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A Parametric 8-Point Approximating Subdivision Scheme for Flexible Geometric Curve Design

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Abstract

This paper presents a parametric stationary binary 8-point approximating subdivision scheme constructed from the difference between the refinements generated by the 6-point Lagrange and B-spline schemes. Three shape parameters are introduced to modify the Lagrange refinement through neighboring difference vectors, yielding a flexible family of 8-point schemes. Several parameter restrictions produce the sub-families S_r and S_t . The support and boundary refinement rules are established, while convergence and smoothness are analyzed using spectral and Laurent polynomial techniques. The resulting schemes attain C^6 and C^8 continuity over their corresponding admissible parameter ranges. Gibbs-free parameter conditions are obtained through a cumulative-sum criterion. Numerical experiments illustrate the geometric effect of the shape parameters, and the Hausdorff distance $d_H(C_k, C_5)$ is used to quantify the approximation behavior. The proposed scheme provides flexible geometric control for curve design and computer-aided geometric modeling.

Keywords: Stationary subdivision scheme, Lagrange subdivision, B-spline subdivision, shape parameters, convergence, smoothness, Gibbs phenomenon, Hausdorff distance
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1 Introduction

Curves are fundamental geometric entities used to represent smooth shapes in Computer-Aided Geometric Design, computer graphics, animation, engineering, image processing, and industrial manufacturing. The construction of smooth curves from a finite set of control points is therefore an important problem in geometric modeling. Classical curve representations, including Bézier curves, B-spline curves, and spline functions, provide desirable properties such as local control, continuity, and geometric flexibility. However, subdivision

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schemes provide a recursive approach for generating smooth curves from an initial set of control points through repeated refinement. By applying prescribed refinement rules, a sequence of refined control polygons is generated, which, under suitable conditions, converges to a limiting curve. These properties have made subdivision schemes an important tool for curve construction and geometric design.

The development of subdivision schemes, beginning with the corner-cutting method of de Rham [1] and followed by the recursive spline surface scheme of Catmull and Clark [2], the discrete B-spline subdivision techniques of Cohen et al. [3], and the smooth curve construction discussed by De Boor [4], has led to extensive research on their convergence, smoothness, approximation, and shape-preserving properties. The convergence of uniform binary subdivision schemes for curve design was analyzed by Dyn et al. [5], while Dyn and Levin [6] investigated stationary and non-stationary binary subdivision schemes. Dyn and Levin [7] discussed various subdivision schemes for geometric modeling and presented different tools for analyzing their convergence, smoothness, and approximation properties. According to Dyn [8], radial functions provide an effective framework for interpolation and approximation. Lian [9] introduced a general family of $2m$ -point a -ary and $(2m + 1)$ -point a -ary interpolatory subdivision schemes for curve design. The butterfly subdivision scheme was introduced for surface interpolation with tension control by Dyn et al. [10]. Hassan et al. [11] proposed a novel 4-point ternary interpolatory subdivision scheme with a tension parameter. Mustafa and Rehman [12] developed general formulae for the masks of $(2b + 4)$ -point n -ary approximating and interpolating subdivision schemes, which generalize and unify several well-known schemes and provide masks for higher-arity schemes. Zhang et al. [13] introduced a non-uniform, non-stationary Loop subdivision scheme for interpolating arbitrary initial triangular meshes, providing finite-step interpolation with a single shape parameter and without requiring additional modifications to the initial mesh or subdivision rules.

Cheng and Zhou [14] investigated necessary conditions for the convergence of subdivision schemes with nonnegative finite masks, focusing on the locations of support points and their convex hull. They also showed that the connectivity of a matrix induced by the mask provides a simple sufficient condition for these properties. Omar and Khan [15] proposed a generalized subdivision surface scheme based on the 2D Lagrange interpolating polynomial, deriving refinement rules for quadrilateral meshes and discussing its error bounds and applications. Building on generalized subdivision constructions, Rehan and Siddiqi [16] introduced families of ternary 3-point and 4-point subdivision schemes with C^1 and C^2 smoothness, along with a 4-point approximating scheme generating C^3 limit curves.

The support of recursive subdivision schemes was studied by Ivriissimtzis et al. [17]. Moreover, Levin [18] presented the Laurent polynomial representation for the analysis of non-uniform binary subdivision schemes and their smoothness properties. These studies provided a theoretical basis for the development and analysis of subdivision schemes with desirable geometric and analytical properties. Building on these fundamental results, subsequent research has focused particularly on approximating subdivision schemes for generating smooth limiting curves.

Approximating subdivision schemes have been widely studied for generating smooth limiting curves from an initial set of control points. Siddiqi and Ahmad [19, 20] developed ap-

proximating subdivision schemes with improved smoothness, including a 3-point C^2 scheme and a C^6 approximating scheme. Ghaffar et al. [21] introduced a 3-point approximating subdivision scheme of varying arity and discussed its application to curve generation. Mustafa and Randhawa [22] analyzed a 3-point binary approximating subdivision scheme and investigated the effect of its shape parameter on the smoothness of the generated curves. Furthermore, Asghar and Mustafa [23] proposed a generalized curve and surface design algorithm based on ridge regression, which is less sensitive to outliers and provides better results than least-squares-based subdivision schemes. These studies demonstrate that the use of shape parameters in approximating subdivision schemes provides additional flexibility in controlling the properties and geometric behavior of the resulting limit curves.

The use of shape parameters has been considered in different subdivision schemes to improve the smoothness and shape control of the generated curves. Tan et al. [24] introduced a new 4-point shape-preserving subdivision scheme with two shape parameters α and β and obtained C^3 -continuous curves for specific parameter values. Zheng et al. [25] designed a class of multi-parameter curve subdivision schemes with high continuity using the relation between subdivision schemes and their corresponding difference schemes. Bari et al. [26] presented quaternary shape-preserving subdivision schemes with multiple parameters. Yang et al. [27] proposed a family of stationary 4-point subdivision schemes with C^2 continuity, fourth-order approximation accuracy, and shape-preserving properties. Ureña and Yáñez [28] developed subdivision schemes based on weighted local polynomial regression and proposed a new technique for the convergence analysis. These studies show that shape parameters can be used to control the smoothness and geometric shape of the curves generated by subdivision schemes. In addition to smoothness and shape control, the suppression of unwanted oscillations near sharp features has also received considerable attention. The Gibbs phenomenon is also an important issue in the analysis of subdivision schemes, particularly in the presence of discontinuities or sharp changes in the data. Gottlieb and Shu [29] studied the Gibbs phenomenon and discussed methods for its resolution. Amat et al. [30] investigated the Gibbs phenomenon in stationary subdivision schemes and established conditions for the absence of Gibbs oscillations. The shape-preserving properties of approximating subdivision schemes have also been studied. Bibi et al. [31] analyzed the conditions for preserving the shape of the limit curve of a binary 3-point approximating subdivision scheme. Zhang et al. [32] proposed a generalized cubic exponential B-spline scheme with shape control. These studies show that the control of shape and Gibbs oscillations is an important consideration in the development of subdivision schemes with desirable curve-generation properties.

