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Density and Graded Balance in q -Rung Orthopair Fuzzy Graphs, with an Application to Medical Examination

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Abstract

We lift the $*$ -density of a fuzzy graph, introduced by Al-Hawary and Almomani, to q -rung orthopair fuzzy graphs. Every such graph carries two underlying Rosenfeld fuzzy graphs, obtained from the membership function and from the complemented non-membership function, and this yields a $*$ -density profile together with a bounded density signature. We prove a sharp upper bound on the $*$ -density in terms of the maximum degree of the underlying crisp graph, which corrects the corresponding statement in the fuzzy case, and a reduction theorem: a strong q -rung orthopair fuzzy graph with constant vertex function is $*$ -balanced if and only if its underlying crisp graph is balanced in the classical sense, which subsumes four separate class theorems in a single argument. The binary property of $*$ -balance then becomes a graded index taking values between zero and one, defined through a logarithmic metric on signatures, equal to one exactly when the graph is $*$ -balanced, and computable in polynomial time in the reduced case. For decision making we induce a q -rung orthopair fuzzy graph on a symptom set from a reference family and prove that the orthopair constraint is preserved. Finally we show that every q -rung orthopair fuzzy distance which aggregates symmetrically over the universe of discourse is degenerate on profiles that permute the same values across different elements. Twenty-four published measures are implemented, and all of them tie on such pairs in two independent benchmarks, whereas the density signature separates them.

Keywords: q -rung orthopair fuzzy graph; $*$ -density; $*$ -balanced graph; logarithmic distance measure; medical diagnosis; multi-criteria decision making.

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1 Introduction

Graph-theoretic models of real systems are almost always built on incomplete information: an edge is rarely simply present or absent, and a vertex rarely simply belongs or does not belong to the structure under study. Following Zadeh's fuzzy sets [1] and Rosenfeld's fuzzy graphs [2], a large body of work has transferred the classical vocabulary of graph theory into the fuzzy setting [3, 4], and onward into the intuitionistic [5] and Pythagorean [6, 7] settings.

Density is one notion that has resisted a fully satisfactory transfer. Al-Hawary [8, 9] introduced balanced fuzzy graphs, and Al-Hawary and Almomani [10] proposed the $*$ -density

$$D^*(G) = \frac{2 \sum_{u,v \in V} \mu(u,v)}{\sum_{u \in V} \sigma(u)}, \quad (1)$$

modifying an earlier definition precisely so that the resulting quantity would agree more closely with what is known about the density of crisp graphs. A fuzzy graph is $*$ -balanced when no non-empty fuzzy subgraph has larger $*$ -density than the graph itself. In [10] it is shown, by four separate arguments, that K_n , C_n , the Petersen graph P_{10} and $K_{n,n}$ all give rise to $*$ -balanced fuzzy graphs when the vertex function is constant, with $*$ -densities $n - 1$, 2 , 3 and n . We note at the outset that [10] remains an unrefereed preprint, still cited as such seven years on; one purpose of Section 4 is to supply the scrutiny it has not received, and Remark 2 identifies an error in it.

Independently, q -rung orthopair fuzzy sets, introduced by Yager [11] as a common generalisation of intuitionistic [12] and Pythagorean [13] fuzzy sets, and since developed in several directions [14, 15, 16], have become a standard vehicle for decision making under uncertainty, and considerable effort has gone into constructing distance and similarity measures on such sets: Minkowski-type [17], information and entropy based [18], cosine [19], Soergel [20], Hausdorff [21] and L_p -norm [22] families, among others. The logarithmic family in particular traces to the weighted induced logarithmic distances of Zeng, Hu and Xie [23]; recent contributions include logarithmic distance measures applied to vehicle selection [24] and new q -ROF distance functions applied to patients' medical status [25, 26], alongside related work in the Pythagorean case [27]; the use of orthopair models for medical assessment goes back to [28]; the present author is a co-author of [24, 25], and this paper is in part a critique of the architecture those measures share. These measures are validated by their metric properties and by their *discrimination power*: their ability to separate pairs on which cruder measures return identical values.

The graph-theoretic side of this framework is by now well populated. Habib, Akram and Farooq [29] introduced q -rung orthopair fuzzy competition graphs; directed hypergraph [30], Hamacher-operator [31] and dual hesitant [32] variants followed. What this literature has concentrated on is the transfer of operations, energies and regularity conditions. Density, and with it the question of which substructures are anomalously concentrated, has not been treated in the q -rung setting at all, and it is what we supply here.

Balance itself, however, has already been carried into several settings adjacent to ours, and the relation to that work needs stating precisely. Karunambigai, Akram, Sivasankar and Palanivel [33] introduced balanced intuitionistic fuzzy graphs; bipolar [34] and interval-valued [35] variants followed, and Ghorai and Pal [36] studied density in the m -polar setting; Amanathulla, Bera and Pal [37] gave balanced picture fuzzy graphs with necessary and sufficient conditions and an algorithm to test them; and Some, Das and Pal [38] reached the spherical case. Four things separate the present paper from all of these.

The object differs. That line builds on the ordinary density of a fuzzy graph, whereas we lift the $*$ -density (1) of [10], whose denominator is the vertex mass rather than an edgewise minimum. The two disagree already at $q = 1$.

Their densities saturate. Some, Das and Pal prove that a complete spherical fuzzy graph has density exactly $(1, 1, 1)$ [38, Thm. 4.1]. Remark 4 below explains why a balance notion built on a normalised density carries little information — every complete graph is then extremal — and it is precisely why our score is defined on the unnormalised average-degree scale. Remark 8 shows the same effect biting in the application.

The decomposition is specific to orthopairs. Lemma 1 works because the two grades are tied by a single constraint, so that $\eta = 1 - \nu$ obeys exactly the Rosenfeld law that μ does. Picture and spherical fuzzy sets carry three independent grades under $\mu^2 + \eta^2 + \nu^2 \leq 1$, and the neutral grade satisfies no analogous law; the decomposition transfers in neither direction, and nor does Theorem 10, which rests on it.

The rung is a scale, not a setting. Theorem 10 holds for every q simultaneously, so the intuitionistic case of [33] is its $q = 1$ slice rather than a parallel result. The rung also raises a design question the fixed settings cannot pose — whether the edge construction should itself be q -powered — which Remark 9 answers. Finally, none of this literature connects balance

to distance measures at all: Theorem 17 concerns a q -ROF distance literature that has no counterpart in the picture or spherical settings.

The two literatures have remained separate, and each has a gap the other fills. The structural theory yields no decision procedure and says nothing about computation. The distance literature, for its part, shares an architecture — the comparison of two q -ROF sets is a symmetric aggregation of per-element deviations — and we show in Section 8 that this architecture has a definite blind spot, reachable in practice, which structural information repairs. Our contributions are:

1. A decomposition lemma (Lemma 1) exhibiting the two Rosenfeld fuzzy graphs underlying a q -ROF graph, and the resulting $*$ -density profile (Section 4).
2. A sharp bound $D^* \leq \Delta(G^*)$ with equality conditions (Theorem 3), which corrects the corresponding statement in [10] — whose hypothesis, we observe in Remark 2, is non-vacuous only for at most three vertices.
3. A reduction theorem (Theorem 10) identifying $*$ -balance in the strong constant case with classical balance of the underlying crisp graph, replacing four case analyses by one argument (Corollary 12).
4. A graded balance index $\beta(G) \in [0, 1]$ built from a logarithmic metric on normalised signatures, with the characterisation $\beta(G) = 1 \iff G$ is $*$ -balanced (Theorem 15) and a resolution of its computability (Remark 7).
5. A construction inducing a q -ROF graph on a symptom set from the disease-symptom knowledge base, with a proof that the orthopair constraint survives it (Lemma 16). We stress in Section 7 that without such a step the construction is vacuous, because that is the natural failure mode.
6. The discrimination theorem (Theorem 17), an instance of it drawn from a published benchmark rather than constructed, and a comparative study against twenty-four implemented measures (Section 10).

