

Boundary-Consistent L1–Legendre Spectral Approximation of Nonlinear Caputo Reaction–Diffusion Problems with Integral Constraints

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Abstract

We develop and verify a boundary-consistent L1–Legendre spectral discretization for nonlinear Caputo reaction–diffusion equations subject to non-local integral endpoint constraints. The boundary lifting and constraint-preserving Legendre space are taken from earlier work; the present contribution concerns the discrete operator, the nonlinear solve, and verification. Because basis functions satisfying integral constraints need not vanish at the endpoints, the usual derivative stiffness form (v_x, w_x) is generally not equivalent to the strong diffusion operator $(-v_{xx}, w)$. We therefore assemble the strong-form matrix directly and formulate the fully discrete L1 system in boundary-adapted coordinates. A residual-damped fixed-point iteration is analyzed under an explicit contractivity condition, and stability and convergence are obtained under discrete accretivity, temporal regularity, and constrained approximation assumptions. The numerical study contains three independent checks. Multi-order tests recover the expected L1 rate $2 - \alpha$; a smooth non-polynomial benchmark exhibits spectral spatial convergence; and, most importantly, replacing the strong matrix by the conventional derivative stiffness matrix produces an L^2 error plateau near 2.8×10^{-2} while the boundary-consistent formulation converges to machine precision. Nonlinear tests retain the predicted

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temporal order and enforce the integral constraints to roundoff.
 Reproducible Python code is supplied.

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1 Introduction

Time-fractional reaction–diffusion equations model diffusion and reaction processes with temporal memory. For $0 < \alpha < 1$, the Caputo derivative is especially convenient because the initial datum retains its classical interpretation while the evolution depends on the complete previous history of the state [1, 2, 3, 4]. A representative problem is

$${}_0^C D_t^\alpha u(x, t) = \kappa u_{xx}(x, t) + f(x, t, u(x, t)), \quad (x, t) \in (a, b) \times (0, T], \quad (1)$$

with $\kappa > 0$ and two integral endpoint constraints

$$u(a, t) + \int_a^b \gamma_1(x) u(x, t) dx = \phi_1(t), \quad (2)$$

$$u(b, t) + \int_a^b \gamma_2(x) u(x, t) dx = \phi_2(t). \quad (3)$$

Such conditions couple endpoint values to global spatial information and therefore differ substantially from Dirichlet or Robin conditions. Classical L1 and spectral discretizations for time-fractional diffusion are well established [5, 6, 7, 8], whereas non-local boundary conditions require a space that respects the boundary operator itself [9, 10].

A closely related earlier paper by Alhejaili, Alsa’di, and Salas [11] developed the boundary lifting, the constraint-preserving Legendre basis, a well-posedness framework, and a high-order Beta-window approximation for the same class of Caputo reaction–diffusion problems. The present manuscript is therefore formulated explicitly as a *follow-up numerical paper*, not as a second claim of novelty for the lifting or the boundary-adapted basis. Its purpose is to answer a different question: how should a classical L1 time discretization be coupled to that non-local boundary space so that the spatial operator remains consistent with the original strong differential operator, and how can the resulting nonlinear algebraic problem be solved and verified reproducibly?

This distinction exposes an important point. Let v, w satisfy the homogeneous integral constraints, but not necessarily $v(a) = v(b) = w(a) = w(b) = 0$. Then

$$(-v_{xx}, w) = (v_x, w_x) - [v_x w]_a^b. \quad (4)$$

Thus the usual stiffness form (v_x, w_x) is *not* automatically equivalent to $(-v_{xx}, w)$ for integral boundary conditions. A method that builds a boundary-adapted trial space but then silently drops the endpoint contribution in (4) can solve a different discrete boundary-value problem. We avoid that inconsistency by assembling the strong-form operator matrix directly.

The contributions are deliberately focused on issues that can be tested independently:

- (C1) we derive and use the boundary-consistent strong matrix $K_{ij} = (-\psi_j'', \psi_i)$ and identify explicitly the endpoint term that is lost by the standard derivative-stiffness replacement;
- (C2) we formulate the fully discrete L1 scheme, including nonhomogeneous lifting terms, and analyze a residual-damped nonlinear fixed-point iteration under an explicit contractivity condition;
- (C3) we obtain conditional stability and convergence bounds for the generally nonsymmetric strong-form discretization under a discrete accretivity hypothesis that can be checked at the assembled-matrix level;
- (C4) we validate temporal convergence for several fractional orders, rather than a single value of α ;
- (C5) we add a smooth non-polynomial benchmark that separates spatial and temporal errors and directly compares the strong matrix with the conventional derivative stiffness matrix; the latter exhibits a non-vanishing error plateau;
- (C6) we provide reproducible code that regenerates all reported tables, the accretivity scan, and the operator-consistency figure.

