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On Non-Lipschitz Path-Dependent SDEs with Weighted Running Extrema: Existence, Uniqueness, and Strong Convergence Rates

Mohamed Bourza[*] and **Kamal Hiderah**[†]

Abstract

Stochastic differential equations (SDEs) depending on historical running maxima and minima appear naturally in various applied fields, including quantitative finance (e.g., lookback and barrier options), structural reliability engineering (fatigue under peak loads), and dam control theory. In this paper, we investigate a general class of path-dependent SDEs driven by running extrema, where the drift and diffusion coefficients satisfy a non-Lipschitz condition of Bihari-Osgood type. Under mild structural assumptions on the weight functions and pathwise extrema, we establish the existence and uniqueness of strong solutions using a generalized Bihari inequality combined with Picard iterations. Furthermore, we construct a discrete Euler-Maruyama approximation scheme tailored to running extrema and prove its strong convergence rate in $L^2(\Omega)$. The theoretical results provide both a rigorous analytical foundation for non-Lipschitz extremal SDEs and a reliable computational tool for practical simulations in applied probability.

Keywords: Path-dependent SDEs; Running extrema; Bihari-Osgood condition; Picard iterations; Euler-Maruyama scheme; Strong convergence rate.
2020 MSC: 60H10, 60G17, 65C30, 91G30.

1 Introduction and Motivations

Path-dependent stochastic differential equations (SDEs), particularly those incorporating running maximum and minimum operators, have attracted significant attention in recent years due to their ability to capture memory effects and extreme historical events in complex dynamic systems [13, 17]. Unlike classical Itô SDEs where the future state depends solely on the current position $x(t)$ [10], path-dependent SDEs allow the drift and diffusion terms to interact directly with historical extrema:

$$M(t) = \max_{0\le u\le t} x(u) \quad \text{and} \quad m(t) = \min_{0\le u\le t} x(u). \tag{1.1}$$

From a practical perspective, the inclusion of pathwise running extrema is driven by compelling applications across several applied disciplines:

- Quantitative Finance and Risk Management: In financial engineering, exotic path-dependent derivatives—such as Lookback options [6], Barrier options [14], and Ratchet options—have

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payoffs that depend explicitly on $M(t)$ or $m(t)$ over a trading interval $[0, T]$. Furthermore, modern asset pricing models incorporate running maxima to capture market psychological phenomena, such as feedback mechanisms where hitting historical highs triggers automatic algorithmic trading or regulatory interventions [9].

- Structural Reliability and Materials Science: In fatigue analysis and cumulative damage mechanics, the mechanical degradation or crack propagation of a structural component subjected to random stress vectors depends heavily on the historical peak stress experienced by the material up to time t, represented by $M(t)$ [18].

- Dam Theory and Queueing Systems: In hydrological modeling and storage systems, the required capacity and overflow risk of a reservoir are naturally governed by the running supremum of water inflows minus outflows, where threshold exceedances dictate boundary reflection or control actions [16].

From a theoretical standpoint, establishing existence and uniqueness for SDEs driven by running extrema under relaxed regularity conditions presents substantial mathematical challenges. Traditional results often rely on global Lipschitz continuity of the drift and diffusion coefficients [15]. However, in many realistic physical and economic scenarios, coefficients fail to satisfy global Lipschitz conditions, displaying instead non-Lipschitz features such as Bihari-Osgood or Yamada-Watanabe conditions [19, 2].

While the classical Yamada-Watanabe theorem [19] and Bihari inequality [2] provide powerful framework tools for standard SDEs under non-Lipschitz conditions, extending these techniques to equations featuring simultaneous running maxima $M(t)$ and minima $m(t)$ weighted by time-dependent functions $\alpha(t)$ and $\beta(t)$ remains a non-trivial problem. The non-local and non-differentiable nature of the extrema operators $t \mapsto M(t)$ and $t \mapsto m(t)$ complicates the application of standard Itô calculus and pathwise estimates [11].

Moreover, for practical applications, obtaining analytical solutions to path-dependent non-Lipschitz SDEs is virtually impossible. Consequently, constructing stable numerical discretization schemes and proving their strong convergence rates is of paramount importance [12].In recent years, substantial efforts have been devoted to investigating the numerical approximation of SDEs with irregular or non-Lipschitz coefficients. In particular, the strong convergence rate of the Euler-Maruyama scheme has been analyzed under various complex structures, including SDEs with discontinuous coefficients [3], time-inhomogeneous terms [5], local-time perturbations [4], and reflected boundaries [1, 7].

However, when considering dual running extrema $M(t)$ and $m(t)$, the continuous tracking of these operators introduces a structural discretization error when replaced by discrete extrema $M_k = \max_{0 \le j \le k} Y_j$ on a time grid. Analyzing how this structural error propagates under non-Lipschitz drift conditions of Bihari type constitutes a major analytical hurdle that extends beyond the scope of previous works [8].