2 Preliminaries

Definition 1 ([11]). For $q \geq 1$, a q -rung orthopair fuzzy set on X is $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}$ with $\mu_A, \nu_A : X \rightarrow [0, 1]$ and $\mu_A(x)^q + \nu_A(x)^q \leq 1$. The hesitation degree is $\pi_A(x) = (1 - \mu_A(x)^q - \nu_A(x)^q)^{1/q}$. The cases $q = 1$ and $q = 2$ give intuitionistic and Pythagorean fuzzy sets.

Definition 2. A q -ROF graph on a finite vertex set V , $|V| = n \geq 2$, is a pair $G = (A, B)$ where $A = (\mu_A, \nu_A)$ is a q -ROF set on V and $B = (\mu_B, \nu_B)$ is a q -ROF set on $E \subseteq V \times V$ with, for every $uv \in E$,

$$\mu_B(uv) \leq \mu_A(u) \wedge \mu_A(v), \quad \nu_B(uv) \geq \nu_A(u) \vee \nu_A(v), \quad \mu_B(uv)^q + \nu_B(uv)^q \leq 1.$$

G is *strong* if the first two hold with equality on E , and *complete* if it is strong with $E = \{uv : u \neq v\}$. We write $G^* = (V, E)$ for the underlying crisp graph and $\Delta(G^*)$ for its maximum degree.

Definition 3. A q -ROF *subgraph* $H = (A', B')$ of G has $W \subseteq V$, $F \subseteq E \cap (W \times W)$, $\mu_{A'} \leq \mu_A$, $\nu_{A'} \geq \nu_A$ on W and likewise on F . The *induced* subgraph $G[W]$ is the special case $F = E \cap (W \times W)$ with all grades inherited. We write $H \subseteq G$ and require $W \neq \emptyset$.

Definition 4 (classical balance). A crisp graph $G^* = (V, E)$ is *balanced* if

$$\max_{\emptyset \neq W \subseteq V} \frac{|E(G^*[W])|}{|W|} = \frac{|E|}{|V|}.$$

Lemma 1 (Decomposition). *Let $G = (A, B)$ be a q -ROF graph on V and put $\eta_A(u) = 1 - \nu_A(u)$, $\eta_B(uv) = 1 - \nu_B(uv)$. Then $G^\mu := (\mu_A, \mu_B)$ and $G^\eta := (\eta_A, \eta_B)$ are both fuzzy graphs on V in the sense of Rosenfeld.*

Proof. For G^μ the requirement is part of Definition 2. For G^η , from $\nu_B(uv) \geq \nu_A(u) \vee \nu_A(v)$,

$$\eta_B(uv) = 1 - \nu_B(uv) \leq 1 - (\nu_A(u) \vee \nu_A(v)) = (1 - \nu_A(u)) \wedge (1 - \nu_A(v)) = \eta_A(u) \wedge \eta_A(v).$$

Symmetry of η_B is inherited from ν_B . $\square$

This elementary observation is what makes the transfer of [10] possible: η obeys exactly the structural law that μ does, so every argument about the membership coordinate applies verbatim to the complemented non-membership coordinate.

3 A logarithmic distance

We record the distance used both pointwise and structurally. It follows the design principle of [23, 24, 25]: all three parameters enter, and the measure is normalised by the cardinality of the universe.

Definition 5. Let A, B be q -ROF sets on $X = \{x_1, \dots, x_n\}$. Their *logarithmic q -ROF distance* is

$$d_q(A, B) = \frac{1}{3n \ln 2} \sum_{i=1}^n \left[\ln(1 + |\mu_A^q(x_i) - \mu_B^q(x_i)|) \right. \\ \left. + \ln(1 + |\nu_A^q(x_i) - \nu_B^q(x_i)|) \right. \\ \left. + \ln(1 + |\pi_A^q(x_i) - \pi_B^q(x_i)|) \right].$$

For $x, y \in [0, 1]^3$ we write the three-coordinate analogue

$$\rho(x, y) = \frac{1}{3 \ln 2} \sum_{j=1}^3 \ln(1 + |x_j - y_j|).$$

Theorem 2. d_q is a metric on the q -ROF sets on X with $0 \leq d_q \leq 1$. The same holds for ρ on $[0, 1]^3$.

Proof. Non-negativity is clear. If $d_q(A, B) = 0$ every summand vanishes, so $\mu_A^q = \mu_B^q$ and $\nu_A^q = \nu_B^q$ pointwise; since $t \mapsto t^q$ is injective on $[0, 1]$, $A = B$. The converse and symmetry are immediate. For the triangle inequality, $s \mapsto \ln(1 + s)$ is increasing and subadditive on $[0, \infty)$, because $(1 + s)(1 + t) = 1 + s + t + st \geq 1 + s + t$, so $\ln(1 + s + t) \leq \ln(1 + s) + \ln(1 + t)$. Given A, B, C and any x_i we have $|\mu_A^q - \mu_C^q| \leq |\mu_A^q - \mu_B^q| + |\mu_B^q - \mu_C^q|$, whence $\ln(1 + |\mu_A^q - \mu_C^q|) \leq \ln(1 + |\mu_A^q - \mu_B^q|) + \ln(1 + |\mu_B^q - \mu_C^q|)$, and similarly for the ν and π terms; summing and dividing gives the claim. Finally each of the $3n$ arguments lies in $[0, 1]$, so each logarithm is at most $\ln 2$ and the total is at most $3n \ln 2$. The argument for ρ is identical with $3n$ replaced by 3. $\square$

Remark 1. The hesitation term is functionally determined by the other two, since $\pi^q = 1 - \mu^q - \nu^q$. It is retained because it re-weights the contribution of a pair according to how much of the orthopair budget is unallocated, which is the standard motivation in this literature [17, 18, 24]. Its presence does not affect the metric axioms.

4 The $\ast$ -density profile and signature

Definition 6. The $\ast$ -density profile of a q -ROF graph G with $E \neq \emptyset$ is $\mathfrak{D}^\ast(G) = (D_\mu^\ast(G), D_\eta^\ast(G))$, where

$$D_\mu^\ast(G) = \frac{2 \sum_{uv \in E} \mu_B(uv)}{\sum_{u \in V} \mu_A(u)}, \quad D_\eta^\ast(G) = \frac{2 \sum_{uv \in E} \eta_B(uv)}{\sum_{u \in V} \eta_A(u)}.$$

If $n < 2$, $E = \emptyset$, or a denominator vanishes, both are set to 0.

Both coordinates are the $\ast$ -density (1) of the corresponding fuzzy graph of Lemma 1. In particular $D_\mu^\ast$ reduces to the $\ast$ -density of [10], so that theory is recovered exactly.

4.1 A sharp upper bound

Theorem 3. Let G be a q -ROF graph with $\Delta = \Delta(G^\ast)$. Then $D_\mu^\ast(G) \leq \Delta$ and $D_\eta^\ast(G) \leq \Delta$. Equality holds in the first if and only if (i) $\mu_B(uv) = \mu_A(u) \wedge \mu_A(v)$ for every $uv \in E$; (ii) $\mu_A(u) = \mu_A(v)$ for every $uv \in E$; and (iii) $\deg_{G^\ast}(u) = \Delta$ for every $u \in V$.

Proof. Write $a_u = \mu_A(u) > 0$. Using $\mu_B(uv) \leq a_u \wedge a_v$ and then $a_u \wedge a_v \leq \frac{1}{2}(a_u + a_v)$,

$$2 \sum_{uv \in E} \mu_B(uv) \leq 2 \sum_{uv \in E} (a_u \wedge a_v) \leq \sum_{uv \in E} (a_u + a_v) = \sum_{u \in V} \deg_{G^\ast}(u) a_u \leq \Delta \sum_{u \in V} a_u,$$

which gives the bound after division. Equality throughout forces $\mu_B = a_u \wedge a_v$ on every edge, which is (i); $a_u \wedge a_v = \frac{1}{2}(a_u + a_v)$, i.e. $a_u = a_v$, on every edge, which is (ii); and $\deg_{G^\ast}(u) = \Delta$ for every u , since $a_u > 0$, which is (iii). Conversely (i)–(iii) give equality at each step. The argument for $D_\eta^\ast$ is identical with $a_u = \eta_A(u) > 0$, using Lemma 1. $\square$

Corollary 4. If $G^\ast = K_p$ then $D_\mu^\ast(G) \leq p - 1$, with equality if and only if G is strong in the membership coordinate and μ_A is constant.

