

Supplementary Material for

Biological degeneracy preserves phase diagrams topology in excitable systems

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Logistic KTz fixed points stability limits

The logistic KTz model is defined by the 3-Dimensional map:

$$x(t+1) = f\left(\frac{x(t) - Ky(t) + z(t) + I_e + I(t)}{T}\right), \quad (\text{S1})$$

$$y(t+1) = x(t), \quad (\text{S2})$$

$$z(t+1) = (1 - \delta)z(t) - \lambda(x(t) - x_r), \quad (\text{S3})$$

where $f(u) = u/(1 + |u|)$ and u is the argument of the logistic activation function. The interpretation of the variables and parameters is presented in the main paper.

If parameters $\delta, \lambda \ll 1$, we can assume that variable z is constant ($z(t+1) = z(t) \approx z^*$). The Jacobian of the fast subsystem x and y is

$$\mathbf{J} = \begin{pmatrix} \frac{1}{TU^2} & \frac{-K}{TU^2} \\ 1 & 0 \end{pmatrix}, \quad (\text{S4})$$

where we define $U(x, y) \equiv 1 + \frac{|x(t) - Ky(t) + z^* + I_e|}{T}$ to simplify the calculation. Observe that we assumed $T > 0$. The characteristic equation is

$$\Lambda^2 - \frac{1}{TU^2}\Lambda + \frac{K}{TU^2} = 0, \quad (\text{S5})$$

with eigenvalues $\Lambda_{\pm} = \frac{1 \pm \sqrt{1 - 4KTU^2}}{2TU^2}$. At the fixed point, we have $x(t+1) = x(t) = x^*$, $y(t+1) = y(t) = x^*$ and $z^* = -\lambda(x^* - x_r)/\delta$. When $\Lambda \in \mathbb{C}$ and $|\Lambda| = 1$ (Andronov-Hopf or Neimark-Sacker bifurcation), the oscillatory attractors appear and the FPs lose stability. Using these conditions, we obtain the relation $U(x^*, y^*) = \sqrt{K/T}$, which can be rewritten as $\sqrt{KT} = T + |x^*(1 - K - \lambda/\delta) + \lambda x_r/\delta + I_e|$. Solving for x^* , we get the expression for the FPs at the stability limit

$$x_{\pm}^* = \frac{\pm(\sqrt{KT} - T) - \lambda x_r/\delta - I_e}{1 - K - \lambda/\delta}. \quad (\text{S6})$$

Using the system equations (Eqs. S1 - S3) in the FP and the relation for $\sqrt{KT}$, we get $x^* = \frac{x^*(1 - K - \lambda/\delta) + \lambda x_r/\delta + I_e}{\sqrt{KT}}$. Solving for x^* ,

$$x^* = \frac{\lambda x_r/\delta + I_e}{\sqrt{KT} + K + \lambda/\delta - 1}. \quad (\text{S7})$$

Assuming $\delta = \lambda$ (as in the phase diagram) and combining Eqs. S6 and S7, we obtain the approximated FPs stability limits

$$I_e + x_r = \pm(K - T). \quad (\text{S8})$$

Supplementary Figures

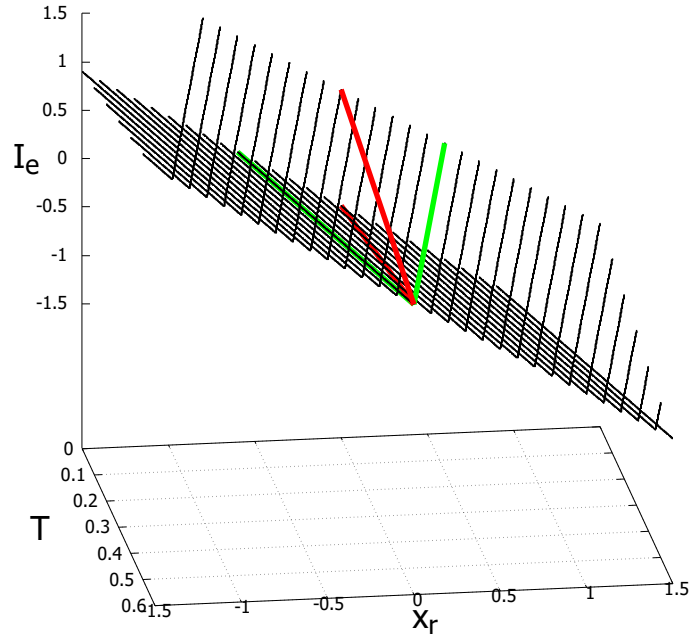


Figure S1: Parameter space with x_r , I_e and T for the logistic KTz. In (a), contour lines (black) are from Eq. S8 for different I_e , the green line is $I_e = 0$ (Fig. S2-(a)) and red is $x_r = 0$ (Fig. S2-(b)).