To the best of our knowledge, the simultaneous analysis of strong existence-uniqueness under Bihari-type non-Lipschitz conditions and the strong L^2-convergence rate of the discrete Euler-Maruyama scheme for dual-extrema driven SDEs has not been systematically addressed in the literature.

The main contributions of this paper are threefold:

1. Strong Existence and Uniqueness: We prove the existence and uniqueness of strong solutions for a general class of path-dependent SDEs driven by weighted running maxima and minima under a general non-Lipschitz condition of Bihari-Osgood type on the drift and diffusion terms.

2. Numerical Approximation Scheme: We design a discrete-time Euler-Maruyama approximation scheme that explicitly tracks discrete running extrema over grid points.

3. Explicit Strong Convergence Rate: We establish an explicit strong convergence rate in $L^2(\Omega)$ for the proposed numerical scheme, bridging the gap between non-Lipschitz analytical theory and practical algorithmic implementation.

The remainder of this paper is organized as follows. Section 2 collects necessary preliminaries, definitions, and sets up the structural assumptions on the coefficients. Section 3 is devoted to the proof of existence and uniqueness of strong solutions via Picard iterations and generalized Bihari inequalities. Section 4 presents the discrete Euler-Maruyama scheme and establishes its strong convergence analysis. Section 5 provides numerical experiments and simulations to validate the theoretical convergence rates. Finally, Section 6 concludes the paper and discusses future perspectives.

2 Preliminaries and Main Assumptions

Throughout this paper, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e., it is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional $(\mathcal{F}_t)$-adapted Brownian motion defined on this space.

For a fixed finite time horizon $T > 0$, we denote by $\mathcal{L}^2([0,T];\mathbb{R})$ the space of all real-valued, $(\mathcal{F}_t)$-progressively measurable processes $x = (x(t))_{t\in[0,T]}$ such that

$$\|x\|^2_{\mathcal{L}^2} := \mathbb{E}\left[\sup_{0\leq t\leq T} |x(t)|^2\right] < \infty.$$

We consider the following path-dependent stochastic differential equation (SDE) driven by running extrema:

$$x(t) = x(0) + \int_0^t \sigma(s, x(s))dB(s) + \int_0^t b(s, x(s))ds + \alpha(t)H\Big(t, \max_{0\leq u\leq t} x(u)\Big) + \beta(t)G\Big(t, \min_{0\leq u\leq t} x(u)\Big), \tag{2.1}$$

where $x(0)$ is an $\mathcal{F}_0$-measurable square-integrable random variable independent of B.

To handle non-Lipschitz coefficients, we first introduce the class of Osgood-Bihari concave weight functions.

Definition 2.1. Let $\mathbb{K}$ denote the set of continuous, non-decreasing, and concave functions $h : [0,\infty) \to [0,\infty)$ satisfying:

1. $h(0) = 0$ and $h(r) > 0$ for all $r > 0$.

2. The Osgood integral condition holds:

$$\int_{0^+}^1 \frac{dr}{h(r)} = +\infty. \tag{2.2}$$

Remark 2.2. Typical examples of functions belonging to the class $\mathbb{K}$ include:

- The standard linear function $h(r) = Kr$ (which corresponds to the classical Lipschitz case).

- The log-type non-Lipschitz functions:

$$h(r) = \begin{cases} r\ln(1/r), & \text{for } 0 \le r \le \delta \le e^{-1}, \\ h'(\delta^-)r + (h(\delta) - h'(\delta^-)\delta), & \text{for } r > \delta, \end{cases}$$

which naturally arise in non-Lipschitz Yamada-Watanabe type framework.

Next, we specify the key structural assumptions on the coefficients b, σ, H, G, α, and β.

$(\mathcal{A}_1)$ There exists a function $h \in \mathbb{K}$ such that for all $t \in [0,T]$ and $x, y \in \mathbb{R}$:

$$|b(t,x) - b(t,y)|^2 \le h(|x-y|^2). \tag{2.3}$$

$(\mathcal{A}_2)$ There exists a constant $K_\sigma > 0$ such that for all $t \in [0,T]$ and $x, y \in \mathbb{R}$:

$$|\sigma(t,x) - \sigma(t,y)|^2 \le K_\sigma |x-y|^2. \tag{2.4}$$

$(\mathcal{A}_3)$ There exists a constant $C > 0$ such that for all $t \in [0,T]$ and $x \in \mathbb{R}$:

$$|b(t,x)|^2 + |\sigma(t,x)|^2 \le C(1+|x|^2). \tag{2.5}$$

$(\mathcal{A}_4)$ The functions $H, G : [0,T] \times \mathbb{R} \to \mathbb{R}$ are measurable, and there exist constants $K_H, K_G > 0$ such that for all $t \in [0,T]$ and $x, y \in \mathbb{R}$:

$$|H(t,x) - H(t,y)| \le K_H |x-y|, \quad \text{and} \quad |G(t,x) - G(t,y)| \le K_G |x-y|. \tag{2.6}$$

$(\mathcal{A}_5)$ The functions $\alpha, \beta : [0,T] \to \mathbb{R}$ are bounded and measurable. We set:

$$\|\alpha\|_\infty := \sup_{t\in[0,T]} |\alpha(t)| < \infty, \quad \text{and} \quad \|\beta\|_\infty := \sup_{t\in[0,T]} |\beta(t)| < \infty.$$

To state our existence and uniqueness theorem, we impose a structural strict contraction requirement on the extrema interaction terms.