Remark 2. Theorem 3 sharpens [10, Thm. 4], which asserts $D^\ast(G) \leq 2$ for a complete fuzzy graph satisfying $|V| \geq |E|$. For $G^\ast = K_p$ that hypothesis reads $p \geq p(p-1)/2$, i.e. $p \leq 3$, so the statement is non-vacuous only for at most three vertices — where it agrees with Corollary 4, since $p-1 \leq 2$. Theorem 3 removes the hypothesis, replaces the constant 2 by the sharp value $\Delta(G^\ast)$, and applies to arbitrary q -ROF graphs.

4.2 Regularity

Definition 7. The μ -degree of u is $d_\mu(u) = \sum_{v \neq u} \mu_B(uv)$ and the total degree is $t_\mu(u) = \mu_A(u) + d_\mu(u)$; G is r_μ -regular if $d_\mu \equiv r_\mu$ and r_μ -totally regular if $t_\mu \equiv r_\mu$. The η versions are analogous.

Theorem 5. Let G be a q -ROF graph on p vertices. If G is r_μ -regular then

$$D_\mu^\ast(G) = \frac{p r_\mu}{\sum_u \mu_A(u)}, \quad \text{which is } r_\mu/c \text{ when } \mu_A \equiv c;$$

and if G is r_μ -totally regular then

$$D_\mu^\ast(G) = \frac{p r_\mu}{\sum_u \mu_A(u)} - 1, \quad \text{which is } r_\mu/c - 1 \text{ when } \mu_A \equiv c.$$

The analogous statements hold for $D_\eta^\ast$.

Proof. Each edge is counted twice in $\sum_u d_\mu(u)$, so $2 \sum_{uv \in E} \mu_B(uv) = \sum_u d_\mu(u) = p r_\mu$, giving the first formula. Summing $t_\mu(u) = \mu_A(u) + d_\mu(u) = r_\mu$ over u gives $p r_\mu = \sum_u \mu_A(u) + 2 \sum_{uv \in E} \mu_B(uv)$, and division by $\sum_u \mu_A(u)$ gives the second. $\square$

4.3 The normalised signature

The profile $\mathfrak{D}^*$ is unnormalised: for the complete crisp graph it equals $n - 1$. The metric of Section 5 needs bounded components, so we also record a normalised version.

Definition 8. The *density signature* of G is $\Delta(G) = (\delta_\mu(G), \delta_\eta(G), \delta_\pi(G))$, where

$$\delta_\mu = \frac{D_\mu^*(G)}{n-1}, \quad \delta_\eta = \frac{D_\eta^*(G)}{n-1}, \quad \delta_\pi = \frac{1}{|E|} \sum_{uv \in E} \pi_B(uv),$$

with $\Delta(G) = (0, 0, 0)$ in the degenerate cases of Definition 6.

Proposition 6. $\delta_\mu(G), \delta_\eta(G), \delta_\pi(G) \in [0, 1]$.

Proof. Immediate from Theorem 3 and $\Delta(G^*) \leq n-1$ for the first two; $\pi_B(uv) \in [0, 1]$ pointwise gives the third. $\square$

The first two components are relational — edge mass against vertex mass — while the third is a mean. The asymmetry is deliberate: π_B admits no monotone relation to π_A , so a ratio would be unbounded. We state this rather than conceal it.

Proposition 7. If $G_1 \cong G_2$ as q -ROF graphs then $\mathfrak{D}^*(G_1) = \mathfrak{D}^*(G_2)$ and $\Delta(G_1) = \Delta(G_2)$.

Proof. An isomorphism is a bijection preserving all vertex and edge grades; every sum in Definitions 6 and 8 ranges over all vertices or all edges. $\square$

4.4 Stability

A decision-support quantity is of little use if a small revision to one expert's grade moves it far. The signature is stable in exactly that sense, and the following theorem makes the bound explicit.

Theorem 8 (Stability). Let G, G' be q -ROF graphs on the same n vertices with the same edge set and the same coefficient c , whose vertex grades satisfy $|\mu_A(u) - \mu_{A'}(u)| \leq \varepsilon$ for every u . Put $m = \sum_u \mu_A(u)$ and assume $m > n\varepsilon$. Then

$$|\delta_\mu(G') - \delta_\mu(G)| \leq \frac{2n\varepsilon}{m - n\varepsilon},$$

and the same bound holds for δ_η with m replaced by $\sum_u \eta_A(u)$.

Proof. Write $\delta_\mu = a/b$ with $a = 2 \sum_{uv \in E} \mu_B(uv)$ and $b = (n-1)m$. Since $x \mapsto x \wedge y$ is 1-Lipschitz and $c \leq 1$, each edge grade moves by at most ε , so $|a' - a| \leq 2|E|\varepsilon \leq n(n-1)\varepsilon$; and $|b' - b| \leq (n-1)n\varepsilon$. Then

$$\left| \frac{a'}{b'} - \frac{a}{b} \right| \leq \frac{|a' - a|}{b'} + \frac{a}{b} \cdot \frac{|b' - b|}{b'} \leq \frac{n(n-1)\varepsilon + n(n-1)\varepsilon}{(n-1)(m - n\varepsilon)} = \frac{2n\varepsilon}{m - n\varepsilon},$$

using $\delta_\mu \leq 1$ (Proposition 6) and $b' \geq (n-1)(m - n\varepsilon)$. The η argument is identical by Lemma 1. $\square$

Corollary 9. If in addition $\delta_\pi(G') = \delta_\pi(G)$, then

$$\rho(\Delta(G'), \Delta(G)) \leq \frac{1}{3 \ln 2} \left(\frac{2n\varepsilon}{m_\mu - n\varepsilon} + \frac{2n\varepsilon}{m_\eta - n\varepsilon} \right),$$

since $\ln(1+t) \leq t$.

Remark 3. The third component is excluded on purpose. Since $\pi = (1 - \mu^q - \nu^q)^{1/q}$ has unbounded derivative as the hesitation approaches 0, δ_π is not uniformly Lipschitz in the vertex grades, and no bound of this form can hold near that boundary. It is Lipschitz on any region where π_B is bounded away from 0, which covers every profile in Section 9.2, but we state the theorem for the two relational components where it holds unconditionally.

5 *-Balance and its reduction to crisp balance

Definition 9. A q -ROF graph G is **-balanced* if for every non-empty q -ROF subgraph $H \subseteq G$ we have $D_\mu^*(H) \leq D_\mu^*(G)$ and $D_\eta^*(H) \leq D_\eta^*(G)$. We write $S(G) = \frac{1}{2}(D_\mu^*(G) + D_\eta^*(G))$ for the combined score.

Remark 4 (Why η , and why not the normalised signature). Two design choices need justification. First, it is tempting to define the second coordinate directly from ν and to require $D_\nu^*(H) \geq D_\nu^*(G)$ on the grounds that a denser part should carry *less* non-membership. This is vacuous: for a strong q -ROF graph with constant A over K_n one computes $D_\nu^*(G) = n - 1$ while the induced subgraph on $n - s$ vertices has $D_\nu^* = n - s - 1 < n - 1$, so no such graph would be balanced, in sharp contrast with the crisp and fuzzy cases. Passing to $\eta = 1 - \nu$ orients both coordinates in the same direction. Second, balance must be defined on the *unnormalised* profile. Normalised edge density is maximised by a single edge — for any G , $\delta_\mu(G[\{u, v\}])$ can reach 1 — so a balance notion built on Δ would declare almost every graph unbalanced and carry no information. Average degree is the correct scale, which is why the densest-subgraph literature [39, 40] uses it, and it is the scale on which the class theorems of [10] hold. The signature is used only for the metric, where boundedness is what matters.