The Lagrange and B-spline basis functions provide different approaches for constructing approximating subdivision schemes. In the present work, a 6-point Lagrange subdivision scheme and a 6-point B-spline subdivision scheme are considered. When applied to the same initial control polygon, the two schemes generally produce different refined points. The resulting difference vectors are used to modify the Lagrange refinement and introduce shape control into a new stationary binary approximating subdivision scheme.

Based on this construction, a stationary binary 8-point approximating subdivision scheme with three shape parameters γ_a , γ_b , and γ_c is proposed and denoted by S_n . The shape parameters control the contribution of neighboring difference vectors and consequently

influence the geometry of the resulting curves. Several parameter restrictions are considered to obtain useful subfamilies and to study their geometric and analytical properties.

The proposed schemes are analyzed with respect to support, convergence, smoothness, and Gibbs oscillations. Necessary spectral conditions and sufficient Laurent polynomial conditions are used to determine the admissible parameter ranges. Boundary refinement rules are developed for open control polygons, while cyclic refinement is used for closed polygons. Numerical experiments are also presented to examine the geometric effect of the shape parameters and the approximation behavior through Hausdorff distance.

The remainder of this paper is organized as follows. Section 2 presents the basic notation and preliminary results used in the analysis. Section 3 describes the construction of the proposed stationary binary 8-point approximating subdivision scheme, its special subfamilies, and the treatment of open and closed control polygons. Section 4 investigates the support, convergence, and smoothness properties of the proposed schemes. Section 5 presents the numerical results and Hausdorff-distance analysis. Section 6 investigates the Gibbs oscillation behavior of the considered subfamilies. Section 7 concludes the paper. The explicit boundary refinement rules are collected in Appendices A and B.

2 Basic notation and results

Let

$$P^k = \{P_i^k : i \in \mathbb{Z}\}$$

denote the control points at the k th refinement level. A binary subdivision scheme with mask $a = \{a_i\}_{i \in \mathbb{Z}}$ is defined by

$$P_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{i-2j}^{[k]} P_j^k, \quad i \in \mathbb{Z},$$

where $a^{[k]} = \{a_i^{[k]}\}_{i \in \mathbb{Z}}$ is the mask at the k th refinement level. The mask is assumed to have finite support. The scheme is called stationary if its mask is independent of the refinement level, that is,

$$a_i^{[k]} = a_i, \quad i \in \mathbb{Z}, \quad k \in \mathbb{Z}^+.$$

Otherwise, it is called non-stationary. For a stationary binary subdivision scheme, the even and odd refined points are written as

$$P_{2i}^{k+1} = \sum_{j \in \mathbb{Z}} a_{2i-2j} P_j^k, \tag{2.1}$$

$$P_{2i+1}^{k+1} = \sum_{j \in \mathbb{Z}} a_{2i+1-2j} P_j^k. \tag{2.2}$$

The subdivision mask of a stationary scheme is represented by the Laurent polynomial

$$a(z) = \sum_{i \in \mathbb{Z}} a_i z^i, \quad z \in \mathbb{C} \setminus \{0\}, \tag{2.3}$$

where $\{a_i\}_{i \in \mathbb{Z}}$ denotes the subdivision mask. This representation is used below in the convergence and smoothness analysis.

For the binary approximating schemes considered in this paper, the mask satisfies

$$\sum_{i \in \mathbb{Z}} a_{2i} = \sum_{i \in \mathbb{Z}} a_{2i+1} = 1. \quad (2.4)$$

The following results concerning convergence, smoothness, and the Gibbs phenomenon are recalled for completeness. The convergence and smoothness analysis is based on the subdivision and difference-scheme framework of Dyn et al. [5, 7], while the Gibbs criterion used in this paper follows Amat et al. [30].

Theorem 2.1. *Let S be a stationary binary subdivision scheme with subdivision matrix M , and let $\{\lambda_i\}$ denote the eigenvalues of M ordered according to*

$$\lambda_0 = 1, \quad |\lambda_i| > |\lambda_{i+1}|, \quad i \in \mathbb{N}.$$

These spectral conditions provide necessary conditions for the convergence and smoothness of the corresponding limit curve. In particular, if the limit curve is C^2 -continuous, then

$$\lambda_1^2 = |\lambda_2| > |\lambda_3|.$$

The convergence and higher-order smoothness can be further investigated through the corresponding difference schemes and their subdivision matrices.

Theorem 2.2. *Let S be a stationary binary subdivision scheme with mask $a^{(0)} = \{a_i^{(0)}\}_{i \in \mathbb{Z}}$. Suppose that its j th-order difference schemes S_j , $j = 1, \dots, n+1$, exist with masks $a^{(j)} = \{a_i^{(j)}\}_{i \in \mathbb{Z}}$ satisfying*

$$\sum_{i \in \mathbb{Z}} a_{2i}^{(j)} = \sum_{i \in \mathbb{Z}} a_{2i+1}^{(j)} = 1, \quad j = 0, 1, \dots, n. \quad (2.5)$$

If there exists an integer $L \geq 1$ such that

$$\left\| \left(\frac{1}{2} S_{n+1} \right)^L \right\|_{\infty} < 1,$$

then S is convergent and its limit curve is C^n -continuous, where

$$\left\| \left(\frac{1}{2} S_{n+1} \right)^L \right\|_{\infty} = \max_{0 \leq i < 2^L} \left\{ \sum_{j \in \mathbb{Z}} |b_{i-2^L j}^{[L]}| \right\}. \quad (2.6)$$

Here,

$$b^{[L]}(z) = b(z)b(z^2) \cdots b(z^{2^{L-1}}), \quad b(z) = \frac{1}{2} a^{(n+1)}(z),$$

with

$$b^{[L]}(z) = \sum_{i \in \mathbb{Z}} b_i^{[L]} z^i.$$

In particular, for $L = 1$,

$$b^{[1]}(z) = b(z),$$

and

$$\left\| \frac{1}{2} S_{n+1} \right\|_{\infty} = \frac{1}{2} \max \left\{ \sum_{i \in \mathbb{Z}} |a_{2i}^{(n+1)}|, \sum_{i \in \mathbb{Z}} |a_{2i+1}^{(n+1)}| \right\}. \quad (2.7)$$

Theorem 2.3. Let $0 \leq \psi \leq h$, and let g be a piecewise-defined function given by

$$\begin{aligned} g(x) &= g_-(x), & g_- &\in C^m((-\infty, \psi]), & x &\leq \psi, \\ g(x) &= g_+(x), & g_+ &\in C^m((\psi, +\infty)), & x &> \psi, \end{aligned}$$

where $n \geq 2$ and $g_-(\psi) > g_+(\psi)$.