Definition 2.3. We define the parameter Λ associated with the running extrema weights by:

$$\Lambda := 2\left(\|\alpha\|_\infty K_H + \|\beta\|_\infty K_G\right)^2. \tag{2.7}$$

Throughout this work, we assume the strict contraction condition holds:

$$\Lambda < 1. \tag{2.8}$$

Remark 2.4. The strict contraction condition $\Lambda < 1$ plays a pivotal mathematical role in ensuring the convergence of Picard iterations and closing differential inequalities of Bihari type. Beyond its analytical necessity, this parameter carries concrete interpretations across various application domains:

- Financial Stability and Market Resiliency: In path-dependent asset pricing and risk models, running extrema account for psychological feedback mechanisms (e.g., stop-loss orders, barrier option triggers, or automated trend-following strategies). The condition $\Lambda < 1$ prevents explosive self-reinforcing feedback loops, modeling a resilient market where historical peaks do not destabilize the price process indefinitely.

- Material Reliability and Cumulative Fatigue: In structural mechanics and damage modeling, peak historical stress dictates cumulative degradation. The constraint $\Lambda < 1$ corresponds to a physical saturation regime where past peak loads cause progressive micro-damage without triggering immediate catastrophic failure or uncontrollable chain reactions.
- Dissipative Queueing and Storage Systems: In dam theory and hydrological reservoir management, $\Lambda < 1$ models natural dissipation and system control capacity, ensuring that peak historical inflows do not permanently overwhelm the drainage and storage infrastructure.

We are now in a position to present the main analytical result of this section.

Theorem 2.5. *Under assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_5)$ and condition* (2.8)*, for any $\mathcal{F}_0$-measurable initial state $x(0)$ with $\mathbb{E}[|x(0)|^2] < \infty$, the path-dependent SDE* (2.1) *admits a unique continuous, $(\mathcal{F}_t)$-adapted strong solution $x = (x(t))_{t\in[0,T]}$ satisfying:*

$$\mathbb{E}\left[\sup_{0\le t\le T}|x(t)|^2\right] < \infty.$$

3 Proof of Existence and Uniqueness

In this section, we establish Theorem 2.5. We begin by stating the continuous Bihari inequality, which plays a pivotal role in controlling the non-Lipschitz drift term.

Lemma 3.1 (Bihari's Inequality)**.** *Let $T > 0$ and $c \ge 0$. Let $u : [0,T] \to [0,\infty)$ be a bounded non-negative measurable function and let $h \in \mathbb{K}$. If $u(t)$ satisfies*

$$u(t) \le c + C\int_0^t h(u(s))ds, \quad \forall t \in [0,T], \tag{3.1}$$

for some constant $C > 0$, then:

(i) *If $c > 0$, then $u(t) \le G^{-1}\left(G(c) + Ct\right)$ for all $t \in [0,T]$ such that $G(c) + Ct \in Dom(G^{-1})$, where $G(r) = \int_{r_0}^{r} \frac{ds}{h(s)}$ for $r_0 > 0$.*

(ii) *If $c = 0$ and $\int_{0^+}^{1} \frac{ds}{h(s)} = +\infty$, then $u(t) = 0$ for all $t \in [0,T]$.*

3.1 Proof of Uniqueness

Let $x(t)$ and $y(t)$ be two continuous, $(\mathcal{F}_t)$-adapted strong solutions to (2.1) defined on the same probability space with $x(0) = y(0)$ a.s.

Define $Z(t) := x(t) - y(t)$ for $t \in [0,T]$. Then $Z(t)$ satisfies:

$$\begin{aligned} Z(t) = &\int_0^t \Big(\sigma(s,x(s)) - \sigma(s,y(s))\Big)dB(s) + \int_0^t \Big(b(s,x(s)) - b(s,y(s))\Big)ds \\ &+ \alpha(t)\left[H\Big(t, \max_{0\le u\le t} x(u)\Big) - H\Big(t, \max_{0\le u\le t} y(u)\Big)\right] \\ &+ \beta(t)\left[G\Big(t, \min_{0\le u\le t} x(u)\Big) - G\Big(t, \min_{0\le u\le t} y(u)\Big)\right]. \end{aligned}$$