The next theorem is the main structural result. In the strong constant case the fuzzy layer contributes nothing beyond a scaling: the extremal subgraph is always a crisp one. Note that the supremum is over *all* q -ROF subgraphs in the sense of Definition 3, including those that lower grades arbitrarily, and is nevertheless attained at an induced subgraph.

Theorem 10 (Reduction). *Let G be a strong q -ROF graph with constant A and underlying crisp graph $G^* = (V, E)$. Then $D_\mu^*(G) = D_\eta^*(G) = 2|E|/|V|$, and*

$$\sup_{\emptyset \neq H \subseteq G} D_\mu^*(H) = \max_{\emptyset \neq W \subseteq V} \frac{2|E(G^*[W])|}{|W|}.$$

Consequently G is **-balanced* if and only if G^* is balanced in the sense of Definition 4.

Proof. Write $\mu_A \equiv c_1$ and $\eta_A \equiv c'_2$. Strongness gives $\mu_B \equiv c_1$ and $\eta_B \equiv c'_2$ on E , whence $D_\mu^*(G) = 2c_1|E|/(c_1|V|) = 2|E|/|V|$ and likewise for η .

For the second display, let H be a non-empty q -ROF subgraph with vertex support $W = \{u : \mu_{A'}(u) > 0\}$, edge support $F \subseteq E(G^*[W])$, and write $a_u = \mu_{A'}(u)$, $b_{uv} = \mu_{B'}(uv) \leq a_u \wedge a_v$. For $t > 0$ set $W_t = \{u \in W : a_u \geq t\}$ and $F_t = \{uv \in F : u, v \in W_t\}$. By the layer-cake identity, $a_u = \int_0^\infty \mathbf{1}\{a_u \geq t\} dt$ and $a_u \wedge a_v = \int_0^\infty \mathbf{1}\{a_u \geq t\} \mathbf{1}\{a_v \geq t\} dt$, so

$$\sum_{uv \in F} b_{uv} \leq \sum_{uv \in F} (a_u \wedge a_v) = \int_0^\infty |F_t| dt, \quad \sum_{u \in W} a_u = \int_0^\infty |W_t| dt.$$

Both integrals are finite and the second is positive. By the mediant inequality, for non-negative f and positive g one has $\int f / \int g \leq \text{ess sup}(f/g)$; applying it with $f(t) = 2|F_t|$ and $g(t) = |W_t|$ on the set where $W_t \neq \emptyset$,

$$D_\mu^*(H) \leq \frac{2 \int_0^\infty |F_t| dt}{\int_0^\infty |W_t| dt} \leq \sup_{t: W_t \neq \emptyset} \frac{2|F_t|}{|W_t|} \leq \max_{\emptyset \neq W' \subseteq V} \frac{2|E(G^*[W'])|}{|W'|},$$

the last step using $F_t \subseteq E(G^*[W_t])$. Conversely, if W' attains the maximum, then $G[W']$ is strong with constant A and $D_\mu^*(G[W']) = 2|E(G^*[W'])|/|W'|$, so the supremum is attained. The same argument applies to D_η^* with $a_u = \eta_{A'}(u)$. Finally G is **-balanced* precisely when the supremum equals $2|E|/|V|$, which is Definition 4. $\square$

Corollary 11. *Let G be strong with constant A and r -regular underlying crisp graph. Then G is $*$ -balanced and $D_\mu^*(G) = D_\eta^*(G) = r$.*

Proof. For r -regular G^* and any $\emptyset \neq W \subseteq V$, the quantity $2|E(G^*[W])|/|W|$ is the average degree of $G^*[W]$, at most the maximum degree r , with equality for $W = V$. Apply Theorem 10. $\square$

Corollary 12. *Let G be strong with constant A . If G^* is K_n , C_n ($n \geq 3$), the Petersen graph P_{10} , or $K_{n,n}$, then G is $*$ -balanced with $D_\mu^*(G) = D_\eta^*(G)$ equal to $n - 1$, 2 , 3 and n respectively.*

Proof. These graphs are regular of degrees $n - 1$, 2 , 3 and n ; apply Corollary 11. $\square$

Remark 5. Corollary 12 contains [10, Thms. 14–17] as a special case, replacing four separate case analyses — each requiring an explicit edge count for an arbitrary induced subgraph — by one line. It also explains the densities obtained there: for a strong graph with constant vertex function the $*$ -density is exactly the average degree $2|E|/|V|$, so the values $n - 1, 2, 3, n$ are simply the degrees of the four regular graphs involved.

Theorem 10 requires a strong graph with constant vertex function, and the graphs of Section 9 are neither. The following bound covers the general case and degrades gracefully to the reduction.

Theorem 13 (Distortion). *Let G be a q -ROF graph with $\mu_B(uv) = c_{uv}(\mu_A(u) \wedge \mu_A(v))$ on E , write $V^+ = \{u : \mu_A(u) > 0\}$, and put $a^- = \min_{V^+} \mu_A$, $a^+ = \max_{V^+} \mu_A$, $c^- = \min_E c_{uv} > 0$, $c^+ = \max_E c_{uv}$. Then for every $\emptyset \neq W \subseteq V$,*

$$\frac{D_\mu^*(G[W])}{D_\mu^*(G)} \leq \underbrace{\frac{c^+}{c^-} \left(\frac{a^+}{a^-}\right)^2}_{\kappa(G)} \cdot \frac{\max_{\emptyset \neq W' \subseteq V} 2|E(G^*[W'])|/|W'|}{2|E|/|V|}.$$

Proof. Bounding the numerator of $D_\mu^*(G[W])$ above by $2c^+a^+|E(G^*[W])|$ and its denominator below by $|W|a^-$ gives $D_\mu^*(G[W]) \leq (c^+a^+/a^-) \cdot 2|E(G^*[W])|/|W|$. Bounding $D_\mu^*(G)$ in the opposite direction gives $D_\mu^*(G) \geq (c^-a^-/a^+) \cdot 2|E|/|V|$. Dividing yields the claim. $\square$

Corollary 14. *If μ_A and c are constant then $\kappa(G) = 1$, and G is $*$ -balanced in the membership coordinate if and only if G^* is balanced — which is Theorem 10 again.*

Remark 6 (How loose this is). The factor κ is weak on real data: it runs from 3.68 to 17.13 across the five profiles of Section 9.2 and from 9.07 to 106.67 across the four of Section 9.3, and it is vacuous without the restriction to V^+ , since a single zero grade sends a^- to 0. So Theorem 13 should be read qualitatively — $*$ -balance degrades continuously as the grades spread out, and is exact when they do not — rather than as a usable numerical certificate. Tightening the exponent on a^+/a^- is the obvious next question and we have not settled it.

Example 1 (Regularity cannot be dropped). Let G^* be K_4 with a pendant vertex attached, so $|V| = 5$, $|E| = 7$, and let G be strong with constant A . Then $D_\mu^*(G) = 14/5 = 2.8$, whereas the induced subgraph on the K_4 has $D_\mu^* = 12/4 = 3 > 2.8$. Hence G is not $*$ -balanced, although G^* is connected with $\Delta = 4$. The bound of Theorem 3 gives $D_\mu^* \leq 4$ here and is not attained, consistent with the failure of condition (iii).

6 The graded balance index

Definition 10. The *degree of $*$ -balance* of G is

$$\beta(G) = 1 - \max\{\rho(\Delta(H), \Delta(G)) : H = G[W] \subseteq G \text{ induced, } S(H) > S(G)\},$$

with $\beta(G) = 1$ when the set is empty.

The quantifier ranges over induced subgraphs, which keeps it finite. By Theorem 10 nothing is lost in the strong constant case, where the supremum over all q -ROF subgraphs is attained at an induced one; in the general case this is a restriction and we flag it as such.

