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Research Article

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Posted Date: October 15th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-10948914/v1>

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Additional Declarations: No competing interests reported.

Research

Feasible-Space Dilution and Objective Alignment in Shallow Penalty-X QAOA: A Controlled RCSP Study

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Abstract

Purpose Poor feasible sampling in full-space Penalty-X quantum approximate optimization algorithm (QAOA) can reflect representation-induced dilution, classical optimizer inadequacy, objective misalignment, or limited ansatz capacity. We separate these explanations on a controlled routing benchmark.

Methods Exact-statevector experiments use 140 resource-constrained shortest-path tasks and a scale-controlled Hamiltonian. Exact depth embeddings diagnose optimizer inadequacy. A frozen discovery/held-out design compares mean-energy training (O0), an exact-feasibility capacity control (O2), and conditional value at risk (CVaR)-0.10 (O3), with post-hoc depth, optimizer, tail-fraction, and finite-shot checks.

Results O2 showed that the fixed ansatz could reach feasible mass not induced by O0. On 84 held-out tasks from 15 graphs, O3 improved graph-level log-feasibility gain over O0 by **+0.3547** decades and was non-inferior to O2: the O3–O2 effect was **−0.0086**, with lower bound **−0.0360** above the frozen **−0.10** margin. Certified nested failures occurred under COBYLA, Nelder–Mead, and SLSQP. Fixed-endpoint sampling

was stable, whereas both finite-shot retraining intervals crossed zero.

Conclusion Optimizer inadequacy and objective misalignment are distinct under the tested diagnostics, and CVaR approaches the mechanistic O2 capacity control on held-out graphs. The classically tractable study supports no quantum-advantage or extrapolative scaling claim and does not test hardware noise.

Keywords: quantum approximate optimization algorithm, constrained optimization, resource-constrained shortest path, feasible-space dilution, conditional value at risk, variational quantum algorithms

1 Introduction

Full-space Penalty- X QAOA starts from the uniform state over all 2^m edge-bit strings and uses a transverse- X mixer that does not preserve feasibility. In constrained routing, feasible routes may occupy only a minute part of that representation. Poor feasible sampling can then reflect several different mechanisms: representation-induced dilution, an uncontrolled penalty scale, classical optimizer inadequacy, a training loss that is misaligned with feasible entry, or insufficient ansatz capacity. An endpoint probability alone does not identify which mechanism is responsible.

We separate these explanations on a controlled resource-constrained shortest- path (RCSP) benchmark. The primary study uses exact CPU statevectors, a uniform $|+\rangle$ initial state, a diagonal scale-controlled penalty Hamiltonian, the transverse- X mixer, and alternating-layer count $p \leq 3$; a post-hoc robustness subset reaches $p = 4$. Here “shallow” refers to p , not compiled hardware depth. The benchmark has 140 distinct-cardinality tasks on 25 synthetic layered directed acyclic graphs and spans raw edge-bit feasible fractions from 1.91×10^{-6} to 2.34×10^{-2} .

The central comparison is the sequence $O0 \rightarrow O2 \rightarrow O3$. $O0$ minimizes mean Hamiltonian energy. $O2$ directly maximizes exact feasible mass and is used only as a mechanistic statevector capacity control: $O2-O0$ asks whether the same ansatz can reach feasibility that $O0$ does not induce.

O3 minimizes lower-tail CVaR without changing the quantum Hamiltonian or ansatz; O3–O2 asks whether an energy-based loss approaches that measured capacity. A graph-disjoint discovery/held-out design freezes CVaR-0.10 before the two held-out tests, including H2 non-inferiority to O2. O2 is not proposed as a deployable objective.

Exact ansatz nesting provides a separate optimizer diagnostic. Adding a zero-angle layer embeds every $p = 2$ state in the $p = 3$ family, so a worse reported deeper objective certifies optimizer inadequacy for that run. This certificate does not establish that passing runs are global optima, nor does it make objective alignment automatic. Post-hoc checks extend the diagnostic across COBYLA, Nelder–Mead, SLSQP, $p = 4$, tested evaluation budgets, CVaR tail fractions, and finite-shot estimation and training. They qualify the frozen study rather than enlarge its confirmatory family.

The paper makes four contributions:

- (C1) a controlled RCSP dilution benchmark and Hamiltonian scale contract that preserve every exact ground state;
- (C2) nested capacity/optimizer diagnosis, with robustness across tested optimizers and depth;
- (C3) objective attribution through the O0–O2–O3 sequence and frozen held-out confirmation of H1 and H2; and
- (C4) explicit robustness boundaries for tail fraction, finite shots, depth–budget interaction, high-size reversal, and classical tractability.

The exact label-setting solver reproduces all 140 frozen optima, so the tasks are classically tractable mechanism probes, not evidence of quantum advantage. Online Resource 1 contains the complete protocols, numerical matrices, theorem statements and proofs, and reproducibility provenance.

