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Analytical Construction of the CKM Matrix Using a Single Electroweak Constant

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Abstract

We present closed analytical expressions for all nine elements of the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix within the $SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ model, based on a *mixing-equilibrium condition* for a quark flavour state, postulated as

$$\alpha_q(1 - \alpha_q) = \mathcal{C}_q \lambda^2,$$

where $0 < \alpha_q < 1 - \alpha_q$ represents the *equilibrium mixing-weight* of the sector, $\lambda^2 = 1/(2\pi^2)$ is the inverse Haar volume measure of $SU(2)_L$, and $\mathcal{C}_q = 1 + (B - L)_q/2$ is the *quark coupling factor*, with $(B - L)_q$ denoting the gauge charge. We define λ as the *fermion mixing constant*, which numerically matches the empirical Cabibbo scale ($\lambda \approx \sin \theta_C$) and serves as the *baseline mixing scale* for both the quark and lepton sectors.

We have constructed the CKM matrix as the Hadamard product of the isospin activity matrix $\mathbf{G}$ and the inter-generational suppression matrix $\mathbf{S}$, followed by exact row unitarization. The matrix elements are expressed directly in terms of λ and α_q , with exact analytical forms provided for every component of $\mathbf{G}$ and $\mathbf{S}$. The proposed closed-form formulation of the CKM matrix achieves sub-sigma agreement with 2024 Particle Data Group (PDG) world averages (mean absolute pull of 0.22σ , maximum pull of 0.59σ at $|V_{ub}|$), offering clear advantages over the standard Wolfenstein expansion.

The mixing scale $\lambda_W = \lambda$ and the Wolfenstein amplitude $A \approx 0.819$ (0.58σ from PDG), alongside the CKM unitarity triangle coordinates $\bar{\rho} \approx 0.143$ and $\bar{\eta} \approx 0.379$ derived from the *equilibrium retention-weight* $1 - \alpha_q = \sin \delta_{13}$, achieve agreement with global fits within 0.02σ (for the unitarity apex). Consequently, the phase angle $\gamma = \arcsin(1 - \alpha_q) \approx 69.54^\circ$ uniquely determines the Jarlskog invariant as $J_{\text{CP}}^q \approx 3.22 \times 10^{-5}$, matching the PDG global fit to within 0.82σ .

The same constant λ underpins the closed-form formulation of the PMNS lepton mixing matrix, where the lepton-sector coupling factor $\mathcal{C}_\ell = 1/2$ yields the corresponding lepton parameter α_ℓ via $\alpha_\ell(1 - \alpha_\ell) = \mathcal{C}_\ell \lambda^2$, demonstrating that $\lambda = 1/(\pi\sqrt{2})$ is the *single universal invariant for quarks as well as leptons*. Regardless of the broader interpretation of electroweak symmetry breaking, the proposed analytical approach provides closed-form, parameter-free expressions for CKM observables, with potential insight into flavour-mixing behaviour.

1 Introduction

The Standard Model describes quark flavour mixing by the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1, 2], which parametrises the misalignment between quark mass eigenstates and weak-interaction eigenstates, governing charged-current flavour-changing transitions. The Wolfenstein

parametrisation [3] makes the hierarchical structure of the CKM matrix manifest through a systematic expansion in the small parameter $\lambda \approx 0.225$, with off-diagonal elements suppressed by successive powers $\mathcal{O}(\lambda)$, $\mathcal{O}(\lambda^2)$, and $\mathcal{O}(\lambda^3)$ as the generation gap increases. The origin of this hierarchy, in particular the value of λ itself and the emergence of its associated coefficients, has not been specified within the Standard Model. Several approaches have attempted to relate the observed mixing pattern to quark masses, flavour symmetries, or more fundamental algebraic structures.

A major class of approaches is based on *quark mass textures*. Fritzsch-type and related strict, original texture-zero models seek to express mixing angles in terms of quark mass ratios, thereby connecting flavour mixing to the observed mass hierarchy [4, 5]. Such constructions successfully reproduce the qualitative hierarchy between the Cabibbo mixing angle and the smaller $2 \leftrightarrow 3$ and $1 \leftrightarrow 3$ mixing angles. However, these original, highly constrained configurations are fundamentally incompatible with modern precision data, particularly in the heavy-quark sector, forcing researchers to introduce modified textures or arbitrary additional non-zero entries [6, 7]. While hierarchical Hermitian and sparse mass matrices provide alternative constructions, their predictions depend on the chosen texture [8].

A second major direction employs *flavour symmetries*. Non-Abelian groups such as S_3 , A_4 , and related discrete groups attempt to replace the arbitrary, unstructured mass matrices of the Standard Model with *structured mass matrices*, generating analytical relations among mixing parameters [9, 10, 11]. In representative S_3 constructions, an approximate block structure arises naturally, while symmetry breaking generates the smaller mixing elements [12, 13]. These models demonstrate that family symmetry can provide a physical origin for the observed hierarchy. However, their fundamental limitation is a lack of predictive uniqueness, where the detailed CKM matrix depends entirely on model-dependent choices, including the selection of the discrete group, the specific symmetry-breaking vacuum alignment, and associated fitted parameters, rather than emerging as a direct consequence of the underlying flavour-mixing process.

Related *hierarchy-building mechanisms*, including Froggatt–Nielsen and modular-flavour constructions, generate quark masses and mixing from charge assignments, modular weights, or flavon expectation values rather than imposing texture zeros directly [14]. Unified constructions similarly connect the flavour states to a larger gauge or flavour symmetry [15]. These approaches can relate several observables within a common framework, but the resulting mixing pattern remains sensitive to the assumed flavour charges, vacuum expectation values, or symmetry-breaking structure.

Another class of approach seeks an *algebraic or geometric origin* for flavour mixing. Constructions based on octonions, exceptional Jordan algebras, or related structures explore whether fermion masses and mixing angles can be derived from an underlying algebraic state space [16, 17]. However, existing constructions still require additional assumptions to connect the algebraic spectrum to the mixing amplitudes, particularly in the $2 \leftrightarrow 3$ and $1 \leftrightarrow 3$ sectors.

Several common features emerge from these studies. First, the observed CKM hierarchy is strongly correlated with the hierarchical structure of the quark masses. Second, adjacent-generation transitions are substantially larger than the direct $1 \leftrightarrow 3$ transitions. Third, successful models generally require an organising principle beyond the numerical values of the CKM elements themselves. An analytical formulation should derive the CKM matrix from the underlying flavour-mixing process and charged-current dynamics, rather than parametrising it with fitted constants. *The core objective of the analytical formulation should be to unfold insight into the distinctive flavour behaviour.*

To address this objective, we propose a closed-form analytic formulation of CKM matrix elements based on a *mixing-equilibrium condition* within the left-right symmetric model (LRSM) [18, 19]. It is built upon a single sector-invariant baseline mixing scale, $\lambda = 1/(\pi\sqrt{2}) \approx 0.225079$, designated as the *fermion mixing constant*. The value of λ is analytically determined by the Haar measure of $SU(2)_L$. The *equilibrium mixing-weight* $\alpha_q \approx 0.063084$ is uniquely derived from λ through the

mixing-equilibrium condition: $\alpha_q(1 - \alpha_q) = 7/(12\pi^2)$. The physical significance of both quantities is discussed in detail, and the analytical association of every element of the constituent matrices with these parameters is explicitly formulated. The resulting CKM matrix achieves sub-sigma agreement with all nine 2024 Particle Data Group (PDG) [20] world averages, exhibiting a maximum pull of 0.59σ without any free parameters.

The paper is organised as follows. Section 2 postulates the mixing-equilibrium condition, defines λ and α_q , and examines their physical significance. Section 3 constructs the isospin activity matrix $\mathbf{G}$, the inter-generational suppression matrix $\mathbf{S}$, and the complete CKM matrix via Hadamard product and row normalisation. Section 4 compares the computed result directly with the Wolfenstein parametrisation. Section 5 concludes the paper. The description and the context of origin of key symbols are compiled in Table 1 for easy reference due to their extensive use.

Table 1: Key symbols, descriptions, and their contexts of origin.

