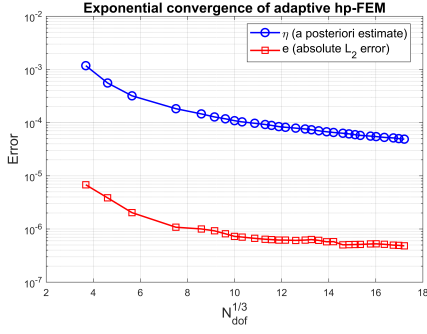
Example 2. The geometric configuration, domain decomposition, initial mesh, and adaptive refinement strategy are the same as those used in Example 1. Hence, only the exact solution and the coefficient functions ϵ and σ are changed.



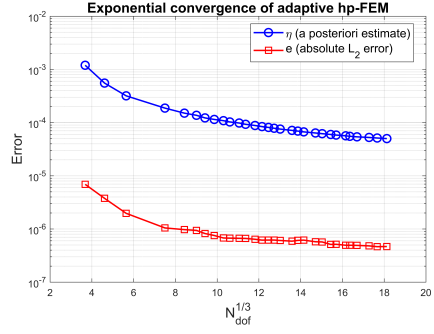
(a) $m=2$



(b) $m=3$



(c) $m=4$



(d) $m=5$

Fig. 4: Exponential convergence of the adaptive hp -FEM for different values of $m = 2, 3, 4, 5$ in (72) and (73). Comparison of the a posteriori error estimate η and the absolute L_2 error e versus the cubic root of the number of degrees of freedom $N_{dof}^{1/3}$ on a semi-logarithmic scale.

The exact solution is prescribed as

$$\begin{aligned} \mathbf{U}_1 &= \frac{\pi}{s^3 \epsilon(x, y)} (6x^5 - 15x^4 + 10x^3) \sin(2\pi y), \\ \mathbf{U}_2 &= -\frac{30}{s^3 \epsilon(x, y)} x^2 (1-x)^2 \sin^2(\pi y). \end{aligned} \quad (74)$$

The functions $\epsilon(\mathbf{x})$ and $\sigma(\mathbf{x})$, $\mathbf{x} := (x, y) \in \mathbb{R}^2$ are defined as

$$\epsilon(x, y) = \begin{cases} 2 & \text{in } \Omega^1, \\ 1 & \text{in } \Omega^2 := \Omega \setminus \Omega^1, \end{cases} \quad (75)$$

$$\sigma(x, y) = \begin{cases} 1 + \sin^2(4\pi x) \cos^2(4\pi y), & \text{in } \Omega^1, \\ 0 & \text{in } \Omega^2 := \Omega \setminus \Omega^1. \end{cases} \quad (76)$$

As shown in Table 5, both the a posteriori error estimator η and the absolute L_2 error e decrease monotonically throughout the 30 adaptive iterations. During this process, the mesh is progressively refined from 16 to 1800 elements, while the maximum polynomial degree increases from 1 to 4. These results demonstrate the effectiveness and robustness of the proposed hp -adaptive strategy used for Example 2.

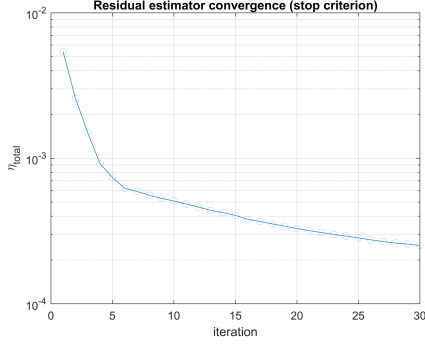
<i>Iteration</i>	<i>N_e(Elements)</i>	<i>N_{dof}(DoFs)</i>	<i>max(p)</i>	<i>η(ErrorEst.)</i>	<i>e(L₂Error)</i>
1	16	50	1	5.385e-03	3.481e-05
5	266	720	3	7.382e-04	1.861e-05
10	500	1452	3	5.081e-04	1.808e-05
15	750	2374	4	4.049e-04	1.783e-05
20	1050	3474	4	3.286e-04	1.764e-05
25	1400	4458	4	2.834e-04	1.757e-05
30	1800	5642	4	2.517e-04	1.751e-05