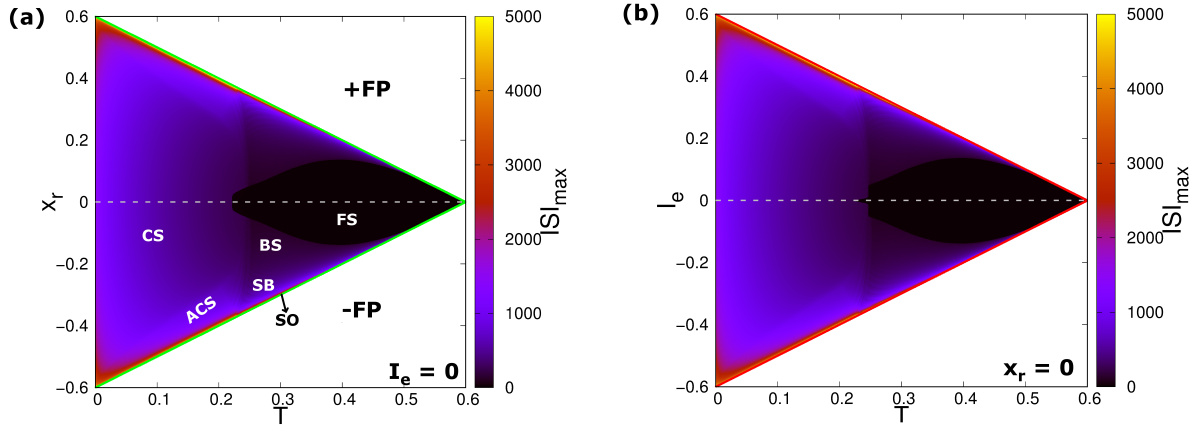


Figure S2: Phase diagrams of the logistic KTz model with $\delta = \lambda = 0.001$. In (a), the original phase diagram with $I_e = 0$. In (b), the excitable phase diagram with $x_r = 0$. Color scale represents the maximum ISI. The dashed grey lines indicate the axis of symmetry of the diagrams. The green and red curves represent the approximated FPs stability limits of Eq. S8. The behaviors are: positive/negative fixed point (+/-FP), cardiac spiking (CS), aperiodic CS (ACS), bursting (BS), slow BS (SB), fast spiking (FS) and subthreshold oscillations (SO). The behaviors and their placement in parameter space in (b) are analogous to (a) and are illustrated in Fig. S3. The dimensionless grid size along both axes is 10^{-3} and 10^5 time steps (ts) were used with a transient of 5×10^4 ts for each point. Initial conditions (ICs) are $x(0) = y(0) = z(0) = 0$.