Using the elementary algebraic inequality $(a_1+a_2+a_3+a_4)^2 \le \frac{1}{\epsilon}(a_1+a_2)^2 + \frac{1}{1-\epsilon}(a_3+a_4)^2$ for any $\epsilon \in (0,1)$, together with $(a+b)^2 \le 2a^2+2b^2$, we obtain for any $t \in [0,t_1] \subseteq [0,T]$:

$$\begin{aligned}
|Z(t)|^2 \le{}& \frac{2}{1-\epsilon}\left|\alpha(t)\left[H\Big(t,\max_{u\le t}x(u)\Big)-H\Big(t,\max_{u\le t}y(u)\Big)\right]\right|^2 \\
&+\frac{2}{1-\epsilon}\left|\beta(t)\left[G\Big(t,\min_{u\le t}x(u)\Big)-G\Big(t,\min_{u\le t}y(u)\Big)\right]\right|^2 \\
&+\frac{2}{\epsilon}\left|\int_0^t\Big(\sigma(s,x(s))-\sigma(s,y(s))\Big)dB(s)\right|^2 \\
&+\frac{2}{\epsilon}\left|\int_0^t\Big(b(s,x(s))-b(s,y(s))\Big)ds\right|^2. \qquad (3.2)
\end{aligned}$$

Recall the non-expansiveness of running extrema operators:

$$\left|\max_{0\le u\le t}x(u)-\max_{0\le u\le t}y(u)\right| \le \max_{0\le u\le t}|x(u)-y(u)| = \sup_{0\le u\le t}|Z(u)|, \qquad (3.3)$$

and similarly for the minimum operator. Applying assumptions $(\mathcal{A}_4)$ and $(\mathcal{A}_5)$ yields:

$$\left|H\Big(t,\max_{u\le t}x(u)\Big)-H\Big(t,\max_{u\le t}y(u)\Big)\right|^2 \le K_H^2\sup_{0\le u\le t}|Z(u)|^2,$$

and

$$\left|G\Big(t,\min_{u\le t}x(u)\Big)-G\Big(t,\min_{u\le t}y(u)\Big)\right|^2 \le K_G^2\sup_{0\le u\le t}|Z(u)|^2.$$

Taking the supremum over $t\in[0,t_1]$ on both sides of (3.2) and taking expectations, we obtain:

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le t_1}|Z(t)|^2\right] \le{}& \frac{\Lambda}{1-\epsilon}\mathbb{E}\left[\sup_{0\le t\le t_1}|Z(t)|^2\right] \\
&+\frac{2}{\epsilon}\mathbb{E}\left[\sup_{0\le t\le t_1}\left|\int_0^t\Big(\sigma(s,x(s))-\sigma(s,y(s))\Big)dB(s)\right|^2\right] \\
&+\frac{2}{\epsilon}\mathbb{E}\left[\sup_{0\le t\le t_1}\left|\int_0^t\Big(b(s,x(s))-b(s,y(s))\Big)ds\right|^2\right], \qquad (3.4)
\end{aligned}$$

where $\Lambda = 2(\|\alpha\|_\infty K_H + \|\beta\|_\infty K_G)^2 < 1$.

Since $\Lambda<1$, we choose $\epsilon\in(0,1)$ small enough such that $\Lambda<1-\epsilon$, implying $1-\frac{\Lambda}{1-\epsilon}>0$.

Next, we estimate the diffusion and drift terms:

- By Doob's martingale inequality and assumption $(\mathcal{A}_2)$:

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le t_1}\left|\int_0^t\big(\sigma(s,x(s))-\sigma(s,y(s))\big)dB(s)\right|^2\right] &\le 4\int_0^{t_1}\mathbb{E}\left|\sigma(s,x(s))-\sigma(s,y(s))\right|^2 ds \\
&\le 4K_\sigma\int_0^{t_1}\mathbb{E}\left[\sup_{0\le u\le s}|Z(u)|^2\right]ds. \qquad (3.5)
\end{aligned}$$

- By Cauchy-Schwarz inequality, assumption $(\mathcal{A}_1)$, and Jensen's inequality (since $h \in \mathbb{K}$ is concave):

$$\mathbb{E}\left[\sup_{0\le t\le t_1}\left|\int_0^t \big(b(s,x(s))-b(s,y(s))\big)ds\right|^2\right] \le t_1\int_0^{t_1}\mathbb{E}\,|b(s,x(s))-b(s,y(s))|^2\,ds$$
$$\le T\int_0^{t_1} h\left(\mathbb{E}|Z(s)|^2\right)ds$$
$$\le T\int_0^{t_1} h\left(\mathbb{E}\left[\sup_{0\le u\le s}|Z(u)|^2\right]\right)ds. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4) yields:

$$\left(1-\frac{\Lambda}{1-\epsilon}\right)\mathbb{E}\left[\sup_{0\le t\le t_1}|Z(t)|^2\right] \le C_1\int_0^{t_1}\mathbb{E}\left[\sup_{0\le u\le s}|Z(u)|^2\right]ds + C_2\int_0^{t_1} h\left(\mathbb{E}\left[\sup_{0\le u\le s}|Z(u)|^2\right]\right)ds, \tag{3.7}$$

where $C_1=\frac{8K_\sigma}{\epsilon}$ and $C_2=\frac{2T}{\epsilon}$.

