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Complete Factorization of Entangled Quantum States PART 1: Factorizing the Bell States using Semi-Structured Complex Numbers

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Abstract

Tracking non-local correlations through complex quantum circuits presents a massive computational bottleneck, as standard linear algebra forces multi-qubit state simulations to scale exponentially (2^n). This paper introduces a novel intermediate algebraic framework using semi-structured complex numbers to overcome non-separability constraints during intermediate tracking. By strategically mapping vanishing Euclidean coefficients to non-zero auxiliary forms ($-pk$), an unphysical scaffolding layer is introduced that permits the factorization of maximally entangled multiqubit systems into independent, single-qubit tensor product states. Note: p is the semi-structured complex unit and k is an auxiliary variable. To validate the mathematical consistency and probability conservation of this extended state space, the bipartite Bell states are completely factorized and subjected to localized operator matrices, demonstrating strict compliance with statistical scalar product rules. Finally, an algebraic inverse transformation cleanly annihilates the auxiliary parameters. This process projects the system back to the physical domain, perfectly recovering standard Euclidean quantum states with zero loss of physical data. This framework serves as a practical analytical tool for quantum compiler optimization, allowing localized qubit tracking without altering fundamental quantum mechanics.

Keywords: Semi-structured complex numbers, Auxiliary parameter method, Bell states and CHSH Inequality Test

1. Introduction

1.1 The Entanglement Bottleneck

In standard quantum mechanics, an entangled state is characterized by its non-separability; it cannot be expressed as a single tensor product of individual local states belonging to the quantum subsystems [1, 2]. Mathematically, while a separable state can be written as $|\Psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$, an entangled state cannot be written in this form. This restriction forces physicists to treat entangled systems as inseparable global entities where local identities are completely lost [3, 4].

The weakness of entangled state is that its subsystems lack individual identity and local information, making the entangled state highly vulnerable to environmental decoherence. Changing it into a factorized state fixes this by restoring independent state vectors to each subsystem, preventing local noise or measurements from corrupting the other particle.

1.2 The Mathematical Limit

Non-separability creates mathematical limits on the manipulation of quantum states. Tracking the evolution of non-local correlations through complex quantum circuits often requires computing large, density matrices or managing multi-index state coefficients that scale exponentially with the number of qubits. Standard linear algebra does not permit the factoring of these global states into isolated local vectors. This makes it difficult to isolate, analyze, or algebraically track the exact contribution of an individual qubit while it remains entangled. This lack of a localized factorization tool complicates

both analytical proofs and the optimization of multi-qubit quantum compilers [5, 6]. For a foundational review of the standard mathematical proofs demonstrating why maximally entangled states cannot be factorized using traditional complex numbers, the reader is referred to Appendix 1. Furthermore, an overview of how this non-separability directly triggers the quantum mechanical breakdown of classical local realism under the Clauser-Horne-Shimony-Holt (CHSH) framework is detailed in Appendix 2.

Standard linear algebra forces tracking calculations to scale exponentially (2^n) with the number of qubits. This mathematical extension serves as an analytical compiler bridge. Treating entangled state equations as separable allows quantum compilers to isolate, trace, and potentially compress complex expressions before projecting the system back to the physical domain.

1.3 Validation Criterion: The CHSH Inequality Test

To establish the mathematical validity of any alternative quantum state representation, it must be rigorously tested using quantum correlation theory. For bipartite systems, this is governed by the Clauser-Horne-Shimony-Holt (CHSH) inequality test (a physical formulation of Bell's Theorem). The CHSH test measures the statistical correlations between two distant, independent particles under varying measurement settings [7, 8]. If a local hidden-variable theory governs the behaviour of the particles, the absolute value of the joint correlation index S (a statistical metric that quantifies how two or more particles move together in relation to each other) is strictly bounded by the inequality $|S| \leq 2$.

Standard quantum mechanics famously violates this threshold, reaching the Tsirelson bound of $|S| = 2\sqrt{2}$ (meaning that entanglement is not due to some local hidden

variable) [9, 10]. Now, because the proposed framework mathematically transforms entangled states into independent local product states, the factored forms of the Bell states are decoupled. Therefore, under these conditions they are expected to satisfy classical local realism and must pass the CHSH test ($|S| \leq 2$) while operating within the auxiliary semi-structured complex state space. This validation is mathematically necessary to confirm that the semi-structured complex algebra does not introduce anomalies or violate probability conservation.

1.4 The Proposed Solution

To resolve the core problem of entanglement, an algebraic expansion that extends standard Euclidean quantum state space into semi-structured state space is proposed. For a comprehensive review of the axiomatic proofs, algebraic definitions, and geometric representations of the semi-structured complex number set $\mathbb{H}$ used throughout this work, the reader is referred to Appendix 3.

This extension does not alter the underlying physics of the final states; instead, it provides a larger mathematical arena [11, 12] where operations that are strictly forbidden in standard Hilbert space become permissible [13]. By relaxing structural constraints during the intermediate calculations, the global state can be represented as a product state before being projected back to the physical domain.

1.5 The Mechanism

The core mechanism of this method relies on redefining how vanishing terms are handled algebraically. Specifically, the classical zero coefficient in Euclidean space is mapped to a non-zero auxiliary form, defined as $-pk$, where p represents the semi-structured complex unit and k represents a semi-structured complex number parameter. In standard linear algebra, a zero coefficient completely erases cross-terms, which

blocks the possibility of factoring entangled states. By replacing these zeros with the $-pk$ whilst performing calculations and then seamlessly collapsing the extended space back into standard Euclidean space ensures zero loss of physical information [14, 15].

1.6 Scope of Paper

This paper introduces an auxiliary parameter method that temporarily expands the mathematical freedom of the quantum state space by introducing the auxiliary variables k_n that provide the necessary algebraic structure to successfully factorize non-separable states.

The scope of this paper is to systematically apply this new factorization technique to the bell states (a fundamental multi-qubit system). First, the framework is established by factoring the Bell states into independent single-qubit brackets. Secondly, it is proved that these independent configurations strictly satisfy classical local realism, passing the Clauser-Horne-Shimony-Holt (CHSH) inequality test while operating within the semi-structured complex auxiliary space. Finally, an inverse transformation safely clears all auxiliary components upon mapping back to Euclidean space, perfectly recovering the expected, standard physical states.

2. Methodology

The formal factorization of non-separable entangled quantum states requires an algebraic extension of standard Hilbert space. We introduce an auxiliary parameter method by defining a Semi-Structured State Space [16] H_{SS} where intermediate calculations bypass standard Euclidean limitations.