Symbol	Description	Defining Context
λ	Fermion mixing constant	$1/\lambda^2 = 2\pi^2$ ($\text{SU}(2)_L$ Haar volume)
D_q	$(B - L)$ quantum number of quarks	$D_q \equiv (B - L) = 1/3$: $B = 1/3$ and $L = 0$
C_q	Quark coupling factor	$C_q = 1 + D_q/2$: $C_q = 7/6$
α_q	Equilibrium mixing-weight of quarks	Mixing equilibrium: $\alpha_q(1 - \alpha_q) = \lambda^2 C_q$
$1 - \alpha_q$	Equilibrium retention-weight of quarks	Mixing equilibrium: $\alpha_q(1 - \alpha_q) = \lambda^2 C_q$

2 The Electroweak Flavour Invariants (λ and α_q)

In this section we define the two interdependent electroweak invariants: (i) *fermion mixing constant* (λ) and (ii) the *quark mixing-weight* (α_q) used in the analytical formulation of the CKM matrix in the next Section.

2.1 Fermion fermion mixing constant (λ)

Definition: The fermion mixing constant λ is defined as:

$$\lambda = \frac{1}{\sqrt{2}\pi}. \quad (1)$$

It characterises the suppression of mixing between the first and second generations, and affects all cross-generation mixing. The significance of λ is discussed below.

2.1.1 Correspondence with Haar Measure Normalisation Factor

For the compact, non-Abelian Lie group $\text{SU}(2)$, the total group volume under the standard bi-invariant metric is $\text{Vol}(\text{SU}(2)) = 2\pi^2$ [21]. The square of the fermion mixing constant (λ^2) equals the normalisation factor of the invariant Haar measure on the $\text{SU}(2)_L$ group manifold:

$$\lambda^2 \equiv \frac{1}{\text{Vol}(\text{SU}(2)_L)} = \frac{1}{2\pi^2}. \quad (2)$$

The cross-generational transition amplitude between the first and second generations matches the inverse square root of the $\text{SU}(2)_L$ group volume.

2.1.2 Correspondence with Charge Transfer Across an Overlapped State

We show that λ arises directly from the charge transfer during a flavour transition (illustrated in Fig. 1). According to the Gell-Mann–Nishijima relation for any quark, the charge $Q = T_3 + D_q/2$, and $D_q = B - L = 1/3$. It is a constant for all quark flavours. When a flavour transition occurs between two adjacent generations (for example $u \leftrightarrow s$ or $c \leftrightarrow d$), the charge modification $\Delta Q = \pm 1$ is realised by $|\Delta T_3| = 1$.

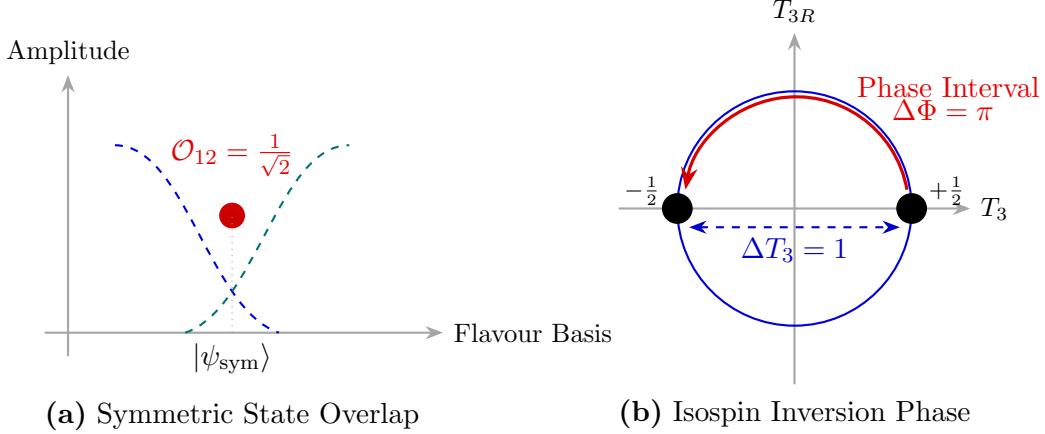


Figure 1: **(a)** Unbiased inter-generation overlap amplitude $\mathcal{O}_{12} = 1/\sqrt{2}$ of the symmetric intermediate state $|\psi_{\text{sym}}\rangle$. **(b)** Weak-isospin inversion under continuous $\text{SU}(2)_L$ generator rotation across the total phase interval $\Delta\Phi = \pi$.

(a) Symmetric State Projection: Consider the two-generation subspace spanned by the orthonormal flavour basis $\{|g_1\rangle, |g_2\rangle\}$ with $\langle g_i | g_j \rangle = \delta_{ij}$. A W boson coupling between two generations has no intrinsic preference for either state; the coupling therefore proceeds through equal amplitudes to both. The corresponding intermediate state is the symmetric superposition

$$|\psi_{\text{sym}}\rangle = \frac{1}{\sqrt{2}}(|g_1\rangle + |g_2\rangle). \quad (3)$$

As shown in Fig. 1(a), the projection of this symmetric state onto each generation basis state gives the inter-generation overlap factor:

$$\mathcal{O}_{12} = |\langle g_1 | \psi_{\text{sym}} \rangle| = \frac{1}{\sqrt{2}} \approx 0.707107. \quad (4)$$

(b) Normalised Charge Transfer Across an Overlapped State: A charged-current weak transition ($W^\pm$ exchange) maps an up-type state ($T_3 = +1/2$) to a down-type state ($T_3 = -1/2$). Under the $\text{U}(1)$ subgroup of $\text{SU}(2)_L$ generated by T_3 , this corresponds to a rotation governed by the operator

$$U(\Phi) = \exp\left(-i\Phi \frac{\sigma_2}{2}\right), \quad \Phi \in [0, \pi], \quad (5)$$

where σ_2 denotes the second Pauli matrix mediating transitions between weak isospin eigenstates. As shown in Fig. 1(b), the total phase interval required to invert T_3 across the unit separation ($\Delta T_3 = 1$) is

$$\Delta\Phi = \int_0^\pi d\Phi = \pi. \quad (6)$$

Normalising the inter-generational overlap factor $\mathcal{O}_{12}$ over this phase interval yields the fermion mixing constant:

$$\lambda \equiv \mathcal{O}_{12} \cdot \frac{\Delta Q}{\Delta \Phi} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\pi} = \frac{1}{\sqrt{2}\pi} \approx 0.225079. \quad (7)$$

The squared value satisfies $\lambda^2 = 1/(2\pi^2)$, consistent with (2).

2.1.3 Correspondence with Cabibbo Mixing Scale:

The quark mixing relation for the first two generations is expressed as

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos \theta_C & \sin \theta_C \\ -\sin \theta_C & \cos \theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}, \quad (8)$$

where θ_C is the Cabibbo angle. In the Standard Model, the Wolfenstein parameter $\lambda \equiv \sin \theta_C$ parametrises the charged-current coupling between the first and second generations. The derived value

$$\lambda = \frac{1}{\sqrt{2}\pi} \approx 0.225079, \quad (9)$$

agrees with the experimentally extracted value $\sin \theta_C \approx 0.22500 \pm 0.00067$ (PDG global CKM fit) to within 0.04%.

2.2 The Mixing Equilibrium Postulate

Let x denote the *mixing-weight*, which measures the probability space accessible for cross-flavour transitions; and its complement, $(1 - x)$, represents the *retention-weight*, the probability of a state maintaining its native flavour in a given sector. The product of these components ($x(1 - x)$) defines the *mixing variance*, which measures the total statistical spread of flavour transitions within a sector. We postulate that at the ground state, the mixing variance is proportional to the *quark coupling factor*, $\mathcal{C}_q$, with the proportionality constant given by the square of the *fermion mixing constant* (λ^2) given by:

$$x(1 - x) \Big|_{\text{equilibrium}} = \lambda^2 \mathcal{C}_q, \quad (10)$$

where $0 \leq x \leq 1/2$. The quark-sector coupling, $\mathcal{C}_q = 1 + D_q/2$, where $D_q = B - L$ gauge charge. We find D_q from the Gell-Mann-Nishijima relation [22, 23]:

$$Q = T_3 + \frac{D_q}{2}, \quad (11)$$

For quarks, we have baryon number $B = \frac{1}{3}$ and lepton number $L = 0$, yielding $D_q = \frac{1}{3}$ for all quark flavours. Consequently, the coupling factor evaluates to

$$\mathcal{C}_q = 1 + \frac{D_q}{2} = 1 + \frac{1}{6} = \frac{7}{6}. \quad (12)$$

The equilibrium mixing root $x \Big|_{\text{equilibrium}} = \alpha_q$ is determined by solving:

$$\alpha_q^2 - \alpha_q + \lambda^2 \mathcal{C}_q = 0, \text{ where } \mathcal{C}_q = 7/6. \quad (13)$$

2.2.1 The Mixing-Equilibrium Condition

The *mixing-equilibrium condition* for a quark flavour state, postulated as

$$\alpha_q(1 - \alpha_q) = \mathcal{C}_q \lambda^2, \quad (14)$$

where α_q represents the intrinsic *mixing-weight* of the quark flavour at the equilibrium point and $(1 - \alpha_q)$ represents the corresponding *retention-weight* constrained by $0 < \alpha_q < 1 - \alpha_q$, while $\mathcal{C}_q$ is the sector-specific *coupling factor*, and $\lambda = \frac{1}{\pi\sqrt{2}}$ is defined as the *fermion mixing constant*.