Table 5: Adaptive hp -FEM convergence: a posteriori error estimate η and absolute L_2 error e at selected iterations

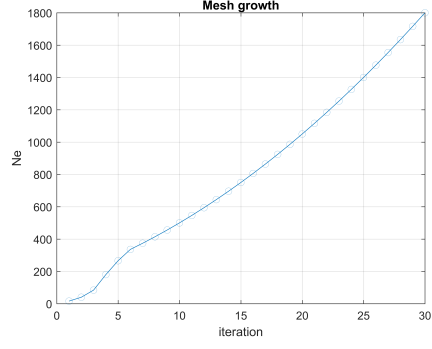
Figure 5a illustrates the evolution of the residual-based a posteriori error estimator η associated with (75) and (76). The estimator decreases monotonically throughout the adaptive process, exhibiting a pronounced reduction during the initial refinement stages followed by a steady decay. This behavior indicates that the adaptive refinement procedure effectively concentrates refinements in regions where the estimated error is largest. The continued decrease of η is consistent with the convergence of the adaptive algorithm and demonstrates the effectiveness of the proposed hp -adaptive strategy in producing increasingly accurate approximations while maintaining computational efficiency.

Figure 5b presents the evolution of the mesh throughout the adaptive process. The number of elements increases progressively from 16 in the initial discretization to 1800 at the final iteration, reflecting the cumulative effect of successive local refinements. The smooth growth of the mesh indicates that refinement is applied in a controlled manner, avoiding unnecessary refinements. Together with the concurrent reduction of the a posteriori error estimator, this behavior suggests that the adaptive strategy effectively concentrates refinement in regions requiring increased resolution. Overall, the observed mesh-growth pattern is consistent with the expected performance of an efficient hp -adaptive refinement strategy.

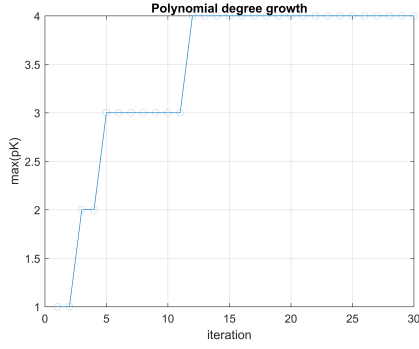
Figure 5c shows the evolution of the maximum polynomial degree employed by the adaptive hp -FE algorithm. The successive increases of $\max(p)$ from 1 to 4 indicate that the adaptive procedure combines local polynomial enrichment with mesh refinement, rather than relying solely on mesh subdivision. Once $p = 4$ is reached, the maximum polynomial degree remains unchanged, suggesting that subsequent improvements in accuracy are achieved primarily through localized mesh refinement instead



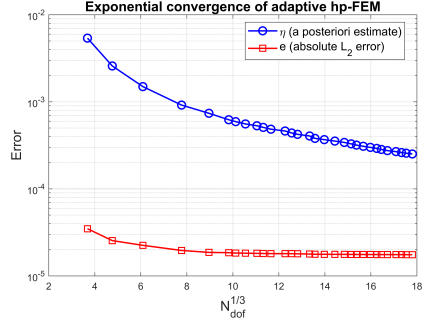
(a) Convergence history of the residual-based a posteriori error estimator η for (75) and (76).



(b) Mesh growth dynamics: number of elements N_e vs. iteration count. The gradual increase from a coarse initial mesh (16 elements) to a final refined mesh indicates controlled adaptive refinement.



(c) Evolution of the maximum polynomial order $\max(p)$ during the adaptive process. The stepwise increase from $p = 1$ to $p = 4$ demonstrates the activation of p -refinement alongside h -refinement.



(d) Exponential convergence of the adaptive hp -FEM. Comparison of the a posteriori error estimate η and the absolute L_2 error e versus the cubic root of the number of degrees of freedom $N_{\text{dof}}^{1/3}$ on a semi-logarithmic scale.

Fig. 5

of further increasing the approximation order. This behavior is consistent with the adaptive nature of hp -FEM, in which h - and p -refinements are selected according to the estimated local regularity of the solution. The stabilization of $\max(p)$ therefore suggests that the current polynomial degree is sufficient to satisfy the prescribed error tolerance, while the continued reduction of both a posteriori error estimator and the discretization error is achieved through further local mesh refinements.