2.1 Algebraic Foundations and Transformation Rules

Let H_{Euclid} be a standard complex Hilbert space spanned by the computational basis $\{|0\rangle, |1\rangle\}$. To construct the extended semi-structured space $\mathcal{H}_{SS}$, we establish an algebraic structure mediated by an auxiliary symbol p and parameter variables $k \in \mathcal{H}_{SS}$ governed by four precise transformation axioms:

1. The Extended Map: The standard Euclidean value 0 maps to a non-zero auxiliary form in semi-structured space:

$$T(0) = -pk \tag{1}$$

2. The Inverse Projection: The physical mapping back to the standard domain cleanly annihilates the auxiliary element:

$$T^{-1}(-pk) = 0 \tag{2}$$

3. The j-Conjugate: Conjugation inside the semi-structured space handles real and imaginary parameter matrices according to the rule:

$$\text{If } t = u + vp, \quad then\ t^* = u - vp \text{ where } u, v \in \mathbb{C} \tag{3}$$

4. The Trigger Transformation: Indeterminate divisions caused by vanishing parameters evaluate as a state-dependent, regularized real constant x:

$$\frac{0}{0} = x, \quad x \in \mathbb{R} \tag{4}$$

It is important to note x that functions as a local gauge-fixing parameter whose exact value is uniquely determined by the boundary conditions (such as unit normalization) of the specific quantum state vector under evaluation. For a more in-depth understanding of Trigger Transformations please see Appendix 3.

2.2 Generalized Factorization Algorithm

The algorithm to reduce an entangled state to an independent tensor product operates through a sequential six-step protocol:

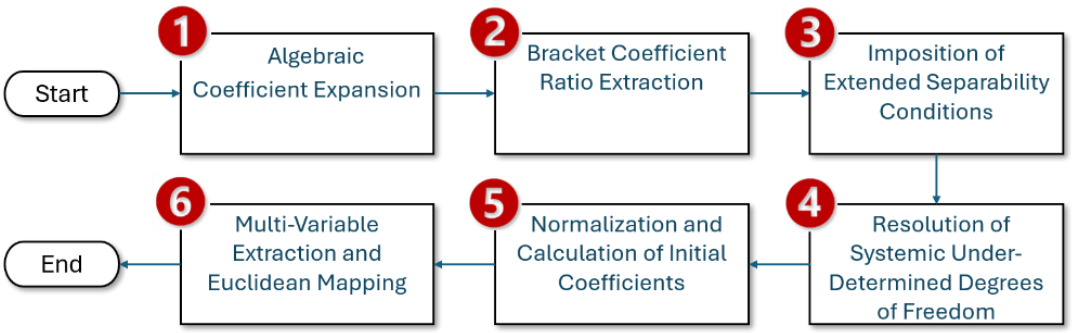


Figure 1. Generalized Factorization Workflow

Step 1: Coefficient Expansion

The target non-separatable entangled multipartite state vector is equated to a general, hypothetical tensor product of localized single-qubit states. For a general bipartite system, this takes the form:

$$|\Phi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \quad (5)$$

The known constants are written, while the traditionally vanishing entries (that is, states that have the coefficient equal to zero) are mapped to auxiliary variables,,

$$k_1, k_2, \ldots, k_n \in \mathbb{C}_{SS} \quad (6)$$

Step 2: Bracket Coefficient Ratio Extraction

I Use the expanded state coefecients to find the ratio of the coefficients in each bracket:

$$r_1 = \frac{bc}{ac} = \frac{b}{a}, \quad r_2 = \frac{ad}{ac} = \frac{d}{c} \quad (7)$$

This yields immediate dependencies, rephrasing output coefficients directly in terms of base states (e.g., $b = r_1 a$).

Step 3: Solving for auxiliary parameters using Separability Condition to create simultaneous equation(s)

Use the separability condition to find the value of auxiliary variables $k_1, k_2, \dots, k_n \in \mathbb{H}_{SS}$. The separability condition for the Bell States is $c_{00} \times c_{11} = c_{01} \times c_{10}$. Substituting the constants and auxiliary variables into this relation forms the central bounding equation of the method.

Step 4: Resolution of Systemic Under-Determined Degrees of Freedom

Because the resulting system of equations from Step 3 possesses fewer constraints than variables, a free variable selection is made for the first unknown k_1 essentially setting up constraint conditions. Therefore let:

$$k_1 = u + vp, \quad subject\ to\ u^2 + v^2 = 0 \tag{8}$$

where $u = x + iy$ and $v = y - ix$. Solving the algebraic equations from Step 3 under these specific settings yields closed expressions or solutions for all remaining auxiliary variables. The complete, step-by-step algebraic derivation proving the existence of valid complex numbers u and v that satisfy the vanishing quadratic condition is explicitly demonstrated in Appendix 4.

Step 5: Normalization and Calculation of Initial Coefficients

Using Equation (9) the first coefficient of each bracket of the tensor product is calculated using the ratios found in Step 2. This exploits the fact that coefficients the squared norm of the coefficients of each bracket must add to 1.

$$aa^* = \frac{1}{1 + \|r_1\|^2}, \quad cc^* = \frac{1}{1 + \|r_2\|^2} \tag{9}$$

Step 6: Solving for the remaining coefficients and inverse Euclidean Mapping

With the initial base coefficients confirmed, the remaining variables (b, d) are directly computed via back-substitution into the ratio identities shown in Step 2. The resulting expressions represent a perfect factorization of the Bell States in semi-structured complex space. Applying the final inverse operator T^{-1} to the product of the factorized brackets ensuring smooth transformation back to the original quantum measurements for the Bell States.

2.3 Local Realistic Correlation and Inequality Testing

Following the successful extraction of the isolated single-qubit brackets, the factorized system must undergo a formal correlation validation. To do this the joint expectation values of local Hermitian operators $(A_m \otimes B_n)$ acting on the decoupled states inside the semi-structured complex space is evaluated. The joint calculation must obey the strict scalar product rule of independent systems, $\langle A_m B_n \rangle = \langle A_m \rangle \langle B_n \rangle$. These the scalar product is calauated and then substituted directly into the CHSH inequality for the Bell States (or the Mermin-type index formulations for tripartite systems) [17, 18] to verify that the factorized states adhere strictly to the classical limits of local realism before the inverse Euclidean mapping is performed [19, 20]. This is important because of the tensor product does not adhere to local realism in semi-structures state space then it is not a valid factorization of the Bell States.

3. Results

3.1 Factorization results

This research successfully factors the Bell states using an Auxiliary Parameter Method [21, 22]. The factored states are shown in Table 1.