The square root of the normalized mixing variance $\alpha_q(1 - \alpha_q)/\mathcal{C}_q$ yields the fermion mixing constant, establishing a universal baseline mixing scale λ :

$$\lambda = \sqrt{\frac{\alpha_q(1 - \alpha_q)}{\mathcal{C}_q}}, \quad (15)$$

Substituting $\lambda^2 = 1/(2\pi^2)$ and $\mathcal{C}_q = 7/6$, Eq. (14) becomes

$$\alpha_q^2 - \alpha_q + 7/(12\pi^2) = 0, \quad (16)$$

The smaller root of this:

$$\alpha_q = \frac{1 - \sqrt{1 - 7/(3\pi^2)}}{2} = \frac{1 - \sqrt{0.76358}}{2} \approx \mathbf{0.063084}. \quad (17)$$

We will show later that $(1 - \alpha_q)$ is the magnitude of the imaginary part of the complex CP-violation phase factor. The equilibrium condition is shown in Fig. 2 in a plot of variance versus mixing-weight.

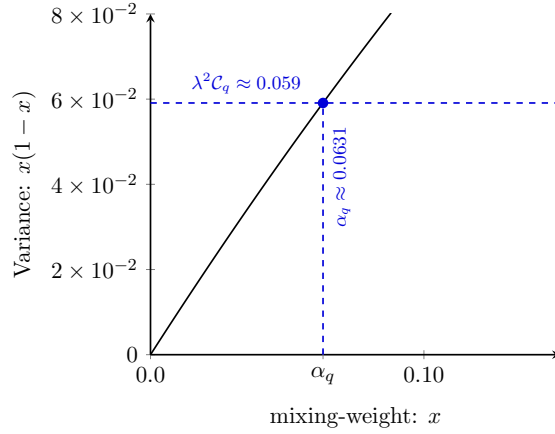


Figure 2: Solution of the mixing-equilibrium condition for the quark sector.

2.2.2 Probabilistic Interpretation

To describe flavour mixing, we introduce an idempotent flavour-mixing operator $\hat{\mathcal{P}}$ satisfying $\hat{\mathcal{P}}^2 = \hat{\mathcal{P}}$. For a normalised flavour state, the mixing probability is $\langle \hat{\mathcal{P}} \rangle = \alpha_q$, the equilibrium mixing-weight with the complementary unmixing probability $1 - \alpha_q$, the CP violation phase parameter:

$$P_{\text{mix}} = \alpha_q, \quad P_{\text{unm}} = 1 - \alpha_q, \quad P_{\text{mix}} + P_{\text{unm}} = 1. \quad (18)$$

Because $\hat{\mathcal{P}}$ is idempotent, $\langle \hat{\mathcal{P}}^2 \rangle = \langle \hat{\mathcal{P}} \rangle = \alpha_q$, and the variance of the binary mixing process is exactly:

$$\text{Var}(\hat{\mathcal{P}}) = \langle \hat{\mathcal{P}}^2 \rangle - \langle \hat{\mathcal{P}} \rangle^2 = \alpha_q(1 - \alpha_q). \quad (19)$$

Equation (14) therefore establishes that the variance of the equilibrium mixing process equals the quark-sector coupling factor $\mathcal{C}_q$ scaled by λ^2 .

2.2.3 Correspondence to Weinberg Angle (θ_W)

The equilibrium mixing-weight α_q defined in Eq. (14) matches the on-shell mass-splitting ratio of the neutral and charged gauge bosons:

$$R_b = \frac{M_Z - M_W}{M_Z + M_W}. \quad (20)$$

Evaluating with precision collider measurements ($M_W = 80.377 \pm 0.012$ GeV and $M_Z = 91.1876 \pm 0.0021$ GeV [20]):

$$R_b = \frac{91.1876 - 80.377}{91.1876 + 80.377} \approx \mathbf{0.063011} \pm \mathbf{0.000069}, \quad (21)$$

within 0.12% of the analytic value $\alpha_q \approx 0.063084$. Using $M_W = M_Z \cos \theta_W$ and the half-angle identity give:

$$\alpha_q = \frac{1 - M_W/M_Z}{1 + M_W/M_Z} = \frac{1 - \cos \theta_W}{1 + \cos \theta_W} = \tan^2 \frac{\theta_W}{2}. \quad (22)$$

Thus $\alpha_q = \tan^2(\theta_W/2)$ is a directly measurable electroweak quantity, confirmed to 0.12% by precision data. The connection between the two fundamental electroweak mixing angles then follows from the mixing-equilibrium condition (14):

$$\tan^2 \frac{\theta_W}{2} \left(1 - \tan^2 \frac{\theta_W}{2} \right) \approx \mathcal{C}_q \sin^2 \theta_C, \quad (23)$$

where the approximation uses $\sin \theta_C \approx \lambda$ (to 0.04%). Neither of these angles is an input, and both are derived consequences of the same equilibrium condition.

2.2.4 The Equilibrium Retention-Weight ($1 - \alpha_q$) and CP Violation Phase

The mixing-equilibrium condition (14) gives two roots that sum to unity. The smaller root is equilibrium mixing-weight, $\alpha_q \approx 0.063084$ identifies with $\tan^2(\theta_W/2)$, confirmed to 0.12% by the measured ratio $R_b = 0.063011$ (Section 2.2.3). We now show that the larger root:

$$1 - \alpha_q \approx 0.936916 \quad (24)$$

also carries an equally precise physical identification.

In the standard Chau-Keung parametrisation of the CKM matrix, quark-sector CP violation is governed by the irreducible phase δ_{13} appearing in the heavy-light mixing element $V_{ub} = |V_{ub}| e^{-i\delta_{13}}$. The PDG global fit gives $\delta_{13} = 68.8^\circ \pm 4.5^\circ$ [20], so that:

$$\sin \delta_{13} = 0.932 \pm 0.015. \quad (25)$$

The complementary root (equilibrium retention-weight), $(1 - \alpha_q) = 0.936916$, is in agreement with $\sin \delta_{13}$ to **0.5%**, well within the 1σ experimental uncertainty of 0.015.

A possible interpretation of CP-violating phase projection: The quadratic constraint of the mixing-equilibrium condition $\alpha_q(1 - \alpha_q) = \mathcal{C}_q\lambda^2$, yields two roots, where the smaller root α_q is the mixing-weight satisfying the equilibrium condition that determines the electroweak mixing scale ($\tan^2(\theta_W/2)$), while the larger root ($1 - \alpha_q$) is the equilibrium retention-weight that determines the magnitude of the CP-violating phase projection. Because both roots emerge from a parameter-free quadratic relation, $\sin \delta_{13}$ is not an independent input parameter; rather, it is identically determined by the equilibrium retention-weight, yielding the explicit relation $\sin \delta_{13} = 1 - \alpha_q \approx 0.937$. This formulation is consistent with the measured value of 0.932 ± 0.015 within current experimental uncertainty, providing a probabilistic origin for quark-sector CP violation under the equilibrium condition.

3 Analytical Modelling and Formulation of the CKM Matrix

In Section 2, we defined the fermion mixing constant λ and the equilibrium mixing-weight α_q to describe flavour mixing across different generations. However, an operator is required to govern flavour mixing within the same generation between up-type and down-type quarks.

At tree level, flavour transitions are mediated exclusively by the charged-current weak interaction through the exchange of a $W^\pm$ gauge boson. Because the $W^\pm$ carries electric charge ± 1 , the corresponding charged-current vertex satisfies the selection rule $\Delta Q = \pm 1$. Consequently, a direct transition must connect an up-type quark (u, c, t with $Q = +2/3$) and a down-type quark (d, s, b with $Q = -1/3$) [24]. Direct tree-level transitions between quarks of the same charge are absent in the Standard Model, as flavour-changing neutral currents (FCNCs) are forbidden at tree level by the GIM mechanism [25].