The rest of the paper is organized as follows. Section 2 states the model and recalls the boundary lifting. Section 3 constructs the boundary-adapted space and the corrected strong-form spatial operator. Section 4 gives the fully discrete L1 scheme. Section 5 analyzes the residual iteration. Section 6 gives stability and convergence results. Section 7 reports the manufactured tests, and Section 9 summarizes the conclusions.

2 Problem, boundary operators, and lifting

Let $\Omega = (a, b) \subset \mathbb{R}$, $T > 0$, and $Q_T = \Omega \times (0, T]$. We consider

$${}_0^C D_t^\alpha u = \kappa u_{xx} + f(x, t, u), \quad (x, t) \in Q_T, \quad (5)$$

with

$$u(x, 0) = u_0(x), \quad x \in [a, b], \quad (6)$$

and boundary functionals

$$B_1 v := v(a) + \int_a^b \gamma_1(x) v(x) dx, \quad (7)$$

$$B_2 v := v(b) + \int_a^b \gamma_2(x) v(x) dx. \quad (8)$$

The boundary data are

$$B_1 u(t) = \phi_1(t), \quad B_2 u(t) = \phi_2(t). \quad (9)$$

For $0 < \alpha < 1$,

$${}_0^C D_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} v'(s) ds. \quad (10)$$

We use the following data assumptions throughout.

Assumption 2.1 (Boundary data and compatibility). The kernels satisfy $\gamma_1, \gamma_2 \in L^2(a, b)$, the boundary data satisfy $\phi_i \in C^1[0, T]$, and $u_0 \in H^2(a, b)$ is compatible with the boundary data:

$$B_i u_0 = \phi_i(0), \quad i = 1, 2.$$

Let

$$p_0(x) = 1, \quad p_1(x) = \frac{2x - a - b}{b - a},$$

and define

$$C_\gamma = \begin{pmatrix} B_1 p_0 & B_1 p_1 \\ B_2 p_0 & B_2 p_1 \end{pmatrix}. \quad (11)$$

Assumption 2.2 (Boundary nondegeneracy). C_γ is invertible.

Under Theorem 2.2, the affine lifting

$$R(x, t) = r_0(t) p_0(x) + r_1(t) p_1(x), \quad \begin{pmatrix} r_0(t) \\ r_1(t) \end{pmatrix} = C_\gamma^{-1} \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}, \quad (12)$$

satisfies $B_i R(t) = \phi_i(t)$. Setting $y = u - R$ gives homogeneous constraints $B_i y = 0$. Since the selected lifting is affine in x , $R_{xx} = 0$. Consequently the transformed problem is

$${}_0^C D_t^\alpha y + \kappa A_\gamma y = \mathcal{F}(t, y), \quad A_\gamma y = -y_{xx}, \quad (13)$$

where

$$\mathcal{F}(t, y) = f(\cdot, t, y + R) - {}_0^C D_t^\alpha R + \kappa R_{xx} = f(\cdot, t, y + R) - {}_0^C D_t^\alpha R. \quad (14)$$

Notice that (14) uses the ordinary second derivative R_{xx} and does not apply the homogeneous-domain operator A_γ to a lifting that need not belong to $D(A_\gamma)$.

For analytical well-posedness, one may assume that the realization

$$A_\gamma v = -v_{xx}, \quad D(A_\gamma) = \{v \in H^2(a, b) : B_1 v = B_2 v = 0\}$$

is sectorial on $L^2(a, b)$ and that f is globally Lipschitz in its last argument, or locally Lipschitz on an invariant bounded range. The associated mild-solution theory is developed in [11]; we do not duplicate it here. We only strengthen the forcing regularity used later to

$$f(\cdot, \cdot, 0) \in L^\infty(0, T; L^2(a, b)), \quad (15)$$

which is sufficient for the uniform-in-time estimates used by the numerical analysis.

