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Amir Kahrom

a.kahrom.physics@gmail.com

independent researcher https://orcid.org/0009-0008-2963-3588

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When Do Probability and Madelung Velocities Differ? An Exact Geometric Criterion

Amir Kahrom
a.kahrom.physics@gmail.com
ORCID: 0009-0008-2963-3588

Abstract

In the Madelung formulation of quantum mechanics, the wave-function phase defines a local hydrodynamic velocity field, while the probability density defines observable geometric features whose motion can be tracked in time. We derive an exact local relation between the velocity v_ρ of a non-degenerate probability extremum and the Madelung velocity v_{hyd} evaluated at the same point. Starting from the continuity equation, we obtain the kinematic identity

$$v_\rho - v_{\mathrm{hyd}} = \frac{\rho}{\partial_x^2 \rho}\, \partial_x^2 v_{\mathrm{hyd}},$$

which implies that $v_\rho = v_{\mathrm{hyd}}$ if and only if the hydrodynamic velocity field is locally affine, $\partial_x^2 v_{\mathrm{hyd}} = 0$. The result is purely kinematic: it follows from the continuity equation and the extremum condition alone, depending only on the local geometry of the density and the velocity field. We illustrate this criterion with closed-form examples: the freely propagating Gaussian packet, coherent states, and the accelerating Airy packet all satisfy the equality $v_\rho = v_{\mathrm{hyd}}$. Conversely, a cubic-phase Gaussian state generates a velocity field with non-vanishing spatial curvature and therefore exhibits a genuine mismatch between the extremum velocity and the local phase flow. A three-dimensional extension shows that the coincidence condition becomes $\nabla(\nabla \cdot \mathbf{v}_{\mathrm{hyd}}) = \mathbf{0}$ at a non-degenerate critical point. The result provides a local geometric diagnostic for when probability extrema follow phase-defined flow.

Keywords: quantum mechanics, probability density, Madelung velocity, phase velocity, probability-extremum velocity, quantum hydrodynamics

1 Introduction

The Madelung formulation of non-relativistic quantum mechanics offers a hydrodynamic representation in which the complex wave function $\psi(\mathbf{r}, t)$ is expressed in polar form, decomposing the dynamics into a continuity equation for the probability density and a quantum Hamilton–Jacobi equation for the phase [1, 2]. In one spatial dimension, the spatial gradient of the phase defines the local convective velocity field

$$v_{\mathrm{hyd}}(x, t) = \frac{1}{m} \partial_x S(x, t), \tag{1.1}$$

which enters the continuity equation via the probability current density $j(x, t) = \rho(x, t) v_{\mathrm{hyd}}(x, t)$. In trajectory-based approaches and quantum fluid dynamics, this representation provides a natural hydrodynamic interpretation of probability transport, treating v_{hyd} as the underlying velocity field of the quantum flow [3].

Concurrently, the probability density $\rho(x, t) = |\psi(x, t)|^2$ enjoys a direct operational status: it is experimentally reconstructible through repeated position measurements on identically prepared ensembles without requiring phase retrieval [4, 5]. Its isolated local extrema—such as wave-packet peaks or interference fringes—constitute well-defined geometric features of the quantum state.

The kinematic tracking of moving extrema and intensity structures is an established methodology across classical wave mechanics and singular optics [6, 7]. Transposed to quantum foundations, this raises a natural and fundamental question: *under what conditions does the motion of a probability extremum faithfully trace the underlying phase-induced Madelung flow?*

Let $x_0(t)$ denote the trajectory of an isolated, non-degenerate spatial extremum of $\rho(x,t)$, defined implicitly by the condition

$$\partial_x \rho(x,t)|_{x=x_0(t)} = 0. \tag{1.2}$$

The geometric velocity of this feature is given by

$$v_\rho(t) \equiv \dot{x}_0(t), \tag{1.3}$$

which is intrinsically determined by the local spatiotemporal curvature of the density profile alone. It has frequently been assumed, based on prototypical states such as rigid wave packets and harmonic-oscillator coherent states, that this geometric velocity coincides with the local hydrodynamic velocity:

$$v_\rho(t) \stackrel{?}{=} v_{\mathrm{hyd}}(x_0(t),t). \tag{1.4}$$

While this coincidence underpins the intuitive association between moving probability peaks and underlying fluid streamlines, treating it as a generic identity conflates the geometric kinematics of the distribution with the convective flow encoded in the phase.

In this work, we demonstrate that the relation between v_ρ and v_{hyd} is governed by an exact, local kinematic identity:

$$v_\rho(t) - v_{\mathrm{hyd}}(x_0(t),t) = \left.\frac{\rho}{\partial_x^2 \rho}\, \partial_x^2 v_{\mathrm{hyd}}\right|_{x=x_0(t)}, \tag{1.5}$$

valid at any non-degenerate extremum where $\rho \neq 0$ and $\partial_x^2 \rho \neq 0$. This identity follows strictly from the continuity equation and the implicit differentiation of the extremum condition, requiring no dynamical assumptions, quantum potentials, or integration of the Schrödinger equation. Consequently, the geometric velocity of a density peak coincides with the local Madelung velocity,

$$v_\rho(t) = v_{\mathrm{hyd}}(x_0(t),t), \tag{1.6}$$

if and only if the hydrodynamic velocity field is locally affine in space:

$$\partial_x^2 v_{\mathrm{hyd}}\big|_{x=x_0(t)} = 0 \quad \Longleftrightarrow \quad \partial_x^3 S\big|_{x=x_0(t)} = 0. \tag{1.7}$$

This criterion sharply delineates the conceptual domains of agreement and departure. Freely dispersing Gaussian packets, coherent states, and notably the accelerating Airy wave packet [6] all satisfy $\partial_x^2 v_{\mathrm{hyd}} = 0$. The Airy packet exemplifies that *temporal acceleration of a density feature does not induce a discrepancy*; rather, the breakdown of equality is governed solely by the spatial curvature of the velocity field. To illustrate a genuine failure of the hydrodynamic tracking, we analyze a state with a cubic phase profile, showing that an initial spatial curvature in v_{hyd} produces an instantaneous and quantifiable mismatch $v_\rho \neq v_{\mathrm{hyd}}$ even for a standard Gaussian density.