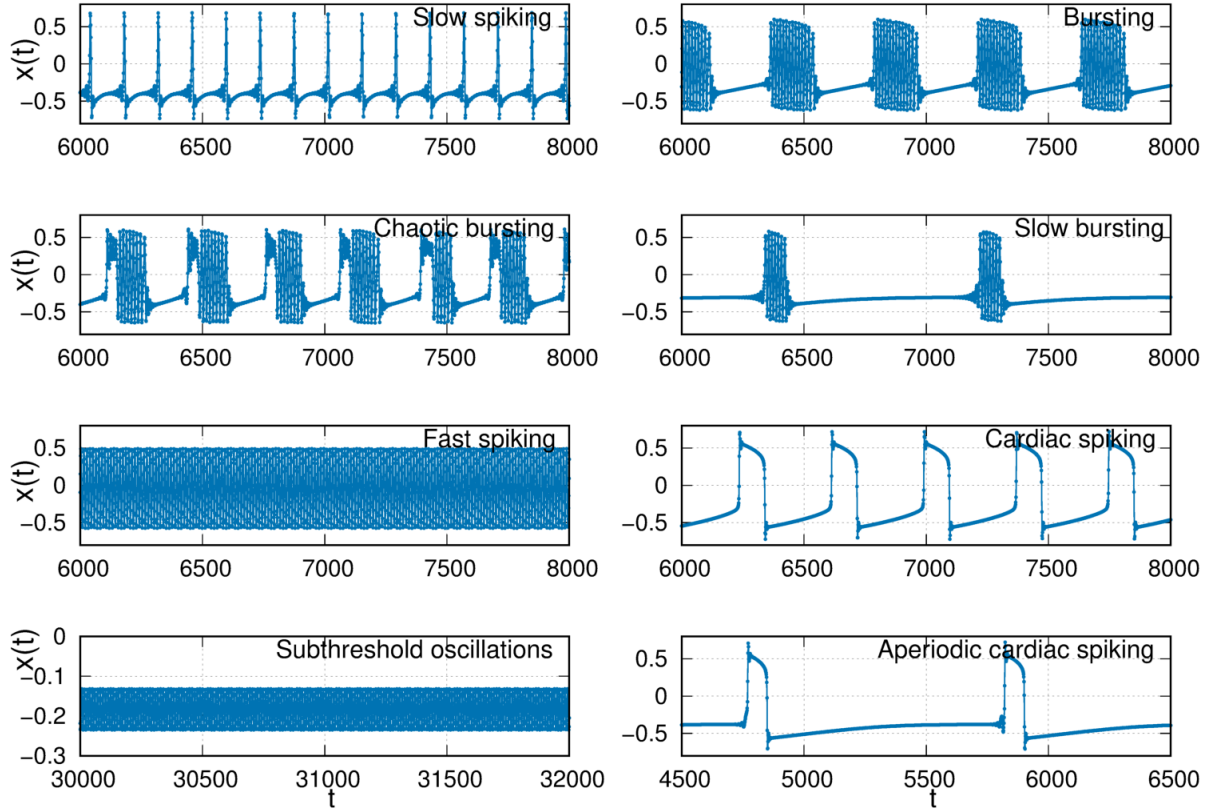


Figure S3: Behaviors found in the excitable ($x_r = 0$) phase diagram for the logistic KTz model in Fig. S2-(b) (slow spiking is the exception, which is outside this diagram). The placement of behaviors in the parameter space is analogous to Fig. S2-(a). Slow spiking (SS): $\delta = \lambda = 0.01$, $T = 0.21$ and $I_e = -0.37$. Bursting (BS): $T = 0.27$ and $I_e = -0.2$. Chaotic bursting: $T = 0.25$ and $I_e = -0.15$. Slow bursting (SB): $T = 0.27$ and $I_e = -0.3$. Fast spiking (FS): $T = 0.3$ and $I_e = -0.05$. Cardiac spiking (CS): $T = 0.2$ and $I_e = -0.2$. Subthreshold oscillations (SO): $T = 0.4$ and $I_e = -0.2$. Aperiodic cardiac spiking (ACS): $T = 0.25$ and $I_e = -0.35$. ICs are $x(0) = y(0) = z(0) = 0$.

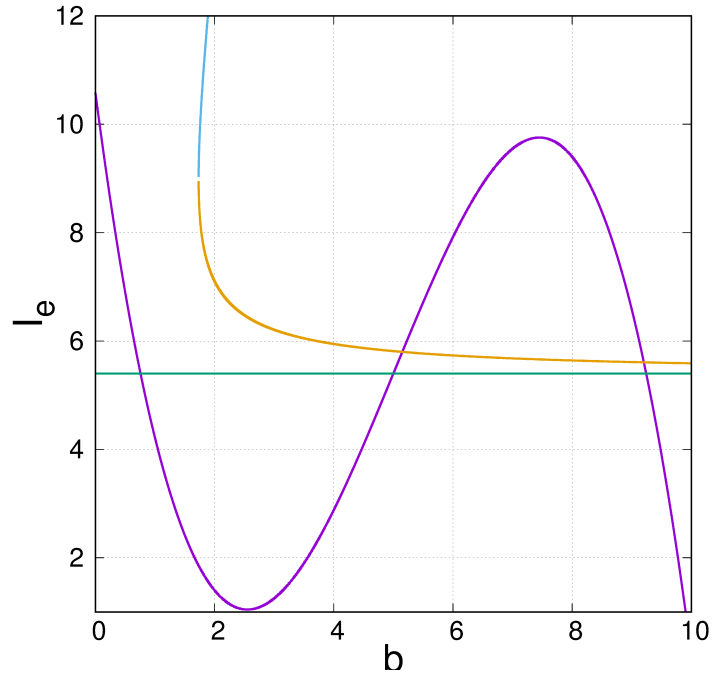


Figure S4: Approximate FPs stability limits of the HR model. The complete set of approximate FPs stability limits presented in the main paper are: Eq. 19 in green, Eq. 20 in purple and the two resulting solutions of Eq. 16 in yellow and blue.