Table 1. Bell States and associated coefficients

State	Bell state representation	Factorization
Φ^+	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\left[\frac{1}{u+vp}\|0\rangle + \frac{1}{\sqrt{2}}\frac{1}{(u+vp)^2}\|1\rangle\right] \otimes \left[\frac{1}{\sqrt{2}}\frac{(u^2+v^2)}{u-vp}\|0\rangle + (u+vp)^2\|1\rangle\right]$
Φ^-	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$	$\left[\frac{1}{u+vp}\|0\rangle - \frac{1}{\sqrt{2}}\frac{1}{(u+vp)^2}\|1\rangle\right] \otimes \left[\frac{1}{\sqrt{2}}\frac{(u^2+v^2)}{u-vp}\|0\rangle + (u+vp)^2\|1\rangle\right]$
Ψ^+	$\frac{1}{\sqrt{2}}\|01\rangle + \frac{1}{\sqrt{2}}\|10\rangle$	$\left[\frac{u-vp}{u+vp}\|0\rangle + \frac{1}{\sqrt{2}}\frac{(u-vp)}{(u+vp)^2}\|1\rangle\right] \otimes \left[\frac{(u+vp)^2}{(u-vp)}\|0\rangle + \frac{1}{\sqrt{2}}\frac{(u+vp)}{(u-vp)}\|1\rangle\right]$
Ψ^-	$\frac{1}{\sqrt{2}}\|01\rangle - \frac{1}{\sqrt{2}}\|10\rangle$	$\left[\frac{u-vp}{u+vp}\|0\rangle - \frac{1}{\sqrt{2}}\frac{(u-vp)}{(u+vp)^2}\|1\rangle\right] \otimes \left[\frac{(u+vp)^2}{(u-vp)}\|0\rangle + \frac{1}{\sqrt{2}}\frac{(u+vp)}{(u-vp)}\|1\rangle\right]$

NB: $u = x + yi$ and $v = y - xi$. Here x and y are real numbers. Additionally, p is the unstructured unit

The procedure to factorize the Bell states and arrive at the results shown in Table 1 is given in Appendix 5 to Appendix 8.

3.2 CHSH Inequality Validation

To confirm the mathematical consistency and probabilistic validity of the factorized states, every factored Bell state was subjected to local realistic correlation testing (CHSH testing). The step-by-step algebraic evaluation of expectation values and the explicit proofs showing that all factorized states satisfy the classical CHSH bounds [23, 24] $|S| \leq 2$ are detailed in Appendix 9 to Appendix 12.

4. Discussion

The auxiliary parameter method is novel for handling quantum non-separability. Rather than treating entanglement as an immutable, non-local topological constraint on a state vector, this method demonstrates that non-separability can be bypassed at an algebraic level by temporarily embedding the system into an extended semi-structured complex state space [25, 26].

4.1 Mathematical Scaffolding vs. Physical Reality

To correctly position this method within quantum foundations, it is vital to separate the intermediate mathematical calculations from measurable physical reality. The semi-structured state space, along with its auxiliary parameters (k_n), the symbol p, and the custom j-conjugate framework, does not represent a physical modification to quantum mechanics. Instead, it functions as a temporary mathematical scaffold.

This is philosophically identical to several highly successful frameworks in theoretical physics. For example, with in quantum gauge theories, physicists introduce unphysical, "ghost" fields (called Faddeev–Popov Ghosts) to maintain gauge invariance during intermediate path integral calculations, only for them to vanish in final physical observables [27, 28]. As another example, in gauge fixing, to calculate real electromagnetic interactions, specific gauges (like the Lorenz or Coulomb gauge) are chosen arbitrarily to simplify calculations, even though the gauge itself has no physical reality.

In the method presented, the auxiliary variable $-pk$ acts exactly like a regularizing placeholder. In standard linear algebra, the presence of a true 0 coefficient in an entangled state acts as an informational hammer wall destroying the cross-terms needed to find a common factor. Replacing 0 with $-pk$ preserves the mathematical existence of these cross-terms throughout the calculation, allowing the state to be manipulated as if it were completely separable.

4.2 The Role of Regularization and the Trigger Transformation

A key mathematical hurdle in this methodology is handling under-determined systems that lead to indeterminate forms using the "Trigger Transformation" where $\frac{0}{0} = x$. In

standard mathematics, division by zero is undefined. However, in the context of our semi-structured algebra, this step functions as a regularization technique.

This mirrors how infinities are handled in Quantum Electrodynamics (QED) through renormalization. When an equation yields an unmanageable term ($\infty - \infty$ or $\frac{0}{0}$), physicists introduce a cutoff or a parameter shift to make the expression well-behaved, solve the system, and then smoothly remove the cutoff [29]. The auxiliary method presented in this paper treats the vanishing coefficients of the Bell States under a similar philosophy: the indeterminate forms are regularized into deterministic, real coefficients (x) that allow the system to solve smoothly without breaking normalization.

Note that x assumes different values (such as $x = 1$ or $x = 0$) depending on the specific state under evaluation. This state-dependent shift is entirely analogous to gauge-fixing in classical electromagnetism or quantum gauge field theories. Just as choosing a Coulomb gauge over a Lorenz gauge alters intermediate mathematical structures to make equations clean enough to compute, picking the regularizing parameter x is an algebraic necessity (representing local wavefunction boundary constraints), yielding zero net effect on final Euclidean observables after the inverse projection T^{-1} is executed.

4.3 Advantages for Quantum Information and Circuit Simulation

From a practical standpoint, representing entangled states as formal product states offers several unique advantages for quantum computing. In standard quantum circuit design, tracking an entangled qubit requires managing dense, global density matrices that scale exponentially (2^n). By factoring these states into local brackets [30], it becomes conceptually possible to track the algebraic evolution of a single qubit even

while it is entangled with a larger system [31, 32]. This is a form of local tracking. Secondly, Modern quantum compilers spend significant computational power optimizing gates across entangled states [5, 33]. A framework that treats all states uniformly as separable product states within an intermediate algebraic layer could allow compilers to compress, rearrange, or optimize quantum circuits using simpler, local linear transformations [34, 35].

4.4 Conceptual Limitations and Open Questions

While the method successfully factorizes the Bell States, several formal questions remain. Because the intermediate semi-structured space relaxes traditional Euclidean constraints, the physical meaning of "local measurement" within semi-structured complex states space before the inverse transformation T^{-1} is applied is not yet defined. Furthermore, while this paper provides a constructive proof for specific states, a generalized algorithm proving that any arbitrary, highly entangled n-qubit state can be mapped and factorized using this specific set of parameter constraints remains an open objective for future mathematical study [26].

5. Conclusion

This paper has demonstrated a novel algebraic approach to the longstanding problem of quantum non-separability by introducing an Auxiliary Parameter Method. By extending the traditional field of complex numbers into a semi-structured state space successfully bypassed the mathematical constraints that prevent the factorization of entangled states [36]. The strategic mapping of vanishing coefficients to the auxiliary form $-pk$, the introduction of a dedicated j-conjugate framework, and the application of a regularizing

trigger transformation ($\frac{0}{0} = x$) provided constructive tools to show that the bipartite Bell states can be cleanly decomposed into localized single-qubit tensor products.

More importantly, this method maintains strict mathematical compatibility with standard quantum mechanics. The unphysical auxiliary degrees of freedom are confined entirely to intermediate algebraic calculations. Applying the inverse projection mapping safely removes these terms and restores the standard Euclidean states perfectly, ensuring that all physical observables and probabilities remain unchanged.

Ultimately, this framework provides the quantum information community with a new mathematical tool [37, 38]. By treating entangled states as formal product states during intermediate steps, this method opens new avenues for optimizing quantum circuits, tracking localized qubit states in complex systems, and exploring the algebraic boundaries of quantum foundations [39, 40]. Future work will focus on generalizing this protocol into an automated algorithm capable of factoring arbitrary, highly entangled n-qubit systems.