2 Problem and Estimands

Let $G = (V, E)$ be a directed graph with source s , target t , positive edge costs c_e , nonnegative resource consumptions r_e , and budget B . The single-resource RCSP is

$$\min_P \sum_{e \in P} c_e \quad \text{subject to} \quad \sum_{e \in P} r_e \leq B, \quad (1)$$

over directed s - t paths P [1]. The controlled instances are synthetic layered directed acyclic graphs.

Each edge has a bit x_e . With signed incidence coefficients a_{ve} and $d_s = 1, d_t = -1, d_v = 0$ otherwise, feasibility requires

$$\sum_e a_{ve} x_e = d_v \quad (v \in V), \quad \sum_e r_e x_e \leq B. \quad (2)$$

Selections that violate flow or resource constraints remain computational-basis states. If $m = |E|$ and M_{state} strings are feasible, define

$$\phi_{\text{state}} = \frac{M_{\text{state}}}{2^m}, \quad D = -\log_{10} \phi_{\text{state}}, \quad G_{\text{feas}}(\theta) = \log_{10} \frac{P_{\text{feas}}(\theta)}{\phi_{\text{state}}}. \quad (3)$$

Thus D measures dilution in this edge-bit representation and G_{feas} measures feasible-mass gain over its uniform starting state. The path-domain fraction M/K among K candidate routes generally differs from ϕ_{state} because most bit strings are not routes.

The normalized cost Hamiltonian H_C is diagonal. At alternating-layer count p , Penalty-X uses

$$|\psi_p(\gamma, \beta)\rangle = \prod_{\ell=1}^p e^{-i\beta_\ell \sum_e X_e} e^{-i\gamma_\ell H_C} |+\rangle^{\otimes m}. \quad (4)$$

Feasibility is encoded only through H_C ; neither the initial state nor the transverse- X mixer preserves it.

The main objective chain applies three classical losses to the same circuit and H_C . O0 is mean energy, $L_{O0} = \langle H_C \rangle$. O2 is $L_{O2} = 1 - P_{\text{feas}}$ and directly measures the tested statevector ansatz's feasible-concentration capacity. O3 is exact lower-tail CVaR_{0.10}(H_C): energy-probability pairs are sorted, the lowest-energy probability mass 0.10 is averaged, and only the required fraction of the cutoff mass is included. CVaR is established in variational optimization [2]; its use here is evaluated, not claimed as new. A secondary expected-penalty loss O1 and the complete objective definitions are reported in Online Resource 1, Sec. S3.

For feasible strings $\mathcal{F}$ and exact-optimal strings $\mathcal{O} \subseteq \mathcal{F}$, the operational endpoints are

$$P_{\text{feas}} = \sum_{x \in \mathcal{F}} p_{\theta}(x), \quad P_{\text{opt}} = \sum_{x \in \mathcal{O}} p_{\theta}(x), \quad P_{\text{opt}} = P_{\text{feas}} P_{\text{opt}|\text{feas}}, \quad (5)$$

where $P_{\text{opt}|\text{feas}} = P_{\text{opt}}/P_{\text{feas}}$. This identity separates entry into the feasible set from selection of an optimum after entry; mean energy is not substituted for either endpoint.

3 Methods

3.1 Controlled benchmark and Hamiltonian

The deterministic generator forms layered directed acyclic graphs with positive integer costs in $[1, 9]$ and resources in $[1, 1000]$. Five base graphs at each target edge count $m \in \{7, 10, 13, 16, 19\}$ contribute up to seven distinct achievable feasible-route cardinalities. Exact route and edge-bit enumeration records feasible and optimal sets independently; the resulting 140 tasks contain no duplicate primary feasible sets. Enumeration is reference computation at these sizes, not a performance baseline.

For an edge string x , let $f_{\text{raw}}(x) = \sum_v (\sum_e a_{ve} x_e - d_v)^2$, $u(x) = \max\{0, \sum_e r_e x_e - B\}$, $S_f = \max_x f_{\text{raw}}(x)$, and $S_r = \sum_e r_e$. We use

$$\begin{aligned} P_{\text{flow}}(x) &= \mathbf{1}[f_{\text{raw}} > 0] + \min\{f_{\text{raw}}/S_f, 1\}, \\ P_{\text{resource}}(x) &= \mathbf{1}[u > 0] + \min\{(u/S_r)^2, 1\}, \\ E_{\text{raw}}(x) &= C(x) + 172(P_{\text{flow}} + P_{\text{resource}}), \quad H_C(x) = E_{\text{raw}}(x)/172. \end{aligned} \quad (6)$$

The coefficient 172 is one above the generator-wide routing-cost bound $19 \times 9 = 171$. Every violation therefore lies above every feasible routing cost. Raw and normalized ground-state sets match the exact RCSP optima on 140/140 tasks; the controlled absolute energy span is 490.51–744.17, and the median within-base maximum/minimum span ratio is 1.07495.