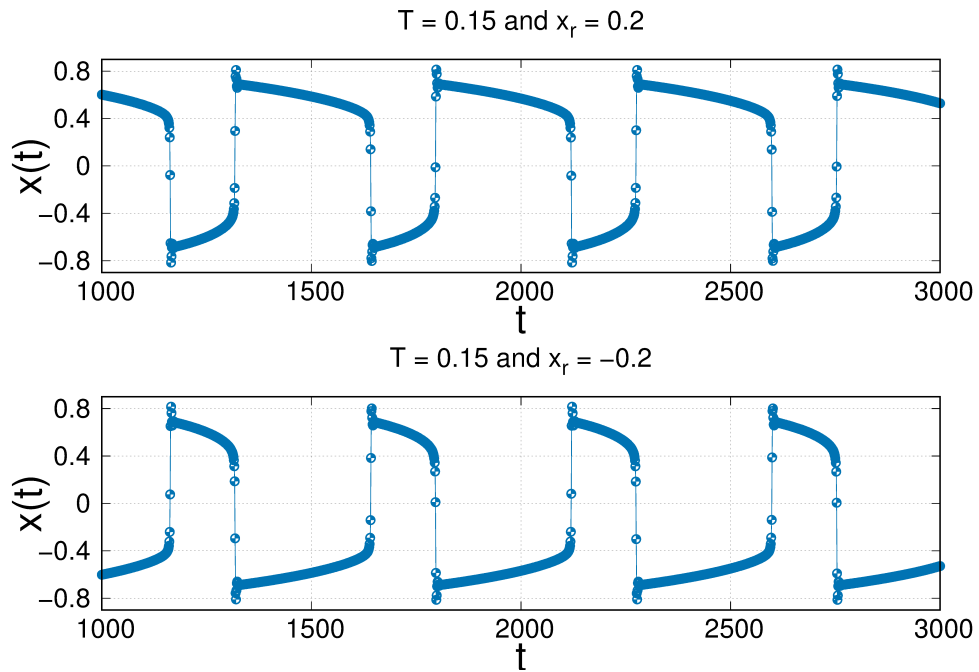


Figure S5: Example of symmetric membrane potential in the logistic KTz phase diagram of Fig. S2-(a). Switching the sign of x_r leads to the same membrane potential time series, but upside down.

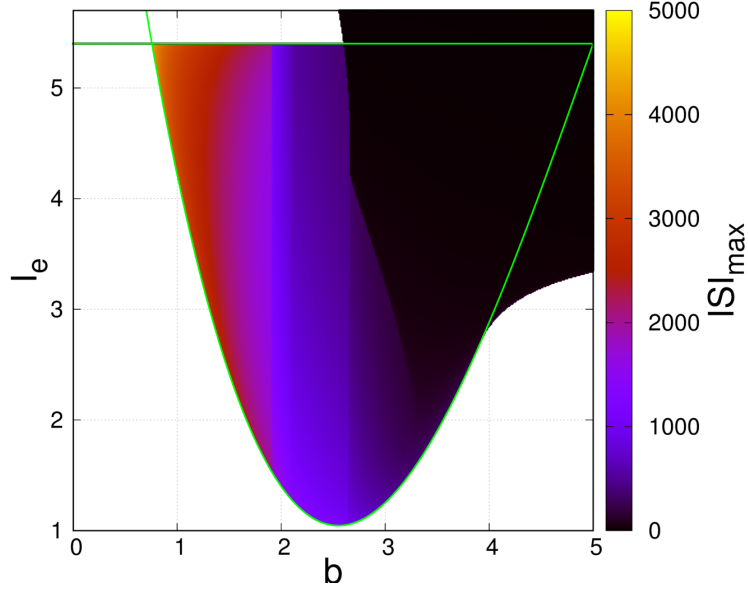


Figure S6: Phase diagram of the HR model for $r = 0.001$ with the approximated FP stability limits. The color scale represents the maximum ISI and the green curves represent the approximated stability limits of Eqs. 19 and 20 of the main paper. The dimensionless grid size along both axes is 10^{-3} and 20^4 tu were used with a transient of 10^4 tu for each point. IC are $x(0) = y(0) = z(0) = 0$.