3 Boundary-adapted spectral space and strong-form operator

Let P_k be the Legendre polynomial of degree k on $[-1, 1]$ and define the shifted modes

$$\tilde{P}_k(x) = P_k\left(\frac{2x - a - b}{b - a}\right), \quad k = 0, 1, \dots \quad (16)$$

For each $k \geq 2$, define

$$\psi_k = \tilde{P}_k - c_{0k} \tilde{P}_0 - c_{1k} \tilde{P}_1, \quad \begin{pmatrix} c_{0k} \\ c_{1k} \end{pmatrix} = C_\gamma^{-1} \begin{pmatrix} B_1 \tilde{P}_k \\ B_2 \tilde{P}_k \end{pmatrix}. \quad (17)$$

Then $B_1\psi_k = B_2\psi_k = 0$ and

$$V_N^\gamma = \text{span}\{\psi_2, \dots, \psi_N\} \quad (18)$$

is a constraint-preserving polynomial space. This construction is recalled from [11] because it is needed to define the fully discrete method.

Write

$$y_N^n(x) = \sum_{k=2}^N U_k^n \psi_k(x), \quad \mathbf{U}^n = (U_2^n, \dots, U_N^n)^T. \quad (19)$$

The mass matrix is

$$M_{ij} = (\psi_j, \psi_i)_{L^2(\Omega)}. \quad (20)$$

The correct strong-form spatial matrix is

$$K_{ij} = (-\psi_j'', \psi_i)_{L^2(\Omega)}. \quad (21)$$

For comparison, the usual derivative stiffness matrix is

$$S_{ij} = (\psi_j', \psi_i')_{L^2(\Omega)}.$$

Integration by parts gives

$$K_{ij} = S_{ij} - [\psi_j'(x)\psi_i(x)]_{x=a}^{x=b}. \quad (22)$$

Because $B_i\psi_k = 0$ does not imply $\psi_k(a) = \psi_k(b) = 0$, the endpoint term in (22) is generally nonzero. Accordingly, K rather than S is used throughout this paper. In the Dirichlet special case, the endpoint values vanish and the two formulations coincide.

Assumption 3.1 (Discrete strong-form accretivity). There exists $\mu \geq 0$, independent of N , such that for every coefficient vector $\mathbf{v}$,

$$\mathbf{v}^T \frac{K + K^T}{2} \mathbf{v} \geq \mu \mathbf{v}^T M \mathbf{v}. \quad (23)$$

Moreover, the matrices appearing in the time-stepping scheme are non-singular.

Remark 3.2. Theorem 3.1 is deliberately stated as a property of the *strong-form discrete operator*. For general integral boundary constraints, a Poincaré inequality by itself does not justify replacing (21) by the derivative stiffness matrix. In applications, (23) can be checked nu-

merically for the selected kernels and basis by inspecting the smallest generalized eigenvalue of $\frac{1}{2}(K + K^T)$ relative to M .

4 L1 time discretization

Let $t_n = n\tau$, $\tau = T/M$, and set

$$D_\tau^\alpha v^n = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n-1} b_j^{(\alpha)} (v^{n-j} - v^{n-j-1}), \quad (24)$$

with

$$b_j^{(\alpha)} = (j+1)^{1-\alpha} - j^{1-\alpha}. \quad (25)$$

The weights are positive and decreasing. If $v \in C^2[0, T]$, then

$$\left| {}_0^C D_t^\alpha v(t_n) - D_\tau^\alpha v^n \right| \leq C_\alpha \tau^{2-\alpha} \max_{0 \leq t \leq t_n} |v''(t)|. \quad (26)$$

The regularity in (26) is important: a manufactured solution proportional to t^β with $1 < \beta < 2$ does not satisfy C^2 regularity at $t = 0$ and should not be used to validate the classical $2 - \alpha$ rate on a uniform mesh.

Let

$$c_\tau = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}, \quad A_\tau = c_\tau M + \kappa K. \quad (27)$$

Expanding the L1 history gives the fully discrete coefficient system

$$A_\tau \mathbf{U}^n = \mathbf{H}^n + \mathbf{F}^n(\mathbf{U}^n), \quad (28)$$

where

$$\mathbf{H}^n = c_\tau \left[\sum_{j=1}^{n-1} (b_{j-1}^{(\alpha)} - b_j^{(\alpha)}) M \mathbf{U}^{n-j} + b_{n-1}^{(\alpha)} M \mathbf{U}^0 \right], \quad (29)$$

and

$$F_i^n(\mathbf{U}) = \int_a^b [f(x, t_n, R^n + y_{\mathbf{U}}(x)) - D_\tau^\alpha R^n + \kappa R_{xx}^n] \psi_i(x) dx. \quad (30)$$

Here $y_{\mathbf{U}} = \sum_{k=2}^N U_k \psi_k$. For the affine lifting (12), $R_{xx}^n = 0$.