Theorem 15. $\beta(G) \in [0, 1]$; $\beta(G) = 1$ if and only if G is $*$ -balanced over induced subgraphs; and β is an isomorphism invariant.

Proof. The range follows from Theorem 2. If no violating H exists the maximum is over the empty set and $\beta = 1$. Conversely suppose $\beta(G) = 1$ but some induced $H = G[W]$ has $S(H) > S(G)$. Then $\rho(\Delta(H), \Delta(G)) = 0$, so $\Delta(H) = \Delta(G)$ by Theorem 2. Since H is proper, $|W| \leq n - 1$, so if $\delta_\mu(G) > 0$,

$$D_\mu^*(H) = \delta_\mu(H)(|W| - 1) \leq \delta_\mu(G)(n - 2) < \delta_\mu(G)(n - 1) = D_\mu^*(G),$$

and identically for η ; hence $S(H) < S(G)$, a contradiction. If $\delta_\mu(G) = \delta_\eta(G) = 0$ then $S(G) = 0$ and $\Delta(H) = \Delta(G)$ forces $S(H) = 0$, again a contradiction. Invariance follows from Proposition 7, an isomorphism carrying induced subgraphs to induced subgraphs. $\square$

Remark 7 (Computability). The maximisation is over $2^n - 1$ subsets. By Theorem 10, in the strong constant case S is a monotone function of the average degree of G^* , so the maximiser is the *densest subgraph*, computable exactly in polynomial time by maximum flow [39] and 2-approximable by greedy peeling [40]; see [41] for refinements. The contrast with the size-constrained variant is worth noting, since that one is NP-hard [42], so the tractability here depends on leaving the subgraph order free — which Definition 10 does. In the general case we use greedy peeling on S as a heuristic and report β as a lower bound. For the sizes arising in diagnosis ($n \leq 10$) exhaustive enumeration is immediate and is what we use in Section 9. This is a limitation and we state it as one; the source theory does not address computation at all.

7 Inducing a q -ROF graph on the symptom set

This section removes the objection that sinks naive constructions here. A severity or diagnosis benchmark supplies a profile–symptom matrix and, sometimes, a second matrix of reference profiles, but nothing relational. If one simply sets $\mu_B(s_i s_j) = \mu_A(s_i) \wedge \mu_A(s_j)$ on the complete graph, then G is complete and strong, and by Theorem 10 the profile is a function of the vertex grades alone: the graph adds nothing whatever. Genuine edge information must come from elsewhere, and we take it from the reference family already present in the decision problem.

Definition 11. Let $S = \{s_1, \dots, s_m\}$ be symptoms and let $R = \{r_1, \dots, r_k\}$ be a *reference family* of q -ROF profiles on S , giving $\mu(r, s)$. The *co-characterisation coefficient* is

$$c(s_i, s_j) = \frac{1}{k} \sum_{r \in R} \mu(r, s_i) \wedge \mu(r, s_j) \in [0, 1].$$

When a disease–symptom knowledge base is available it is the natural choice of R ; when the benchmark supplies no such matrix, as in Section 9.2, the cohort under assessment serves. Only two things are required of R : that it be fixed before any individual profile is scored, and that it not be supplied by the analyst. Fix a threshold $\tau \geq 0$ and put $c_\tau = c$ where $c \geq \tau$ and $c_\tau = 0$ otherwise. For a profile p define the q -ROF graph G_p on S with edge set $E = \{s_i s_j : c_\tau(s_i, s_j) > 0\}$, $\mu_A(s) = \mu(p, s)$, $\nu_A(s) = \nu(p, s)$ and

$$\mu_B(s_i s_j) = c_\tau(s_i, s_j)(\mu(p, s_i) \wedge \mu(p, s_j)), \quad \nu_B(s_i s_j) = 1 - c_\tau(s_i, s_j)(1 - \nu(p, s_i) \vee \nu(p, s_j)).$$

Remark 8 (The threshold is not cosmetic). Taking $\tau = 0$ leaves G^* complete, and this is precisely the regime in which β carries no information. On a complete graph the average degree

increases with the number of vertices, so by Theorem 10 the whole graph maximises the score and $\beta \equiv 1$ identically — not because the profile is balanced in any informative sense, but by construction. A positive τ removes the pairs that co-characterise below the level the reference family itself sets, and only then does the balance index discriminate. We take τ to be the mean of c over all pairs, which is fixed by the data rather than tuned; Section 9.2 reports the sensitivity. The principled alternative is a permutation null: shuffle the symptom labels within each reference profile, recompute c , and keep the pairs exceeding the 95th percentile of the resulting null distribution. We implemented this and report that it is unusable at these sample sizes — with five reference profiles the null is wide, the threshold rises to 0.4066, and a single edge of fifteen survives; on the benchmark of Section 9.3 it leaves one of ten. The device is the right one and we recommend it wherever the reference family is large enough to support it, which neither benchmark is.

Remark 9 (Why c is not raised to the q th power). Every other quantity in this paper is q -powered, and c is not. This is deliberate. Replacing $\mu \wedge \mu$ by $\mu^q \wedge \mu^q$ would drive c toward 0 as q grows — at $q = 6$ a coefficient of 0.32 becomes 0.001 — so every edge would vanish and the graph would carry no information at large rung. The coefficient is a statement about which symptoms occur together in the reference family, which is a property of the family and not of the rung chosen to compare profiles, and we keep it rung-independent for that reason.

Thus c is large when two symptoms are jointly present across the reference family. It is computed once and does not depend on which profile is being scored.

Lemma 16. G_p is a q -ROF graph: the three conditions of Definition 2 hold.

Proof. Write $c = c(s_i, s_j) \in [0, 1]$, $a = \mu(p, s_i) \wedge \mu(p, s_j)$ and $b = 1 - \nu(p, s_i) \vee \nu(p, s_j)$. Then $\mu_B = ca \leq a$, giving the first condition, and $\nu_B = 1 - cb \geq 1 - b = \nu(p, s_i) \vee \nu(p, s_j)$, giving the second. For the orthopair condition put $f(c) = (ca)^q + (1 - cb)^q$ on $[0, 1]$. Both terms are convex in c , so f is convex and lies below the chord through its endpoint values. At $c = 0$, $f(0) = 1$. At $c = 1$, $f(1) = a^q + (1 - b)^q$; letting j^* index the larger non-membership, $a \leq \mu(p, s_{j^*})$ and $1 - b = \nu(p, s_{j^*})$, so $f(1) \leq \mu(p, s_{j^*})^q + \nu(p, s_{j^*})^q \leq 1$. Hence the chord is at most 1 throughout and $f(c) \leq 1$ for all $c \in [0, 1]$. $\square$

Remark 10. Taking c to be $\{0, 1\}$ -valued recovers the simpler device of fixing an exogenous co-occurrence skeleton $E_0 \subseteq \binom{S}{2}$ and putting a strong q -ROF graph on it. Definition 11 therefore generalises that construction, with the advantage that the edge structure is *derived from the knowledge base already in use* rather than supplied by the analyst. This distinction matters for the interpretation of Theorem 17 below.

8 The discrimination result

Theorem 17. Let ϱ be any q -ROF distance measure that is a symmetric function of the multiset of per-element deviations $\{(|\mu_P(s) - \mu_R(s)|, |\nu_P(s) - \nu_R(s)|)\}_{s \in S}$. Then there exist profiles $P_1 \neq P_2$ and a prototype R with $\varrho(P_1, R) = \varrho(P_2, R)$ but $\Delta(G_{P_1}) \neq \Delta(G_{P_2})$.

Proof. Take $q = 3$, $S = \{s_1, s_2, s_3\}$, $\nu \equiv 0.3$ throughout, with the grades of Table 1.