Let S_a be a univariate stationary subdivision scheme with a finitely supported mask $a = \{a_j\}_{j \in \mathbb{Z}}$ satisfying

$$a_j = 0, \quad j < M \text{ or } j > M + N,$$

where $M < 0$, $N \geq 2$, and $M + N > 0$. Let the iterated mask $a^{[k]} = \{a_j^{[k]}\}_{j \in \mathbb{Z}}$ be defined recursively by

$$a_j^{[k]} = \sum_{i \in \mathbb{Z}} a_i^{[k-1]} a_{j-2i}, \quad a_j^{[1]} = a_j.$$

For $0 \leq l < 2^k$, define

$$\nu_l^{[k]}(i) = \begin{cases} \sum_{\tau \leq i} a_{2^k \tau + l}^{[k]}, & i < 0, \\ 0, & i = 0, \\ \sum_{\tau \geq i} a_{2^k \tau + l}^{[k]}, & i > 0. \end{cases}$$

If

$$\nu_l^{[k]}(i) \geq 0, \quad \forall i, k,$$

and h is sufficiently small, then the following properties hold.

(P1) If

$$|x| \geq \max \left\{ \left| \frac{M-1}{2} \right|, \left| \frac{M+N}{2} + 1 \right| \right\} h,$$

then

$$|g(x) - (S_a^{\infty} g_h)(x)| = O(h^n).$$

(P2) If

$$|x| \leq \max \left\{ \left| \frac{M-1}{2} \right|, \left| \frac{M+N}{2} + 1 \right| \right\} h,$$

then there exists $\beta_h = O(h)$ such that

$$g_{1,h} - \beta_h \leq g_+(h) - \beta_h \leq (S_a^{\infty} g_h)(x) \leq g_-(0) + \beta_h = g_{0,h} + \beta_h.$$

The following definition introduces the Hausdorff distance used to evaluate the convergence behavior of the proposed subdivision schemes. Following the definition given in Huang et al. [33], the Hausdorff distance is defined by

Definition 2.1 (Hausdorff Distance). Let $A, B \subset \mathbb{R}^2$ be two nonempty compact sets. The Hausdorff distance between A and B is defined by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|b - a\| \right\}, \quad (2.8)$$

where $\|\cdot\|$ denotes the Euclidean norm. The Hausdorff distance measures the maximum deviation between two sets.

The notation and results presented in this section provide the foundation for the construction and analysis of the proposed subdivision scheme. The next section presents the construction of the proposed stationary binary 8-point approximating subdivision scheme.

3 Construction of the proposed stationary binary 8-point approximating subdivision scheme

In this section, a new stationary binary 8-point approximating subdivision scheme is constructed by combining two different 6-point binary approximating subdivision schemes. The first scheme is based on Lagrange interpolation, while the second scheme is constructed using B-spline basis functions. The difference between the corresponding refined points of these two schemes is used to construct difference vectors. These vectors are then used to modify the Lagrange refinement rules through three shape parameters.

We first consider the 6-point binary approximating subdivision scheme proposed by Mustafa and Rehman [12]. The scheme is based on Lagrange interpolation and is given by

$$\begin{cases} Q_{2i}^{k+1} = \sum_{j=-2}^3 l_j P_{i+j}^k, \\ Q_{2i+1}^{k+1} = \sum_{j=-3}^2 l_{-j} P_{i+j+1}^k. \end{cases} \quad (3.1)$$

where

$$\begin{aligned} l_{-2} &= \frac{77}{8192}, & l_{-1} &= -\frac{693}{8192}, & l_0 &= \frac{3465}{4096}, \\ l_1 &= \frac{1155}{4096}, & l_2 &= -\frac{495}{8192}, & l_3 &= \frac{63}{8192}. \end{aligned}$$

The coefficients of the Lagrange scheme satisfy

$$\sum_{j=-2}^3 l_j = 1,$$

while some coefficients are positive and some are negative. Therefore, the refined points generated by (3.1) are affine combinations of the corresponding control points. Unlike a convex combination, an affine combination may contain negative coefficients and can therefore produce a refined point outside the convex hull of the local control points. This signed affine combination distinguishes the Lagrange refinement from a convex-combination refinement and plays an important role in the resulting geometric behavior.

Next, we consider the 6-point binary approximating subdivision scheme constructed using B-spline basis functions. Its even and odd refinement rules are given by

$$\begin{cases} R_{2i}^{k+1} = \sum_{j=-2}^3 m_j P_{i+j}^k, \\ R_{2i+1}^{k+1} = \sum_{j=-3}^2 m_{-j} P_{i+j+1}^k. \end{cases} \quad (3.2)$$

where

$$\begin{aligned} m_{-2} &= \frac{11}{1024}, & m_{-1} &= \frac{165}{1024}, & m_0 &= \frac{462}{1024}, \\ m_1 &= \frac{330}{1024}, & m_2 &= \frac{55}{1024}, & m_3 &= \frac{1}{1024}. \end{aligned}$$

In this case,

$$\sum_{j=-2}^3 m_j = 1, \quad m_j > 0, \quad j = -2, \dots, 3.$$

Hence, the refined points generated by (3.2) are convex combinations of the corresponding local control points. Thus, the Lagrange and B-spline schemes provide two different types of refinement from the same initial control polygon: the Lagrange scheme uses an affine combination, whereas the B-spline scheme uses a convex combination.

Both schemes are applied to the same initial control polygon. Figure 1 illustrates the refined curves obtained after the first refinement level using the 6-point Lagrange and 6-point B-spline subdivision schemes.

5 Numerical results and Hausdorff distance analysis

This section presents numerical results for the proposed stationary binary 8-point subdivision scheme S_n and the two subfamilies S_r and S_t . The graphical results illustrate the effect of the shape parameters on the generated limit curves. The geometric approximation behavior of the schemes is then evaluated using the Hausdorff distance defined in Section 2.

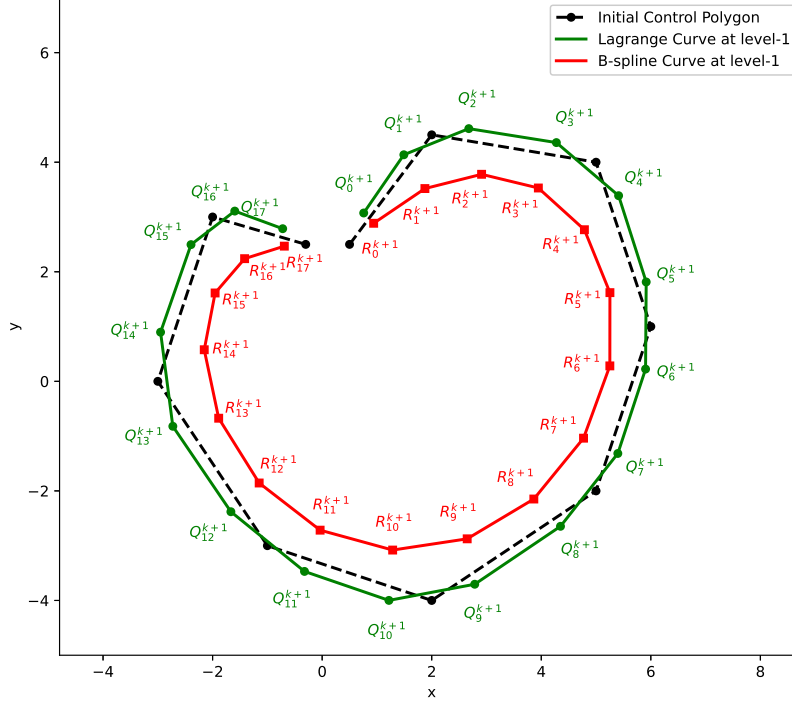


Figure 1: Curves obtained after the first refinement level using the 6-point Lagrange and 6-point B-spline subdivision schemes.