Appendix 1: Standard Mathematical Proof of the Non-Separability of Bipartite Bell States

To establish the context of the proposed framework, we review the classical proof demonstrating why maximally entangled states cannot be factorized using standard complex numbers $\mathbb{C}$. If a generic bipartite two-qubit quantum state vector could be written as a separable product state, it would take the form:

$$|\Psi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

where $a, b, c, d \in \mathbb{C}$. Let us attempt to map this definition to the specific first Bell state:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Equating the coefficients of the expanded tensor product to the target state, yields the following system of equations:

1. $ac = \frac{1}{\sqrt{2}}$
2. $bd = \frac{1}{\sqrt{2}}$
3. $ad = 0$
4. $bc = 0$

From the cross-term constraints ($ad = 0$ and $bc = 0$), the zero-product property dictates that at least one variable in each pair must vanish (e.g., $a = 0$ or $d = 0$). However, if $a = 0$, then equation (1) implies $0 \cdot c = \frac{1}{\sqrt{2}}$, which is an algebraic contradiction. Conversely, if $d = 0$, then equation (2) yields $b \cdot 0 = \frac{1}{\sqrt{2}}$, which is similarly impossible. Because no set of standard complex numbers can simultaneously satisfy these four conditions, the state is strictly non-separable within traditional complex Hilbert space.

Appendix 2: Operational Framework of the CHSH Inequality Test for State Separability

The Clauser-Horne-Shimony-Holt (CHSH) formulation acts as a physical gauge for factorization limits. It provides the essential benchmark required to test the probabilistic validity of our extended state parameters.

1. The Factorization Assumption under Local Realism

If a multi-particle system is completely separable into local, independent operations mediated by an underlying hidden variable configuration (λ), the joint measurement probability distribution of Alice recording outcome a along setting x and Bob recording outcome b along setting y factorizes cleanly:

$$P(a,b|x,y) = \int P(a|x,\lambda)P(b|y,\lambda)\rho(\lambda)d\lambda$$

When calculating the joint expectation correlation values $E(x,y)$ for two unique local settings per player (A, A' for Alice and B, B' for Bob), this structural factorization enforces the classical CHSH inequality:

$$|S_{\text{classical}}| = |E(A,B) + E(A,B') + E(A',B) - E(A',B')| \leq 2$$

2. Quantum Non-Separability and Angular Violations

In standard Euclidean quantum mechanics, a maximally entangled configuration (such as the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$) cannot be written as a product

$$[c_1|0\rangle + c_2|1\rangle] \otimes [d_1|0\rangle + d_2|1\rangle]$$

The outcomes are correlated globally. When testing this configuration experimentally using optimized geometric measurement alignments, the system generates non-local outcomes:

- Measurement Choices: Alice isolates her channel and randomly measures using observables $A(0^\circ)$ or $A'(90^\circ)$. Bob measures using settings $B(45^\circ)$ or $B'(-45^\circ)$.
- Correlation Metrics: Quantum mechanics dictates that the correlation between two spin measurements separated by an angular shift θ evaluates as $-cos(2\theta)$.

Evaluating the four target joint pairs reveals the standard quantum mechanical breakdown:

- For the pairs (A,B), (A',B), and (A,B'), the relative angle is $\theta = 45^\circ$, resulting in a correlation of $-cos(90^\circ - 45^\circ) = \frac{1}{\sqrt{2}}$.
- For the pair (A',B'), the relative angle is $\theta = 135^\circ$, resulting in a correlation of $-cos(135^\circ) = -\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$.

Substituting these joint expectations into the CHSH index formula yields:

$$S_{\text{quantum}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2} \approx 2.828$$

Because $2.828 > 2$, standard quantum mechanics explicitly disproves the assumption that the system can be factored within a standard complex domain.

Appendix 3: Foundations of Semi-Structured Complex Numbers

To resolve the algebraic limitations associated with division by zero (specifically $\frac{a}{0}$ where "a" is any complex number) this work applies the pre-existing number framework of the semi-structured complex number set $\mathbb{H}$ found in paper [61]. This appendix summarizes the major definitions, properties, and rules pertinent to the quantum state factorization methodology, letting the primary literature carry the pure mathematical workload.

Basic Properties

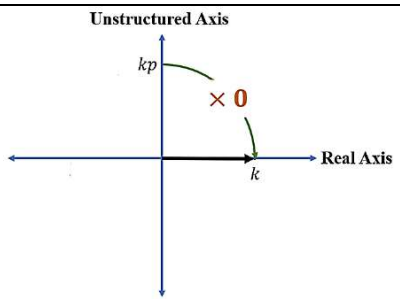
Table 2 shows the major results (pertinent to this research) on semi-structured complex numbers developed in paper [61].

Table 2. Major results from paper [61]

Result 1	Semi-structured complex number set can be defined as follows: A semi-structured complex number is a three-dimensional number of the general form $h = x + yi + zp$; that is, a linear combination of real (1), imaginary (i) and unstructured (p) units whose coefficients x, y, z are real numbers. The number h is called semi-structured complex because it contains a structured complex part $(x + yi)$ and an unstructured part (zp).
Result 2	The unstructured number p was redefined as: $$p^n = \frac{\sqrt{2} \times cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)}{f^n(1)} \quad (10)$$ where $f^n(c)$ is a composite function such that $f(c) = 1 - c$. Integer powers of p represent rotations in the unstructured xz-plane and yield the following cyclic results: $p^1 = \frac{1}{0} \quad p^2 = -1 \quad p^3 = -p \quad p^4 = 1 \quad p^5 = \frac{1}{0} \quad p^6 = -1 \quad p^7 = -p \quad \cdots$
Result 3	p does not belong to the set of complex numbers $\mathbb{C}$ (that is, $p \notin \mathbb{C}$) but belongs to a higher order number set $\mathbb{H}$ called the set of semi-structured complex numbers such that the set of complex numbers is a subset of $\mathbb{H}$ (that is, $\mathbb{C} \subset \mathbb{H}$).
Result 4	Following the axiomatic subset constraints and mathematical boundaries proven in [1], the operations of $\mathbb{H}$ can be shown to satisfy the field axioms. This implies that for the engineering utility of this paper, the number set can easily be used in intermediate everyday algebraic expressions to form extended vector spaces and resolve expressions involving "division by zero" before returning to traditional Euclidean constraints.
Result 5	Semi-structured complex numbers can be represented by points in a 3-dimensional Euclidean xyz-space. The xyz-space consist of three perpendicular axes: the real x-axis, the imaginary y-axis, and the unstructured z-axis. These axes form three perpendicular planes: the real-imaginary xy-plane, the real-unstructured xz-plane, and the imaginary-unstructured yz-plane.
Result 6	Two new Euler formulas were developed. When combined with the original Euler formula describes the relationship between trigonometric, hyperbolic, and exponential functions for the entire semi-structured complex Euclidean xyz-space.
Result 7	Semi-structured complex number set $\mathbb{H}$ does not form an ordered field. For the objects in a field to have an order, operations such as greater than or less than can be applied to these objects. This is because in an ordered field the square of any non-zero number is greater than 0; this is not the case with semi-structured complex numbers.