3.2 Optimization and matched attribution

All reported circuit states are exact CPU statevectors. The discovery pilot runs $p = 1, 2, 3$ from three fixed random seeds. Pilot γ coordinates

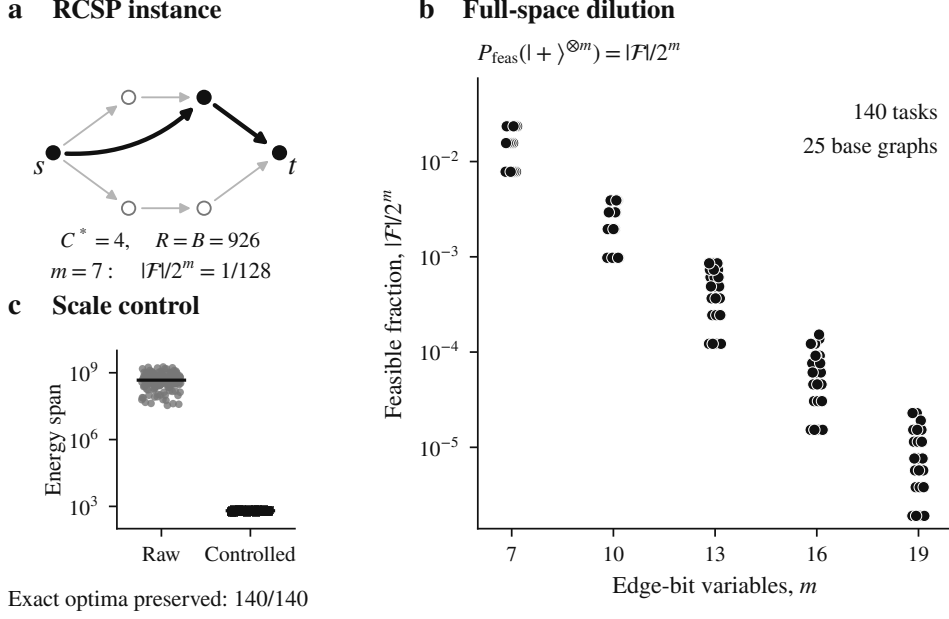


Fig. 1 Controlled RCSP benchmark and representation-induced dilution. **a**, one frozen seven-edge task; the heavy path is its unique feasible and optimal route, with cost 4 and resource use equal to the budget 926. **b**, all 140 distinct-cardinality tasks on 25 base graphs. Uniform edge-bit states assign feasible mass $|\mathcal{F}|/2^m$. **c**, every raw and controlled energy span; horizontal marks are medians. Scale control preserves every exact ground state

are initialized in $[0, 2\pi]$ and β coordinates in $[0, \pi]$; after initialization the optimizer coordinates are unbounded. The implementation neither bounds nor reduces γ modulo 2π , and we do not assume fundamental 2π periodicity for the generally noninteger spectrum of the normalized Hamiltonian.

COBYLA receives 120 objective evaluations for the pilot and nested diagnostic and 240 for each matched objective arm. For $(\gamma_1, \gamma_2, \beta_1, \beta_2)$, the exact embedding

$$\iota(\theta_2) = (\gamma_1, \gamma_2, 0, \beta_1, \beta_2, 0) \quad (7)$$

reproduces the $p = 2$ state inside $p = 3$. A deeper terminal objective worse than this known point is therefore a run-specific certificate of optimizer inadequacy, not proof of limited ansatz capacity. Passing this necessary test does not certify global optimality.

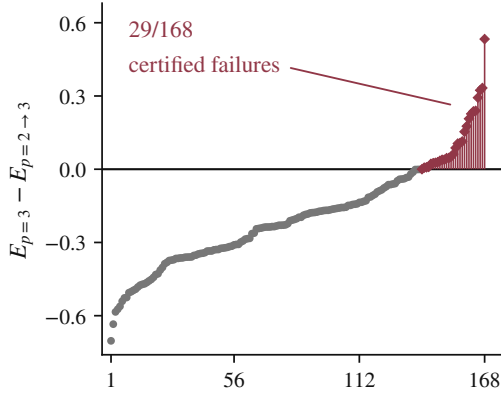
All O0/O2/O3 matched arms use one common mean-energy-selected $p = 2$ state, embedded with a zero third layer. The common starting point is deliberate: it holds initialization fixed so that subsequent differences are attributable to the classical loss. The design does not estimate performance under objective-specific initialization policies.