Therefore, the weak isospin projection T_3 plays a crucial role in intra-generational flavour mixing. To formalise this description, we introduce the *weak-isospin activity operator* Γ_q , defined relative to the unit span ($|\Delta_3| = 1$) of the isospin interval. This operator quantifies the uncommitted isospin capacity available for a quark state to respond to W -boson exchange, evaluating to $1/2$ for up-type quarks and extending to $3/2$ for down-type quarks, with a mixing enhancement by α_q for the bottom quark. Building upon this foundation, we present the complete analytical construction of the CKM matrix elements in this section.

3.1 Flavour Mixing and Transition in Isospin Space

Based on the LRSM, each quark is characterised by two orthogonal projections on the isospin plane. The gauge-active component T_3 (coupling to the charged weak current) and the right-handed isospin projection T_{3R} (coupling only through the $(B - L)$ charge). These two components satisfy the Casimir invariant

$$T_3^2 + T_{3R}^2 = \frac{1}{4}, \quad (26)$$

placing every quark state on a circle of radius $\frac{1}{2}$ in the (T_3, T_{3R}) plane. The four first-generation quark configurations occupy the four cardinal points of this isospin circle, shown in Fig. 3: u_L at $T_3 = +\frac{1}{2}$, d_L at $T_3 = -\frac{1}{2}$, u_R at $T_{3R} = +\frac{1}{2}$, and d_R at $T_{3R} = -\frac{1}{2}$.

Charged-current transition as T_3 rotation. The $W^\pm$ boson couples exclusively to T_3 . An intra-generation flavour transition, say $u_L \rightarrow d_L$, is a pure rotation along the T_3 axis, so that T_{3R} remains frozen at zero while T_3 is driven from $+\frac{1}{2}$ to $-\frac{1}{2}$ along the horizontal diameter of the isospin circle (arrow in Fig. 3). This gives the physical behaviour of intra-generation flavour mixing: the quark changes identity by moving between the two T_3 endpoints of the isospin doublet. No inter-

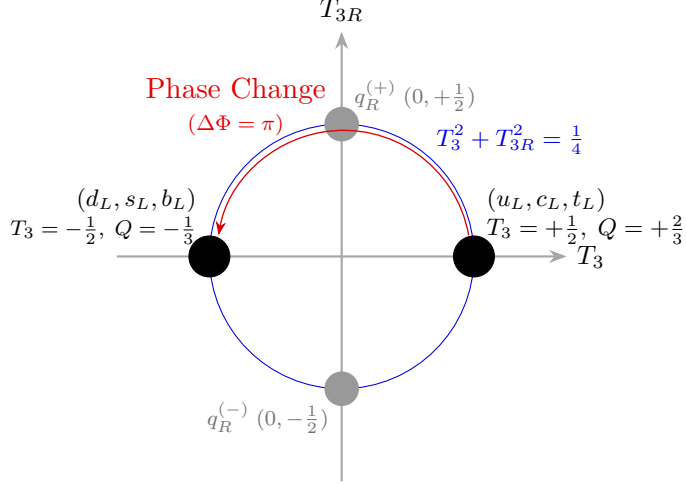


Figure 3: The states of all three quark generations on the invariant Casimir circle ($R = 1/2$). Flavour transitions proceed via phase rotation ($\Delta\Phi = \pi$) across the neutral point ($T_3^{(0)} = -1/6$).

generation mechanism is involved. The T_{3R} axis plays no role because the charged W boson carries no right-handed isospin.

T_3 position and down-type flavour affinity. The weak isospin projection T_3 plays a significant role in flavour transitions. It changes from $-\frac{1}{2}$ to $+\frac{1}{2}$ when a down-type state changes to an up-type state, yielding a net change of $\Delta T_3 = 1$, whereas a transition from an up-type to a down-type state results in a net change of $\Delta T_3 = -1$. Flavour transitions exhibit a greater affinity originating from down-type to up-type states associated with this net positive change in ΔT_3 . To quantify this behaviour such that a negative projection yields a higher numerical value, we define the weak-isospin enhancement operator as

$$\Gamma_q = 1 - T_3^L(q), \quad (27)$$

valid for all up-type quarks (u, c, t) and down-type quarks (d, s) of the first two generations. For the third-generation down-type quark (b) , it is defined as

$$\Gamma_b = 1 - T_3^L(b) + \alpha_q = \frac{3}{2} + \alpha_q. \quad (28)$$

Accordingly, up-type quarks (u, c, t) , which sit at $T_3^L = +\frac{1}{2}$, yield $\Gamma_{u,c,t} = \frac{1}{2}$. All down-type quarks (d, s, b) sit at $T_3^L = -\frac{1}{2}$, yielding an enhanced capacity of $\frac{3}{2}$. Specifically, for the first two generations of down-type quarks (d, s) we have $\Gamma_{d,s} = \frac{3}{2}$, while for the third-generation down-type quark (b) it becomes $\Gamma_b = \frac{3}{2} + \alpha_q$.

The flavour transition amplitude between quarks i and j at a W vertex is proportional to $\sqrt{\Gamma_i \Gamma_j}$, determined by these T_3 positions and higher-generation boundary parameters. This factor enters the mixing matrix elements across generations.

3.2 Conceptual Formulation

The CKM matrix is constructed as the Hadamard (element-wise) product of two matrices $\mathbf{G}$ and $\mathbf{S}$, followed by row normalisation:

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix} \circ \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}, \quad (29)$$

where the row and column indices refer to the specific states of the generations: $i = 1, 2, 3$ correspond to the up-type flavour states u, c, t respectively, and $j = 1, 2, 3$ correspond to the down-type flavour states d, s, b respectively.

- (i) **G** is the **isospin activity matrix**, which arises due to the uneven electroweak coupling observed from the weak-isospin availability operator $\hat{\Gamma} = \mathbb{I} - \hat{T}_{3L}$.
- (ii) **S** is the **inter-generational suppression matrix**, which describes the suppression of flavour mixing across the three generations of quarks.

The physical transition probability amplitude $|V_{ij}|$ is obtained by unitary row normalisation:

$$\mathcal{M}_{ij} = G_{ij} \cdot S_{ij}, \quad |V_{ij}| = \frac{\mathcal{M}_{ij}}{\mathcal{N}_i}, \quad \mathcal{N}_i = \sqrt{\sum_{j=1}^3 |\mathcal{M}_{ij}|^2}. \quad (30)$$

3.3 The Isospin Activity Matrix (G)

The isospin activity factor $\Gamma_q = 1 - T_{3L}(q)$ was defined in (27) in Section 3.1. For all the left-handed quarks, the operator form is $\hat{\Gamma} = \mathbb{I} - \hat{T}_{3L}$ except the bottom quark. Since $T_{3L} = +1/2$ for all up-type quarks (u, c, t), and $T_{3L} = -1/2$ for all down-type quarks (d, s, b), with the dynamical α_q -enhancement carried exclusively by the bottom-quark activity operator, we obtain:

$$\Gamma_u = \Gamma_c = \Gamma_t = \frac{1}{2}, \quad \Gamma_d = \Gamma_s = \frac{3}{2}, \quad \Gamma_b = \frac{3}{2} + \alpha_q. \quad (31)$$

Since $\alpha_q = 0.063084$ as derived in (17) of Section 2.2, we have $\Gamma_b = \frac{3}{2} + \alpha_q = 1.563084$. Using these values, we define the isospin activity matrix element $G_{ij} = \sqrt{\Gamma_{u_i} \Gamma_{d_j}}$, giving:

$$\begin{aligned} \mathbf{G} &= \begin{pmatrix} \sqrt{\Gamma_u \Gamma_d} & \sqrt{\Gamma_u \Gamma_s} & \sqrt{\Gamma_u \Gamma_b} \\ \sqrt{\Gamma_c \Gamma_d} & \sqrt{\Gamma_c \Gamma_s} & \sqrt{\Gamma_c \Gamma_b} \\ \sqrt{\Gamma_t \Gamma_d} & \sqrt{\Gamma_t \Gamma_s} & \sqrt{\Gamma_t \Gamma_b} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{3}/2 & \sqrt{3}/2 & \sqrt{(3+2\alpha_q)}/2 \\ \sqrt{3}/2 & \sqrt{3}/2 & \sqrt{(3+2\alpha_q)}/2 \\ \sqrt{3}/2 & \sqrt{3}/2 & \sqrt{(3+2\alpha_q)}/2 \end{pmatrix} \approx \begin{pmatrix} 0.866025 & 0.866025 & 0.884049 \\ 0.866025 & 0.866025 & 0.884049 \\ 0.866025 & 0.866025 & 0.884049 \end{pmatrix}. \end{aligned} \quad (32)$$

Because all up-type quarks share the same isospin coupling factor $\Gamma_u = \Gamma_c = \Gamma_t = 1/2$, all rows of **G** are identical. The sole differentiation across columns comes from the higher-generation mixing parameter α_q , which elevates the third-column entries from $\sqrt{3}/2$ to $\sqrt{(3+2\alpha_q)}/2$.