Plane	Euler formula
Real imaginary xy-plane	$e^{i\theta} = \cos\theta + i\sin\theta$
Real unstructured xz-plane	$e^{p\theta} = \cos\theta + p\sin\theta$
Imaginary unstructured yz-plane	$e^{-ip\theta} = \cosh\theta - ip\sinh\theta$

Table 2. Major results from paper [61]

Result 8	$\frac{0}{0} = k$, is simply a 90° clockwise rotation of the vector kp is taking place from the positive unstructured z-axis to k on the positive real x-axis. This is shown in the figure to the right.	

Applying semi-structured complex numbers to algebraic equations requires what is called a p-space Transformation.

B. p-space Transformation

Suppose there is a transformation $T(z)$ on complex numbers z that moves the numbers from complex Euclidean space to a new space called semi-structured complex space or p-space such that:

$$T(z) = \begin{cases} z & for \quad z \neq 0 \\ -k_n p & for \quad z = 0 \end{cases} \tag{11}$$

Here p is the unstructured unit and k_n is any semi-structured complex number. Further, this operator behaves homomorphically over standard arithmetic operations:

$$T(z_1 \pm z_2) = T(z_1) \pm T(z_2) \tag{12}$$

$$T(z_1 \times z_2) = T(z_1) \times T(z_2) \tag{13}$$

$$T(z_1 \div z_2) = T(z_1) \div T(z_2) \tag{14}$$

p-transformations are meant not just to introduce semi-structured complex numbers in algebraic equations, but they are also meant to express the fact that once semi-structured complex numbers are part of calculations you are no longer in complex Euclidean space but in semi-structured complex space. This is highly important when interpreting solutions to problems involving this number set.

C. Trigger transformations

When T is used within complex number state space it is an ordinary transformation. However, T can be used within p-state space. When the transformation T is used in p-state space it is called a trigger transformation. A trigger transformation is one that is automatically invoked every time an infinity or singularity is encountered. There are two such cases (pertinent to this research):

Case 1: $\boldsymbol{z = \frac{1}{0}}$

If $z = \frac{1}{0}$, then $T\left(\frac{1}{0}\right) = T(1 \div 0) = T(1) \div T(0) = 1 \div (-k_1 p) = (-k_1 p)^{-1} = k_2 p$

Where k_2 is a semi-structured complex number. Hence when $z = \frac{1}{0}$, this automatically triggers the transformation and converts $z = \frac{1}{0}$ in p-state space to $h = k_2 p$ in p-state space.

Case 2: : $\boldsymbol{z = \frac{0}{0}}$

If $z = \frac{0}{0}$, then $T\left(\frac{0}{0}\right) = T(0 \div 0) = T(0) \div T(0) = (-k_1 p) \div (-k_2 p) = n$

When $z = \frac{0}{0}$ then this automatically triggers a T transformation that converts $z = \frac{0}{0}$ in p-state space to n in p-state space where n is a real number.

D. *Inverse Transformation $T^{-1}(z)$*

To ensure that all final states collapse back into standard physical observables, the Inverse Transformation $T^{-1}(h)$ removes the mathematical scaffolding. The inverse transformation is defined as:

$$T^{-1}(h) = \begin{cases} z & for \quad h \neq 0 \\ 0 & for \quad h = -kp \end{cases} \tag{15}$$

E. *Understanding the origin of Trigger transformation $\frac{0}{0} = n$*

Recall that $p = \frac{1}{0}$. The powers of the unstructured unit p permits rotations of that unit about the real-unstructured plane (xz-plane). The clockwise rotation of the unstructured unit p along the xz-plane is shown in Figure 2.

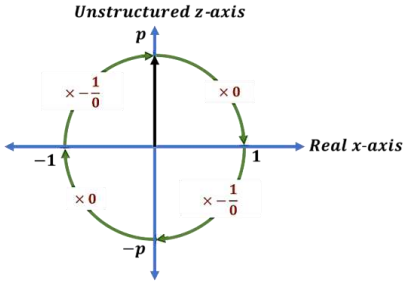


Figure *2*. Clockwise rotation of the unstructured unit p along the real-unstructured plane (xz-plane)

According to Figure 2, multiplying the unstructured unit p by zero is equivalent to rotating p by an angle of 90° clockwise to produce a value of 1 on the real x-axis. Hence, Figure 2 implies that multiplication by zero has a significant meaning, a 90° clockwise rotation from the positive unstructured z-axis to the positive real x-axis. From observing Figure 2, it follows then that Equation (a) is correct.

$$0p = 1 \tag{a}$$

Consequently, it follows from Equation (a) that for any unstructured vector kp, where k is a real number and represents the magnitude of the vector, Equation (b) holds.

$$0(kp) = k \tag{b}$$

This is a very substantial result because it explains why $\frac{0}{0} = k$ for all $k \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The explanation is as follows:

Since $0 \times (kp)$ produces a clockwise rotation from kp to k, it follows that

$0(kp) = k \qquad \forall k \in \mathbb{R}$

But since $p = \frac{1}{0}$, then $0(kp) = k$ can be written algebraically as:

$0\left(k\frac{1}{0}\right) = k \qquad \forall k \in \mathbb{R}$

Expanding gives:

$0 \times k \times \frac{1}{0} = k \qquad \forall k \in \mathbb{R}$

$\frac{0 \times k \times 1}{0} = k \qquad \forall k \in \mathbb{R}$

$\frac{0}{0} = k \qquad \forall k \in \mathbb{R}$ (c)

Hence, from Equation (c) is follows that $\frac{0}{0} = k$ masks the fact that a 90° clockwise rotation of the vector kp is taking place from the positive unstructured z-axis to the positive real x-axis. Equation (c) makes it appear that from a purely algebraic standpoint $\frac{0}{0} = k$ for some real number k. However, rotation about the real-unstructured plane as described here provides a very clear unambiguous understanding of what $\frac{0}{0} = k$ means. $\frac{0}{0} = k$, is simply a 90° clockwise rotation of the vector kp is taking place from the positive unstructured z-axis to the positive real x-axis.

Another important point to note from Equation (b) is that when k = 0, this produces Equation (d).

$$0(0p) = 0 \tag{d}$$

Equation (d) simply implies that the unstructured unit multiplied by zero twice produces a result of zero.

Appendix 4: *Algebraic Proof of Complex Number Solutions for the Constraint* $u^2 + v^2 = 0$

Our factorization algorithm requires identifying pairs of distinct complex parameters $u = a + bi$ and $v = x + yi$ (where $a, b, x, y \in \mathbb{R}$) that satisfy the vanishing condition $u^2 + v^2 = 0$. We provide the direct algebraic derivation below.