3.3 Frozen confirmation and robustness

The discovery set contains 56 tasks from ten base graphs; the held-out set is its task-and-graph complement, 84 tasks from 15 graphs. CVaR-0.10 was selected on discovery only. Before held-out execution, H1 ($O3-O0 > 0$) and H2 ($O3-O2$ non-inferiority margin -0.10 decades) were frozen with the graph as analysis unit. Task contrasts are averaged within graph; one-sided 95% lower bounds use 10,000 graph bootstrap resamples, exact sign-flip tests use all 2^{15} graph sign assignments, and Holm adjustment covers H1/H2. Discovery and held-out observations are never pooled for a new inferential estimate.

Post-hoc protocols were frozen before their formal runs and do not alter H1 or H2. They cover the nested diagnostic under COBYLA, Nelder-Mead, and SLSQP; an outcome-blind 24-task $p \in \{2, 3, 4\}$ subset at 120, 240, and 480 calls; six discovery CVaR tail fractions; fixed-endpoint sampling; and finite-shot retraining on ten preselected held-out graphs. An exact label-setting solver checks classical tractability [1]. Scaling uses a separate prespecified size-conditioned design and a pre-optimization computational- budget guard. Complete task selection, optimizer options, seed derivations, stopping rules, statistical details, and result matrices are in Online Resource 1, Secs. S2–S8.

Runs retain deterministic identities and all failure or censoring states. Canonical data and frozen outputs are hash protected. OpenAI Codex assisted with code, audits, and document preparation; the human author remains responsible for protocols, results, interpretations, and the submitted text.

a Certified optimizer failure

Seed-level comparisons, ranked
Continuation lowers the objective in all 29 certified failures.

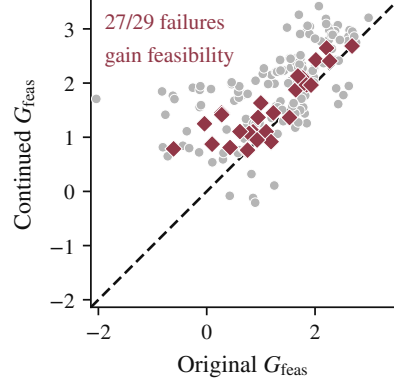
b Continuation

Fig. 2 Nested optimizer diagnosis and continuation. **a**, the original $p = 3$ objective minus the exactly embedded $p = 2$ objective for all 168 task-seed pairs, ranked by signed gap. The 29 certified failures are diamonds; the remaining pairs are grey circles. **b**, matched feasibility before and after continuation; the dashed line is equality and diamonds identify the same original failure set. Continuation lowers the objective in 29/29 failures and increases G_{feas} in 27/29

4 Results

4.1 Controlled dilution and optimizer diagnosis

The scale contract in [figure 1](#) removed the large numerical-span confound while preserving all exact optima. The nested results are summarized in [figure 2](#) and [table 1](#). The 56-task depth pilot then completed 504/504 planned optimizations. Its independently initialized $p = 3$ fit appeared worse than $p = 2$, but the nesting test changed that interpretation: all 168 zero-layer identities held exactly, and 29/168 reported $p = 3$ mean-energy objectives were worse than their embedded $p = 2$ points. Continuation lowered the objective in 29/29 certified failures and increased G_{feas} in 27/29. The remaining 139 runs simply passed this necessary test; they were not certified global optima.

Table 1 Optimizer attribution on the 56-task discovery set. Counts are seed-level unless stated otherwise.

Diagnostic	Result
Nested $p = 2 \rightarrow p = 3$ zero-layer identity	168/168
Original $p = 3$ objective worse than embedded $p = 2$	29/168
Continuation recovered lower objective in that failure set	29/29
Continuation increased G_{feas} in that failure set	27/29
Random $p = 3$ had lower energy but lower G_{feas}	82/168

Mean-energy improvement was not equivalent to feasible-mass improvement. In 82/168 matched comparisons, the state with lower mean energy also had lower G_{feas} ; component accounting showed redistribution among infeasible energy levels. Certified optimizer inadequacy also occurred under every tested local optimizer. O0/O3 nested-failure counts were 28/168 and 38/168 for COBYLA, 54/168 and 61/168 for Nelder–Mead, and 46/168 and 72/168 for SLSQP. Accordingly, certified optimizer inadequacy occurs under several tested optimizers and evaluation budgets; this is not a claim about arbitrary classical budgets or extensive multi-start search.

4.2 Capacity control and objective alignment

On discovery, O2 increased P_{feas} over O0 on 53/56 tasks. The median O2–O0 capacity gap was 0.2193 decades in G_{feas} , demonstrating feasible-concentration capacity that mean-energy training did not induce under the matched design. O3 increased P_{feas} over O0 on 50/56 tasks, left a median O2–O3 residual of 0.00690 decades, and closed a median 97.9% of the taskwise O2–O0 gap. These discovery results selected O3 and $\alpha = 0.10$; O2 remains an exact-statevector capacity control, not a deployable objective.