5.1 Graphical results

We first consider the proposed three-parameter scheme S_n . Three parameter configurations are used to illustrate the influence of the shape parameters on the resulting limit curves:

- Set 1: $(\gamma_a, \gamma_b, \gamma_c) = (0, 1, 0.01)$,
- Set 2: $(\gamma_a, \gamma_b, \gamma_c) = (0.05, 0.4, 0)$,
- Set 3: $(\gamma_a, \gamma_b, \gamma_c) = (0.2, 0, 0.04)$.

The corresponding curves are shown in Figure 2. The parameter changes produce noticeable variations in the geometric shape while the same initial control polygon is retained. In particular, the magnitude and sign of the shape parameters alter the local displacement introduced by the resultant vectors, resulting in different limit curves. The corresponding curves obtained after five refinement levels are shown in Figure 2.

The first subfamily S_r is obtained by setting $\gamma_a = \gamma_c = 0$. To examine the effect of the remaining parameter γ_b , three values are considered in the graphical experiments, namely

$$\gamma_b = 0.52, \quad \gamma_b = 0.72, \quad \gamma_b = 1.90.$$

The corresponding curves obtained after five refinement levels are shown in Figure 3. The

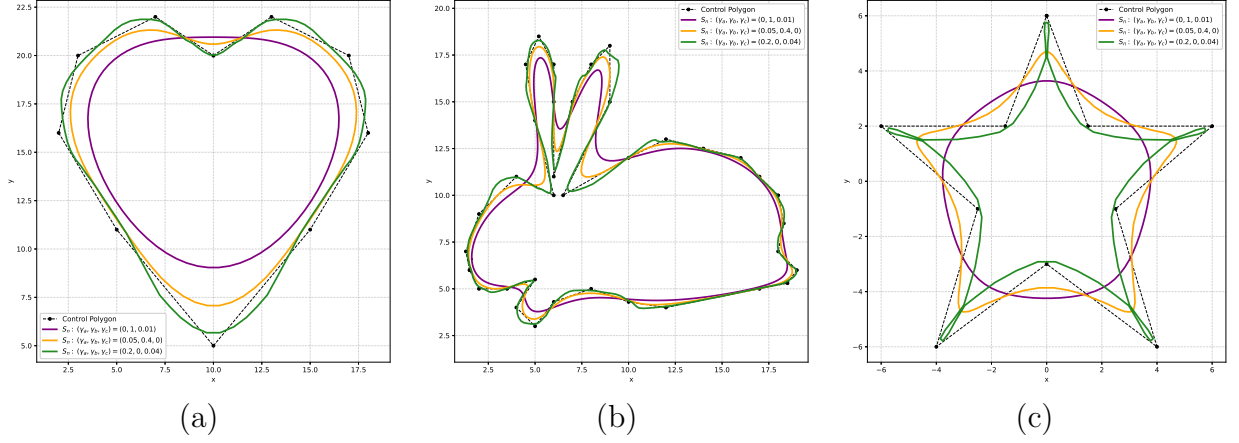


Figure 2: Curves generated by the proposed scheme S_n for different parameter configurations.

change in γ_b produces visible changes in the shape of the resulting curves, demonstrating the shape-control capability of the one-parameter subfamily.

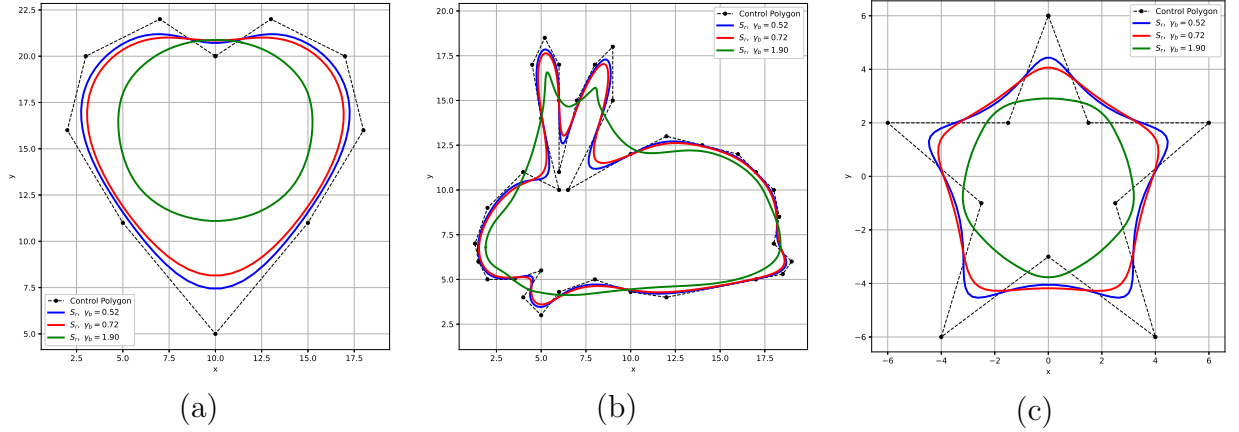


Figure 3: Curves generated by the subfamily S_r for different values of γ_b .

The one-parameter 8-point subfamily S_t is considered next. The parameter configurations are selected as

$$\gamma_a = 0.4, \quad \gamma_a = -0.4, \quad \gamma_a = 0.1.$$

The corresponding curves obtained after five refinement levels are shown in Figure 4. The figures demonstrate that both the magnitude and sign of the free shape parameter affect the geometric behavior of the resulting curves.