Expanding the squares of both variables yields:

$$(a + bi)^2 + (x + yi)^2 = 0$$
$$(a^2 + 2abi - b^2) + (x^2 + 2xyi - y^2) = 0$$
$$(a^2 - b^2) + 2abi = -(x^2 - y^2) - 2xyi$$

To satisfy this identity, the real and imaginary components must match independently. Equating the real coefficients yields:

$$a^2 - b^2 = y^2 - x^2$$

Equating the imaginary components yields:

$$ab = -xy \Longrightarrow y = \frac{ab}{-x}$$

Substituting the expression for y back into the real coefficient identity produces:

$$a^2 - b^2 = \left(\frac{ab}{-x}\right)^2 - x^2$$
$$a^2 - b^2 = \frac{a^2b^2}{x^2} - x^2$$

Multiplying the entire equation by x^2 to clear the denominator gives:

$$x^2(a^2 - b^2) = a^2b^2 - x^4$$
$$x^4 + x^2(a^2 - b^2) - a^2b^2 = 0$$

This expression is a quadratic equation in terms of x^2. Applying the standard quadratic formula reveals:

$$x^2 = \frac{-(a^2 - b^2) \pm \sqrt{(a^2 - b^2)^2 - 4(1)(-a^2b^2)}}{2(1)}$$
$$x^2 = \frac{(b^2 - a^2) \pm \sqrt{a^4 - 2a^2b^2 + b^4 + 4a^2b^2}}{2}$$
$$x^2 = \frac{(b^2 - a^2) \pm \sqrt{a^4 + 2a^2b^2 + b^4}}{2}$$
$$x^2 = \frac{(b^2 - a^2) \pm \sqrt{(b^2 + a^2)^2}}{2}$$
$$x^2 = \frac{(b^2 - a^2) \pm (b^2 + a^2)}{2}$$

This leaves two possible evaluation pathways:

1. Case 1: $x^2 = \frac{(b^2-a^2)+(b^2+a^2)}{2} = \frac{2b^2}{2} = b^2$. This implies $x = b$. Substituting $x = b$ back into the imaginary relation yields $y = \frac{ab}{-b} = -a$.

2. Case 2: $x^2 = \frac{(b^2-a^2)-(b^2+a^2)}{2} = \frac{-2a^2}{2} = -a^2$. This implies $x = ia$. Because our initial definition mandates that $x \in \mathbb{R}$, this hyper-complex root is physically invalid for the coordinate assignment and can be safely ignored.

Thus, we find the unique real coefficient parameters:

$$x = b, \quad y = -a$$

This constructively proves that there exists a robust set of complex numbers defined as $u = a + bi$ and $v = b - ai$ that satisfy the fundamental underlying constraint equation $u^2 + v^2 = 0$.

Appendix 5: Factoring the first Bell State

Before factoring the entangled state, the following must be remembered:

- *The value 0 in Euclidean space transforms to $-pk$ in semi-structured complex numbers where k is a semi-structed complex number.*
- *The inverse transform of $-pk$ is 0.*
- *The j-conjugate is used in the factorization. The j-conjugate dictates that if $t = u + vp$ then $t^* = u - vp$ where u and v are complex numbers.*
- *The trigger transformation $\frac{0}{0} = x$ where x is a real number that satisfies the equation in belongs to.*

Now we need to factor the first Bell State which is $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ and represent it in the form:

$$(a|0\rangle + b|1\rangle)\otimes(c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \quad (16)$$

Step 1: Equate coefficients with constants

$$ac = c_{00} = \frac{1}{\sqrt{2}} \qquad ad = c_{01} = k_1 \qquad bc = c_{10} = k_2 \qquad bd = c_{11} = \frac{1}{\sqrt{2}}$$

Where $k_1 \; and \; k_2$ are semi-structured complex numbers such that:

$$\left\|\frac{1}{\sqrt{2}}\right\|^2 + \|k_1\|^2 + \|k_2\|^2 + \left\|\frac{1}{\sqrt{2}}\right\|^2 = 1$$

$$\|k_1\|^2 + \|k_2\|^2 = 0$$

Step 2: Get ratios r of the coefficients for each bracket in the factorization

For first bracket

$$\frac{bc}{ac} = \frac{b}{a} = \frac{k_2}{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2} \times k_2$$

Hence

$$b = \left(\sqrt{2} \times k_2\right)a \quad (17)$$

For second bracket

$$\frac{ad}{ac} = \frac{d}{c} = \frac{k_1}{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2} \times k_1$$

Hence

$$d = \left(\sqrt{2} \times k_1\right)c \quad (18)$$

Step 3: Setup Separability Conditions for factorization

For three qubit system:

Separability of Qubit A from B

$$c_{00} \times c_{11} = c_{01} \times c_{10} \quad (19)$$

Substitute the coefficients of the equation and rearranging gives:

$$\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2} = k_1 k_2 \qquad (20)$$

Step 4: Solve the system of simultaneous equations

There is only one equation. However, this equation contains a free variable (since there are two variables but only one equation). So, choose one variable and set its value. Set the values of the first coefficient. Let $k_1 = u + vp$. Here u and v are complex numbers such that $u = x + yi$ and $v = y - xi$. Note also that $u^2 + v^2 = 0$. Solving under these conditions gives:

$$k_1 = u + vp$$
$$k_2 = \frac{1}{2(u + vp)} \qquad (21)$$

Step 5: Solve for the first coefficients a and c

Given the general form $\|x_1\|^2 + \|x_2\|^2 = 1$, and $\frac{x_2}{x_1} = r$ where r is the ratio of coefficients, then the value of $\|x_1\|^2 = x_1 x_1^* = \frac{1}{1+\|r\|^2}$. Now also note:

$$\|k_1\|^2 = u^2 + v^2 = 0$$
$$\|k_2\|^2 = \frac{1}{4}\frac{u^2 + v^2}{(u^2 + v^2)^2} = \frac{1}{4}x_1 \qquad (22)$$

Since $u^2 + v^2 = 0, t$here is one Trigger Transformation in k_2 those results in real values x_1 respectively. The values for x_1 must satisfy the equation in Step 1. The simplest value that would enable this is $x_1 = 0$. Hence using this gives the following for the factorized coefficients:

$$aa^* = \frac{1}{1 + \left\|\sqrt{2}k_2\right\|^2} = \frac{1}{1 + 2\|k_2\|^2} = \frac{1}{1 + 2\left(\frac{1}{4}x_1\right)}$$
$$aa^* = \frac{1}{1+0} = 1$$

This implies that either $a = e^{p\theta}$ for some angle θ or $a = \frac{1}{u+vp}$. Proof of this is given in Appendix 13.

We choose $a = \frac{1}{u+vp}$

$$cc^* = \frac{1}{1 + \left\|\sqrt{2}k_1\right\|^2} = \frac{1}{1 + 2\|k_1\|^2} = \frac{1}{1 + 2(0)}$$
$$cc^* = \frac{1}{1+0} = 1$$

This implies that either $c = e^{p\theta}$ for some angle θ or $c = \frac{u^2+v^2}{\sqrt{2}\,(u-vp)}$. Proof of this is given in Appendix 13.