Both preregistered held-out criteria passed (figure 3 and table 2). Across 15 held-out graph units and 84 tasks, H1 estimated O3–O0 G_{feas} at +0.3547 decades, with one-sided 95% lower bound +0.2374 and Holm-adjusted $p = 2.44 \times 10^{-4}$. H2 estimated O3–O2 at −0.0086 decades; its lower bound −0.0360 exceeded the frozen −0.10-decade

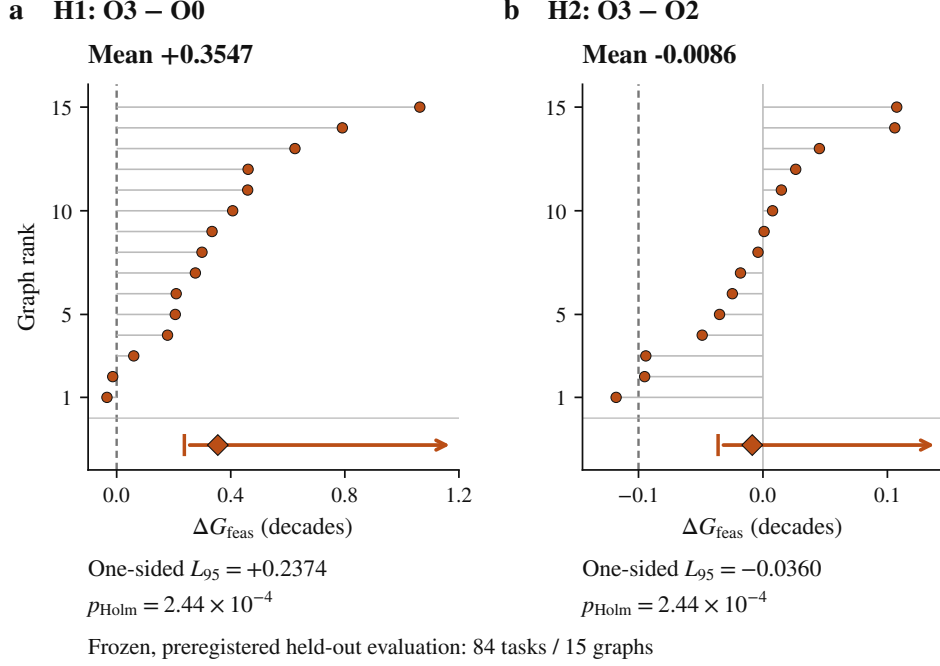


Fig. 3 Frozen, preregistered held-out objective alignment on 84 tasks and 15 graphs. **a**, H1 (O3–O0); **b**, H2 (O3–O2). Each circle is a complete graph mean, ranked independently within its panel; stems extend from zero. Diamonds below the graph points are the mean effects. Each lower cap and rightward arrow denotes a one-sided 95% interval extending to $+\infty$, not a two-sided interval. Dashed lines mark zero for H1 and the frozen -0.10 non-inferiority margin for H2; the thin solid line in **b** marks zero. L_{95} is the one-sided lower bound and both displayed p -values are Holm adjusted

non-inferiority margin, with the same adjusted p -value. Thus CVaR approached the measured exact-feasibility capacity at the graph analysis unit. O3 also increased P_{opt} on 80/84 tasks, while changes in $P_{\text{opt|feas}}$ remained heterogeneous.

The distinction survives exclusion of certified failures. Requiring both O0 and O3 to pass the nested test leaves 113, 72, and 70 pairs under COBYLA, Nelder–Mead, and SLSQP; their graph-cluster O3–O0 G_{feas} intervals are $[0.5770, 0.9971]$, $[0.5796, 1.2473]$, and $[0.6096, 1.3458]$. This

Table 2 Preregistered held-out graph-level tests (15 base graphs; 84 tasks). Lower bounds are one-sided grouped-bootstrap 95% bounds; p -values are Holm adjusted.

Test	Contrast	in	Effect	Lower bound	Null/margin	Holm p
	G_{feas}					
H1	O3	CVaR	+0.3547	+0.2374	0	0.000244
	– O0	mean				
		energy				
H2	O3	CVaR	– 0.0086	–0.0360	–0.10	0.000244
	O2	capacity				
		control				

supports separation of optimizer inadequacy from objective misalignment under the tested diagnostic, not universal independence.