The remainder of this paper is structured as follows. In Section 2, we formalize the kinematic definition of the extremum velocity. In Section 3, we derive the fundamental identity (1.5) and discuss its geometric content. In Section 4, we analyze closed-form examples spanning both affine and non-affine flows. In Section 5, we generalize the criterion to three spatial dimensions using the density Hessian. Section 6 addresses the foundational implications for the Madelung and de Broglie–Bohm pictures, the operational accessibility of the criterion, and its physical domain of validity. Section 7 provides concluding remarks.

2 Geometric Velocity of Probability Extrema

The probability density $\rho(x,t)$ is, in principle, reconstructible from repeated position measurements performed on identically prepared ensembles [4]. Its local extrema thus constitute observable geometric features of the quantum state, the evolution of which can be tracked kinematically.

Let $x_0(t)$ denote a smooth trajectory of a non-degenerate extremum of $\rho(x,t)$, defined implicitly by the conditions:

$$\partial_x \rho\big(x_0(t),t\big) = 0, \qquad \partial_x^2 \rho\big(x_0(t),t\big) \neq 0. \tag{2.1}$$

The sign of $\partial_x^2\rho$ distinguishes a local maximum from a minimum. We define the *geometric velocity* of this density feature as:

$$v_\rho(t) \equiv \dot{x}_0(t). \tag{2.2}$$

An explicit expression for v_ρ is obtained by differentiating the extremum condition in Eq. (2.1) along the trajectory $x_0(t)$. Requiring that $\partial_x\rho(x_0(t),t) = 0$ holds identically in time implies $\frac{d}{dt}[\partial_x\rho(x_0(t),t)] = 0$. Applying the chain rule, we obtain:

$$\partial_x^2 \rho\big(x_0(t),t\big)\, \dot{x}_0(t) + \partial_x \partial_t \rho\big(x_0(t),t\big) = 0. \tag{2.3}$$

Given the non-degeneracy condition $\partial_x^2\rho \neq 0$, the extremum velocity is:

$$v_\rho(t) = -\left.\frac{\partial_x\partial_t\rho}{\partial_x^2\rho}\right|_{x=x_0(t)}. \tag{2.4}$$

Equation (2.4) is a purely kinematic result. It expresses the motion of a probability extremum solely in terms of the local spatiotemporal geometry of the density, independent of the wavefunction phase or any specific dynamical equation. Consequently, v_ρ is operationally distinct: it may be inferred from successive density measurements without phase reconstruction. This leads to the fundamental question of whether this geometric velocity coincides with the phase-defined flow velocity:

$$v_\rho(t) \stackrel{?}{=} v_{\mathrm{hyd}}\big(x_0(t),t\big). \tag{2.5}$$

3 Derivation of the Geometric Criterion

We now derive an exact local relation between the geometric velocity and the Madelung velocity, defined via the polar form $\psi = \sqrt{\rho}e^{iS/\hbar}$ as $v_{\mathrm{hyd}} = \frac{1}{m}\partial_x S$. Our derivation relies exclusively on the continuity equation, $\partial_t\rho + \partial_x(\rho v_{\mathrm{hyd}}) = 0$, and requires no dynamical assumptions (such as the quantum Hamilton–Jacobi equation).

3.1 Exact Identity

Expanding the continuity equation yields $\partial_t\rho = -v_{\mathrm{hyd}}\partial_x\rho - \rho\partial_x v_{\mathrm{hyd}}$. Differentiating this with respect to x gives:

$$\partial_x\partial_t\rho = -2(\partial_x v_{\mathrm{hyd}})(\partial_x\rho) - v_{\mathrm{hyd}}\,\partial_x^2\rho - \rho\,\partial_x^2 v_{\mathrm{hyd}}. \tag{3.1}$$

At a non-degenerate extremum $x_0(t)$, where $\partial_x\rho = 0$, this simplifies to:

$$\partial_x\partial_t\rho\big|_{x_0} = -v_{\mathrm{hyd}}\,\partial_x^2\rho - \rho\,\partial_x^2 v_{\mathrm{hyd}}. \tag{3.2}$$

Substituting this into our definition of v_ρ (Eq. 2.4), we obtain the exact identity:

$$\boxed{v_\rho - v_{\mathrm{hyd}} = \frac{\rho}{\partial_x^2\rho}\,\partial_x^2 v_{\mathrm{hyd}}}. \tag{3.3}$$

3.2 Local Criterion

Equation (3.3) implies that $v_\rho = v_{\rm hyd}$ if and only if the second spatial derivative of the hydrodynamic velocity vanishes at the extremum:

$$\boxed{v_\rho = v_{\rm hyd} \quad \Longleftrightarrow \quad \partial_x^2 v_{\rm hyd} = 0} \quad \text{at } x = x_0(t). \tag{3.4}$$

The discrepancy between the two velocities is governed by the product of a density-geometry factor and the velocity-curvature factor:

$$v_\rho - v_{\rm hyd} = \left(\frac{\rho}{\partial_x^2 \rho}\right) \partial_x^2 v_{\rm hyd}. \tag{3.5}$$

If the velocity field is locally affine ($\partial_x^2 v_{\rm hyd} = 0$), the geometric velocity coincides with the Madelung flow.

3.3 Physical Interpretation

The correction term in Eq. (3.5) reveals that the coincidence of probability-peak motion and hydrodynamic flow is not an inherent law of quantum evolution, but rather a specific geometric condition satisfied by states with affine-like velocity fields. This decoupling demonstrates that tracking probability extrema is, at a fundamental level, an independent observational procedure that does not universally map onto the Madelung or Bohmian trajectories. The result applies locally to any one-dimensional quantum state admitting a regular Madelung representation, highlighting that the breakdown of hydrodynamic tracking is controlled solely by the spatial curvature of the velocity field, irrespective of temporal acceleration.

4 Exact Consequences of the Geometric Criterion

To substantiate the physical and operational implications of the geometric criterion established in Sec. 3, we now investigate representative exact solutions of the Schrödinger equation. These examples demonstrate how the vanishing or non-vanishing of the spatial curvature of the velocity field dictates the correspondence between the geometric tracking of probability extrema and the convective Madelung flow.