3.4 The Inter-Generational Flavour Mixing Matrix (S)

We now propose analytical expressions for each element of the inter-generational suppression matrix **S** in terms of λ and α_q , with each factor motivated by the physical process governing that transition.

3.4.1 Adjacent-Generation Mixing

(a) Mixing of Generations 1 and 2. The $1 \leftrightarrow 2$ mixing refers to the physically allowed charged-current transitions $u \leftrightarrow s$ and $c \leftrightarrow d$. In Section 2 we have defined the fermion mixing constant (λ) which characterises the suppression across two successive generations. To determine how this

scale enters the amplitude of an actual flavour transition, we construct a normalised flavour mixing amplitude. The amplitude complementary to λ is:

$$\lambda_{\perp} = \sqrt{1 - \lambda^2}, \quad (33)$$

satisfying the normalisation condition $\lambda^2 + \lambda_{\perp}^2 = 1$. A transition from one flavour to another is naturally specified relative to the amplitude of this complementary configuration. The corresponding transition strength is:

$$\frac{\lambda}{\lambda_{\perp}} = \frac{\lambda}{\sqrt{1 - \lambda^2}}. \quad (34)$$

We consequently define the transition amplitude associated with the elementary $1 \leftrightarrow 2$ flavour sector as:

$$S_{12} \equiv \lambda_{12} = \frac{\lambda}{\sqrt{1 - \lambda^2}}. \quad (35)$$

This relation is a direct geometric consequence of expressing the fundamental mixing scale λ relative to its normalised complementary component. It relates to the Cabibbo mixing angle θ_C since $\lambda = \sin \theta_C$ gives:

$$S_{12} = \tan \theta_C. \quad (36)$$

Thus, the fermion mixing constant λ acts as the baseline scale and specifies the underlying mixing mechanism, while the mixing amplitude S_{12} defines the strength relative to the unmixed component. For small mixing, a Taylor expansion reveals:

$$\lambda_{12} = \lambda + \frac{\lambda^3}{2} + \mathcal{O}(\lambda^5), \quad (37)$$

demonstrating that while the distinction is negligible at leading order, it becomes relevant when the full normalisation of the flavour configuration is retained. Assuming inter-generational suppression to be symmetric across the $1 \leftrightarrow 2$ sector, the bidirectional transition amplitudes are:

$$S_{12} = S_{21} = \lambda_{12} = \tan \theta_C = \frac{1/(\pi\sqrt{2})}{\sqrt{1 - 1/(2\pi^2)}} = \frac{1}{\sqrt{2\pi^2 - 1}} \approx 0.231007. \quad (38)$$

Therefore, the fermion mixing constant λ alone determines the amplitude for adjacent-generation transitions.

(b) Mixing of Generations 2 and 3. The physical charged-current manifestations of the $2 \leftrightarrow 3$ generation mixing refer specifically to $c \leftrightarrow b$ and $t \leftrightarrow s$. We propose the following symmetric closed form for the $2 \leftrightarrow 3$ suppression ($S_{23} = S_{32}$):

$$S_{23} = S_{32} = \sigma^2 \lambda (1 + \alpha_q) \approx 0.174418 \times 0.225079 \times 1.063084 \approx 0.041734, \quad (39)$$

where

$$\sigma = \sqrt{\lambda(1 - \lambda)}, \quad \sigma^2 = \lambda(1 - \lambda) \approx 0.174418. \quad (40)$$

The origin of the factors σ^2 , λ , and $(1 + \alpha_q)$ in (39) is as follows.

- i. The second generation acts as an intermediate state in the hierarchical cascade. Its participation in the primary $1 \leftrightarrow 2$ mixing channel limits the residual amplitude available for the subsequent $2 \leftrightarrow 3$ transition. The complementary weights of the participating and residual components are

λ and $(1 - \lambda)$ respectively. The geometric-mean amplitude, which accounts for the equilibrium participation of the intermediate second-generation state, is:

$$\sigma = \sqrt{\lambda(1 - \lambda)}, \quad \sigma^2 = \lambda(1 - \lambda) \approx 0.174418. \quad (41)$$

This factor is the Bernoulli variance $\sigma^2 = \lambda(1 - \lambda)$, which is the probability that a cross-generational step both occurs (with probability λ) and retains residual capacity for the next link (with probability $1 - \lambda$). It quantifies the intermediate-state probability of the second generation and should be understood as a mixing-suppression factor.

- ii. The fermion mixing constant λ represents the baseline scale, setting the scale of suppression across successive generations. The transition between the second and third generations therefore retains the same fundamental mixing scale.
- iii. The bottom quark, as the third-generation down-type state, carries an isospin activity factor $\Gamma_b = \frac{3}{2} + \alpha_q$ relative to $\Gamma_c = \frac{1}{2}$ for the up-type charm quark (31). The effective coupling interval for the $2 \leftrightarrow 3$ transition is therefore $\Gamma_b - \Gamma_c = (1 + \alpha_q)$ relative to the unit interval of the $1 \leftrightarrow 2$ transition. It is thus believed that the equilibrium mixing-weight α_q enters the $2 \leftrightarrow 3$ transition as an enhancement factor $(1 + \alpha_q)$.

Thus, the $2 \leftrightarrow 3$ mixing combines three factors: the intermediate-state probability factor σ^2 , the fermion mixing constant λ modified by the $2 \leftrightarrow 3$ enhancement by a factor of $(1 + \alpha_q)$.

3.4.2 Skip-Generation Flavour Mixing

The skip-generation flavour mixing refers to transitions between the first and third generations, such as $u \leftrightarrow b$ and $t \leftrightarrow d$. These transitions are governed by the CKM matrix elements V_{ub} and V_{td} , respectively, which are both of order λ^3 in the Wolfenstein parametrisation. Because the masses of the first- and third-generation quarks differ by several orders of magnitude, the corresponding mixing amplitudes are not described by the same suppression scale used for adjacent-generation transitions. Instead, non-adjacent transitions are effectively mediated through the intermediate second-generation flavour states s and c .

(a) Mixing of u and b : Formulation of S_{13} . The $u \leftrightarrow b$ mixing is modelled by hierarchical mixing in which the $s \leftrightarrow c$ link connects the $u \leftrightarrow s$ and $c \leftrightarrow b$ transitions, giving a four-state mixed process with three connected flavour links:

$$u \leftrightarrow s, \quad s \leftrightarrow c, \quad c \leftrightarrow b. \quad (42)$$

The $u \leftrightarrow s$ and $c \leftrightarrow b$ links connect adjacent generations and therefore carry the fundamental mixing scale λ . The intermediate $s \leftrightarrow c$ link, which connects the pair of flavours within the second generation, provides the connecting intermediate states linking the u quark flavour in generation 1 and the b quark flavour in generation 3:

$$u \longleftrightarrow s \longleftrightarrow c \longleftrightarrow b. \quad (43)$$

The intermediate $s \leftrightarrow c$ link carries the three-channel factor associated with the three connected flavour directions across the three generations. Its effective mixing factor is:

$$\lambda_{sc} = \frac{\lambda}{3}. \quad (44)$$

The effective $1 \leftrightarrow 3$ suppression factor is consequently given by the product of the three constituent flavour-link factors in (43):

$$S_{13} = \lambda_{us} \cdot \lambda_{sc} \cdot \lambda_{cb}. \quad (45)$$

With $\lambda_{us} = \lambda$, $\lambda_{sc} = \lambda/3$, $\lambda_{cb} = \lambda$, we obtain:

$$S_{13} = \lambda \left(\frac{\lambda}{3} \right) \lambda = \frac{\lambda^3}{3} \approx 0.003801. \quad (46)$$

The factor $\lambda/3$ provides the additional suppression from the intermediate second-generation link in the $1 \leftrightarrow 3$ cascade.