The initial vector $\mathbf{U}^0$ is obtained by a stable projection of $u_0 - R(\cdot, 0)$ into V_N^γ . Because all ψ_k satisfy the two homogeneous boundary functionals exactly, the reconstruction

$$u_N^n = R^n + y_N^n \quad (31)$$

satisfies the prescribed integral boundary data up to assembly and floating-point errors.

5 Residual-damped nonlinear iteration

At each time level define

$$\mathcal{R}_n(\mathbf{V}) = A_\tau \mathbf{V} - \mathbf{H}^n - \mathbf{F}^n(\mathbf{V}). \quad (32)$$

The nonlinear discrete solution is a root of $\mathcal{R}_n$. Define the fixed-point map

$$G_n(\mathbf{V}) = A_\tau^{-1}(\mathbf{H}^n + \mathbf{F}^n(\mathbf{V})). \quad (33)$$

The correction used in the algorithm solves

$$A_\tau \mathbf{d}^{(k)} = -\mathcal{R}_n(\mathbf{V}^{(k)}), \quad (34)$$

so that $\mathbf{d}^{(k)} = G_n(\mathbf{V}^{(k)}) - \mathbf{V}^{(k)}$. We update

$$\mathbf{V}^{(k+1)} = \mathbf{V}^{(k)} + \lambda_k \mathbf{d}^{(k)}, \quad 0 < \lambda_k \leq 1. \quad (35)$$

In the implementation, λ_k starts at one and is halved until the dual residual norm decreases.

Let $\|\mathbf{v}\|_M = (\mathbf{v}^T M \mathbf{v})^{1/2}$ and $\|\mathbf{r}\|_{M^{-1}} = (\mathbf{r}^T M^{-1} \mathbf{r})^{1/2}$.

Assumption 5.1 (Discrete load Lipschitz property). On the bounded range visited by the iteration,

$$\|\mathbf{F}^n(\mathbf{V}) - \mathbf{F}^n(\mathbf{W})\|_{M^{-1}} \leq L_h \|\mathbf{V} - \mathbf{W}\|_M. \quad (36)$$

Theorem 5.2 (Contractive nonlinear solve). Suppose Theorem 5.1 holds and

$$q_n := L_h \|A_\tau^{-1}\|_{M^{-1} \rightarrow M} < 1. \quad (37)$$

Then G_n has a unique fixed point. The undamped iteration $\mathbf{V}^{(k+1)} = G_n(\mathbf{V}^{(k)})$ converges linearly with factor at most q_n . More generally, if the accepted damping parameters satisfy $\lambda_k \geq \lambda_{\min} > 0$, then (35) converges with factor at most

$$1 - \lambda_{\min}(1 - q_n) < 1.$$

Proof. By (33), (36), and the induced operator norm,

$$\|G_n(\mathbf{V}) - G_n(\mathbf{W})\|_M \leq q_n \|\mathbf{V} - \mathbf{W}\|_M.$$

Thus G_n is a contraction and Banach's fixed-point theorem gives existence and uniqueness of the discrete nonlinear solution. For a damped step,

$$\begin{aligned}\left\| \mathbf{V}^{(k+1)} - \mathbf{U}^n \right\|_M &\leq (1 - \lambda_k) \left\| \mathbf{V}^{(k)} - \mathbf{U}^n \right\|_M \\ &\quad + \lambda_k \left\| G_n(\mathbf{V}^{(k)}) - G_n(\mathbf{U}^n) \right\|_M \\ &\leq (1 - \lambda_k(1 - q_n)) \left\| \mathbf{V}^{(k)} - \mathbf{U}^n \right\|_M.\end{aligned}$$

which yields the stated bound. $\square$

Remark 5.3. Backtracking is a robustness device, not a substitute for a convergence hypothesis. Theorem 5.2 gives a regime in which convergence is guaranteed. Outside that regime, residual decrease can still be useful computationally, but a global convergence claim would require additional monotonicity or globalization assumptions.

Stopping rule. For accuracy control, we stop when

$$\left\| A_\tau^{-1} \mathcal{R}_n(\mathbf{V}^{(k)}) \right\|_M \leq \varepsilon_{\text{alg}}, \quad (38)$$

rather than comparing a raw Euclidean residual with the discretization error. This makes the algebraic tolerance consistent with the coefficient correction induced by the discrete operator.