4.1 Freely Propagating Gaussian Wave Packet

The standard benchmark for localized quantum dynamics is the freely propagating Gaussian wave packet. Its complex amplitude is given by

$$\psi(x,t) = \frac{1}{(2\pi\sigma_t^2)^{1/4}} \exp\left[-\frac{(x - X_c(t))^2}{4\sigma_t^2} + \frac{i}{\hbar} S(x,t)\right], \tag{4.1}$$

where $X_c(t) = X_0 + v_0 t$ represents the mean position, and $\sigma_t = \sigma_0 \sqrt{1 + \hbar^2 t^2/(4m^2\sigma_0^4)}$ denotes the position uncertainty at time t. The quantum phase associated with this state is quadratic in space:

$$S(x,t) = \frac{m}{2}\left[\frac{\dot\sigma_t}{\sigma_t}\big(x - X_c(t)\big)^2 + 2\dot X_c(t)\big(x - X_c(t)\big)\right] + \Theta(t), \tag{4.2}$$

with $\Theta(t)$ being a purely time-dependent phase factor.

Evaluating the Madelung velocity field via Eq. (1.1) yields

$$v_{\rm hyd}(x,t) = \frac{1}{m}\partial_x S(x,t) = \dot X_c(t) + \frac{\dot\sigma_t}{\sigma_t}\big(x - X_c(t)\big). \tag{4.3}$$

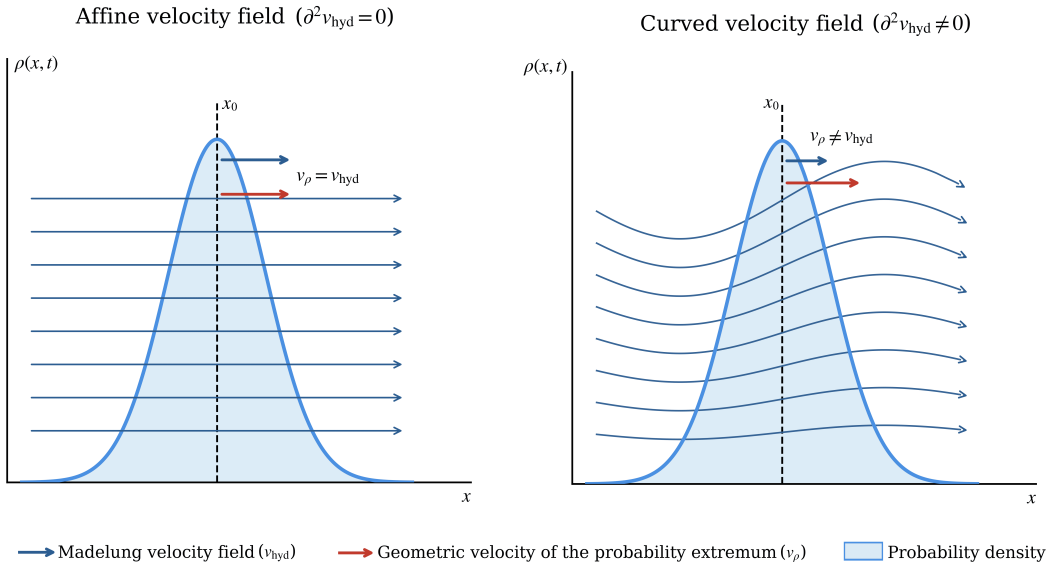


Figure 1: Conceptual comparison between the geometric velocity of a probability extremum and the local Madelung flow. The shaded profile represents the probability density $\rho(x,t)$, while the vector field indicates the local hydrodynamic velocity $v_{\rm hyd}(x,t) = \frac{1}{m}\partial_x S(x,t)$. The prominent arrow designates the geometric velocity $v_\rho = \dot{x}_0(t)$ of the density maximum $x_0(t)$. **Left panel:** The velocity field is locally affine ($\partial_x^2 v_{\rm hyd} = 0$), ensuring exact kinematic coincidence between the peak motion and the local flow ($v_\rho = v_{\rm hyd}$). **Right panel:** The velocity field possesses non-zero spatial curvature ($\partial_x^2 v_{\rm hyd} \neq 0$), generating an exact local kinematic discrepancy $\Delta v_{\rm geom} \equiv v_\rho - v_{\rm hyd} = (\rho/\partial_x^2\rho)\partial_x^2 v_{\rm hyd}$.

Crucially, $v_{\rm hyd}(x,t)$ is strictly affine with respect to the spatial coordinate x. Its second spatial derivative vanishes identically across the entire domain:

$$\partial_x^2 v_{\rm hyd}(x,t) = 0 \quad \forall\, x \in \mathbb{R}. \tag{4.4}$$

According to the local criterion in Eq. (3.4), the vanishing of $\partial_x^2 v_{\rm hyd}$ guarantees that any non-degenerate extremum of $\rho(x,t)$ must propagate at the exact local Madelung velocity.

To verify this directly from the operational definition of v_ρ, consider the probability density:

$$\rho(x,t) = |\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left[-\frac{\big(x - X_c(t)\big)^2}{2\sigma_t^2}\right]. \tag{4.5}$$

The spatial gradient of the density,

$$\partial_x \rho(x,t) = -\frac{x - X_c(t)}{\sigma_t^2}\rho(x,t), \tag{4.6}$$

reveals an isolated, non-degenerate maximum at $x_0(t) = X_c(t)$. At this extremum, the spatial curvature is strictly negative and non-vanishing:

$$\partial_x^2 \rho\big|_{x=X_c} = -\frac{\rho(X_c,t)}{\sigma_t^2} \neq 0. \tag{4.7}$$

Computing the mixed spatiotemporal derivative at the peak gives

$$\partial_x \partial_t \rho\big|_{x=X_c} = \frac{\dot{X}_c(t)}{\sigma_t^2}\rho(X_c,t). \tag{4.8}$$

Substituting Eqs. (4.7) and (4.8) into the geometric velocity formula (2.4) confirms that

$$v_\rho(t) = -\left.\frac{\partial_x \partial_t \rho}{\partial_x^2 \rho}\right|_{x=X_c} = \dot{X}_c(t). \tag{4.9}$$

Concurrently, evaluating the hydrodynamic velocity field (4.3) at the trajectory of the peak yields

$$v_{\mathrm{hyd}}\big(x_0(t), t\big) = v_{\mathrm{hyd}}\big(X_c(t), t\big) = \dot{X}_c(t). \tag{4.10}$$

Hence, we recover exact agreement:

$$v_\rho(t) = v_{\mathrm{hyd}}\big(x_0(t), t\big) = \dot{X}_c(t). \tag{4.11}$$

From a foundational perspective, the Gaussian wave packet illuminates why the tracking of probability peaks is often conflated with hydrodynamic flow: because quadratic phases generate purely affine velocity fields, the geometric peak motion and the fluid velocity coincide identically. However, as demonstrated by the geometric identity (3.3), this agreement is an artifact of the affine nature of the velocity profile ($\partial_x^2 v_{\mathrm{hyd}} = 0$) rather than an intrinsic equivalence between geometric tracking and convective quantum transport.