4.3 Depth and robustness boundaries

In the preselected 24-task depth–budget subset (figure 4), pooled $p = 2 \rightarrow 3$ nested- failure rates were 20.8%, 21.5%, and 26.4% at 120, 240, and 480 calls; $p = 3 \rightarrow 4$ rates were 31.2%, 31.2%, and 32.6%. Increasing the tested evaluation budget did not monotonically reduce nested failures in this controlled subset. Yet at 120 calls, graph-mean $p = 4$ minus $p = 3$ G_{feas} was +0.7206 for O0 and +0.2021 for O3, positive on 10/10 graphs in each arm; the analogous effects remained positive at 240 and 480 calls. Depth can therefore add usable capacity while making the tested optimization problem harder. We do not claim that these instances remain difficult under arbitrarily large classical optimization budgets or extensive multi-start search.

The post-hoc discovery scan also bounded tail-fraction sensitivity. Equal- weight graph means were similar around the frozen choice for $\alpha = 0.05, 0.10$, and 0.25 , but optimized responses were non-monotone on 54/56 tasks. At $\alpha = 1$, exact weighted CVaR equaled mean energy to within 2.42×10^{-13} ; separately executed optimizer trajectories need not coincide. No post-hoc tail fraction revises the frozen H1/H2 result.

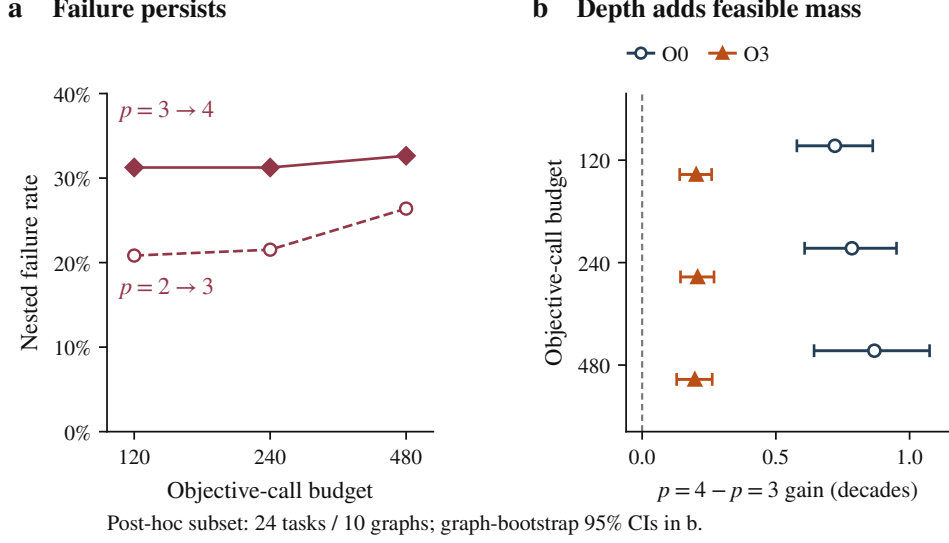


Fig. 4 Depth and budget boundaries on the preselected post-hoc subset of 24 tasks and 10 graphs. **a**, certified nested-failure rates pool O0 and O3 over three seeds (144 comparisons per transition and budget). Increasing the tested objective-call budget does not monotonically reduce failures. **b**, equal-weight graph means and 95% graph-bootstrap intervals for the $p = 4$ minus $p = 3$ feasibility gain; circles denote O0 and triangles O3. Intervals retain the precision reported in the frozen depth–budget report. Finite-shot training and the full depth summaries are shown in Online Resource 1

4.4 Finite-shot and scaling limits

Fixed-endpoint sampling recovered the exact sign of non-tied O3–O0 P_{feas} comparisons in 94.8%, 99.3%, and 100.0% of cases at 10^3 , 10^4 , and 10^5 shots. End-to-end finite-shot training was less decisive (Online Resource 1, finite-shot training analysis). The matched exact-trained effect on ten held-out graphs was 0.3599 (95% graph-cluster CI [0.1641, 0.5697]); effects were 0.1706 ([−0.0106, 0.3862]) at 10^4 shots per call and 0.1246 ([−0.0050, 0.3351]) at 10^3 . Both point estimates retained the O3–O0 direction, but both intervals crossed zero.

The completed size-conditioned study also prevents extrapolation. At $m = 20$, mean response exponents were 0.6150 for O0, 0.7885 for O2, and 1.2821 for O3, reversing the earlier O3–O0 ordering. At $m = 22$, runs exceeded the preregistered computational budget and were administratively censored. This is a limitation of the study’s computational budget, not a principled quantum- algorithm barrier, a claim that 22 qubits cannot be simulated, or a statevector-memory limit. We therefore make no extrapolative scaling claim beyond the completed range.

Finally, an exact label-setting solver reproduced 140/140 frozen optima with small label sets and microsecond-scale descriptive timings. The benchmark is classically tractable by design. Complete optimizer, depth, alpha, finite-shot, classical, and scaling matrices are in Online Resource 1, Secs. S4–S9.