Figure 5d presents the convergence behavior of the proposed adaptive hp -FEM in terms of the a posteriori error estimator η and the absolute L_2 error e as functions of $N_{\text{dof}}^{1/3}$. As discussed in Example 1, optimal convergence for adaptive hp finite element methods is characterized by an approximately linear decay on a semi-logarithmic plot with respect to $N_{\text{dof}}^{1/3}$. The nearly linear decay of the error estimator on the semi-logarithmic scale is consistent with optimal convergence with respect to the number of degrees of freedom, a characteristic feature of hp -FEM. The absolute L_2 error decreases in a similar manner and remains in close agreement with the trend predicted by the error estimator, supporting the effectiveness of the adaptive refinement criterion. This convergence behavior reflects the combined effect of localized h -refinement and selective p -enrichment, enabling accurate approximations to be achieved with a moderate increase in the number of degrees of freedom. Overall, the numerical results are consistent with the expected performance of an efficient hp -adaptive refinement strategy.

Example 3. The computational setting is the same as in Example 1. In particular, the geometry, partitioning, boundary conditions, initial mesh, and adaptive refinement strategy remain unchanged. The only modification concerns the exact solution and the material coefficient functions ϵ and σ , which are given by

$$\begin{aligned} \mathbf{U}_1 &= \frac{1}{s^3 \epsilon(x, y)} \cos(\pi x) \cosh(y) e^{-2y} (1 + y) \sin^2(\pi y), \\ \mathbf{U}_2 &= -\frac{1}{s^3 \epsilon(x, y)} \cos(2\pi x) \sinh(y) e^{-y} (1 + y^2) \sin^2(\pi y), \end{aligned} \quad (77)$$

$$\epsilon(x, y) = \begin{cases} 1 + \exp(-20 [(x - 0.5)^2 + (y - 0.5)^2]), & \text{in } \Omega^1, \\ 1, & \text{in } \Omega^2 := \Omega \setminus \Omega^1, \end{cases} \quad (78)$$

$$\sigma(x, y) = \begin{cases} 1.01 + \exp(-20 [(x - 0.5)^2 + (y - 0.5)^2]), & \text{in } \Omega^1, \\ 0 & \text{in } \Omega^2 := \Omega \setminus \Omega^1. \end{cases} \quad (79)$$

Table 6 presents convergence history of the proposed hp -adaptive finite element method. Both the a posteriori error estimator η and the absolute L_2 error e decrease monotonically throughout the iterations. Meanwhile, the number of elements and the maximum polynomial degree increase progressively, reflecting the adaptive refinement process. This coordinated evolution of the mesh and approximation order leads to increasingly accurate numerical approximations.

Figure 6a shows the evolution of the residual-based a posteriori error estimator η associated with (78) and (79). The estimator decreases rapidly during the initial adaptive iterations, followed by a more gradual decay as the refinement process progresses. This behavior indicates that the adaptive strategy effectively reduces the estimated discretization error in the early stages while continuing to improve the approximation

<i>Iteration</i>	$N_e(\text{Elements})$	$N_{dof}(\text{DoFs})$	$\max(p)$	$\eta(\text{ErrorEst.})$	$e(L_2\text{Error})$
1	16	50	1	8.516e-04	4.960e-06
5	224	560	2	1.249e-04	1.420e-06
10	418	1066	4	8.475e-05	1.242e-06
15	648	1930	6	6.743e-05	1.173e-06
20	928	2684	6	5.704e-05	1.151e-06
25	1258	3626	6	4.778e-05	1.139e-06
30	1638	5216	6	4.128e-05	1.121e-06

Table 6: Adaptive hp -FEM convergence: a posteriori error estimate η and absolute L_2 error e at selected iterations

through localized refinement. The sustained reduction of η throughout the adaptive process is consistent with the convergence of the proposed hp -adaptive refinement strategy.