5.2 Hausdorff distance analysis

Let C_k denote the subdivision curve obtained after k refinement levels, and let C denote the corresponding limit curve whenever the subdivision scheme is convergent. In the numerical experiments, the fifth-level curve C_5 is used as a sufficiently refined numerical approximation

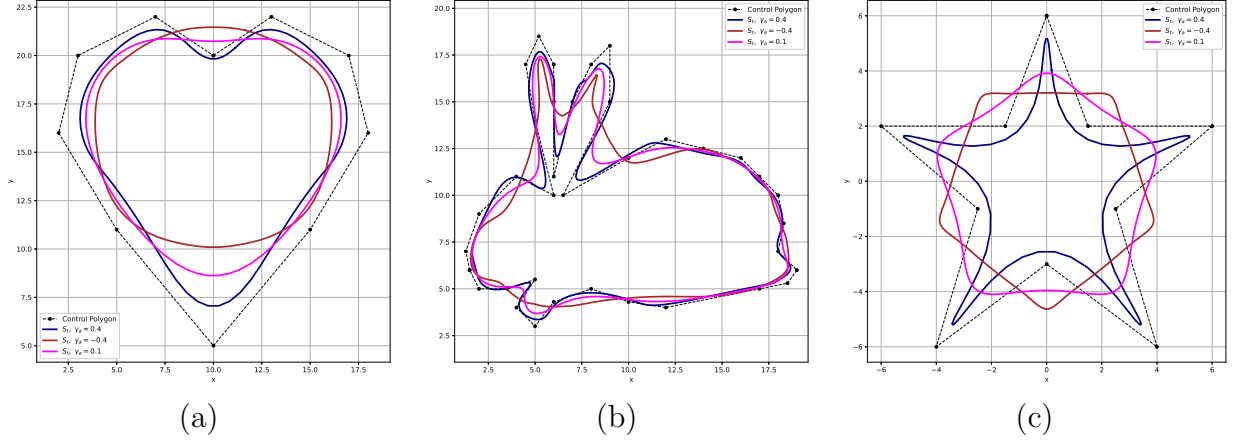


Figure 4: *Curves generated by the subfamily S_t for different parameter configurations.*

to C . Therefore, for $k = 0, 1, \dots, 4$, the geometric difference between C_k and the numerical reference curve C_5 is measured by

$$d_H(C_k, C_5).$$

A smaller value of $d_H(C_k, C_5)$ indicates that the refined curve C_k is geometrically closer to the fifth-level reference curve.

For the proposed scheme S_n , the Hausdorff distances corresponding to the three parameter configurations used in Figure 2 are listed in Table 1. The values decrease with successive refinement levels for all three test configurations, showing a clear reduction in the geometric approximation error.

The corresponding Hausdorff error-decay plots are presented in Figure 5. The logarithmic vertical scale clearly illustrates the reduction in the Hausdorff distance as the refinement level increases. Different parameter configurations exhibit different decay rates, indicating that the shape parameters influence not only the geometry of the limit curve but also its numerical approximation behavior.

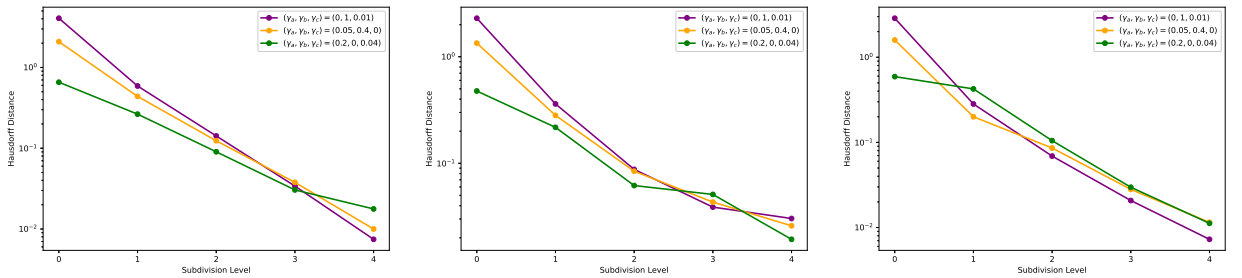


Figure 5: *Hausdorff error decay for the proposed scheme S_n under different parameter configurations.*

For the subfamily S_r , the Hausdorff distance is computed for the parameter values used in the quantitative experiment. The corresponding results are reported in Table 2. The Hausdorff distance decreases with refinement for all three test cases, although the decay rate depends on the selected value of γ_b and on the initial test curve.

Table 1: Hausdorff distances for the three S_n test cases at different subdivision levels.

Figure 2(a)			
Level	Set 1	Set 2	Set 3
0	4.066890057407e+00	2.086522565075e+00	6.563525679785e-01
1	5.894794392887e-01	4.383628708234e-01	2.641979941186e-01
2	1.419111371378e-01	1.237725635605e-01	9.034647734935e-02
3	3.401029552693e-02	3.781214935621e-02	3.046240496931e-02
4	7.451003114428e-03	9.997042644764e-03	1.771635449022e-02

Figure 2(b)			
Level	Set 1	Set 2	Set 3
0	2.297685840776e+00	1.341927897382e+00	4.768739627103e-01
1	3.606672309992e-01	2.817413331377e-01	2.174863804692e-01
2	8.769806885425e-02	8.427950007370e-02	6.182746467819e-02
3	3.880214947705e-02	4.325100654275e-02	5.102755649369e-02
4	3.033620057084e-02	2.592704953173e-02	1.934603397641e-02

Figure 2(c)			
Level	Set 1	Set 2	Set 3
0	2.871181499326e+00	1.594031830384e+00	5.921606915458e-01
1	2.825347260138e-01	1.999777950877e-01	4.247145078298e-01
2	6.901331929656e-02	8.587190175468e-02	1.047899118590e-01
3	2.075666807879e-02	2.808631304137e-02	2.974279393965e-02
4	7.268542104239e-03	1.150617564731e-02	1.120546239138e-02

The corresponding Hausdorff error-decay plots are shown in Figure 6.

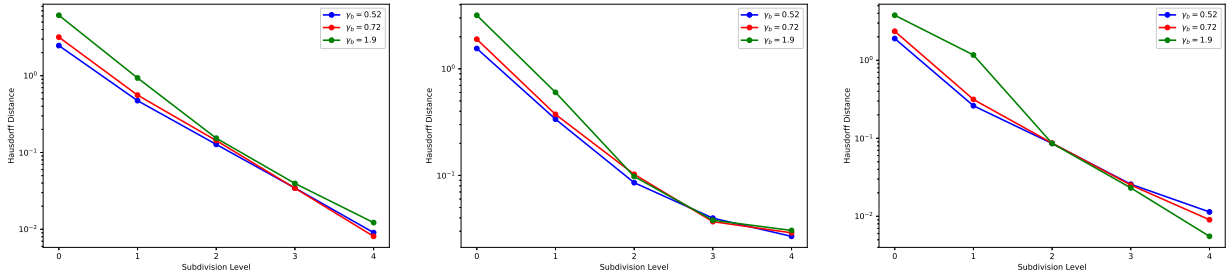


Figure 6: *Hausdorff error decay for the subfamily S_r under different values of γ_b .*

Finally, the Hausdorff distance is evaluated for the one-parameter 8-point subfamily S_t . The three parameter configurations used in Figure 4 are also considered in the quantitative analysis. The corresponding Hausdorff distances are reported in Table 3. All three parameter configurations show a clear decrease in Hausdorff distance as the subdivision level increases.