4.2 Harmonic-Oscillator Coherent State

A paradigmatic realization of rigid quantum transport is provided by the canonical coherent state of a one-dimensional harmonic oscillator with mass m and natural frequency ω. The corresponding probability density retains a constant spatial width and follows the classical trajectory:

$$\rho(x,t) = \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left[-\frac{m\omega}{\hbar}\big(x - \bar{x}(t)\big)^2\right], \tag{4.12}$$

where the centroid position and classical momentum evolve as

$$\bar{x}(t) = A\cos(\omega t + \phi), \qquad p(t) = m\dot{\bar{x}}(t) = -m\omega A \sin(\omega t + \phi). \tag{4.13}$$

Up to a spatially uniform, time-dependent phase factor $\Theta(t)$, the quantum phase associated with the coherent wave packet is strictly linear in the displacement coordinate:

$$S(x,t) = p(t)x + \Theta(t). \tag{4.14}$$

Evaluating the hydrodynamic velocity field via Eq. (1.1) gives

$$v_{\mathrm{hyd}}(x,t) = \frac{1}{m}\partial_x S(x,t) = \frac{p(t)}{m} = \dot{\bar{x}}(t). \tag{4.15}$$

Consequently, the velocity field is spatially homogeneous throughout the entire configuration space:

$$\partial_x v_{\mathrm{hyd}}(x,t) = 0 \quad \Longrightarrow \quad \partial_x^2 v_{\mathrm{hyd}}(x,t) = 0 \quad \forall\, x \in \mathbb{R}. \tag{4.16}$$

This spatial uniformity represents the simplest subclass of affine flows, in which deformation rates vanish identically ($\partial_x v_{\mathrm{hyd}} = 0$).

From the probability density (4.12), the spatial gradient is

$$\partial_x \rho(x,t) = -\frac{2m\omega}{\hbar}\big(x - \bar{x}(t)\big)\rho(x,t), \tag{4.17}$$

which yields a unique, isolated extremum located at $x_0(t) = \bar{x}(t)$. The second spatial derivative at the peak is strictly negative:

$$\partial_x^2 \rho\big|_{x=\bar{x}(t)} = -\frac{2m\omega}{\hbar}\rho\big(\bar{x}(t), t\big) \neq 0, \tag{4.18}$$

confirming non-degeneracy. Differentiating the extremum condition yields the geometric velocity:

$$v_\rho(t) = \dot{x}_0(t) = \dot{\bar{x}}(t) = \frac{p(t)}{m}. \tag{4.19}$$

Comparing Eq. (4.19) with the local Madelung velocity evaluated at the peak trajectory,

$$v_{\mathrm{hyd}}\big(x_0(t), t\big) = \dot{\bar{x}}(t), \tag{4.20}$$

we obtain exact coincidence:

$$v_\rho(t) = v_{\mathrm{hyd}}\big(x_0(t), t\big) = \dot{\bar{x}}(t). \tag{4.21}$$

The harmonic-oscillator coherent state demonstrates the geometric criterion in its most transparent, non-dispersive form. Because the velocity field is spatially uniform ($\partial_x v_{\mathrm{hyd}} = 0$), the wave packet undergoes pure rigid translation without internal convective shear. In this regime, the Madelung velocity field coincides globally with the geometric velocity of the density maximum, providing a classic example where the convective fluid motion and the observable trajectory of the probability profile are indistinguishable.

4.3 Airy Wave Packet: Accelerated Motion Without Velocity Curvature

The accelerating Airy wave packet serves as a decisive conceptual test for the geometric criterion. In this state, the probability maxima undergo constant temporal acceleration in the absence of any external potential, while the associated Madelung velocity field remains strictly uniform across all space. This example sharply decouples the temporal acceleration of a density feature from the spatial curvature of the flow, demonstrating that the latter alone governs the discrepancy between v_ρ and v_{hyd}.

The non-spreading Airy solution to the free 1D Schrödinger equation, originally constructed by Berry and Balazs [6], takes the form

$$\psi(x,t) = \mathrm{Ai}\left(\frac{x}{L} - \frac{\hbar^2 t^2}{4m^2 L^4}\right) \exp\left[\frac{i}{\hbar}\left(\frac{\hbar^2 x t}{2mL^3} - \frac{\hbar^4 t^3}{12 m^3 L^6}\right)\right], \tag{4.22}$$

where L denotes a characteristic spatial width. The resulting probability density is

$$\rho(x,t) = \mathrm{Ai}^2\left(\frac{x}{L} - \frac{\hbar^2 t^2}{4m^2 L^4}\right). \tag{4.23}$$

Although the idealized solution (4.22) is not square-integrable over $\mathbb{R}$, its local density structure, the regular distribution of its nodes, and the trajectories of its non-degenerate local maxima are mathematically well-posed and locally gauge-invariant. It therefore provides a rigorous local realization of the kinematic identity (3.3).

Introducing the dimensionless similarity variable

$$z(x,t) \equiv \frac{x}{L} - \frac{\hbar^2 t^2}{4m^2 L^4}, \tag{4.24}$$

the density depends solely on z, such that $\rho(x,t) = R(z) \equiv \mathrm{Ai}^2(z)$. Every isolated local extremum of the Airy profile is defined by a stationary point z_0 satisfying $R'(z_0) = 0$ with $R''(z_0) \neq 0$. The spatial trajectory of any chosen maximum satisfies $z\big(x_0(t), t\big) = z_0$, leading directly to the parabolic worldline:

$$x_0(t) = x_0(0) + \frac{\hbar^2}{4m^2 L^3} t^2. \tag{4.25}$$

Differentiating this trajectory with respect to time yields the geometric velocity:

$$v_\rho(t) = \dot{x}_0(t) = \frac{\hbar^2 t}{2m^2 L^3}. \tag{4.26}$$

Alternatively, evaluating v_ρ directly from the implicit kinematic expression (2.4) confirms this result. Computing the spatial and mixed spatiotemporal derivatives yields

$$\partial_x^2 \rho(x,t) = \frac{1}{L^2} R''(z), \qquad \partial_x \partial_t \rho(x,t) = -\frac{\hbar^2 t}{2m^2 L^5} R''(z). \tag{4.27}$$

Evaluating the ratio at any non-degenerate extremum z_0 gives

$$v_\rho(t) = -\left.\frac{\partial_x \partial_t \rho}{\partial_x^2 \rho}\right|_{z=z_0} = \frac{\hbar^2 t}{2m^2 L^3}, \tag{4.28}$$

in exact concordance with Eq. (4.26).