6 Stability and convergence

We first recall the L1 energy inequality.

Lemma 6.1 (L1 energy inequality). *For any Hilbert-space sequence $\{w^n\}_{n=0}^M$,*

$$(D_\tau^\alpha w^n, w^n) \geq \frac{1}{2} D_\tau^\alpha \|w^n\|^2. \quad (39)$$

Proof. The L1 derivative can be written as a positive convex combination of increments $w^n - w^{n-j}$. The identity

$$(p, p - q) = \frac{1}{2} (\|p\|^2 - \|q\|^2 + \|p - q\|^2) \geq \frac{1}{2} (\|p\|^2 - \|q\|^2)$$

and positivity of the L1 weights give (39). $\square$

Theorem 6.2 (Linear stability). *Assume Theorem 3.1 and let the right-hand side be independent of y_N^n . Then*

$$\|y_N^n\|_{L^2} \leq C \left(\|y_N^0\|_{L^2} + \max_{1 \leq k \leq n} \|g_N^k\|_{L^2} \right), \quad (40)$$

where C is independent of τ and N on any fixed finite interval on which the discrete comparison estimate is valid.

Proof. Testing the strong-form scheme with y_N^n , using Theorem 6.1 and (23), yields

$$\frac{1}{2} D_\tau^\alpha \|y_N^n\|_{L^2}^2 + \kappa \mu \|y_N^n\|_{L^2}^2 \leq \|g_N^n\|_{L^2} \|y_N^n\|_{L^2}.$$

Young's inequality and the standard discrete fractional comparison estimate imply (40). No integration by parts is used, so no boundary term is discarded. $\square$

Theorem 6.3 (Nonlinear stability on a bounded Lipschitz range). *Suppose Theorem 3.1 holds and f is Lipschitz with constant L_* on a bounded interval containing the exact and numerical solutions. Then*

$$\|y_N^n\|_{L^2} \leq C \left(\|y_N^0\|_{L^2} + \max_{1 \leq k \leq n} \|\mathcal{F}(t_k, 0)\|_{L^2} \right), \quad (41)$$

where C may depend on T, α, κ, μ , and L_* , but not on τ or N .

Proof. Write $\mathcal{F}(t_n, y_N^n) = \mathcal{F}(t_n, 0) + [\mathcal{F}(t_n, y_N^n) - \mathcal{F}(t_n, 0)]$, test by y_N^n , and apply Theorem 6.1, (23), the Lipschitz bound, and Young's inequality. This gives

$$D_\tau^\alpha \|y_N^n\|_{L^2}^2 \leq C \|y_N^n\|_{L^2}^2 + C \|\mathcal{F}(t_n, 0)\|_{L^2}^2.$$

A discrete fractional Gronwall inequality yields (41). $\square$

For convergence, it is useful to separate the temporal, spatial, and algebraic errors rather than hard-wire a spatial rate that may depend on the constraint kernels.

Assumption 6.4 (Stable constrained approximation). *There exists a projection Π_N^γ into V_N^γ and a quantity $\varepsilon_N \rightarrow 0$ such that, uniformly in time,*

$$\|y - \Pi_N^\gamma y\|_{L^2} + \|A_\gamma y - A_N \Pi_N^\gamma y\|_{L^2} \leq C \varepsilon_N, \quad (42)$$

where A_N is the strong-form discrete operator induced by (21).

Theorem 6.5 (Fully discrete error estimate). *Assume that y is twice continuously differentiable in time with values in $L^2(\Omega)$. Let f be Lipschitz on the bounded solution range. Assume also 3.1 and 6.4. At every time level, impose the stopping criterion (38). Then*

$$\max_{0 \leq n \leq M} \|u(t_n) - u_N^n\|_{L^2} \leq C \left(\tau^{2-\alpha} + \varepsilon_N + \varepsilon_{\text{alg}} \right). \quad (43)$$

Proof sketch. Set $y = u - R$ and split

$$y(t_n) - y_N^n = \underbrace{y(t_n) - \Pi_N^\gamma y(t_n)}_{\xi^n} + \underbrace{\Pi_N^\gamma y(t_n) - y_N^n}_{\eta^n}.$$