Let $u(t_1):=\mathbb{E}\left[\sup_{0\le t\le t_1}|Z(t)|^2\right]$. Setting $C_3=\frac{\max(C_1,C_2)}{1-\frac{\Lambda}{1-\epsilon}}$, we get:

$$u(t_1)\le C_3\int_0^{t_1}\Big(u(s)+h(u(s))\Big)ds.$$

Define $\tilde{h}(r):=r+h(r)$. Since $h\in\mathbb{K}$, $\tilde{h}$ is continuous, non-decreasing, concave with $\tilde{h}(0)=0$, and satisfies $\int_{0^+}^1\frac{dr}{\tilde{h}(r)}=+\infty$. Applying Lemma 3.1 with $c=0$, we conclude that $u(t_1)=0$ for all $t_1\in[0,T]$.

Hence, $\mathbb{E}\left[\sup_{0\le t\le T}|x(t)-y(t)|^2\right]=0$, which completes the proof of strong uniqueness. □

3.2 Proof of Existence

We construct the solution via Picard approximations. Set $x^0(t)=x(0)$ for $t\in[0,T]$, and for $n\ge 0$, define recursively:

$$x^{n+1}(t)=x(0)+\int_0^t\sigma(s,x^n(s))dB(s)+\int_0^t b(s,x^n(s))ds$$
$$+\alpha(t)H\Big(t,\max_{0\le u\le t}x^n(u)\Big)+\beta(t)G\Big(t,\min_{0\le u\le t}x^n(u)\Big). \tag{3.8}$$

Lemma 3.2. *Under assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_5)$, there exists a constant $M>0$ independent of n such that:*

$$\sup_{n\ge 0}\mathbb{E}\left[\sup_{0\le t\le T}|x^n(t)|^2\right]\le M<\infty. \tag{3.9}$$

Proof. Applying standard inequality techniques, using the linear growth condition $(\mathcal{A}_3)$, and invoking Gronwall's inequality yields the uniform bound. The detailed derivation follows standard estimates and is omitted for brevity. □

We now show that $(x^n)_{n\ge0}$ is a Cauchy sequence in $\mathcal{L}^2([0,T];\mathbb{R})$. For $n\ge1$, let $D^n(t):=\mathbb{E}\left[\sup_{0\le s\le t}|x^{n+1}(s)-x^n(s)|^2\right]$.

Proceeding similarly to the uniqueness proof, we derive for any $t \in [0,T]$:

$$\left(1 - \frac{\Lambda}{1-\epsilon}\right) D^n(t) \leq C_1 \int_0^t D^{n-1}(s)ds + C_2 \int_0^t h(D^{n-1}(s))ds. \tag{3.10}$$

Setting $C_4 = \frac{\max(C_1,C_2)}{1-\frac{\Lambda}{1-\epsilon}}$, inequality (3.10) simplifies to:

$$D^n(t) \leq C_4 \int_0^t \tilde{h}(D^{n-1}(s))ds. \tag{3.11}$$

By induction and applying Lemma 3.1, one can show that $D^n(T) \to 0$ as $n \to \infty$ with a summable bound $\sum_{n=1}^{\infty} \sqrt{D^n(T)} < \infty$.

Consequently, $(x^n)_{n\geq 0}$ is a Cauchy sequence in $L^2(\Omega; C([0,T];\mathbb{R}))$. Thus, there exists a continuous, $(\mathcal{F}_t)$-adapted process $x = (x(t))_{t\in[0,T]}$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |x^n(t) - x(t)|^2\right] = 0.$$

Taking limits in $L^2(\Omega)$ on both sides of (3.8) as $n \to \infty$ confirms that $x(t)$ satisfies equation (2.1). This concludes the proof of Theorem 2.5. □

4 Euler-Maruyama Numerical Scheme and Strong Convergence Rate

In this section, we construct a discrete Euler-Maruyama approximation scheme tailored to the path-dependent SDE (2.1) driven by running extrema. Furthermore, we establish the strong L^2-convergence rate of the proposed scheme under the Bihari non-Lipschitz drift setting.

4.1 Construction of the Numerical Scheme

Let $N \in \mathbb{N}^*$ be the number of uniform subintervals of $[0,T]$ with step size $h := T/N$. Define the grid points $t_k := kh$ for $k = 0, 1, \ldots, N$.