5 Scope of the Dilution Interpretation

Raw edge-bit feasible density is not intrinsic RCSP hardness. An explicit m -edge directed chain has one route and hence $\phi_{\text{state}} = 2^{-m}$, yet its solution is read in $O(m)$ time. Subdividing an edge similarly changes the encoding statistic while preserving route order and optima:

$$\phi_{\text{state}}(I^{(r)}) = 2^{-(r-1)}\phi_{\text{state}}(I). \quad (8)$$

Thus ϕ_{state} is operationally relevant to the tested full-space encoding, but does not alone imply an explicit-input lower bound; graph-query conclusions depend on representation and access [3, 4].

A narrower statement applies when the only unknown-set information is a counted membership query. For a uniformly random size- M marked subset of an N -label domain, a protocol with a pathwise hard cap of q such queries satisfies

$$\mathbb{E}_{\mathcal{F}} P_{\text{success}} \leq \min\{1, (2q + 1)^2 M/N\}. \quad (9)$$

Training and feedback queries count if they use that interface. This is an average-case oracle statement, not a pointwise claim for every natural feasible set. The membership-query arguments adapt standard Grover/BBBV-style search bounds [5–9] and are used here to delimit

the information-access model rather than as a claim of a new generic quantum- search lower bound.

The experiment exposes the full multilevel RCSP energy, and CVaR uses its ordering and weights. The membership-only bound therefore does not lower-bound the rich-energy O3 experiment. Candidate lists, advice, warm starts, feasible- state loaders, oracles, and structured mixers also change the accessible information. Search-with-advice results make this dependence explicit [10–15]. Such structure can reduce the search-space burden while relocating cost into preprocessing, information storage, state preparation, mixer or oracle construction, validation, and reuse. Full theorem statements, assumptions, boundary cases, proofs, posterior/advice formulations, and counterexamples are in Online Resource 1, Sec. S1.

6 Related Work

Constrained QAOA spans penalty encodings and algorithms that restrict the state support or dynamics. The alternating-operator framework formalizes problem-aware initial states and feasible-subspace mixers [16]; broader constructions cover constraint encodings and preserving mixers [17, 18]. Grover mixers transfer part of the burden to feasible-state preparation [19], Zeno dynamics uses repeated projections [20], and indicator-function methods evaluate inequalities in ancillas [21]. These are alternatives to, rather than matched variants of, the uniform full-space Penalty-X design studied here.

Recent structured approaches sharpen that contrast. Onah, Firt, and Michielsen introduce a constraint-enhanced one-hot construction with W states and block- XY mixing [22]; related analysis studies shallow feasibility limits for permutation constraints [23]. Bucher et al. combine iterative warm starts with one-hot XY mixers [24], extending the wider warm-start and initial-state/mixer-alignment literature [25, 26]. RCSP flow and resource feasibility is not a simple fixed-Hamming-weight constraint, so those constructions are not treated as automatic RCSP baselines.

Classical loss design provides a different intervention. CVaR emphasizes a low-energy tail [2]; validation-oracle objectives can target feasibility directly [27]; and feasibility- driven penalty scheduling jointly varies constraint penalties [28]. Onah and Michielsen give finite-depth,

finite-shot guarantees for a Fejér-filtered constrained model with explicit phase-normalization assumptions [29]. These studies motivate careful separation of objective information, spectral assumptions, and sampling resources. Our O2 instead serves only as an exact-statevector capacity control, while O3 retains the same H_C and ansatz as O0.

Finally, QAOA initialization and continuation are standard responses to outer-loop difficulty [30]; general training hardness and reachability results warn against identifying a terminal local-optimizer result with the global variational optimum [31, 32]. The exact zero-layer test is narrower: it certifies only that a reported deeper point is worse than a known embedded point. The contribution of this study is the combined attribution design—scale control, nested optimizer diagnosis, O2 capacity measurement, and graph-held-out O3 confirmation—not a new ansatz, CVaR definition, or generic optimizer-hardness theorem. Extended study-by-study comparisons are in Online Resource 1, Tables S14–S15.

7 Discussion and Limitations

Optimizer inadequacy is not objective misalignment

The nested test changes what a poor deeper run means: a terminal objective above an embedded shallower point is certified optimizer inadequacy under that run’s budget. It does not certify global success when passed. Conversely, lower mean energy can accompany lower feasible mass, and the O3–O0 contrast remains positive after both arms pass the diagnostic under each tested optimizer. Optimization quality and choice of loss must therefore be assessed separately.

O2 identifies unused feasible-concentration capacity

O2 supplies the missing counterfactual: what can the same ansatz and matched optimizer do when feasible mass itself is requested? Its improvement over O0 prevents the O0 shortfall from being attributed wholly to ansatz capacity. The held-out H2 result then gives O3 a sharper interpretation than a simple O3–O0 win: CVaR approached this exact-feasibility capacity control within the frozen margin. O2 requires exact membership aggregation and is neither a shot-efficient nor deployment-ready objective.