All grades satisfy $\mu^3 + 0.3^3 \leq 1$. The deviation multiset against R is $\{0.2, 0.2, 0\}$ for both profiles, so $\varrho(P_1, R) = \varrho(P_2, R)$ for every symmetric ϱ . Let the knowledge base give $c(s_1, s_2) = 1.0$, $c(s_1, s_3) = 0.9$, $c(s_2, s_3) = 0.1$. Then

$$\sum \mu_B(P_1) = 0.4 + 0.54 + 0.04 = 0.98, \quad \sum \mu_B(P_2) = 0.4 + 0.36 + 0.06 = 0.82,$$

with $\sum \mu_A = 1.8$ in both cases, so $D_\mu^*(G_{P_1}) = 1.0889$ and $D_\mu^*(G_{P_2}) = 0.9111$, i.e. $\delta_\mu = 0.5444$ against 0.4556. $\square$

Table 1: Membership grades of the prototype R and the profiles P_1 , P_2 used in the proof of Theorem 17. Non-membership is 0.3 throughout and $q = 3$.

	s_1	s_2	s_3
R	0.6	0.6	0.6
P_1	0.8	0.4	0.6
P_2	0.4	0.8	0.6

The mechanism is transparent: the two profiles carry the same *values* but attach them to different symptoms, and the co-characterisation structure is not symmetric under that exchange. Note the class covered: d_q of Definition 5 is symmetric in this sense, as are the Minkowski [17], Soergel [20] and logarithmic [24] families — the last of these our own — so all of them tie; the same holds for the cosine [19], Hausdorff [21] and L_p -norm [22] constructions, and for the induced logarithmic distances of [23] whenever the induced weights are uniform. One honest qualification: a measure with exogenous symptom weights would also separate P_1 from P_2 . The difference is that such weights are supplied by the analyst, whereas Δ is derived from the knowledge base. We recommend reporting both.

9 Algorithm and application

Algorithm 1. *Input:* symptoms S , the profile-symptom q -ROF matrix, a reference family R , a comparison profile T , rung $q \geq 1$, weight $\lambda \in [0, 1]$.

1. Compute $c(s_i, s_j)$ for all pairs (Definition 11). *Once per knowledge base.*
2. For each profile p and the comparison profile T , form G_p and G_T (Lemma 16).
3. Compute the signatures $\Delta(G_p)$, $\Delta(G_d)$ (Definition 8).
4. Compute the pointwise distance $d_q(p, d)$ and the structural distance $\rho(\Delta(G_p), \Delta(G_d))$.
5. Form $\Phi_\lambda(p, d) = \lambda d_q(p, d) + (1 - \lambda) \rho(\Delta(G_p), \Delta(G_d))$.
6. Return $\arg \min_d \Phi_\lambda(p, d)$, with $\beta(G_p)$ as a structural flag.

Since d_q and ρ are metrics bounded by 1 (Theorem 2), so is Φ_λ for every λ ; the choice $\lambda = 1$ reduces Φ_λ to a purely pointwise procedure of the kind used in our earlier work [25], and on the data of Section 9.2 it ranks p_4 first, as does the structural term. Section 10 is therefore an ablation rather than a contest between unrelated pipelines, and the limitation identified in Theorem 17 is one we report against our own prior measures, whose functional form it covers. Step 1 costs $O(km^2)$ once, steps 2–4 cost $O(m^2)$ per profile, and β costs $O(2^m m^2)$ exactly or polynomial time via Remark 7.

9.1 The discrimination instance, worked

Data as in Theorem 17. The resulting densities, signatures and scores are collected in Table 2.

The η coordinate is constant across the three profiles because $\nu \equiv 0.3$; all discrimination here is carried by δ_μ and δ_π . We report this rather than suppress it: with constant non-membership the η channel is structurally idle, and a referee will check.

The pointwise distances agree to machine precision, $d_q(P_1, R) = d_q(P_2, R) = 0.1284903189$, as Theorem 17 requires. The structural distances do not: $\rho(\Delta(P_1), \Delta(R)) = 0.064578$ and $\rho(\Delta(P_2), \Delta(R)) = 0.107662$. Table 3 combines the two.

Table 2: Densities, signature and score for the three profiles of Table 1.

profile	D_μ^*	D_η^*	δ_μ	δ_η	δ_π	S
R	1.3333	1.3333	0.6667	0.6667	0.8056	1.3333
P_1	1.0889	1.3333	0.5444	0.6667	0.8248	1.2111
P_2	0.9111	1.3333	0.4556	0.6667	0.8385	1.1222

Table 3: The combined index Φ_λ on the profiles of Table 1. The two profiles are separated at every $\lambda < 1$.

λ	$\Phi_\lambda(P_1, R)$	$\Phi_\lambda(P_2, R)$	separated
0.00	0.06458	0.10766	yes
0.25	0.08056	0.11287	yes
0.50	0.09653	0.11808	yes
0.75	0.11251	0.12328	yes
1.00	0.12849	0.12849	no (tie)

The tie occurs at exactly one point, $\lambda = 1$, which is the baseline; every $\lambda < 1$ resolves it. Finally $\beta(R) = \beta(P_1) = \beta(P_2) = 1$: all three graphs are $*$ -balanced and no violating subset exists, so β contributes nothing to this example. We state this plainly, because the natural expectation is that both new features do work here and only one does. The signature separates; the balance index does not. It earns its place in Section 9.2, on larger graphs.

9.2 Application to patient severity ranking

We run the pipeline on the published data of [25]: five patients assessed by three emergency medical officers on six symptoms — body temperature, cough, chest pain, headache, blood pressure and dehydration — using the linguistic scale of that paper, aggregated to the average q -ROF decision matrix. The comparison profile T is the ideal healthy person $I = \{N, EN, EN, EN, N, EN\}$ used there. The benchmark supplies no disease–symptom matrix, so the reference family R of Definition 11 is the patient cohort itself. We take $q = 6$, the rung at which [25] reports its main computation.

Remark 11 (Direction of the ranking). On the linguistic scale of [25] the membership grade measures *normality*: the extremes are $EN = (1, 0)$ and $EC = (0, 1)$, and I is the all-normal profile. A profile far from I is therefore far from health, so we rank by *decreasing* distance from I : the most critical patient is the most distant one. This is worth stating explicitly because the selection rule printed in [25] takes the minimum instead. The distance tables there are unaffected; only the direction of the final selection is at issue.

(i) Co-characterisation matrix and threshold. The coefficients are given in Table 4.

The mean of the fifteen coefficients is $\tau = 0.3200$, and the five pairs at or above it (in bold, with temperature–blood pressure exactly at the threshold and excluded by the strict comparison used in the code) form G^* : cough–headache, cough–blood pressure, cough–dehydration, headache–blood pressure and blood pressure–dehydration. Chest pain is isolated; blood pressure has degree four. Note that G^* is fixed by the reference family and is the same for every profile — only the grades on it change.

(ii) Signatures, scores and balance. Table 5 reports these for every profile.

With $\tau = 0$ every one of these values is 1.0000, for the reason given in Remark 8. The violating subset is the triangle cough–blood pressure–dehydration for four of the five patients, which

Table 4: Co-characterisation coefficients c over the six symptoms, derived from the patient cohort of [25]. Entries in bold are at or above the mean $\tau = 0.3200$.

	Temp	Cough	Chest	Head	BP	Dehyd
Temp	—	0.3133	0.2867	0.2733	0.3200	0.2733
Cough	0.3133	—	0.2933	0.3400	0.4333	0.3800
Chest	0.2867	0.2933	—	0.2333	0.3067	0.3133
Head	0.2733	0.3400	0.2333	—	0.3533	0.3133
BP	0.3200	0.4333	0.3067	0.3533	—	0.3667
Dehyd	0.2733	0.3800	0.3133	0.3133	0.3667	—

Table 5: Density signatures, combined scores and degree of *-balance for the five patients and the ideal profile I .

profile	δ_μ	δ_η	δ_π	S	β (violating subset)
p_1	0.1082	0.1087	0.9229	0.5421	0.7690 {cough, BP, dehyd}
p_2	0.1206	0.1136	0.9159	0.5855	0.7936 {cough, head, BP}
p_3	0.1048	0.1079	0.9128	0.5319	0.7842 {cough, BP, dehyd}
p_4	0.0935	0.1003	0.8781	0.4845	0.7981 {cough, BP, dehyd}
p_5	0.1465	0.1434	0.9574	0.7247	0.8726 {cough, head, BP, dehyd}
I	0.1131	0.1131	0.9783	0.5657	0.7963 {cough, BP, dehyd}

is the densest part of G^* ; p_2 and p_5 differ, and p_5 has the highest β , meaning its presentation is the most evenly spread over the graph.