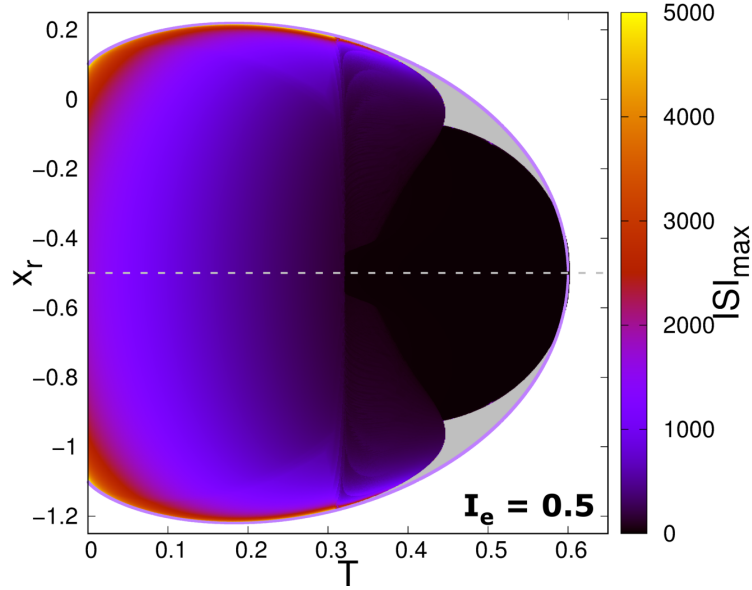


Figure S7: Phase diagram for the hyperbolic KTz model with $I_e = 0.5$. This diagram is in the plane orthogonal to the corresponding excitable diagram with $x_r = -0.5$ in Fig. 5-(a) of the main paper (see Fig. 4-(a) of the main paper). The purple curve represents the approximate stability limit of Eq. 12 of the main paper. The dashed grey line indicates the axis of symmetry of the diagram. Color scale represents the maximum ISI. The dimensionless grid size, total ts, transient and ICs are the same as in Fig. 1 of the main paper.

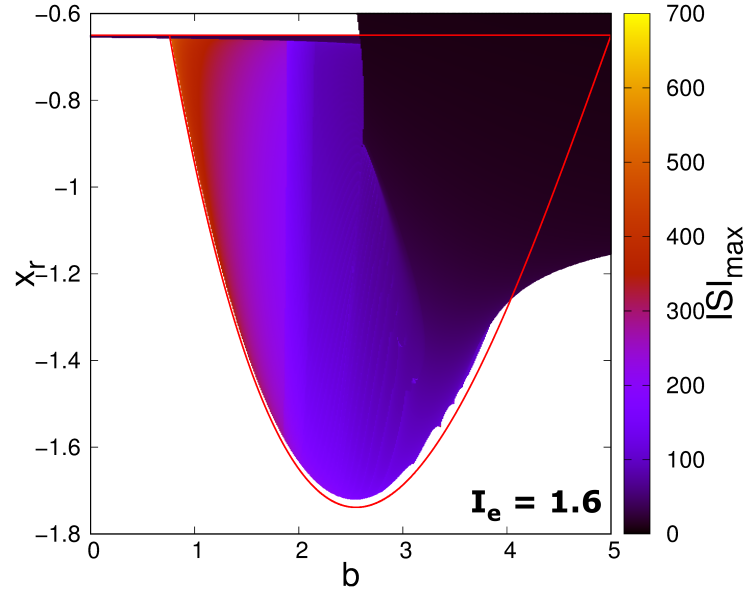


Figure S8: Phase diagrams with fixed external current ($I_e = 1.6$) for the HR model. Red lines are the approximate FPs stability limits from Eqs. 19 and 20 of the main paper. Color scale represents the maximum ISI. The dimensionless grid size along both axes is 10^{-3} and 10^4 tu were used with a transient of 5×10^3 tu for each point. ICs are $x(0) = y(0) = z(0) = 0$. The phases are analogous to Fig. 1-(b) of the main paper.