Next, we inspect the Madelung velocity field generated by the quantum phase in Eq. (4.22):

$$S(x,t) = \frac{\hbar^2 x t}{2mL^3} - \frac{\hbar^4 t^3}{12 m^3 L^6}. \tag{4.29}$$

The hydrodynamic velocity field is given by

$$v_{\mathrm{hyd}}(x,t) = \frac{1}{m}\partial_x S(x,t) = \frac{\hbar^2 t}{2m^2 L^3}. \tag{4.30}$$

Crucially, $v_{\mathrm{hyd}}(x,t)$ is spatially homogeneous at each instant:

$$\partial_x v_{\mathrm{hyd}}(x,t) = 0 \quad \Longrightarrow \quad \partial_x^2 v_{\mathrm{hyd}}(x,t) = 0 \quad \forall\, x \in \mathbb{R}. \tag{4.31}$$

Because the spatial curvature of the velocity field vanishes identically, our criterion (3.4) guarantees exact correspondence:

$$\boxed{v_\rho(t) = v_{\mathrm{hyd}}\big(x_0(t), t\big) = \frac{\hbar^2 t}{2m^2 L^3}}. \tag{4.32}$$

The physical significance of this result lies in the presence of finite temporal acceleration:

$$a_\rho \equiv \dot{v}_\rho(t) = \frac{\hbar^2}{2m^2 L^3} \neq 0. \tag{4.33}$$

Although the probability maximum accelerates continuously under the internal action of the Bohmian quantum potential, the geometric tracking velocity v_ρ and the convective fluid velocity v_{hyd} remain strictly identical at all times. This proves that temporal acceleration of probability extrema ($\dot{v}_\rho \neq 0$) does not generate a departure from hydrodynamic convection. The sole mathematical source of kinematic slippage between density peaks and the Madelung flow is the spatial curvature of the velocity field ($\partial_x^2 v_{\mathrm{hyd}} \neq 0$), not its temporal variation.

4.4 Cubic-Phase Wave Packet: Nonzero Geometric Discrepancy

To provide an explicit counterexample demonstrating that probability extrema do not generically follow the local Madelung flow, we construct an initial quantum state characterized by a cubic phase profile. Consider the wave function at $t = 0$:

$$\psi(x,0) = A \exp\!\left(-\frac{x^2}{4\sigma_0^2}\right) \exp\left(i\alpha x^3\right), \tag{4.34}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ parametrizes the strength of the cubic phase modulation, σ_0 characterizes the spatial localization width, and $A = (2\pi\sigma_0^2)^{-1/4}$ is the normalization prefactor.

The initial probability density is completely independent of the phase parameter α:

$$\rho(x,0) = |\psi(x,0)|^2 = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\!\left(-\frac{x^2}{2\sigma_0^2}\right). \tag{4.35}$$

The density profile possesses an isolated, non-degenerate maximum at the origin:

$$x_0(0) = 0, \qquad \partial_x^2 \rho\big|_{(0,0)} = -\frac{\rho(0,0)}{\sigma_0^2} < 0. \tag{4.36}$$

Evaluating the geometric curvature factor appearing in the discrepancy identity yields

$$\frac{\rho(0,0)}{\partial_x^2 \rho(0,0)} = -\sigma_0^2. \tag{4.37}$$

The phase function associated with the initial state (4.34) is given by

$$S(x,0) = \hbar \alpha x^3, \tag{4.38}$$

which generates a quadratic Madelung velocity field:

$$v_{\mathrm{hyd}}(x,0) = \frac{1}{m}\partial_x S(x,0) = \frac{3\hbar\alpha}{m} x^2. \tag{4.39}$$

At the location of the probability maximum, the hydrodynamic velocity vanishes identically:

$$v_{\mathrm{hyd}}(0,0) = 0. \tag{4.40}$$

However, the velocity field possesses a strictly non-zero spatial curvature:

$$\partial_x^2 v_{\mathrm{hyd}}(x,0) = \frac{6\hbar\alpha}{m} \neq 0. \tag{4.41}$$

Because the velocity field is intrinsically non-affine ($\partial_x^2 v_{\mathrm{hyd}} \neq 0$), the geometric criterion (3.4) immediately predicts a finite kinematic discrepancy between the tracking velocity of the peak and the convective flow.

Substituting Eqs. (4.37) and (4.41) into the exact identity (3.3) yields

$$v_\rho(0) - v_{\mathrm{hyd}}(0,0) = \left(-\sigma_0^2\right)\left(\frac{6\hbar\alpha}{m}\right) = -\frac{6\hbar\alpha\sigma_0^2}{m}. \tag{4.42}$$

Since $v_{\mathrm{hyd}}(0,0) = 0$, the instantaneous geometric velocity of the probability maximum evaluates to

$$\boxed{v_\rho(0) = -\frac{6\hbar\alpha\sigma_0^2}{m} \neq v_{\mathrm{hyd}}(0,0)}. \tag{4.43}$$

Equation (4.43) delivers an exact analytical counterexample to the conventional presumption that local probability features are transported by the Madelung velocity field. For $\alpha > 0$, the probability maximum instantly migrates to the left ($v_\rho(0) < 0$), even though the local fluid element at the origin is strictly stationary ($v_{\mathrm{hyd}}(0,0) = 0$).

The physical mechanism underlying this apparent paradox is purely kinematic and is governed by the continuity equation. Although the convective velocity at $x = 0$ is zero, the non-uniform gradient of v_{hyd} across the envelope causes an asymmetric mass evacuation. Specifically, because $v_{\mathrm{hyd}}(x,0) \geq 0$ everywhere, fluid elements for $x > 0$ stream to the right with speed $\propto x^2$, depleting the right flank of the wave packet more rapidly than the left flank. This differential convective deformation dynamically tilts the profile, displacing the instantaneous peak toward the left.