By Theorem 6.4, $\|\xi^n\| \leq C\varepsilon_N$. Inserting the projected exact solution into the discrete scheme produces a consistency defect bounded by

$$C(\tau^{2-\alpha} + \varepsilon_N + \varepsilon_{\text{alg}}),$$

using the L1 estimate (26), the spatial consistency in (42), and the algebraic stopping rule. Subtracting the numerical scheme, testing by η^n , and using Theorem 6.1, strong-form accretivity, and the Lipschitz estimate yields a discrete fractional Gronwall inequality for $\|\eta^n\|^2$. Taking square roots and adding the projection error gives (43). $\square$

Corollary 6.6. *If, for the selected kernels and regular solution, the constrained approximation satisfies $\varepsilon_N \leq CN^{-s}$ for some $s > 0$, and the nonlinear tolerance is chosen as $\varepsilon_{\text{alg}} = O(\tau^{2-\alpha} + N^{-s})$, then*

$$\max_{0 \leq n \leq M} \|u(t_n) - u_N^n\|_{L^2} \leq C(\tau^{2-\alpha} + N^{-s}). \quad (44)$$

7 Verification and operator-consistency tests

The numerical study is designed to answer four distinct questions: whether the L1 temporal rate is recovered for several values of α ; whether the constrained Legendre space exhibits spectral spatial convergence for a smooth non-polynomial solution; whether the strong matrix K is genuinely necessary; and whether the nonlinear iteration keeps the algebraic error below the discretization error. All computations use $\Omega = (0, 1)$, $\kappa = 1$, and

$$\gamma_1(x) = \gamma_2(x) = -\frac{1}{20}. \quad (45)$$

The associated boundary functionals are therefore

$$B_1 v = v(0) - \frac{1}{20} \int_0^1 v(x) dx, \quad B_2 v = v(1) - \frac{1}{20} \int_0^1 v(x) dx.$$

For $p_0 = 1$ and $p_1 = 2x - 1$,

$$C_\gamma = \begin{pmatrix} 19/20 & -1 \\ 19/20 & 1 \end{pmatrix}, \quad \det C_\gamma = \frac{19}{10} \neq 0. \quad (46)$$

Gauss–Legendre quadrature with sufficiently high order is used for all matrix, load, and error integrals.

7.1 Temporal convergence for several fractional orders

To isolate the temporal error, define

$$w_p(x) = 4 \left[x(1-x) + \frac{1}{114} \right]. \quad (47)$$

Then $B_1 w_p = B_2 w_p = 0$ and w_p belongs exactly to the boundary-adapted polynomial space for every $N \geq 2$. We take

$$u_{\text{ex}}(x, t) = t^2 w_p(x),$$

for which

$${}_0^C D_t^\alpha t^2 = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}, \quad w_p''(x) = -8.$$

The linear forcing is therefore

$$f_{\text{lin}}(x, t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} w_p(x) + 8t^2. \quad (48)$$

Because the spatial shape is represented exactly, the observed error is dominated by the L1 time discretization. Table 1 reports the final-time L^2 errors for $N = 8$. The rates approach $2 - \alpha$ for all three tested orders. The approach is slower for $\alpha = 0.25$, but the monotone trend is consistent with the asymptotic value 1.75.

Table 1: Multi-order temporal convergence for the linear benchmark, $N = 8$, $T = 1$.

α	$2 - \alpha$	E_{80}	r_{80}	E_{160}	r_{160}	E_{320}	r_{320}
0.25	1.75	8.466×10^{-6}	1.698	2.592×10^{-6}	1.707	7.895×10^{-7}	1.715
0.50	1.50	4.845×10^{-5}	1.485	1.725×10^{-5}	1.490	6.130×10^{-6}	1.493
0.75	1.25	2.294×10^{-4}	1.247	9.656×10^{-5}	1.248	4.063×10^{-5}	1.249

7.2 A smooth non-polynomial spatial benchmark

A polynomial exact solution cannot demonstrate spectral spatial convergence. We therefore introduce

$$w_s(x) = \sin(\pi x) + \frac{2}{19\pi}. \quad (49)$$

Since

$$\int_0^1 w_s(x) dx = \frac{40}{19\pi}, \quad w_s(0) = w_s(1) = \frac{2}{19\pi},$$

we again have $B_1 w_s = B_2 w_s = 0$. Set

$$u_{\text{ex}}(x, t) = t w_s(x).$$

The L1 formula is exact for the linear function t , so this choice removes temporal truncation error. Using

$${}_0^C D_t^\alpha t = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, \quad w_s''(x) = -\pi^2 \sin(\pi x),$$

the required forcing is

$$f_s(x, t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} w_s(x) + \pi^2 t \sin(\pi x). \quad (50)$$

Table 2 gives the final-time error for $\alpha = 0.75$ and eight time steps. The strong-form method converges spectrally until roundoff is reached. In contrast, replacing K by

$$S_{ij} = (\psi_j', \psi_i')$$

produces an error that stagnates near 2.792×10^{-2} . The plateau is not a temporal or spatial-resolution effect; it is the consequence of solving the modified discrete problem obtained after dropping the endpoint term in

(22).