The corresponding Hausdorff error-decay plots are shown in Figure 7.

Overall, the numerical results show a consistent decrease in the Hausdorff distance with increasing subdivision level for the proposed scheme and the considered subfamilies. The results also demonstrate that the shape parameters influence both the geometric form

Table 2: Hausdorff distances for the S_r test cases at different subdivision levels.

Figure 3(a)			
Level	$\gamma_b = 0.52$	$\gamma_b = 0.72$	$\gamma_b = 1.90$
0	2.468485657069e+00	3.178812738292e+00	6.127885328435e+00
1	4.733113080699e-01	5.588053715560e-01	9.348701992029e-01
2	1.275968892859e-01	1.431092603197e-01	1.535757614388e-01
3	3.451448586281e-02	3.431011121188e-02	3.956629428668e-02
4	9.052870448169e-03	8.095079336644e-03	1.222086355883e-02

Figure 3(b)			
Level	$\gamma_b = 0.52$	$\gamma_b = 0.72$	$\gamma_b = 1.90$
0	1.552698271205e+00	1.897636412702e+00	3.183114835769e+00
1	3.373290506613e-01	3.739885705484e-01	6.018176757041e-01
2	8.551991263683e-02	1.025025715321e-01	9.769198366637e-02
3	3.975107932984e-02	3.685083496395e-02	3.798490557833e-02
4	2.682594583383e-02	2.893514847132e-02	3.048230430881e-02

Figure 3(c)			
Level	$\gamma_b = 0.52$	$\gamma_b = 0.72$	$\gamma_b = 1.90$
0	1.899954446653e+00	2.355610748757e+00	3.777365163872e+00
1	2.611874848156e-01	3.147120948776e-01	1.169434355565e+00
2	8.586158396673e-02	8.666614448182e-02	8.605553501483e-02
3	2.575779007453e-02	2.517494141328e-02	2.313214814304e-02
4	1.137417668037e-02	9.028707051270e-03	5.533563467253e-03

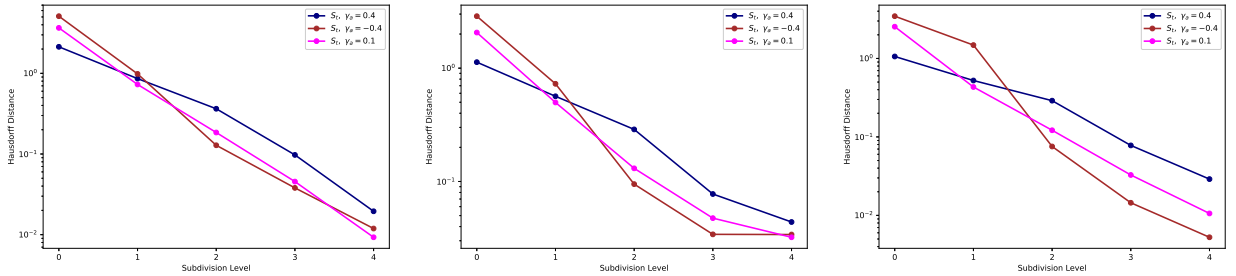


Figure 7: Hausdorff error decay for the subfamily S_t under different parameter configurations.

of the limit curves and the rate at which the refined curves approach the corresponding reference curves. The graphical comparisons and the Hausdorff-distance measurements therefore provide complementary evidence of the approximation behavior of the proposed subdivision schemes.

6.2 Numerical experiments

The analytical results are further illustrated through three numerical experiments.

Experiment 6.1: Parameter variation.

Table 3: Hausdorff distances for the S_t test cases at different subdivision levels.

Figure 4(a)			
Level	$\gamma_a = 0.4$	$\gamma_a = -0.4$	$\gamma_a = 0.1$
0	2.132550093376e+00	5.109613082889e+00	3.661885033277e+00
1	8.604674993372e-01	9.841903325200e-01	7.281239215917e-01
2	3.639201425532e-01	1.283128614763e-01	1.847951442201e-01
3	9.745567814212e-02	3.801038536945e-02	4.551947914858e-02
4	1.949710611959e-02	1.191843416236e-02	9.285277128154e-03
Figure 4(b)			
Level	$\gamma_a = 0.4$	$\gamma_a = -0.4$	$\gamma_a = 0.1$
0	1.129731593735e+00	2.874604633907e+00	2.062784047046e+00
1	5.646263424412e-01	7.277749743600e-01	4.981495119671e-01
2	2.881099552901e-01	9.476484263788e-02	1.306335262863e-01
3	7.752435883416e-02	3.415761649127e-02	4.748377888039e-02
4	4.389460996625e-02	3.403315357595e-02	3.218925906472e-02
Figure 4(c)			
Level	$\gamma_a = 0.4$	$\gamma_a = -0.4$	$\gamma_a = 0.1$
0	1.059432556270e+00	3.454992386202e+00	2.538900925133e+00
1	5.233263670399e-01	1.479788244738e+00	4.328057837250e-01
2	2.896596303587e-01	7.551165212377e-02	1.215976418089e-01
3	7.793153399892e-02	1.447901076863e-02	3.273012272769e-02
4	2.900301591197e-02	5.259689733274e-03	1.058735562408e-02

Consider the initial control polygon

$$\{P_i\}_{i=0}^7 = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 1), (5, 1), (6, 1), (7, 0)\}.$$

The schemes S_r and S_t are applied for different values of their respective shape parameters. The resulting curves are generated after five refinement levels, together with magnified views near the left endpoint.

For S_r , the parameter values are

$$\gamma_b = 0.1, \quad 0.4909090909, \quad 0.7, \quad 0.9, \quad 1.145454545, \quad 1.9,$$

whereas for S_t they are

$$\gamma_a = -0.8, \quad -0.7, \quad 0, \quad 0.3, \quad 0.5, \quad 0.6.$$

The corresponding curves and magnified views are shown in Figure 8. The results illustrate the effect of the shape parameters on the local behavior of the generated curves.

Experiment 6.2: A trigonometric function with a jump.