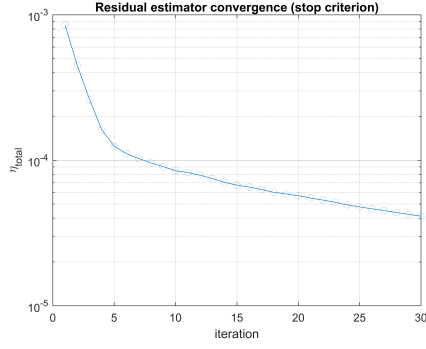
Figure 6b shows the evolution of the computational mesh throughout the adaptive process. As the adaptive iterations progress, the number of elements increases progressively from the initial coarse discretization to the final refined mesh, reflecting the cumulative effect of successive local refinements. Steady increase in the number of elements suggests that the refinements are introduced in a controlled manner, where additional resolution provided only in regions identified by the adaptive procedure. This refinement pattern is consistent with the objective of improving the FE approximation while avoiding unnecessary refinement of the computational domain.

Figure 6c shows the evolution of the maximum polynomial degree over the adaptive iterations. The polynomial degree increases in a stepwise manner, indicating that p -refinement is applied selectively during the adaptive process. The polynomial degree increases from $p = 1$ to $p = 6$, and remains unchanged for the remaining iterations. This behavior is consistent with the adaptive refinement strategy, where polynomial enrichment is introduced only when it is expected to improve the local approximation while maintaining a controlled increase of approximation order.

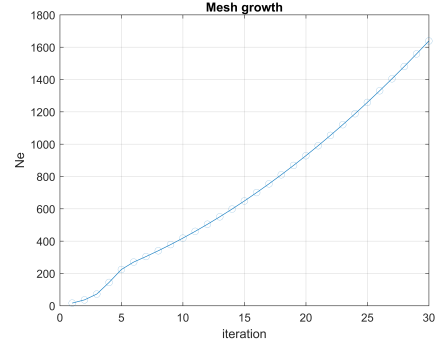
Figure 6d presents the convergence behavior of the adaptive approximation in terms of the residual-based error estimator η and the absolute L_2 error e as functions of $N_{dof}^{1/3}$. Following the exponential convergence criterion introduced in Example 1, an approximately linear decay on the semi-logarithmic scale indicates exponential convergence with respect to $N_{dof}^{1/3}$. Both quantities exhibit a steady decrease throughout the adaptive process. The nearly linear decay of the error estimator on the semi-logarithmic scale is consistent with the expected convergence of an hp -adaptive finite element method, while the corresponding reduction in the absolute L_2 error reflects the progressive improvement of the numerical approximation. Overall, the observed convergence behavior indicates that the adaptive strategy achieves increasingly accurate solutions with a moderate increase of degrees of freedom.

Declarations:

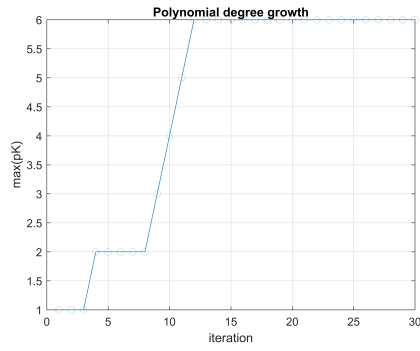
Funding: The authors declare that there was no funding associated with this research.



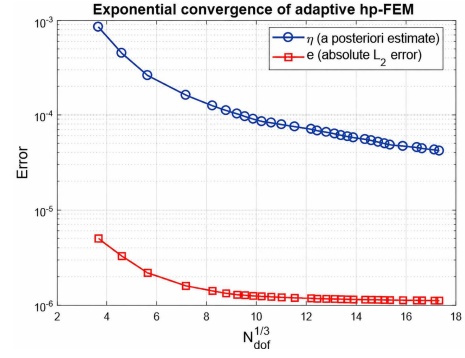
(a) Convergence history of the residual-based a posteriori error estimator η for (78) and (79).



(b) Growth of the computational mesh during the adaptive refinement process, represented by the number of elements N_e versus the iteration count.



(c) Evolution of the maximum polynomial degree $\max(p)$ during the adaptive refinement process.



(d) Convergence history of the residual-based a posteriori error estimator η and the absolute L_2 error e versus $N_{\text{dof}}^{1/3}$ on a semi-logarithmic scale.

Fig. 6

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