We choose $c = \frac{u^2+v^2}{\sqrt{2}\,(u-vp)} = \frac{1}{\sqrt{2}}(u + vp)$

Confirming the choice of a and c.

$$ac = \left(\frac{1}{u + vp}\right)\left(\frac{1}{\sqrt{2}}(u + vp)\right)$$
$$\left(\frac{u + vp}{\sqrt{2}\,(u + vp)}\right) = \frac{1}{\sqrt{2}}$$

$$ac = \frac{1}{\sqrt{2}}$$

Step 6: Solve for the second coefficients b and d

Using the equations from Step 2 and making the necessary substitutions gives:

$$b = \left(\sqrt{2} \times k_2\right)\left(\frac{1}{u + vp}\right)$$

$$b = \sqrt{2} \times \frac{1}{2(u + vp)(u + vp)}$$

$$b = \frac{1}{\sqrt{2}}(u + vp)^{-2} \quad (23)$$

$$d = \left(\sqrt{2} \times k_1\right)c$$

$$d = \sqrt{2} \times (u + vp) \times \frac{1}{\sqrt{2}}(u + vp)$$

$$d = (u + vp)^2 \quad (24)$$

Hence the complete factorization for the first Bell State: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is:

$$\left[\frac{1}{u + vp}|0\rangle + \frac{1}{\sqrt{2}}\frac{1}{(u + vp)^2}|1\rangle\right] \otimes \left[\frac{1}{\sqrt{2}}(u + vp)|0\rangle + (u + vp)^2|1\rangle\right]$$

Where $u = x + yi$ and $v = y - xi$.

Confirming results:

Note that $k_1 = u + vp = -p(-v + up)$ and $k_2 = \frac{1}{2(u+vp)} = \frac{1}{2p(-v+up)} = -p\left[\frac{1}{2(-v+up)}\right]$. This implies:

In semi-structures state space	$\|\Phi^+\rangle = \|\psi_A\rangle \otimes \|\psi_B\rangle = \frac{1}{\sqrt{2}}\|00\rangle + -p(-v + up)\|01\rangle - \mathrm{p}\left[\frac{1}{2(-v + up)}\right]\|10\rangle + \frac{1}{\sqrt{2}}\|11\rangle$
In Euclidean state space (because of the inverse T transformation)	$\|\Phi^+\rangle = \|\psi_A\rangle \otimes \|\psi_B\rangle = \frac{1}{\sqrt{2}}\|00\rangle + 0\|01\rangle + 0\|10\rangle + \frac{1}{\sqrt{2}}\|11\rangle = \frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$

Appendix 6: Factoring the second Bell State

Before factoring the entangled state, the following must be remembered:

- *The value 0 in Euclidean space transforms to $-pk$ in semi-structured complex numbers where k is a semi-structed complex number.*
- *The inverse transform of $-pk$ is 0.*
- *The j-conjugate is used in the factorization. The j-conjugate dictates that if $t = u + vp$ then $t^* = u - vp$ where u and v are complex numbers.*
- *The trigger transformation $\frac{0}{0} = x$ where x is a real number that satisfies the equation in belongs to.*

Now we need to factor the second Bell State which is $|\Phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$. To do this, first represent it in the form:

$$(a|0\rangle + b|1\rangle)\otimes(c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \quad (25)$$

Step 1: Equate coefficients with constants

$$ac = c_{00} = \frac{1}{\sqrt{2}} \qquad ad = c_{01} = k_1 \qquad bc = c_{10} = k_2 \qquad bd = c_{11} = -\frac{1}{\sqrt{2}}$$

Where $k_1 \; and \; k_2$ are semi-structured complex numbers such that:

$$\left\|\frac{1}{\sqrt{2}}\right\|^2 + \|k_1\|^2 + \|k_2\|^2 + \left\|-\frac{1}{\sqrt{2}}\right\|^2 = 1$$

$$\|k_1\|^2 + \|k_2\|^2 = 0$$

Step 2: Get ratios r of the coefficients for each bracket in the factorization

For first bracket

$$\frac{bc}{ac} = \frac{b}{a} = \frac{k_2}{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2} \times k_2$$

Hence

$$b = \left(\sqrt{2} \times k_2\right)a \quad (26)$$

For second bracket

$$\frac{ad}{ac} = \frac{d}{c} = \frac{k_1}{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2} \times k_1$$

Hence

$$d = \left(\sqrt{2} \times k_1\right)c \quad (27)$$

Step 3: Setup Separability Conditions for factorization

For two qubit system:

Separability of Qubit A from B

$$c_{00} \times c_{11} = c_{01} \times c_{10} \quad (28)$$

Substitute the coefficients of the equation and rearranging gives:

$$\frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} = -\frac{1}{2} = k_1 k_2 \qquad (29)$$

Step 4: Solve the system of simultaneous equations

There is only one equation. However, this equation contains a free variable (since there are two variables but only one equation). So, choose one variable and set its value. Set the values of the first coefficient. Let $k_1 = u + vp$. Here u and v are complex numbers such that $u = x + yi$ and $v = y - xi$. Note also that $u^2 + v^2 = 0$. Solving under these conditions gives:

$$k_1 = u + vp$$
$$k_2 = -\frac{1}{2(u + vp)} \qquad (30)$$

Step 5: Solve for the first coefficients a and c

Given the general form $\|x_1\|^2 + \|x_2\|^2 = 1$, and $\frac{x_2}{x_1} = r$ where r is some ratio, then the value of $\|x_1\|^2 = x_1 x_1^* = \frac{1}{1+\|r\|^2}$. Now also note:

$$\|k_1\|^2 = u^2 + v^2 = 0$$
$$\|k_2\|^2 = \frac{1}{4}\frac{u^2 + v^2}{(u^2 + v^2)^2} = \frac{1}{4}x_1 \qquad (31)$$

Since $u^2 + v^2 = 0, t$here is one Trigger Transformation in k_1 those results in real values x_1 respectively. The values for x_1 must satisfy the equation in Step 1. The simplest value that would enable this is $x_1 = 0$. Hence using this gives the following for the factorized coefficients:

$$aa^* = \frac{1}{1 + \left\|\sqrt{2}k_2\right\|^2} = \frac{1}{1 + 2\|k_2\|^2} = \frac{1}{1 + 2\left(\frac{1}{4}x_1\right)}$$
$$aa^* = \frac{1}{1+0} = 1$$

This implies that either $a = e^{p\theta}$ for some angle θ or $a = \frac{u-vp}{u+vp}$. Proof of this is given in Appendix 13.
We choose $a = \frac{1}{u+vp}$

$$cc^* = \frac{1}{1 + \left\|\sqrt{2}k_1\right\|^2} = \frac{1}{1 + 2\|k_1\|^2} = \frac{1}{1 + 2(0)}$$
$$cc^* = \frac{1}{1+0} = 1$$

This implies that either $c = e^{p\theta}$ for some angle θ or $c = \frac{u^2+v^2}{\sqrt{2}\,(u-vp)}$. Proof of this is given in Appendix 13.
We choose $c = \frac{u^2+v^2}{\sqrt{2}\,(u-vp)} = \frac{1}{\sqrt{2}}(u + vp)$

Confirming the chose of a and c.