Consider the piecewise-defined function

$$g(x) = \begin{cases} \sin(\pi x), & 0 \leq x < 0.5, \\ -\sin(\pi x), & 0.5 \leq x \leq 1. \end{cases}$$

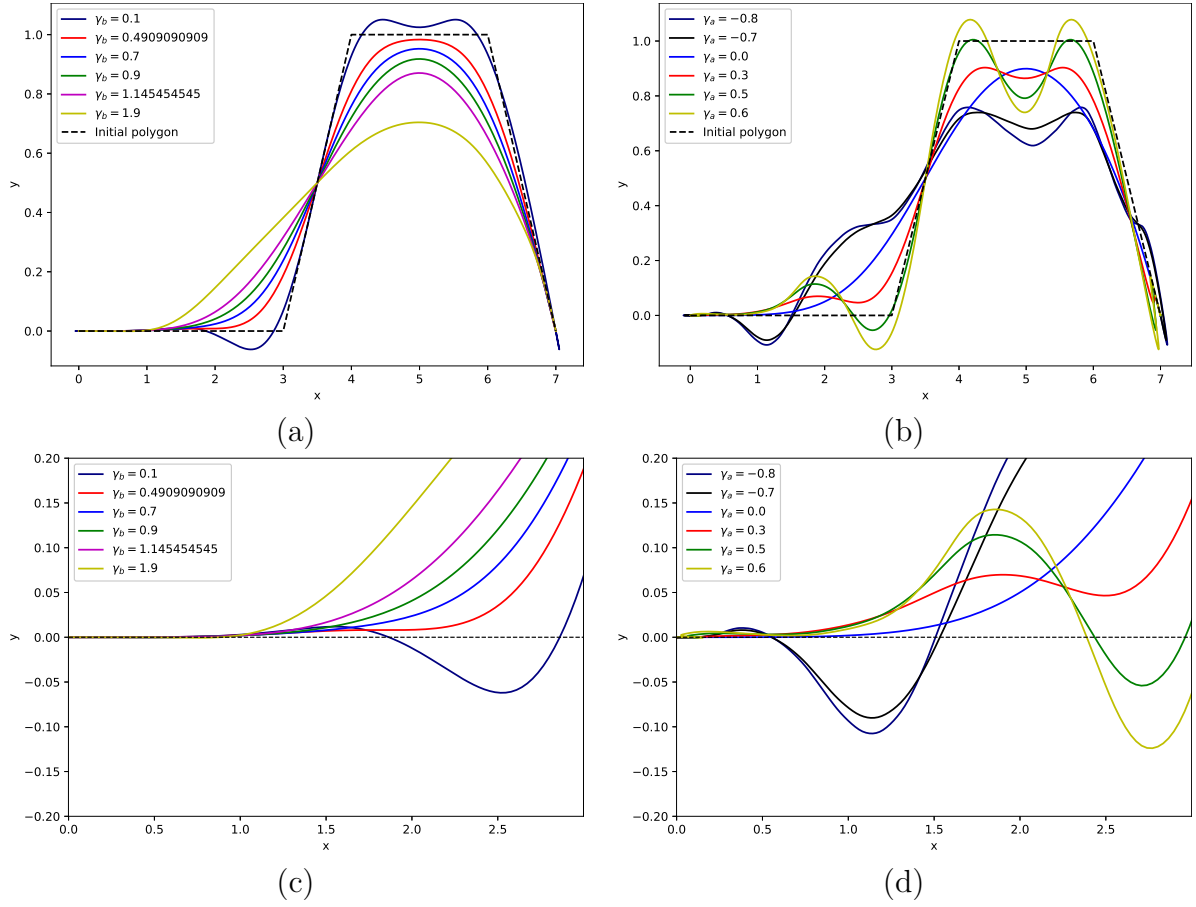


Figure 8: *Curves and left-end magnified views of the schemes S_r and S_t for different values of the shape parameters.*

The function has a jump discontinuity at $x = 0.5$. The two one-sided values at the discontinuity are represented separately in the sampled data. The initial control points are obtained from

$$P_i = (x_i, g(x_i)), \quad x_i = \frac{i}{10},$$

together with the two one-sided values at $x = 0.5$. The corresponding sampled values are listed in Table 4.

Table 4: *Sampled values for the function $g(x)$.*

x	0	0.1	0.2	0.3	0.4	0.5 ⁻	0.5 ⁺	0.6	0.7	0.8	0.9	1
$g(x)$	0	0.309	0.588	0.809	0.951	1	-1	-0.951	-0.809	-0.588	-0.309	0

The corresponding limit curves after five refinement levels, together with their left-end magnified views, are shown in Figure 9. The numerical results illustrate the effect of the selected shape parameters on the behavior of the curves near the discontinuity.

Experiment 6.3: A piecewise-linear function with a jump.

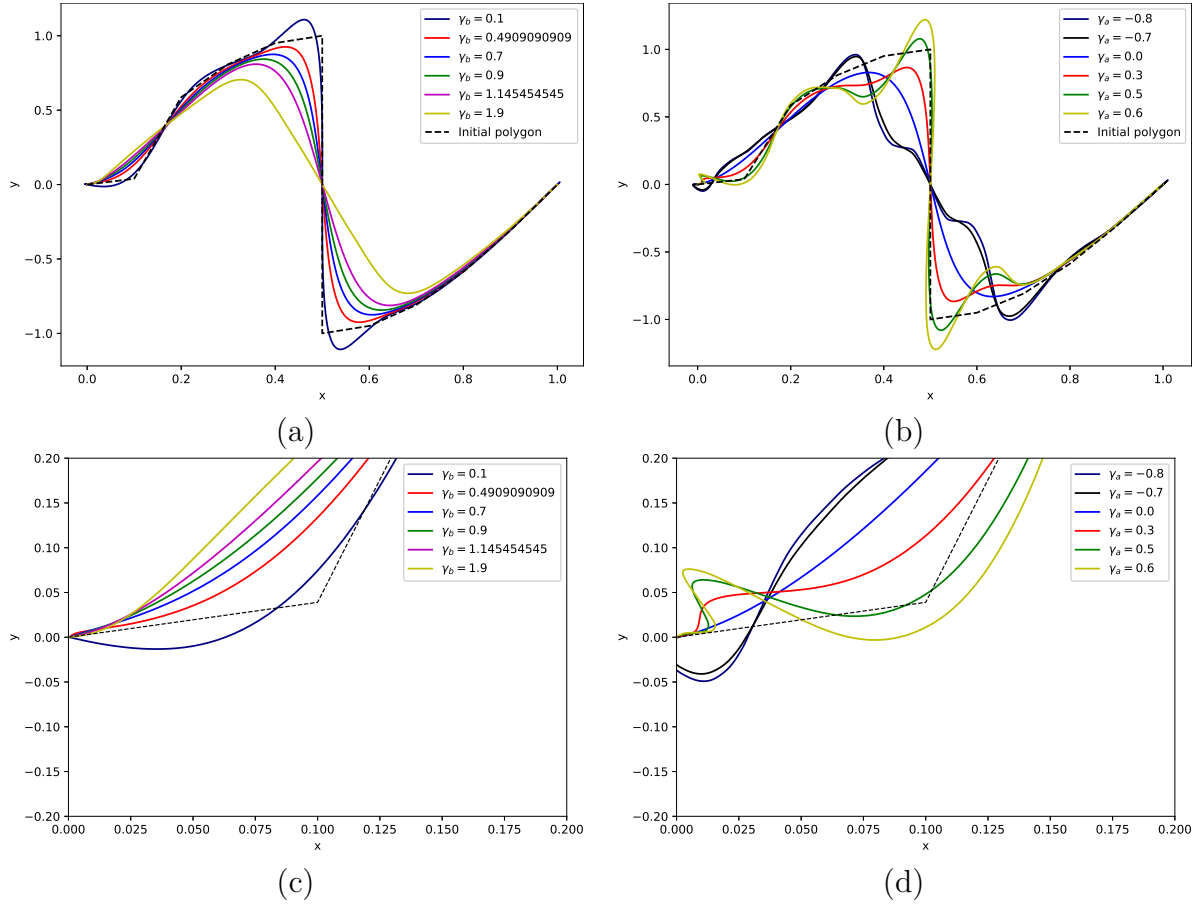


Figure 9: Curves and left-end magnified views of the schemes S_r and S_t for the discontinuous function $g(x)$.