(b) Mixing of t and d : Formulation of S_{31} . The $t \leftrightarrow d$ mixing is similar to $u \leftrightarrow b$ mixing in that both are skip-generation. It is modelled through the chain:

$$t \longleftrightarrow b \longleftrightarrow c \longleftrightarrow d, \quad (47)$$

with three flavour mixing processes among four flavour states:

$$t \leftrightarrow b, \quad b \leftrightarrow c, \quad c \leftrightarrow d. \quad (48)$$

The $t \leftrightarrow d$ mixing is, however, found to be less suppressed relative to $u \leftrightarrow b$ mixing. The asymmetry arises because the top quark resides at the charge-and-mass saturated boundary of the third generation, at which the mass hierarchy terminates, and does not impose an additional factor of $1/3$ on the intermediate $s \leftrightarrow c$ amplitude. Instead, the available amplitude for the $3 \leftrightarrow 1$ skip is governed by a cascade of the S_{32} channel, the S_{21} factor for the residual $c \leftrightarrow d$ transition, and the factor $(1 - \alpha_q)$ reflecting the $t \leftrightarrow d$ transition. Therefore we can have:

$$S_{31} = (1 - \alpha_q) \cdot S_{32} \cdot \lambda. \quad (49)$$

where $(1 - \alpha_q)$ suppression appears for the $t \leftrightarrow b$ mixing, S_{32} for the $b \leftrightarrow c$ mixing, and λ for the $c \leftrightarrow d$ mixing. Substituting $S_{32} = 0.041734$, $\lambda = 0.225079$, and $1 - \alpha_q = 0.936916$:

$$S_{31} = 0.041734 \times 0.225079 \times 0.936916 \approx 0.008801.$$

The asymmetry $S_{31} \neq S_{13}$, despite both being skip-generation transitions, reflects the distinct mixing modes of the quarks and mixing pathway.

3.5 Analytical Form of the Inter-Generational Suppression Matrix

The complete inter-generational suppression matrix $\mathbf{S}$ can be expressed in closed analytical form in terms of the two geometric invariants λ and α_q alone as:

$$\mathbf{S} = \begin{pmatrix} 1 & \frac{\lambda}{\sqrt{1 - \lambda^2}} & \frac{\lambda^3}{3} \\ \frac{\lambda}{\sqrt{1 - \lambda^2}} & 1 & \lambda^2(1 - \lambda)(1 + \alpha_q) \\ \lambda^3(1 - \lambda)(1 - \alpha_q^2) & \lambda^2(1 - \lambda)(1 + \alpha_q) & 1 \end{pmatrix}. \quad (50)$$

The elements of the $\mathbf{S}$ matrix in the $(1, 2)$ sector are $\mathcal{O}(\lambda)$, the $(2, 3)$ sector are $\mathcal{O}(\lambda^2)$, and both skip-generation elements are $\mathcal{O}(\lambda^3)$. The distinct mixing pathways governing the $u \leftrightarrow b$ and $t \leftrightarrow d$ skip-generation transitions give rise to the only asymmetry observed between the off-diagonal corner elements, with $S_{31} = \lambda^3(1 - \lambda)(1 - \alpha_q^2)$ and $S_{13} = \lambda^3/3$. It is interesting to note that the matrix is constructed using effectively only the fermion mixing constant $\lambda = 1/(\sqrt{2}\pi)$ since α_q is derived from it.

3.6 Construction of the CKM Matrix: Numerical Results and Comparison

Combining the intra-generational baseline mixing matrix $\mathbf{G}$ and the inter-generational suppression matrix $\mathbf{S}$ via the Hadamard product (29), the pre-normalised amplitude matrix $\mathbf{M} = \mathbf{G} \circ \mathbf{S}$ is:

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} 0.866025 & 0.866025 & 0.884049 \\ 0.866025 & 0.866025 & 0.884049 \\ 0.866025 & 0.866025 & 0.884049 \end{pmatrix} \circ \begin{pmatrix} 1.000000 & 0.231007 & 0.003801 \\ 0.231007 & 1.000000 & 0.041734 \\ 0.008801 & 0.041734 & 1.000000 \end{pmatrix} \\ &= \begin{pmatrix} 0.866025 & 0.200058 & 0.003360 \\ 0.200058 & 0.866025 & 0.036895 \\ 0.007622 & 0.036143 & 0.884049 \end{pmatrix}. \end{aligned} \quad (51)$$

Note the asymmetry between $M_{23} = 0.036895$ and $M_{32} = 0.036143$. The former involves the bottom-quark column entry $G_{23} = \sqrt{\Gamma_c \Gamma_b} = 0.884049$, while the latter involves the strange-quark column entry $G_{32} = \sqrt{\Gamma_t \Gamma_s} = 0.866025$, reflecting the non-symmetry of the $\mathbf{G}$ matrix.

Applying row-unitarity normalisation (30), the complete theoretical CKM matrix is:

$$|V_{\text{CKM}}^{\text{formulation}}| = \begin{pmatrix} 0.974334 & 0.225077 & 0.003780 \\ 0.224885 & 0.973502 & 0.041474 \\ 0.008614 & 0.040848 & 0.999128 \end{pmatrix}. \quad (52)$$

To evaluate the predictive power of this parameter-free framework, we compare against the 2024 PDG world averages [20]. The statistical pull for each element is defined as:

$$\text{Pull}_{ij} = \frac{|V_{ij}^{\text{theory}}| - |V_{ij}^{\text{PDG}}|}{\sigma_{\text{PDG},ij}}. \quad (53)$$

Table 2 presents the element-by-element comparison of the values computed using the proposed analytical expression and PDG world averages [20]. All nine CKM matrix elements achieve sub-sigma agreement with the experimental data. The maximum deviation is 0.59σ at $|V_{ub}|$, with a mean absolute pull of 0.22σ and an RMS pull of 0.28σ .

Table 2: Comparison of the computed CKM matrix elements against 2024 PDG world averages [20].

Element	Theoretical	PDG value	σ_{PDG}	Pull (σ)
$ V_{ud} $	0.974334	0.97435	0.00014	-0.12
$ V_{us} $	0.225077	0.22500	0.00067	+0.12
$ V_{ub} $	0.003780	0.003732	0.000082	+0.59
$ V_{cd} $	0.224885	0.22486	0.00067	+0.04
$ V_{cs} $	0.973502	0.97349	0.00014	+0.09
$ V_{cb} $	0.041474	0.04182	0.00085	-0.41
$ V_{td} $	0.008614	0.00857	0.00020	+0.22
$ V_{ts} $	0.040848	0.04110	0.00083	-0.30
$ V_{tb} $	0.999128	0.99912	0.00013	+0.06
Maximum Pull				0.59
Mean Pull				0.22
RMS Pull				0.28

The remarkable agreement with PDG data, achieved without free parameters or numerical fitting, provides compelling support for the $\mathbf{G} \circ \mathbf{S}$ framework, strongly indicating that

the proposed construction captures the fundamental mechanism of quark flavour mixing. Furthermore, replacing the analytic value $\alpha_q = 0.063084$ with the directly measured ratio $(M_Z - M_W)/(M_Z + M_W) = 0.063012$ changes the maximum pull by less than 0.001σ , confirming that the model's predictions are remarkably robust against this 0.12% variation.

4 Comparison with the Wolfenstein Parametrisation

The Wolfenstein parametrisation [3, 26] expresses the CKM matrix using four parameters λ_W , A , ρ , η fitted to experimental data to derive the CKM matrix elements:

$$V_{\text{CKM}}^{\text{Wolf}} \simeq \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \quad (54)$$

4.1 Evaluation and Comparison with Proposed Derived Matrix

Using the 2024 PDG values $\lambda_W = 0.22500$, $A = 0.826$, $\bar{\rho} = 0.159$, $\bar{\eta} = 0.348$ in matrix (54) produces:

$$|V_{\text{CKM}}^{\text{Wolf}}| \simeq \begin{pmatrix} 0.97469 & 0.22500 & 0.003600 \\ 0.22500 & 0.97469 & 0.04182 \\ 0.008562 & 0.04182 & 1.000000 \end{pmatrix}. \quad (55)$$

Table 3: Comparison between the proposed computed values and the Wolfenstein leading-order expansion against PDG 2024 values [20]. Pull = (theoretical - $|V^{\text{PDG}}|$)/ σ_{PDG} .