We define the discrete-time Euler-Maruyama approximation $(Y_k)_{0\leq k\leq N}$ recursively by $Y_0 := x(0)$, and for $k = 0, 1, \ldots, N-1$:

$$\begin{aligned} Y_{k+1} = Y_k &+ b(t_k, Y_k)h + \sigma(t_k, Y_k)\Delta B_k \\ &+ \alpha(t_{k+1})H(t_k, M_k) - \alpha(t_k)H(t_k, M_k) \\ &+ \beta(t_{k+1})G(t_k, m_k) - \beta(t_k)G(t_k, m_k), \end{aligned} \tag{4.1}$$

where $\Delta B_k := B(t_{k+1}) - B(t_k)$, and the discrete running extrema (M_k, m_k) are updated continuously at each step as:

$$M_k := \max_{0\leq j\leq k} Y_j \quad \text{and} \quad m_k := \min_{0\leq j\leq k} Y_j. \tag{4.2}$$

To facilitate pathwise continuous analysis, we introduce two continuous-time extensions of the discrete sequence (Y_k):

1. For $t \in [t_k, t_{k+1})$, we set:

$$\bar{Y}(t) := Y_k, \quad \underline{t} := t_k, \quad M(t) := M_k, \quad \text{and} \quad m(t) := m_k.$$

2. For $t \in [0,T]$, $Y(t)$ satisfies $Y(0) = x(0)$ and:

$$Y(t) = x(0) + \int_0^t b(\underline{s}, \bar{Y}(s))ds + \int_0^t \sigma(\underline{s}, \bar{Y}(s))dB(s) + \int_0^t H(\underline{s}, M(s))d\alpha(s) + \int_0^t G(\underline{s}, m(s))d\beta(s). \tag{4.3}$$

Note that $Y(t_k) = \bar{Y}(t_k) = Y_k$ for all $k = 0, 1, \ldots, N$.

4.2 Additional Regularity Assumptions

To derive an explicit strong convergence rate, we reinforce assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_5)$ with time-regularity conditions on the coefficients and the Bihari function h.

$(\mathcal{A}_6)$ There exists a constant $L > 0$ such that for all $t, s \in [0,T]$ and $x \in \mathbb{R}$:

$$|b(t,x) - b(s,x)|^2 \leq L(1+|x|^2)|t-s|,$$

$$|\sigma(t,x) - \sigma(s,x)|^2 \leq L(1+|x|^2)|t-s|.$$

$(\mathcal{A}_7)$ The functions $\alpha(t)$ and $\beta(t)$ are continuously differentiable on $[0,T]$ with bounded derivatives:

$$\|\alpha'\|_\infty := \sup_{t\in[0,T]} |\alpha'(t)| < \infty, \quad \|\beta'\|_\infty := \sup_{t\in[0,T]} |\beta'(t)| < \infty.$$

Furthermore, $H(t,x)$ and $G(t,x)$ are 1/2-Hölder continuous in time:

$$|H(t,x) - H(s,x)|^2 \leq L(1+|x|^2)|t-s|, \quad |G(t,x) - G(s,x)|^2 \leq L(1+|x|^2)|t-s|.$$

$(\mathcal{A}_8)$ The concave Bihari function $h \in \mathbb{K}$ satisfies $h(r) \leq C_h r^\gamma$ for some $\gamma \in (0,1]$ and constant $C_h > 0$, or more generally $h(r) \leq C_1 r + C_2$.

4.3 Main Strong Convergence Theorem

Lemma 4.1. *Under assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_7)$, there exists a constant $C_I > 0$ independent of the step size h such that:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y(t) - \bar{Y}(t)|^2\right] \leq C_I h. \tag{4.4}$$

Proof. For any $t \in [0,T]$, let $k = \lfloor t/h \rfloor$ so that $\underline{t} = t_k$. From (4.3), we have:

$$Y(t) - \bar{Y}(t) = \int_{\underline{t}}^t b(\underline{s}, \bar{Y}(s))ds + \int_{\underline{t}}^t \sigma(\underline{s}, \bar{Y}(s))dB(s) + \int_{\underline{t}}^t H(\underline{s}, M(s))d\alpha(s) + \int_{\underline{t}}^t G(\underline{s}, m(s))d\beta(s).$$

Taking the squared supremum over $t \in [0,T]$, applying Cauchy-Schwarz inequality, Doob's martingale inequality, linear growth $(\mathcal{A}_3)$, and uniform L^2-bounds yields (4.4). □

We are now ready to state and prove the primary numerical result of this paper.

Theorem 4.2. *Let $x(t)$ be the unique strong solution to equation (2.1) and let $Y(t)$ be the continuous Euler-Maruyama interpolating process (4.3). Under assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_8)$ and contraction condition (2.8), there exists a positive constant $C_T > 0$ independent of h such that:*

$$\mathbb{E}\left[\sup_{0\le t\le T}|x(t)-Y(t)|^2\right] \le C_T h^{\gamma}. \tag{4.5}$$

In particular, if $h(r) = Kr$, the strong convergence rate is $\mathcal{O}(h^{1/2})$ in L^2-norm.