The magnitude of this geometric discrepancy,

$$|\Delta v_{\mathrm{geom}}| = |v_\rho(0) - v_{\mathrm{hyd}}(0,0)| = \frac{6\hbar|\alpha|\sigma_0^2}{m}, \tag{4.44}$$

scales linearly with the phase curvature parameter $|\alpha|$ and the packet variance σ_0^2, while scaling inversely with mass m. Importantly, this discrepancy cannot be attributed to the Gaussian nature of the envelope, since an identical Gaussian packet endowed with a quadratic phase generates an affine flow with $\Delta v_{\mathrm{geom}} \equiv 0$ (Sec. 4.1). Rather, the discrepancy is fundamentally sourced by the spatial curvature $\partial_x^2 v_{\mathrm{hyd}}$ of the underlying quantum flow.

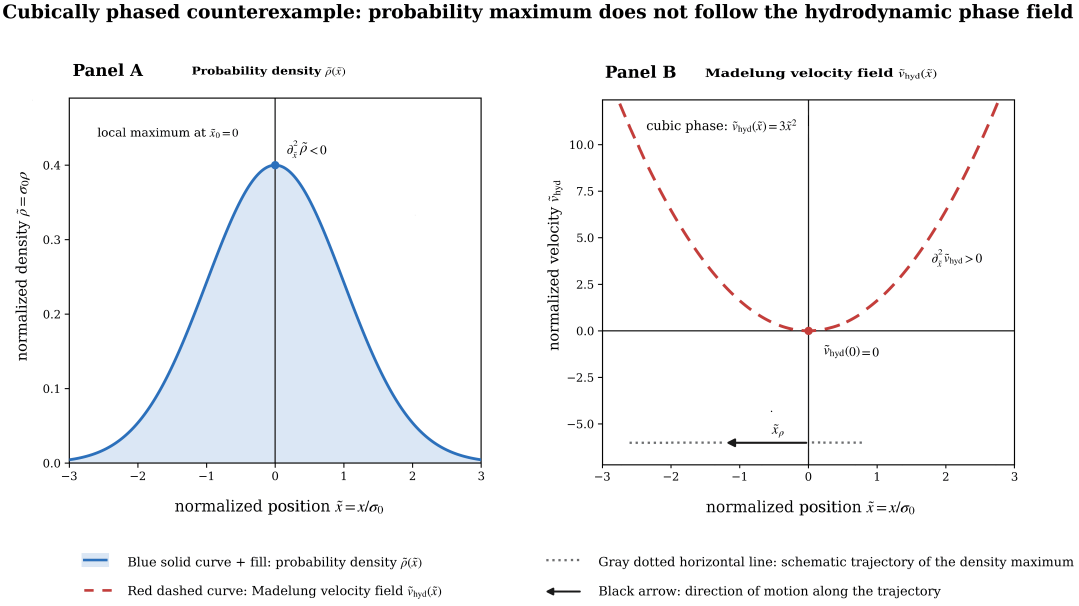


Figure 2: Kinematic breakdown of the hydrodynamic peak tracking for the cubic-phase state (4.34) at $t = 0$. **(A)** The normalized probability density $\rho(x, 0)$ features a non-degenerate maximum at $x_0 = 0$, where the density curvature $\partial_x^2 \rho < 0$ is finite. **(B)** The corresponding Madelung velocity field is quadratic in position, $v_{\mathrm{hyd}}(x, 0) \propto x^2$, which yields a vanishing local flow at the peak, $v_{\mathrm{hyd}}(0, 0) = 0$, but a strictly positive spatial curvature $\partial_x^2 v_{\mathrm{hyd}} > 0$. In accordance with the exact kinematic identity (3.3), this curvature induces an instantaneous backward peak velocity $v_\rho(0) = -6\hbar\alpha\sigma_0^2/m < 0$ (indicated by the directional arrow), demonstrating that probability extrema do not follow the local Madelung streamlines in non-affine flows.

5 Multidimensional Generalization

The one-dimensional geometric identity derived in Sec. 3 admits a natural and rigorous extension to multidimensional quantum systems in $\mathbb{R}^d$. The foundational principle remains invariant: a non-degenerate critical point of the probability density defines an isolated geometric entity whose trajectory can be dynamically tracked by differentiating the critical-point condition with respect to time.

Let the spatial probability density and the hydrodynamic (Madelung) velocity field be defined by

$$\rho(\mathbf{x}, t) = |\psi(\mathbf{x}, t)|^2, \qquad \mathbf{x} \in \mathbb{R}^d, \tag{5.1}$$

$$\mathbf{v}_{\mathrm{hyd}}(\mathbf{x}, t) = \frac{1}{m} \nabla S(\mathbf{x}, t), \tag{5.2}$$

where $S(\mathbf{x}, t)$ denotes the single-valued phase field in regions where $\rho > 0$. Conservation of probability is governed by the continuity equation:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}_{\mathrm{hyd}}) = 0. \tag{5.3}$$

Consider an isolated trajectory $\mathbf{x}_0(t)$ of critical points of the density, defined implicitly by $\nabla \rho(\mathbf{x}_0(t), t) = \mathbf{0}$. We assume that the critical point is non-degenerate along the trajectory, meaning that the Hessian matrix of the density,

$$H_\rho(\mathbf{x}, t) \equiv \nabla \nabla \rho(\mathbf{x}, t), \qquad (H_\rho)_{ij} = \partial_i \partial_j \rho, \tag{5.4}$$

is invertible, i.e., $\det H_\rho(\mathbf{x}_0(t), t) \neq 0$. By the implicit function theorem, this guarantees that the critical point evolves along a unique, differentiable spatial curve $\mathbf{x}_0(t)$.

5.1 Derivation of the Kinematic Identity

Since the gradient $\nabla\rho$ vanishes identically along $\mathbf{x}_0(t)$, its total time derivative must also vanish:

$$\frac{d}{dt}\nabla\rho(\mathbf{x}_0(t),t) = H_\rho(\mathbf{x}_0,t)\,\dot{\mathbf{x}}_0(t) + \nabla\partial_t\rho(\mathbf{x}_0,t) = \mathbf{0}. \tag{5.5}$$

Solving for the instantaneous geometric velocity $\mathbf{v}_\rho \equiv \dot{\mathbf{x}}_0(t)$ yields:

$$\mathbf{v}_\rho = -H_\rho^{-1}\nabla\partial_t\rho. \tag{5.6}$$

Applying the gradient operator to the continuity equation (5.3) and evaluating at the critical point ($\nabla\rho = \mathbf{0}$) allows us to eliminate $\partial_t\rho$, resulting in the exact identity:

$$\boxed{\mathbf{v}_\rho - \mathbf{v}_{\text{hyd}} = \rho H_\rho^{-1}\nabla\left(\nabla\cdot\mathbf{v}_{\text{hyd}}\right)}. \tag{5.7}$$