Table 2: Spatial convergence and operator-consistency test for the smooth benchmark.

N	strong matrix K : E_{L^2}	derivative stiffness S : E_{L^2}
4	6.2093×10^{-4}	2.7928×10^{-2}
6	7.6050×10^{-6}	2.7920×10^{-2}
8	6.0303×10^{-8}	2.7920×10^{-2}
10	3.3208×10^{-10}	2.7920×10^{-2}
12	1.3301×10^{-12}	2.7920×10^{-2}
14	4.8583×10^{-15}	2.7920×10^{-2}
16	6.8342×10^{-16}	2.7920×10^{-2}

7.3 Discrete accretivity scan

The analysis in Section 6 is conditional on the symmetric part of the assembled strong operator being uniformly positive relative to the mass matrix. For the kernel in (45), we therefore report the smallest generalized eigenvalue

$$\lambda_{\min}\left(\frac{1}{2}(K + K^T), M\right).$$

The values in Table 3 remain positive over a substantially wider degree range than that used in the convergence experiments. This does not replace an analytic kernel-dependent proof, but it verifies the stated discrete hypothesis for the reported computations and shows that the $N = 8$ observation is not isolated.

Table 3: Smallest generalized eigenvalue of the symmetric part of the strong matrix.

N	4	6	8	10	12	16	20	24	30
$\lambda_{\min}$	9.447	9.427	9.397	9.358	9.311	9.192	9.040	8.856	8.517

7.4 Nonlinear manufactured problem

For a nonlinear test, return to $u_{\text{ex}} = t^2 w_p$ and choose

$$f_{\text{nl}}(x, t, u) = u^2 + g(x, t), \quad (51)$$

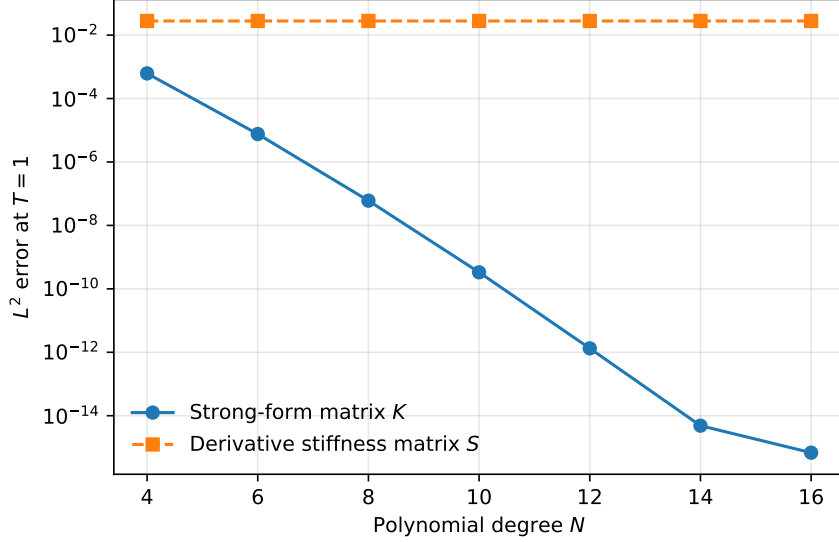


Figure 1: Final-time L^2 error for the smooth spatial benchmark. The boundary-consistent strong matrix converges spectrally, whereas the derivative-stiffness replacement reaches a nonzero error plateau.

with

$$g(x, t) = \frac{2}{\Gamma(3 - \alpha)} t^{2-\alpha} w_p(x) + 8t^2 - t^4 w_p(x)^2. \quad (52)$$

The reaction u^2 is locally Lipschitz on every bounded interval. Table 4 shows the results for $\alpha = 0.75$ and $N = 8$. The temporal rate again approaches 1.25, the nonlinear iteration count decreases as the time step is refined, and the boundary residual remains at roundoff level.

Table 4: Nonlinear benchmark, $\alpha = 0.75$, $N = 8$, $T = 1$.