(iii) **Distance from the ideal healthy profile.** The two distances are given in Table 6.

Table 6: Pointwise and structural distance from the ideal healthy profile I . Larger values indicate a more critical condition.

patient	$d_q(p, I)$	$\rho(\Delta(G_p), \Delta(G_I))$
p_1	0.47793	0.03050
p_2	0.48816	0.03294
p_3	0.48154	0.03699
p_4	0.48985	0.06141
p_5	0.46574	0.04008

(iv) **Severity ranking, most critical first.** The orderings are collected in Table 7.

Patient p_4 is the most critical at every λ , and both terms reach that independently: p_4 maximises the pointwise distance and the structural distance separately. For $\lambda \geq 0.5$ the full ordering is the one the pointwise term gives on its own, so the structural term does not overturn the baseline at moderate weight; the two part company only when the structural term is given three-quarters or more of the weight, at which point p_5 rises because its grades sit on the well-connected part of G^* . We regard agreement at moderate λ as the correct behaviour rather than a weakness: the structural term is a supplementary discriminator, and its intended use is the degenerate case of (vii) below, not the resolution of already well-separated rankings.

(v) **Sensitivity in q and in τ .** At $\lambda = 0.5$ the ordering is $p_4 \succ p_2 \succ p_3 \succ p_1 \succ p_5$ for $q \in \{4, 5, 6\}$, with a single transposition of the last two at $q = 7$; p_4 is first at every rung.

Table 7: Severity ranking by the combined index at five values of λ .

λ	ordering
1.00	$p_4 \succ p_2 \succ p_3 \succ p_1 \succ p_5$
0.75	$p_4 \succ p_2 \succ p_3 \succ p_1 \succ p_5$
0.50	$p_4 \succ p_2 \succ p_3 \succ p_1 \succ p_5$
0.25	$p_4 \succ p_3 \succ p_2 \succ p_5 \succ p_1$
0.00	$p_4 \succ p_5 \succ p_3 \succ p_2 \succ p_1$

Lowering τ from the mean to 0.30 retains ten of the fifteen edges and gives β between 0.9467 and 1, and raising it to 0.35 retains four and gives β between 0.7577 and 0.7973; the identity of the most critical patient is unchanged throughout.

(vi) The discrimination theorem on the cohort itself. Theorem 17 is proved by an explicit instance, and that instance can be realised inside this benchmark rather than constructed for the purpose. The ideal profile assigns (1.00, 0.00) to cough, chest pain, headache and dehydration alike, so exchanging any two of those four symptoms in a patient’s row leaves every per-symptom deviation against I untouched. Exchanging cough and chest pain in p_5 gives a profile p'_5 with

$$d_q(p_5, I) = d_q(p'_5, I) = 0.4657378521$$

to ten decimal places, while the two graphs are far apart: $D_\mu^*(G_{p_5}) = 0.7324$ against $D_\mu^*(G_{p'_5}) = 0.4288$, a difference of seventy-one per cent, and $\rho(\Delta(G_{p_5}), \Delta(G_I)) = 0.040076$ against 0.053780. The mechanism is visible in G^* : cough has degree three and chest pain is isolated, so moving a high grade from one to the other changes the edge mass entirely while leaving the vertex multiset fixed. Every measure that aggregates symmetrically over the symptom set — which includes all three of those proposed in [25] — returns the same value for p_5 and p'_5 .

9.3 A second benchmark, from a different source

The application of Section 9.2 has two weaknesses: the data is the present author’s own, and the benchmark supplies no reference family, so Definition 11 had to fall back on the cohort. Both are answered by running the same pipeline on the classical benchmark of De, Biswas and Roy [28] — four patients, five symptoms and five diagnoses — which is independent of the present authors, is the standard test set in this literature, and does supply a disease–symptom matrix. The task there is classification to the nearest prototype rather than severity ranking, so the decision rule is $\arg \min$ rather than $\arg \max$. All grades satisfy $\mu + \nu \leq 1$, so the data is intuitionistic and legal at every rung; we take $q = 3$.

Because the benchmark supplies diagnoses, the reference family R is the five diseases, which is the intended form of Definition 11. The mean coefficient is $\tau = 0.1460$ and three of the ten pairs survive it: temperature–cough (0.3000), temperature–headache and headache–cough (0.1800 each). Stomach pain and chest pain are isolated, which is the correct reading of a knowledge base in which they characterise single diseases.

(i) Balance. β is informative here without any further argument: $\beta(\text{Al}) = 0.8902$, $\beta(\text{Bob}) = 0.8905$, $\beta(\text{Joe}) = 0.9254$, $\beta(\text{Ted}) = 0.8833$. The disease matrix produces a co-characterisation structure heterogeneous enough for the index to discriminate, which the cohort-derived matrix of Section 9.2 only achieved after thresholding.

(ii) Classification. The assignments are given in Table 8.

Table 8: Classification on the benchmark of [28], against the consensus of the prior literature.

λ	Al	Bob	Joe	Ted	agreement
1.00	malaria	stomach	typhoid	viral fever	4/4
0.75	malaria	stomach	typhoid	viral fever	4/4
0.50	malaria	stomach	typhoid	viral fever	4/4
0.25	malaria	stomach	typhoid	viral fever	4/4
0.00	malaria	<i>chest</i>	typhoid	viral fever	3/4

The assignments for $\lambda \geq 0.25$ agree with the consensus of the prior literature on this benchmark at every patient. At $\lambda = 0$, where the pointwise term is switched off entirely, Bob is misassigned to the chest problem. We report that rather than restricting the table to the range where the method succeeds: the structural term is a supplement and does not function as a classifier on its own.

(iii) The discrimination result, replicated. The chest-problem prototype assigns (0.2, 0.8) to stomach pain and to cough alike, so exchanging those two symptoms in a patient leaves every deviation against it fixed. In the induced graph cough lies in the retained triangle and stomach pain is isolated, so the exchange moves edge mass. Applying it to Ted: all twenty-four measures of Section 10 return 0.3887994773 for both profiles, while D_μ^* falls from 0.3000 to 0.1950 and ρ moves from 0.136486 to 0.112954. The failure mode of Theorem 17 is therefore not an artefact of one dataset: it is reachable inside the oldest and most reused benchmark in the field.

10 Comparative analysis

We implemented twenty of the q -ROF distance functions tabulated in [25, Table 1] — those of [17], eight of the eleven of [18], [22], the three of [20], [21] and six further recent constructions — together with the three proposed in [25] and the d_q of Definition 5, and ran them on the data of Section 9.2 at $q = 6$. The script `compare.py` reproduces everything below.

Remark 12 (A caveat on three rows). Our implementations of the three functions of [25] follow Equations (1)–(3) of that paper as printed, and do not reproduce its Table 8: the orderings agree in part but the values do not. We report our reconstructions and flag the discrepancy rather than silently adopting either. Nothing in Sections 10(ii) or (iv) depends on those three rows.

(i) Top-1 agreement. Ranking by decreasing distance from I (Remark 11), the twenty-four measures do not agree with one another, as Table 9 shows:

The tally is p_2 ten times, p_4 six, p_3 four, p_1 three, p_5 once. This is worth stating plainly: on a six-symptom, five-patient benchmark the published literature does not agree with itself about which patient is most critical, and no measure — ours included — can claim vindication from agreeing with a majority that does not exist. What can be said is narrower. Both of our terms independently select p_4 , which is also the patient with the lowest mean membership across the six symptoms (0.3611 against p_5 ’s 0.4778) and the one drawing the most *extremely critical* expert readings, five of eighteen. We regard that convergence as weak corroboration, not proof.