In terms of the quantum phase field, this discrepancy vector assumes the transparent form:

$$\Delta\mathbf{v}_{\text{geom}} \equiv \mathbf{v}_\rho - \mathbf{v}_{\text{hyd}} = \frac{\rho}{m}H_\rho^{-1}\nabla\left(\nabla^2 S\right). \tag{5.8}$$

5.2 Exact Multidimensional Criterion

For a non-vanishing probability density at the extremum, the geometric discrepancy vector vanishes if and only if the driving vector field vanishes, yielding the criterion:

$$\boxed{\mathbf{v}_\rho = \mathbf{v}_{\text{hyd}} \quad\Longleftrightarrow\quad \nabla\left(\nabla\cdot\mathbf{v}_{\text{hyd}}\right) = \mathbf{0}} \qquad \text{at } \mathbf{x}_0(t). \tag{5.9}$$

This condition reveals that exact peak tracking does not require the full velocity-gradient tensor $\nabla\mathbf{v}_{\text{hyd}}$ to be uniform, but specifically requires the *volumetric divergence* to be locally stationary.

5.3 Foundational Interpretation and Validity

From a foundational perspective, Eq. (5.7) underscores the distinct ontological statuses of these velocities: $\mathbf{v}_{\text{hyd}}$ governs the convective transport of the probability fluid, while $\mathbf{v}_\rho$ governs the morphological migration of the density features themselves. The discrepancy $\Delta\mathbf{v}_{\text{geom}}$ arises because the position of a density extremum is determined not only by the local flow, but by the spatial gradient of the local compression rate, modulated by the anisotropy of the density Hessian H_ρ^{-1}.

This relation is dimension-independent; the one-dimensional theory is merely the simplest scalar projection of this broader geometric structure. Regarding validity, the derivation is purely kinematic and remains valid provided:

1. **Non-degeneracy:** $\det H_\rho \neq 0$, preventing structural bifurcations (coalescence/splitting) of the critical point.

2. **Analyticity:** The derivation requires $\mathbf{x}_0(t)$ to reside in an open domain $\mathbb{R}^d\backslash\{\mathbf{x} \mid \rho = 0\}$. At nodal points where $\rho \to 0$, the Madelung phase often develops singular vortex structures, and the current-based derivation requires careful regularization beyond the standard hydrodynamic limit.

6 Discussion

The principal contribution of this work is the formulation of an exact, purely kinematic relation connecting the spatial migration of probability extrema to the underlying phase-generated Madelung velocity field. Rather than proposing modifications to quantum dynamics or postulating new equations of motion, our results delimit the precise domain of validity of the hydrodynamic description when applied to the motion of wave-packet profiles. By casting the discrepancy in terms of local differential geometry, we resolve when and why the convective flow of probability diverges from the movement of its observable features.

6.1 Ontological Distinction: Hydrodynamic Flow vs. Morphological Kinematics

The Madelung hydrodynamic reformulation of quantum mechanics [1, 3] endows the wave function with a local velocity field,

$$\mathbf{v}_{\mathrm{hyd}}(\mathbf{x},t) = \frac{1}{m}\nabla S(\mathbf{x},t), \tag{6.1}$$

which defines the convective transport of probability and provides the fundamental guidance law in de Broglie–Bohm mechanics. In contrast, the geometric velocity $\mathbf{v}_\rho(t)$, defined implicitly by $\nabla\rho(\mathbf{x}_0(t),t) = \mathbf{0}$, tracks the spatial trajectory of a structural, macroscopic feature of the probability distribution itself.

These two notions of motion belong to distinct ontological categories:

1. $\mathbf{v}_{\mathrm{hyd}}$ represents the Eulerian velocity of the probability current at a fixed point in space-time (or the Lagrangian velocity of a hypothetical corpuscle).

2. $\mathbf{v}_\rho$ is an emergent kinematic property of the global wave-packet envelope, governed by the local geometry of the density landscape.

The multidimensional identity derived in Eq. (5.7),

$$\mathbf{v}_\rho - \mathbf{v}_{\mathrm{hyd}} = \rho H_\rho^{-1}\nabla\left(\nabla\cdot\mathbf{v}_{\mathrm{hyd}}\right), \tag{6.2}$$

with its one-dimensional reduction $v_\rho - v_{\mathrm{hyd}} = \frac{\rho}{\partial_x^2\rho}\partial_x^2 v_{\mathrm{hyd}}$, establishes that the coincidence $\mathbf{v}_\rho = \mathbf{v}_{\mathrm{hyd}}$ is strictly non-generic. Evaluating the phase gradient $\frac{1}{m}\nabla S$ at the peak of a probability distribution does not, in general, yield the velocity of that peak.

The geometric discrepancy vector factorizes into two structurally independent components:

- **The Incompressibility Gradient $\nabla(\nabla\cdot\mathbf{v}_{\mathrm{hyd}})$:** This vector isolates the spatial non-uniformity of the local compression/expansion rate of the probability fluid. The continuity equation naturally filters out pure shear and rigid-body rotation, rendering the velocity discrepancy sensitive solely to the gradient of the velocity divergence (or higher-order phase curvature $\frac{1}{m}\nabla(\nabla^2 S)$).

- **The Density Curvature Tensor ρH_ρ^{-1}:** The inverse Hessian serves as a local geometric transducer, determining how the non-uniform compression rate translates into a physical shift of the maximum, weighted by the local density scale.

6.2 Locality and Diagnostic Value

Tracking the motion of critical points has established foundations in wave physics, singularity optics, and catastrophe theory [7, 8]. Applying this methodology to the quantum probability density yields a diagnostic tool of remarkable analytical economy.

Crucially, the criterion

$$\mathbf{v}_\rho = \mathbf{v}_{\mathrm{hyd}} \quad \Longleftrightarrow \quad \nabla\left(\nabla \cdot \mathbf{v}_{\mathrm{hyd}}\right) = \mathbf{0} \tag{6.3}$$

is strictly local. It eliminates the need to integrate the time-dependent Schrödinger equation across extended spatial domains or calculate complete hydrodynamic stream lines. To determine whether an extremum follows the convective flow, one only needs to evaluate a single differential operator on the phase field at the instantaneous location $\mathbf{x}_0(t)$.