Element	PDG value	σ	Analytic	$\Delta_{\text{Analytic}}/\sigma$	Wolfenstein	$\Delta_{\text{Wolf}}/\sigma$
$ V_{ud} $	0.97435	0.00014	0.974334	-0.12	0.97469	+2.43
$ V_{us} $	0.22500	0.00067	0.225077	+0.12	0.22500	0.00
$ V_{ub} $	0.003732	0.000082	0.003780	+0.59	0.003600	-1.61
$ V_{cd} $	0.22486	0.00067	0.224885	+0.04	0.22500	+0.21
$ V_{cs} $	0.97349	0.00014	0.973502	+0.09	0.97469	+8.57
$ V_{cb} $	0.04182	0.00085	0.041474	-0.41	0.04182	0.00
$ V_{td} $	0.00857	0.00020	0.008614	+0.22	0.008562	-0.04
$ V_{ts} $	0.04110	0.00083	0.040848	-0.30	0.04182	+0.87
$ V_{tb} $	0.99912	0.00013	0.999128	+0.06	1.00000	+6.77
Maximum $ \Delta/\sigma $				0.59		8.57
Mean $ \Delta/\sigma $				0.22		2.23
Free parameters				0 (derived)		4 (fitted)

An element-by-element pull comparison between the values computed from the proposed analytic expressions and the Wolfenstein leading-order expansion against PDG 2024 values [20] is presented in Table 3. The four-parameter Wolfenstein expansion exhibits deviations of pull as high as 8.57σ and a mean pull of 2.23σ . The proposed parameter-free formulation constrains all element pulls to a maximum of 0.59σ and a mean of 0.22σ . This quantitative comparison of experimental data provides compelling support for the proposed formulation.

4.2 Differences Between Proposed Formulation and Wolfenstein Construction

At first glance, the proposed matrix resembles the Wolfenstein parametrisation: both organise the CKM hierarchy as a systematic expansion in the Cabibbo scale λ , with off-diagonal elements suppressed by increasing powers of λ across successive generation gaps. The structural similarity reflects a deeper connection, since the Wolfenstein expansion is an algebraic projection of the same underlying isospin geometry. The decisive difference is that the proposed formulation contains *no free parameters*, as every matrix element is fixed by $\lambda = 1/(\pi\sqrt{2})$ alone, since the equilibrium mixing-weight $\alpha_q \approx 0.063084$ is not an independent input but is uniquely determined by the *mixing-equilibrium condition* $\alpha_q(1 - \alpha_q) = \mathcal{C}_q\lambda^2$. In contrast, the Wolfenstein parametrisation requires four fitted parameters $(\lambda_W, A, \bar{\rho}, \bar{\eta})$ that carry no derived relation to one another. The geometric construction differs from the Wolfenstein parametrisation in structure, not merely in parameter count; the key distinctions are as follows:

- (i) **Adjacent-generation amplitude:** In the proposed formulation, the elements of the adjacent-generation suppression amplitude are

$$S_{12} = S_{21} = \lambda_{12} = \frac{\lambda}{\sqrt{1 - \lambda^2}} \approx 0.231007, \quad (56)$$

which are close to the Cabibbo *tangent*, whereas the standard expansion uses $\lambda = \sin \theta_C \approx 0.22500$. After row normalisation, the physical matrix elements $|V_{us}|$ and $|V_{cd}|$ come out consistent with λ , but the pre-normalised mixing amplitude is λ_{12} , a distinction invisible to the Wolfenstein expansion.

- (ii) **Distinct diagonal elements from row normalisation:** The Wolfenstein leading order gives a single value $1 - \lambda^2/2 = 0.97469$ for both $|V_{ud}|$ and $|V_{cs}|$, and $|V_{tb}| = 1$ exactly. The proposed analytic approach after row normalisation produces three distinct diagonal values: $|V_{ud}| = 0.974334$, $|V_{cs}| = 0.973502$, $|V_{tb}| = 0.999128$, in agreement with experiment. The difference $|V_{ud}| - |V_{cs}| = 8.3 \times 10^{-4}$ is reproduced because $S_{12} \neq S_{23}$, meaning the adjacent-generation amplitudes differ and the unitarity budget is distributed differently in each row.

- (iii) **Closed analytical expressions for all nine elements:** Every element of the geometric matrix has an explicit closed form in terms of λ and α_q alone. For example:

$$|V_{us}| = \frac{\lambda_{12} G_{12}}{N_1}, \quad |V_{cb}| = \frac{S_{23} G_{23}}{N_2} = \frac{\lambda^2(1 - \lambda)(1 + \alpha_q) G_{23}}{N_2}, \quad (57)$$

$$|V_{ub}| = \frac{S_{13} G_{13}}{N_1} = \frac{(\lambda^3/3) G_{13}}{N_1}, \quad |V_{td}| = \frac{S_{31} G_{31}}{N_3} = \frac{\lambda^3(1 - \lambda)(1 - \alpha_q^2) G_{31}}{N_3}. \quad (58)$$

No element is left as a free combination. Inserting $\lambda = 1/(\pi\sqrt{2})$ and $\alpha_q = 0.063084$ yields the full matrix numerically, with zero adjustable inputs.

- (iv) **Parameter count, interdependence, and predictive power:** The present construction uses two constants, λ and α_q , both derived from the gauge group and confirmed by independent data. Crucially, they are not independent; the relation $\alpha_q(1 - \alpha_q) = \mathcal{C}_q\lambda^2$ ties them so that either one determines the other. The complete nine-element CKM matrix is therefore expressible as a function of a *single sector-invariant universal constant* $\lambda = 1/(\pi\sqrt{2})$. The Wolfenstein parametrisation uses four parameters: $\lambda_W \equiv |V_{us}|$ by definition, $A \equiv |V_{cb}|/\lambda_W^2$ by definition, and $\bar{\rho}, \bar{\eta}$ fitted to B -meson observables. Each parameter absorbs exactly one class

of observable, making the parametrisation self-consistently descriptive but not predictive, with no derived relation between the parameters.

- (v) **Precision comparison:** The Wolfenstein leading order gives $|V_{tb}| = 1$ exactly, deviating from the PDG value 0.99912 ± 0.00013 by 6.8σ . In the proposed approach, row normalisation distributes the unitarity budget correctly, giving $|V_{tb}|_{\text{geom}} = 0.999128$ with a pull of $+0.06\sigma$.

4.3 The Wolfenstein Parametrisation as an Algebraic Bridge

The Wolfenstein parametrisation is a compact algebraic form that organises the empirical CKM hierarchy into a power series in λ , with four fitted parameters making the generation-gap suppression pattern visually transparent. Its four parameters are tunable, with each adjusted freely to match experimental data, so the framework reproduces observations by construction rather than by prediction. Specifically, λ is defined as $|V_{us}|$, A is defined as $|V_{cb}|/\lambda^2$, and $\bar{\rho}$, $\bar{\eta}$ are fitted to B -meson and kaon observables. No element of the CKM matrix is *predicted* by the Wolfenstein parametrisation; every parameter absorbs exactly the observable it is used to describe. The parametrisation makes the hierarchy manifest but offers no explanation of why $\lambda \approx 0.225$, why suppression grows as successive powers of this particular value, or why the two diagonal blocks differ. It only indicates that the elements of the CKM matrix are of specific orders of λ .

The proposed formulation effectively uses only one constant $\lambda = 1/(\pi\sqrt{2})$, which is nearly equal to $\sin\theta_C$, while α_q is expressed in terms of λ and quark coupling factor $\mathcal{C}_q$. It explains the significance of each constant and provides a physically reasonable explanation for each element of the matrices based on the underlying flavour-mixing mechanism. The sub-sigma accuracy of each CKM element calculated from the proposed analytical expression provides compelling support for the proposed flavour-mixing model.

4.4 Determination of Wolfenstein Parameters

We can find all four Wolfenstein parameters without fitting. The scale λ_W is identified as the Haar-measure amplitude $\lambda = 1/(\pi\sqrt{2})$, as established in Section 2. The remaining three parameters follow directly from the analytic construction.

Amplitude parameter A . The Wolfenstein amplitude $A \equiv |V_{cb}|/\lambda_W^2$ is determined by the row-2 structure of the geometric matrix. Expressed through the isospin-activity weight G_{23} and the row-2 Euclidean normalisation $N_2 = \sqrt{\sum_j (G_{2j}S_{2j})^2}$:

$$A = \frac{(1 - \lambda)(1 + \alpha_q) G_{23}}{N_2} \approx 0.819, \quad (59)$$

against the PDG global fit $A_{\text{PDG}} = 0.826 \pm 0.012$, a deviation of 0.58σ .