M	E_{L^2}	rate	E_∞	max. iterations
20	1.5306×10^{-3}	—	2.1427×10^{-3}	12
40	6.4552×10^{-4}	1.246	9.0368×10^{-4}	10
80	2.7192×10^{-4}	1.247	3.8066×10^{-4}	9
160	1.1446×10^{-4}	1.248	1.6023×10^{-4}	8
320	4.8156×10^{-5}	1.249	6.7414×10^{-5}	7

The maximum boundary residual over all time levels and all runs in Table 4 is below 2.5×10^{-16} . The dual residual of the nonlinear algebraic

system remains below 3×10^{-11} at the final time for the reported tolerance.

7.5 Reproducibility

The supplied script `JAMC_validation.py` regenerates the multi-order temporal table, the strong-versus-stiffness comparison, the accretivity scan, the nonlinear convergence table, and Fig. 1. The implementation uses direct $O(M^2)$ L1 history accumulation to keep the reference code transparent; fast-memory acceleration is deliberately left outside the scope of the present verification study.

8 Discussion

The numerical results clarify the role of each component of the method. The boundary-adapted basis enforces the integral constraints, while the L1 formula resolves the Caputo memory. Neither fact, by itself, guarantees consistency of the spatial differential operator. The operator comparison in Table 2 is therefore the central verification experiment: both discretizations use the same time formula, the same constrained basis, the same quadrature, and the same forcing, yet only the strong matrix converges to the exact solution. The nonzero plateau produced by S is the numerical signature of the missing endpoint term in (22).

A second point concerns what is, and is not, proved. For general integral kernels the strong matrix is nonsymmetric and its coercivity cannot be inferred from the usual Dirichlet Poincaré argument. We consequently state the analysis under a discrete accretivity hypothesis and verify that hypothesis numerically for the kernel used in the experiments over degrees $4 \leq N \leq 30$. This is more transparent than silently replacing the operator by a symmetric stiffness form. A fully analytic characterization of kernels that guarantee uniform accretivity remains an open problem.

The temporal tests also avoid a common validation ambiguity. The polynomial benchmark uses t^2 , which has the C^2 regularity needed for the classical uniform-mesh L1 estimate, and the multi-order experiment shows the expected dependence of the rate on α . The smooth spatial benchmark instead uses a linear time factor, for which L1 is exact; this separates spectral spatial accuracy from time-discretization effects.

Finally, residual backtracking is interpreted only as a robustness device. The convergence guarantee for the nonlinear solve is the contractivity condition in Theorem 5.2; outside that regime, residual decrease

alone should not be advertised as an unconditional global convergence theorem. For weakly singular temporal solutions, graded meshes or corrected L1 rules would be preferable to the uniform-grid analysis considered here.

9 Conclusion

We presented a boundary-consistent L1–Legendre spectral approximation for nonlinear Caputo reaction–diffusion equations with non-local integral endpoint constraints. The method retains the previously established boundary lifting and constraint-preserving polynomial space, but assembles the diffusion operator in strong form as $K_{ij} = (-\psi_j'', \psi_i)$. This distinction is essential because integral constraints do not generally force the basis functions to vanish at the endpoints.

The analysis separates temporal, spatial, and algebraic errors and gives a contractive regime for the nonlinear fixed-point solve. The expanded verification study supports each part independently. Multi-order experiments recover the expected L1 rate, the smooth non-polynomial benchmark displays spectral spatial convergence, and the direct operator comparison demonstrates that the derivative-stiffness substitution can converge to the wrong discrete limit even while satisfying the integral constraints exactly. For the test kernel, the generalized eigenvalue scan also confirms positivity of the symmetric part of the assembled strong operator over a broad degree range.

The principal unresolved analytical question is a kernel-dependent characterization of strong-form accretivity. Computational extensions include graded or corrected time meshes for initial singularities, fast Caputo history compression, multidimensional integral constraints, and adaptive polynomial degree selection.

Declarations

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Data availability. No external dataset is required. The numerical values in this article are generated by the mathematical model and the supplied reference implementation.

Code availability. The file `JAMC_validation.py`, supplied with the submission package, regenerates all numerical tables and the operator-consistency figure.

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Relationship to prior work. The boundary lifting and boundary-adapted Legendre construction used here were previously reported in Ref. [11]. The present manuscript does not claim those components as new. Its distinct contribution is the boundary-consistent strong-form L1 discretization, residual-damped nonlinear solver analysis, multi-order temporal verification, smooth spectral benchmark, and direct strong-versus-stiffness consistency test described above.

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