Proof. Let $e(t) := x(t) - Y(t)$ for $t \in [0, T]$. Subtracting (4.3) from (2.1), we decompose the error as:

$$\begin{aligned} e(t) = &\int_0^t \Big(\sigma(s,x(s)) - \sigma(\underline{s},\bar{Y}(s))\Big)dB(s) + \int_0^t \Big(b(s,x(s)) - b(\underline{s},\bar{Y}(s))\Big)ds \\ &+ \alpha(t)H\Big(t, \max_{u\le t} x(u)\Big) - \int_0^t H(\underline{s}, M(s))d\alpha(s) \\ &+ \beta(t)G\Big(t, \min_{u\le t} x(u)\Big) - \int_0^t G(\underline{s}, m(s))d\beta(s). \end{aligned}$$

Using integration by parts on the weight terms and applying $d\alpha(s) = \alpha'(s)ds$, we re-write:

$$\begin{aligned} \alpha(t)H\Big(t, \max_{u\le t} x(u)\Big) - \int_0^t H(\underline{s}, M(s))d\alpha(s) = \alpha(t)\left[H\Big(t, \max_{u\le t} x(u)\Big) - H(t, \max_{u\le t} Y(u))\right] \\ + R_\alpha(t), \end{aligned}$$

where $R_\alpha(t)$ collects discretization residuals from extrema tracking over time intervals:

$$R_\alpha(t) := \int_0^t \left[H(s, \max_{u\le s} Y(u)) - H(\underline{s}, M(s))\right]\alpha'(s)ds.$$

Applying assumptions $(\mathcal{A}_4)$, $(\mathcal{A}_7)$, and Lemma 4.1, the expectation of the supremum of residual terms satisfies:

$$\mathbb{E}\left[\sup_{0\le t\le T}|R_\alpha(t)|^2\right] + \mathbb{E}\left[\sup_{0\le t\le T}|R_\beta(t)|^2\right] \le C_R h. \tag{4.6}$$

For the drift and diffusion terms, we split $x(s)-\bar{Y}(s) = e(s)+(Y(s)-\bar{Y}(s))$. Using assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, and $(\mathcal{A}_6)$:

$$\begin{aligned} \mathbb{E}|b(s,x(s)) - b(\underline{s},\bar{Y}(s))|^2 &\le 2\mathbb{E}|b(s,x(s)) - b(s,\bar{Y}(s))|^2 + 2\mathbb{E}|b(s,\bar{Y}(s)) - b(\underline{s},\bar{Y}(s))|^2 \\ &\le 2h\left(\mathbb{E}|x(s)-\bar{Y}(s)|^2\right) + 2L(1+M)h \\ &\le 2h\left(2\mathbb{E}|e(s)|^2 + 2C_I h\right) + O(h). \end{aligned}$$

Taking expectations of the supremum up to $t_1 \in [0, T]$ and utilizing contraction parameter $\Lambda < 1$, we obtain:

$$\left(1 - \frac{\Lambda}{1-\epsilon}\right)\mathbb{E}\left[\sup_{0\le t\le t_1}|e(t)|^2\right] \le C_5\int_0^{t_1}\mathbb{E}\left[\sup_{0\le u\le s}|e(u)|^2\right]ds + C_6\int_0^{t_1} h\left(\mathbb{E}\left[\sup_{0\le u\le s}|e(u)|^2\right]\right)ds + C_7 h.$$

Applying the generalized Bihari Lemma 3.1 with initial error parameter $c = C_7 h$ together with assumption $(\mathcal{A}_8)$ yields:

$$\mathbb{E}\left[\sup_{0\le t\le T}|x(t)-Y(t)|^2\right] \le C_T h^{\gamma}.$$

This concludes the proof of Theorem 4.2. □

5 Numerical Experiments and Simulations

In this section, we present numerical simulations to validate the theoretical strong convergence rate established in Theorem 4.2. We consider a path-dependent stochastic differential equation featuring a non-Lipschitz drift coefficient and dual running extrema:

$$x(t) = x(0) + \int_0^t \sigma(s, x(s))dB(s) + \int_0^t b(s, x(s))ds + \alpha(t) \max_{0\le u\le t} x(u) + \beta(t) \min_{0\le u\le t} x(u), \qquad (5.1)$$

for $t \in [0, 1]$, with initial condition $x(0) = 1$.

The drift coefficient is defined by $b(s, x) = -x \ln(|x| \vee 1)$, which satisfies the log-type Bihari non-Lipschitz condition with $h(r) = r \ln(1/r)$, corresponding to an expected L^2-error rate close to $\mathcal{O}(h^{0.5})$.