Depth adds capacity and optimization difficulty

The $p = 4$ subset illustrates why depth should not be read from one endpoint. It improves terminal feasible mass while producing more $p = 3 \rightarrow 4$ than $p = 2 \rightarrow 3$ nested failures. The tested 120–480-call trajectories do not show a monotone failure reduction, but they cannot establish behavior under much larger budgets, broad multi-start policies, or other optimizers. Depth expands the variational family and can simultaneously make its classical navigation less reliable.

Endpoint robustness exceeds training robustness

Sampling fixed exact-trained endpoints recovers the objective ordering reliably as shots increase. Retraining with a freshly sampled loss is a harder problem: both shot-trained point estimates preserve the O3–O0 direction, yet both graph intervals cross zero and the 1,000-shot condition is less consistent. The evidence supports finite-shot endpoint robustness and only a qualified, mixed training result; it supplies no device-noise or hardware- readiness evidence.

Scope

The benchmark uses synthetic layered DAGs, one additive resource, exact CPU statevectors, and small classically tractable tasks. The exact classical solver confirms that the study is a mechanism probe, not a quantum–classical performance contest. Its conclusions concern uniform-initialized, transverse- X Penalty- X QAOA and do not extend to all constrained QAOA. The $m = 20$ reversal and administratively censored $m = 22$ cells preclude a global scaling claim; the latter records this study’s preregistered computational budget, not a fundamental limit at 22 qubits.

Structured or feasibility-preserving approaches change the information and resource model by moving cost into state preparation, mixer construction, or constraint-aware operators; they are therefore contextual comparators rather than matched baselines for the present Penalty- X attribution study. In particular, no XY, Dicke, Grover, or path mixer is assumed to solve general RCSP feasibility. The membership-query discussion likewise does not lower-bound the rich-energy CVaR

experiment. Real-network validity, hardware noise, compilation, and independent external reproduction remain outside the tested scope, and no quantum-advantage claim is made.

8 Conclusion

In shallow full-space Penalty-X QAOA, poor feasible sampling did not have a single cause. Exact depth nesting exposed run-specific optimizer inadequacy, while matched lower-energy/lower-feasibility states showed that optimizer repair does not resolve objective misalignment. O2 then demonstrated that the same ansatz retained feasible-concentration capacity not induced by mean-energy training.

This makes the O0→O2→O3 sequence the central result. On graph-disjoint held-out data, CVaR-0.10 improved feasibility compensation over O0 and met the preregistered non-inferiority criterion against the exact-feasibility O2 capacity control. CVaR therefore used the available energy distribution more effectively for feasible entry in this controlled setting; O2 itself remains mechanistic and non-deployable.

The robustness evidence sets equally important boundaries. Greater depth can improve terminal feasibility while increasing certified optimization failures. Fixed-endpoint sampling is more stable than finite-shot retraining, whose intervals cross zero. The $m = 20$ reversal and administrative $m = 22$ censoring rule out extrapolative scaling claims.

These synthetic RCSP tasks are classically tractable, and the study tests neither structured mixers, objective-specific initialization, hardware noise, real networks, nor quantum advantage. Its practical message is diagnostic: constrained-QAOA evaluations should distinguish representation dilution, ansatz capacity, optimizer adequacy, objective alignment, and sampling effects before treating a poor endpoint as a fundamental algorithmic limitation.

Supplementary information

Online Resource 1 (ESM_1.pdf): Supplementary methods, proofs, robustness analyses, additional figures and tables, and reproducibility details.

Online Resource 2 (ESM_2.zip): Reproducibility metadata and machine-readable result summaries.

Statements and Declarations

Funding

The author received no specific funding for this work.

Competing interests

The author declares no competing interests.

Data availability

Benchmark definitions, task manifests, compact frozen result tables, and reproducibility documentation are publicly available at <https://github.com/ZhilinChen02/Penalty-X-QAOA-A-Controlled-RCSP-Study>.

Online Resource 2 provides compact claim-to-evidence mappings, numerical audit records, the execution census, and task-stratum summaries. The public repository does not include raw optimizer traces or the complete internal experiment tree; raw statevectors and individual samples were not persisted.

Code availability

The implementation, analysis and figure-reproduction scripts, and environment specifications are available in the public repository linked above.

Author contributions

Zhilin Chen conceived and designed the study, developed the software, conducted the computational experiments and analyses, developed the theoretical arguments, prepared the figures, and wrote and approved the manuscript.

Clinical trial registration

Not applicable.

Consent to Publish declaration

Not applicable.

Ethics and Consent to Participate declarations

Not applicable.

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Supplementary Files

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