Consider the piecewise-defined function

$$f(x) = \begin{cases} x + 1, & x < 1, \\ x + 5, & x \geq 1. \end{cases}$$

The function has a jump discontinuity of magnitude 4 at $x = 1$. The corresponding initial control points are listed in Table 5.

Table 5: Initial control points obtained from the piecewise-defined function $f(x)$.

x	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	-3	-2	-1	0	1	6	7	8	9	10

The schemes S_r and S_t are applied to these initial data using the shape parameters considered in the numerical experiments. The resulting curves after five refinement levels, together with their left-end magnified views, are shown in Figure 10. The results provide a second test of the Gibbs behavior for a discontinuity in a piecewise-linear setting.

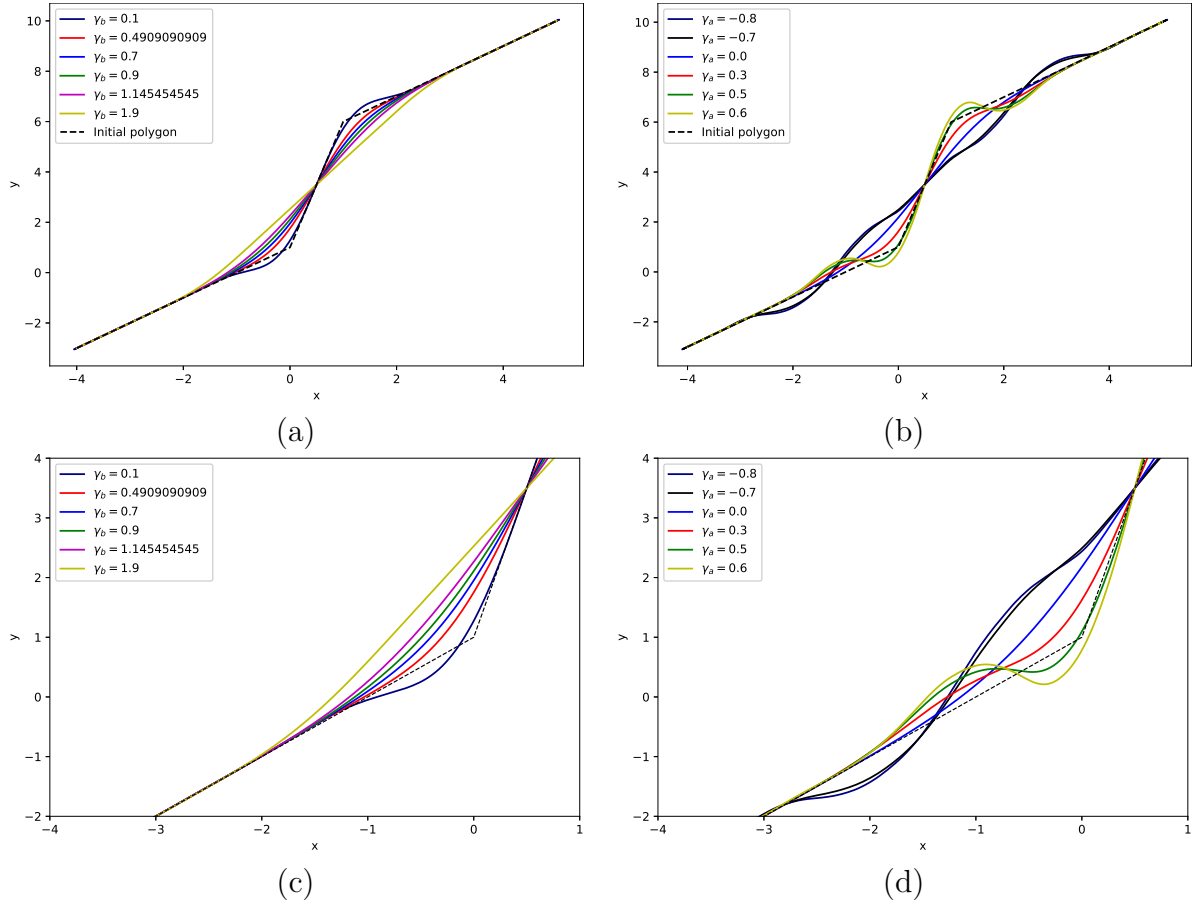


Figure 10: Curves and left-end magnified views of the schemes S_r and S_t for the piecewise-defined function $f(x)$.

7 Conclusion

In this work, a new parametric stationary binary 8-point approximating subdivision scheme is constructed by combining the 6-point Lagrange and B-spline approximating schemes. Three shape parameters are introduced through weighted difference-vector corrections, which provide additional flexibility in controlling the positions of the newly generated points and, consequently, the geometry of the resulting limit curves. Several subfamilies are obtained by imposing suitable restrictions on these parameters, including reduced-support and reduced-parameter schemes.

The support and boundary treatments of the proposed schemes are also analyzed. In particular, the proposed 8-point family has a maximal support width of 15, and suitable reflection and cyclic procedures are given for open and closed control polygons, respectively. The Gibbs phenomenon is investigated using the cumulative-sum criterion, leading to explicit parameter conditions for the considered subfamilies. The numerical experiments further illustrate the influence of the shape parameters on the local behavior of the generated curves and their response near discontinuities.

The Hausdorff distance is also employed to quantify the geometric approximation at succes-

sive refinement levels. The numerical results show a progressive decrease in $d_H(C_k, C_5)$ for the tested parameter configurations, indicating increasing agreement between the refined curves and the fifth-level reference curve. The numerical results show a progressive decrease in the Hausdorff distance for the tested parameter configurations, indicating improved geometric approximation as the refinement level increases. Overall, the proposed construction provides a flexible parametric framework for controlling the support and shape of binary approximating subdivision curves.

As future work, the proposed construction can be extended to higher dimensional settings and surface subdivision schemes. Further investigation of optimal parameter selection, higher-order smoothness, and parameter-dependent shape optimization may also provide useful extensions for geometric design and computer-aided geometric modeling.

Author's contributions

Conceptualization: Rabia Hameed; Formal analysis: Rabia Hameed and Shazia Hassan; Methodology: Rabia Hameed and Shazia Hassan; Supervision: Rabia Hameed; Writing original draft, Rabia Hameed and Shazia Hassan; Writing, review and editing: Rabia Hameed, Shazia Hassan and Misbah Mudassar.

Disclosure statement

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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