This locality clarifies several canonical wave packets analyzed in this work:

- **Gaussian Packets and Coherent States:** The velocity field is globally affine ($\nabla \mathbf{v}_{\mathrm{hyd}} = \mathbf{M}(t)$), implying $\nabla(\nabla \cdot \mathbf{v}_{\mathrm{hyd}}) \equiv \mathbf{0}$. The peak therefore tracks the local current identically, even while the packet spreads anisotropically.
- **Airy Wave Packets:** Despite possessing an intrinsically accelerated trajectory ($\dot{\mathbf{v}}_\rho \neq \mathbf{0}$), the phase remains strictly quadratic in space ($\partial_x^2 v_{\mathrm{hyd}} = 0$). Consequently, the peak of an Airy packet moves in exact unison with the local Madelung flow at all times.
- **Cubic-Phase States:** The cubic spatial dependence of the phase induces a non-vanishing velocity curvature ($\partial_x^2 v_{\mathrm{hyd}} \neq 0$), generating an irreducible kinematic discrepancy where $v_\rho \neq v_{\mathrm{hyd}}$.

Because the proof depends solely on the continuity equation and the implicit differentiation of non-degenerate critical points, the criterion holds universally for arbitrary potentials, conservative or dissipative, linear or non-linear.

6.3 Experimental Signatures and Operational Accessibility

A compelling feature of the geometric velocity $\mathbf{v}_\rho$ is its operational directness. In standard experimental protocols, reconstructed density profiles $\rho(\mathbf{x}, t)$ are readily obtained via repeated position measurements on identically prepared ensembles. The instantaneous velocity $\mathbf{v}_\rho$ can be tracked through successive high-resolution spatial frames without requiring phase-retrieval algorithms.

In contrast, the hydrodynamic velocity $\mathbf{v}_{\mathrm{hyd}}$ is phase-dependent. However, recent advances in experimental quantum mechanics make direct comparison feasible:

1. **Quantum Gas Microscopy and In-Situ Imaging:** Ultra-cold atom platforms enable single-site, real-space imaging of density evolutions [4], providing the spatial resolution necessary to compute local density curvatures H_ρ and peak velocities $\mathbf{v}_\rho$.
2. **Weak-Value Trajectory Reconstruction:** Techniques utilizing weak measurements of momentum followed by strong post-selection of position allow the direct reconstruction of Bohmian velocity fields and local currents [5, 9].

By combining weak-measurement reconstruction of $\mathbf{v}_{\mathrm{hyd}}(\mathbf{x}_0, t)$ with direct photographic tracking of $\mathbf{v}_\rho(t)$ in non-affine systems (such as cubic-phase optical beams or shaped electron wave packets), Eq. (6.2) can be subjected to direct empirical verification.

6.4 Outlook and Broader Kinematic Scope

Several fundamental extensions emerge naturally from this geometric framework:

1. **Degenerate Extrema and Catastrophes:** When $\det H_\rho = 0$, the implicit function theorem ceases to apply, signaling bifurcation events such as packet splitting, coalescence, or the birth/annihilation of critical points. Extending this framework via higher-order singularity theory [7] will provide exact scaling laws for feature velocities near catastrophe manifolds.

2. **Universality Across Classical and Wave Fields:** Because the derivation relies strictly on the continuity equation $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$, identical kinematic identities govern intensity extrema in paraxial physical optics (Poynting flow), classical fluid mechanics (incompressible and compressible tracer dynamics), and plasma density envelopes.

In summary, the criterion establishes that the agreement between peak tracking and hydrodynamic velocity is not an intrinsic feature of quantum mechanics, but a specific consequence of affine phase kinematics. Equation (6.2) provides the precise mathematical boundary separating local fluid flow from the global morphological evolution of wave fields.

7 Conclusion

We have established an exact, purely kinematic criterion governing the spatial migration of non-degenerate probability extrema relative to the local Madelung hydrodynamic flow. For an isolated, non-degenerate critical point in arbitrary spatial dimensions $\mathbb{R}^d$, the discrepancy between the geometric feature velocity $\mathbf{v}_\rho$ and the convective velocity $\mathbf{v}_{\text{hyd}}$ is given by the exact identity:

$$\mathbf{v}_\rho - \mathbf{v}_{\text{hyd}} = \rho H_\rho^{-1} \nabla \left(\nabla \cdot \mathbf{v}_{\text{hyd}} \right) = \frac{\rho}{m} H_\rho^{-1} \nabla \left(\nabla^2 S \right), \tag{7.1}$$

which reduces in one dimension to the scalar relation:

$$v_\rho - v_{\text{hyd}} = \frac{\rho}{\partial_x^2 \rho} \, \partial_x^2 v_{\text{hyd}}. \tag{7.2}$$

Consequently, $\mathbf{v}_\rho = \mathbf{v}_{\text{hyd}}$ if and only if the volumetric divergence of the velocity field is locally stationary at the extremum,

$$\nabla(\nabla \cdot \mathbf{v}_{\text{hyd}}) = \mathbf{0} \quad (\text{or } \partial_x^2 v_{\text{hyd}} = 0 \text{ in 1D}).$$

This geometric identity follows exclusively from the continuity equation and the implicit kinematics of non-degenerate critical points. It is completely independent of specific Hamiltonian dynamics, potential landscapes, or linear/non-linear wave equations, carrying no representation-dependent approximations.

The analytical examples developed in this work systematically validate both regimes of the criterion:

- **Affine Phase Flows:** Standard Gaussian wave packets, coherent states, and intrinsically accelerating Airy peaks all satisfy $\nabla(\nabla \cdot \mathbf{v}_{\text{hyd}}) = \mathbf{0}$, ensuring that their density maxima track the local Madelung velocity identically at all times.
- **Non-Affine Phase Flows:** Wave packets with cubic spatial phase profiles exhibit non-zero velocity curvature ($\partial_x^2 v_{\text{hyd}} \neq 0$), realizing a minimal analytical demonstration of the physical and numerical separation between peak tracking and local convective transport.

Beyond its immediate relevance to foundational questions in quantum mechanics and the interpretation of de Broglie–Bohm trajectories, this formulation provides a universal diagnostic tool. Because the derivation relies solely on local conservation, Eq. (7.1) applies universally across physical domains governed by a continuity equation—including optical beam propagation (Poynting flow), classical compressible fluids, and plasma physics—providing a rigorous geometric boundary separating the motion of observable field features from underlying convective currents.

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