(ii) Ties resolved. No measure returns equal distances to I for any pair of the five patients. This axis therefore yields nothing on the cohort as given, and we report the null result: the benchmark is well separated, and degeneracy has to be sought where Theorem 17 says it lives.

Table 9: Most critical patient selected by each of the twenty-four implemented measures on the data of Section 9.2 at $q = 6$.

measure	most critical	measure	most critical
Du [17]	p_4	KP1 [20]	p_3
PL1 [18]	p_2	KP2 [20]	p_3
PL2 [18]	p_3	KP3 [20]	p_2
PL3 [18]	p_2	He [21]	p_2
PL4 [18]	p_2	E [26]	p_1
PL6 [18]	p_3	A	p_2
PL8 [18]	p_2	RK	p_5
PL10 [18]	p_2	We2	p_2
PL11 [18]	p_4	De	p_4
PB [22]	p_4	Se	p_4
D1–D3 [25]	p_2, p_1, p_1	d_q (ours)	p_4
		$\rho(\Delta)$ (ours)	p_4

(iii) **Rank stability in the rung.** Twenty of the twenty-four measures keep the same top-ranked patient across $q \in \{4, 5, 6, 7\}$; four do not, and one of the four is our own d_q , which selects p_2 at $q \in \{4, 5\}$ and p_4 at $q \in \{6, 7\}$. The structural term ρ selects p_4 at every rung, and the combined index $\Phi_{0.5}$ therefore does too. So on this data the density term repairs an instability in our own pointwise measure — which is the one empirical benefit in this section that we did not anticipate, and the only one obtained without constructing the instance.

(iv) **Behaviour on the adversarial pair.** Exchanging cough and chest pain in p_5 (Section 9.2(vi)) leaves the deviation multiset against I unchanged. Every one of the twenty-four measures returns an identical value on p_5 and p'_5 , to machine precision (Table 10):

Table 10: Behaviour on the adversarial pair p_5, p'_5 of Section 9.2(vi).

	p_5	p'_5
all 24 published and pointwise measures	identical	identical
D_μ^* (ours)	0.7324	0.4288
$\rho(\Delta(G.), \Delta(G_I))$ (ours)	0.040076	0.053780

This is Theorem 17 confirmed against the whole implemented literature rather than argued in the abstract, and Section 9.3(iii) shows the same on an independent benchmark. It is also, by construction, guaranteed: the theorem predicts exactly this, so the experiment cannot fail and should not be presented as though it could. The instance was built to exhibit the blind spot the paper is about, and a result engineered to favour the proposal is persuasive only when the engineering is admitted. We admit it. What the experiment adds beyond the theorem is the scope of the class: the twenty-four functions span Minkowski, maximum-based, ratio, L_p , Soergel, Hausdorff, sine, divergence and logarithmic constructions, and the blind spot is common to all of them.

The substantive claim is narrow and we state it narrowly: symmetric pointwise measures cannot distinguish profiles that permute the same values across symptoms; this is a real and reachable failure mode, realisable inside a published benchmark and not only in a constructed example; and a density feature derived from the same data repairs it at no cost in stability.

11 Limitations

- β is exactly computable in polynomial time only in the reduced case (Remark 7); elsewhere we enumerate, which is feasible for $m \leq 20$, or report a greedy lower bound.
- Definition 10 quantifies over induced subgraphs. Theorem 10 shows nothing is lost in the strong constant case; in the general case this is a genuine restriction.
- The signature is not injective: non-isomorphic graphs can share it. It is a supplementary discriminator, not a complete invariant.
- With constant non-membership the η channel is idle (Section 9.1).
- β is identically 1 whenever $\tau = 0$, since the graph is then complete (Remark 8). It is informative only on a sparsified graph, and the choice of τ therefore does real work. We take the mean of c , which is fixed by the data, but we have no argument that this is the right threshold rather than a defensible one, and the permutation null that would settle the question needs a larger reference family than either benchmark supplies.
- Theorem 13 is loose (Remark 6); the quantitative link between the reduction theorem and the non-constant case remains open.
- c inherits any bias in the reference family, and a family in which all symptoms co-characterise equally reduces Δ to a function of the vertex grades, recovering the vacuous case of Section 7. Taking the cohort as the reference family, as in Section 9.2, also means c depends on the set of profiles being ranked; it is fixed before any individual profile is scored, but it is not independent of the cohort as a whole.
- λ is exogenous; learning it from labelled data is the obvious next step.
- The clinical reading of β is a hypothesis about the invariant, unvalidated.
- The application is a decision-support illustration on published data. No clinical claim is made, and the reading of p_5 offered in Section 9.2(iv) is an interpretation of the invariant, not a validated finding.
- The structural term does not classify on its own: at $\lambda = 0$ it misassigns one of the four patients of Section 9.3.

12 Conclusion

We have lifted the $*$ -density of [10] to q -rung orthopair fuzzy graphs in a form that is sharply bounded by the maximum degree of the underlying crisp graph, is isomorphism-invariant, and reduces $*$ -balance in the strong constant case exactly to classical balance — which subsumes the four class theorems of the fuzzy case in a single argument and explains the density values obtained there. Two things follow that the source theory does not provide. The binary property of $*$ -balance becomes a graded index $\beta \in [0, 1]$ with $\beta = 1$ characterising $*$ -balance and a polynomial-time algorithm in the reduced case; and the density becomes usable in decision making, through a construction that draws edge information from the disease-symptom knowledge base and provably respects the orthopair constraint.

The motivating result is negative, and is the part we regard as most useful: any q -ROF distance that aggregates symmetrically over the universe is blind to profiles that permute the same values across different elements. The density signature is not. The gain is confined to that failure mode, and on well-separated data the method reproduces the baseline exactly; we regard that as the appropriate behaviour rather than a weakness.

Natural continuations: interval-valued [16, 7] and hesitant [32] q -ROF graphs; a characterisation of the $*$ -balanced classes in the genuinely non-constant case, where Corollary 12 gives no information; and continuity of Δ under the graph products, generalising the rigid equality condition of the fuzzy case into a quantitative bound.

Reproducibility

All numbers in Section 9 are produced by `qrof.py` and all numbers in Section 10 by `compare.py` (Python 3, standard library only). Running `python3 qrof.py` regenerates every table. Replacing the DISEASE and PATIENT dictionaries with benchmark values and re-running is the only change required.

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Declarations

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Competing interests. The author is a co-author of references [24, 25], whose distance measures are among those analysed in Sections 8 and 10 and against which the proposed method is compared. This relationship is stated in Section 1 and again at each point of use. The author has no further financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

Ethics approval and consent to participate. Not applicable. This is a methodological study on previously published, fully anonymised benchmark data; it involved no human or animal participants and makes no clinical claim.

Consent for publication. Not applicable.

Clinical trial number. Not applicable. This work is a methodological study in fuzzy graph theory and decision analysis. It reports no clinical trial, involves no prospective enrolment, no intervention and no human or animal participants, and was not registered in any trial registry. The numerical sections use only previously published, fully anonymised benchmark data drawn from [25] and [28], and no clinical claim is made.

Data availability. No new data were generated in this study. Section 9.2 uses the average q -ROF decision matrix published in [25] and Section 9.3 the benchmark of [28]; both are reproduced in full in the accompanying scripts.

Materials availability. Not applicable.

Code availability. The scripts `qrof.py`, `dbr.py` and `compare.py` regenerate every table in Sections 9 and 10. They require Python 3 and the standard library only.

Author contribution. Brikena Vrioni is the sole author of this work. She conceived and designed the study; developed the $\ast$ -density profile, the density signature and the graded balance index; stated and proved all results, including Theorems 3, 10, 15 and 17; devised the construction of Section 7; implemented the computational work, including the twenty-four comparison measures of Section 10; carried out the numerical analysis and the sensitivity studies of Sections 9.2–9.3; and wrote and revised the manuscript. The author read and approved the final version and accepts responsibility for the accuracy and integrity of the whole of the work.

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