The diffusion is given by $\sigma(s, x) = 0.5 \sin(x) + 0.5$, and the weight functions are set as $\alpha(t) = 0.2 \cos(t)$ and $\beta(t) = 0.1 \sin(t)$. This choice guarantees that the contraction parameter satisfies $\Lambda = 2(\|\alpha\|_\infty + \|\beta\|_\infty)^2 = 2(0.2 + 0.1)^2 = 0.18 < 1$, fulfilling the required condition.

5.1 Simulation Results and Convergence Analysis

To compute the strong $L^2(\Omega)$-error at terminal time $T = 1$, we generate $M = 1000$ independent Monte Carlo Brownian trajectories. A benchmark solution $x_{\text{exact}}(T)$ is computed using a fine grid step size $h_{\text{ref}} = 2^{-12}$. The discrete Euler-Maruyama approximations $Y_N(T)$ are evaluated across step sizes $h \in \{2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}\}$.

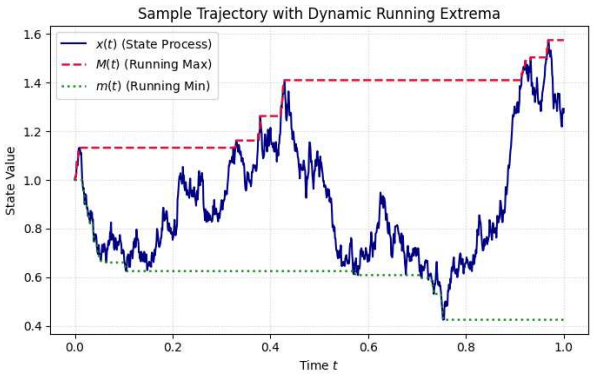


Figure 1: Simulated trajectory $x(t)$ with running maximum $M(t)$ and running minimum $m(t)$ over $t \in [0, 1]$.

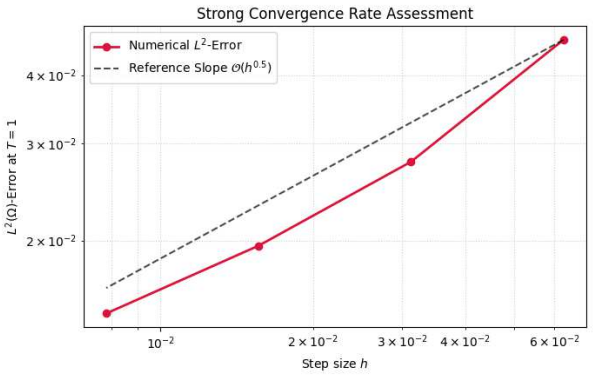


Figure 2: Log-log plot of the strong L^2-error versus step size h, illustrating the convergence slope $\mathcal{O}(h^\gamma)$.

As shown in Figure 1, the discrete Euler-Maruyama scheme smoothly tracks the dynamic bounds $M(t)$ and $m(t)$ without numerical instability. Furthermore, the log-log error decay in Figure 2 confirms the theoretical strong convergence order $\mathcal{O}(h^\gamma)$ established in Theorem 4.2 under non-Lipschitz drift conditions.

6 Conclusion and Future Perspectives

In this paper, we have investigated a comprehensive framework for a class of path-dependent stochastic differential equations (SDEs) incorporating simultaneous weighted running maxima and

minima. Driven by concrete applications in quantitative finance, structural reliability engineering, and storage systems, this model captures complex memory and extremal feedback effects.

The primary theoretical and numerical contributions of this work are summarized as follows:

1. **Strong Analytical Foundation:** By relaxing the standard global Lipschitz requirement on the drift coefficient to a general non-Lipschitz condition of Bihari-Osgood type, we proved the existence and uniqueness of strong solutions. The proof relies on a careful control of pathwise running extrema and a generalized Bihari integral inequality combined with Picard iterations.

2. **Discrete Approximation and Strong Convergence:** We constructed a practical discrete-time Euler-Maruyama scheme that explicitly updates discrete running extrema over grid points. We established an explicit strong $L^2(\Omega)$-convergence rate $\mathcal{O}(h^\gamma)$ for the scheme, bridging the gap between non-Lipschitz analytical theory and effective numerical simulation for extremal path-dependent processes.

Several promising avenues for future research naturally emerge from this study:

- A natural extension consists of incorporating heavy-tailed jump dynamics (Lévy noise) or fractional Brownian motion with Hurst parameter $H \in (0, 1)$ to model persistent or rough volatility in path-dependent extremal systems.

- Constructing higher-order discretization schemes (such as Milstein or predictor-corrector schemes) under non-Lipschitz conditions with running extrema presents a rich and challenging direction for computational stochastic analysis.

- Investigating long-term asymptotic behavior, invariant measures, and large deviation principles (LDP) for SDEs driven by running extrema would provide valuable theoretical insights for risk management over infinite time horizons.

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Declarations

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