Unitarity Triangle apex $(\bar{\rho}, \bar{\eta})$. Section 2.2.4 established that $(1 - \alpha_q)$ the equilibrium retention-weight satisfies $\sin\delta_{13} = 1 - \alpha_q \approx 0.9369$, in agreement with the PDG value 0.932 ± 0.015 at 0.33σ . This immediately yields the unitarity triangle apex angle:

$$\frac{\bar{\eta}}{\bar{\rho}} = \tan\delta_{13} = \frac{1 - \alpha_q}{\sqrt{2\alpha_q - \alpha_q^2}} \approx 2.680. \quad (60)$$

Combined with the predicted $|V_{ub}| = 0.003780$ (Table 3), which fixes the apex radius $\sqrt{\bar{\rho}^2 + \bar{\eta}^2} = |V_{ub}|/(A\lambda^3) \approx 0.409$, the apex coordinates are uniquely determined:

$$\bar{\rho} \approx 0.143, \quad \bar{\eta} \approx 0.379. \quad (61)$$

The PDG direct measurement $\delta_{13} = 68.8^\circ \pm 4.5^\circ$ [20] implies $\tan \delta_{13} = 2.578 \pm 0.643$, placing the predicted ratio 2.680 at 0.16σ . Propagating this uncertainty through Eq. (61) gives $\bar{\rho} = 0.146 \pm 0.088$ and $\bar{\eta} = 0.377 \pm 0.088$ from the direct δ_{13} measurement alone; Eq. (61) deviates from these by less than 0.02σ . The Wolfenstein parameters are therefore reproduced by the proposed analytic model without free parameters, and at the level of precision warranted by direct measurements of δ_{13} . The tighter constraints on $\bar{\rho}$ and $\bar{\eta}$ from the PDG global CKM fit ($\bar{\rho} = 0.159 \pm 0.010$, $\bar{\eta} = 0.348 \pm 0.010$) incorporate B -meson mixing and rare-decay observables beyond the scope of the present $SU(2)_L$ construction; those constraints will be addressed when the $SU(2)_R$ sector is incorporated.

4.5 Unitarity Triangle Angle and the Jarlskog Invariant

The equilibrium mixing-weight α_q directly determines the CP-violating phase scale through the relation $\sin \gamma = 1 - \alpha_q$. Using the evaluated quark equilibrium root $\alpha_q \approx 0.0631$, this yields:

$$\sin \gamma = 1 - \alpha_q \approx 0.9369, \quad (62)$$

corresponding to an effective phase angle of $\gamma \approx 69.54^\circ$.

The Jarlskog CP-violation invariant J_{CP}^q quantifies the area of the unitarity triangle [27]. Incorporating the geometric amplitudes $|V_{us}| \approx \lambda$, $|V_{cb}| \approx \lambda^2(1 - \lambda)(1 + \alpha_q)G_{23}/N_2$, and $|V_{ub}| \approx \lambda^3/3$ (representing equal-weight routing across $N_g = 3$ generations), together with the dynamically derived phase factor $\sin \gamma = 1 - \alpha_q$:

$$J_{\text{CP}}^q = c_{12}^2 c_{13}^2 c_{23} s_{12} s_{13} s_{23} \sin \gamma \approx 3.22 \times 10^{-5}, \quad (63)$$

where $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$ are extracted from the row-unitarised CKM matrix elements, and $\sin \gamma = 1 - \alpha_q$. Comparing this prediction against the PDG global fit $J_{\text{CP}}^q = (3.12 \pm 0.12) \times 10^{-5}$ [20] yields agreement of 0.82σ without free parameters.

5 Summary and Conclusions

We have presented closed analytical expressions for all nine elements of the CKM quark mixing matrix within the LRSM model via a *mixing-equilibrium condition* postulated as

$$\alpha_q(1 - \alpha_q) = \mathcal{C}_q \lambda^2,$$

where $0 < \alpha_q < 1 - \alpha_q$ represents the *equilibrium mixing-weight* of the quark sector, $\lambda^2 = 1/(2\pi^2)$ is the Haar volume measure of $SU(2)_L$, and $\mathcal{C}_q = 1 + (B - L)_q/2$ is the *quark-sector coupling factor*, with $(B - L)_q$ denoting the gauge charge of the quark sector. We have used $\lambda = 1/(\pi\sqrt{2})$ as the fermion mixing constant, which is virtually identical to the mixing scale $\lambda = \sin \theta_C$ used in the Wolfenstein parameterisation. We also identify α_q with $\tan^2(\theta_W/2)$, a relation confirmed independently by precision electroweak data.

The CKM matrix elements are expressed in terms of λ and α_q . All nine elements agree with the 2024 PDG world averages [20] at the sub-sigma level, with maximum pull 0.59σ at $|V_{ub}|$, mean absolute pull 0.22σ , and RMS pull 0.28σ , with zero free parameters.

Replacing the analytic value $\alpha_q = 0.063084$ with the directly measured ratio $(M_Z - M_W)/(M_Z + M_W) = 0.063012$ changes the maximum pull by less than 0.001σ , confirming that the results are insensitive to the 0.12% uncertainty in α_q . The larger root of the mixing-equilibrium condition $\alpha_q(1 - \alpha_q) = \mathcal{C}_q\lambda^2$ yields the sine projection of the CKM CP-violating phase, $\sin\delta_{13} = 0.932 \pm 0.015$, to within 0.5% (0.17 σ). This result indicates that both roots of the foundational equation play a significant role in flavour mixing and CP violation.

Compared with the Wolfenstein leading-order expansion, which uses four fitted parameters but yields large deviations on specific elements (such as $|V_{cs}|$ at 8.6σ and $|V_{tb}|$ at 6.8σ) due to diagonal truncation, the matrix construction achieves substantially better internal consistency because row unitarization resums all-order diagonal corrections automatically.

Moreover, we have predicted all four Wolfenstein parameters without fitting, together with two further observables of the CKM unitarity triangle. The CP-violating phase angle is natively determined by the equilibrium retention-weight via $\sin\gamma = 1 - \alpha_q \approx 0.9369$, yielding $\gamma \approx 69.54^\circ$. The resulting Jarlskog CP-violation invariant evaluates to $J_{\text{CP}}^q \approx 3.22 \times 10^{-5}$, achieving agreement of 0.82σ with the PDG global fit, without free parameters.

The mixing scale λ_W is identified as the Haar-measure amplitude $\lambda = 1/(\pi\sqrt{2})$, and the amplitude parameter A is obtained from the row-2 as $A \approx 0.819$ (PDG: 0.826 ± 0.012 ; agreement within 1%). By identifying $(1 - \alpha_q) = \sin\delta_{13}$ as the sine projection of the CP-violating phase, we determine the unitarity triangle apex ratio $\bar{\eta}/\bar{\rho} = \tan\delta_{13} \approx 2.680$. Combined with the predicted $|V_{ub}|$, this yields the apex coordinates:

$$\bar{\rho} \approx 0.143, \quad \bar{\eta} \approx 0.379 \quad (64)$$

with zero free parameters. These tree-level predictions show excellent agreement with empirical benchmarks, while subsequent incorporation of the $\text{SU}(2)_R$ sector will naturally handle higher-order loop constraints and rare-decay observables to refine global fit boundaries.

A deeper implication points toward quark-lepton unification. The universal constant λ governs the lepton sector through the analogous equilibrium condition

$$\alpha_\ell(1 - \alpha_\ell) = \mathcal{C}_\ell\lambda^2,$$

where $\mathcal{C}_\ell = \frac{1}{2}$, mirroring the quark-sector relation $\alpha_q(1 - \alpha_q) = \mathcal{C}_q\lambda^2$. Consequently, λ operates across both sectors as a fundamental property of fermions. The companion formulation of the PMNS lepton mixing matrix [28] derives all three mixing angles, the Dirac CP phase, and the leptonic Jarlskog invariant from this shared invariant. Together, these works provide analytic expressions for all quark and lepton mixing parameters, including nine CKM elements and four independent PMNS observables using the single universal constant $\lambda = 1/(\pi\sqrt{2})$, without free parameters.

An analytic model or resulting closed-form expressions like the one presented here gain true utility only when they provide insight into the underlying physical mechanism. The parameter-free formulation presented in this work meets this standard by offering direct insight into the quark flavour mixing mechanism

Declarations

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