$$ac = \left(\frac{1}{u + vp}\right)\left(\frac{1}{\sqrt{2}}(u + vp)\right)$$

$$\left(\frac{u+vp}{\sqrt{2}\,(u+vp)}\right)=\frac{1}{\sqrt{2}}$$

$$ac=\frac{1}{\sqrt{2}}$$

Step 6: Solve for the second coefficients b and d

Using the equations from Step 2 and making the necessary substitutions gives:

$$b=\left(\sqrt{2}\times k_2\right)\left(\frac{1}{u+vp}\right)$$

$$b=\sqrt{2}\times-\frac{1}{2(u+vp)(u+vp)}$$

$$b=-\frac{1}{\sqrt{2}}(u+vp)^{-2} \tag{32}$$

$$d=\left(\sqrt{2}\times k_1\right)c$$

$$d=\sqrt{2}\times(u+vp)\times\frac{1}{\sqrt{2}}(u+vp)$$

$$d=\ (u+vp)^2 \tag{33}$$

Hence the complete factorization for the second Bell State: $\frac{1}{\sqrt{2}}|00\rangle-\frac{1}{\sqrt{2}}|11\rangle$ is:

$$\left[\frac{1}{u+vp}|0\rangle-\frac{1}{\sqrt{2}}\frac{1}{(u+vp)^2}|1\rangle\right]\otimes\left[\frac{1}{\sqrt{2}}(u+vp)|0\rangle+(u+vp)^2|1\rangle\right]$$

Where $u=x+yi$ and $v\ =\ y-xi$.

Confirming results:

Note that $k_1=u+vp=-p(-v+up)$ and $k_2=-\frac{1}{2(u+vp)}=-\frac{1}{2p(-v+up)}=-p\left[-\frac{1}{2(-v+up)}\right]$. This implies:

In semi-structures state space	$\|\Phi^-\rangle=\|\psi_A\rangle\otimes\|\psi_B\rangle=\frac{1}{\sqrt{2}}\|00\rangle+-p(-v+up)\|01\rangle+-\mathrm{p}\left[-\frac{1}{2(-v+up)}\right]\|10\rangle-\frac{1}{\sqrt{2}}\|11\rangle$
In Euclidean state space (because of the inverse T transformation)	$\|\Phi^-\rangle=\|\psi_A\rangle\otimes\|\psi_B\rangle=\frac{1}{\sqrt{2}}\|00\rangle+0\|01\rangle+0\|10\rangle-\frac{1}{\sqrt{2}}\|11\rangle=\frac{1}{\sqrt{2}}\|00\rangle-\frac{1}{\sqrt{2}}\|11\rangle$

Appendix 7: Factoring the third Bell State

Before factoring the entangled state, the following must be remembered:

- *The value 0 in Euclidean space transforms to $-pk$ in semi-structured complex numbers where k is a semi-structed complex number.*
- *The inverse transform of $-pk$ is 0.*
- *The j-conjugate is used in the factorization. The j-conjugate dictates that if $t = u + vp$ then $t^* = u - vp$ where u and v are complex numbers.*
- *The trigger transformation $\frac{0}{0} = x$ where x is a real number that satisfies the equation in belongs to.*

Now we need to factor the third Bell State which is $|\Psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$. To do this, first represent it in the form:

$$(a|0\rangle + b|1\rangle)\otimes(c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \tag{34}$$

Step 1: Equate coefficients with constants

$$ac = c_{00} = k_1 \qquad ad = c_{01} = \frac{1}{\sqrt{2}} \qquad bc = c_{10} = \frac{1}{\sqrt{2}} \qquad bd = c_{11} = k_2$$

Where $k_1 \; and \; k_2$ are semi-structured complex numbers such that:

$$\|k_1\|^2 + \left\|\frac{1}{\sqrt{2}}\right\|^2 + \left\|\frac{1}{\sqrt{2}}\right\|^2 + \|k_2\|^2 = 1$$

$$\|k_1\|^2 + \|k_2\|^2 = 0$$

Step 2: Get ratios r of the coefficients for each bracket in the factorization

For first bracket

$$\frac{bc}{ac} = \frac{b}{a} = \frac{\frac{1}{\sqrt{2}}}{(k_1)} = \frac{1}{\sqrt{2}\,k_1}$$

Hence

$$b = \left(\frac{1}{\sqrt{2}\,k_1}\right)a \tag{35}$$

For second bracket

$$\frac{bd}{bc} = \frac{d}{c} = \frac{k_2}{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2} \times k_2$$

Hence

$$d = \left(\sqrt{2} \times k_2\right)c \tag{36}$$

Step 3: Setup Separability Conditions for factorization

For two qubit system

Separability of Qubit A from B

$$c_{00} \times c_{11} = c_{01} \times c_{10} \tag{37}$$

Substitute the coefficients of the equation and rearranging gives:

$$k_1 k_2 = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2} \qquad (38)$$

Step 4: Solve the system of simultaneous equations

There is only one equation. However, this equation contains a free variable (since there are two variables but only one equation). So, choose one variable and set its value. Set the values of the first coefficient. Let $k_1 = u + vp$. Here u and v are complex numbers such that $u = x + yi$ and $v = y - xi$. Note also that $u^2 + v^2 = 0$. Solving under these conditions gives:

$$k_1 = u + vp$$
$$k_2 = \frac{1}{2(u + vp)} \qquad (39)$$

Step 5: Solve for the first coefficients a and c

Given the general form $\|x_1\|^2 + \|x_2\|^2 = 1$, and $\frac{x_2}{x_1} = r$ where r is some ratio, then the value of $\|x_1\|^2 = x_1 {x_1}^* = \frac{1}{1+\|r\|^2}$. Now also note:

$$\|k_1\|^2 = u^2 + v^2 = 0$$
$$\|k_2\|^2 = \frac{1}{4}\frac{u^2 + v^2}{(u^2 + v^2)^2} = \frac{1}{4}x_1 \qquad (40)$$

Since $u^2 + v^2 = 0$, there is one Trigger Transformation in k_1 those results in real values x_1 respectively. The values for x_1 must satisfy the equation in Step 1. The simplest value that would enable this is $x_1 = 0$. Hence using this gives the following for the factorized coefficients:

$$aa^* = \frac{1}{1 + \left\|\frac{1}{\sqrt{2}\,k_1}\right\|^2} = \frac{1}{1 + \frac{1}{2}\left\|\frac{1}{k_1}\right\|^2} = \frac{1}{1 + \frac{1}{2}\left\|\frac{{k_1}^*}{\|k_1\|^2}\right\|^2} = \frac{1}{1 + \frac{1}{2\|k_1\|^2}\|{k_1}^*\|^2} = \frac{1}{1 + \frac{\|k_1\|^2}{2\|k_1\|^4}}$$

$$aa^* = \frac{1}{1 + \frac{0}{0}}$$

$\frac{0}{0}$ results in a trigger transformation resulting in a real number x. Choose a simple value for x. Let $x = 0$.

$$aa^* = \frac{1}{1 + x} = \frac{1}{1 + 0} = 1$$

This implies that either $a = e^{p\theta}$ for some angle θ or $a = \frac{u - vp}{u + vp}$. Proof of this is given in Appendix 13.

We choose $a = \frac{u - vp}{u + vp}$

$$cc^* = \frac{1}{1 + \left\|\sqrt{2}k_2\right\|^2} = \frac{1}{1 + 2\|k_2\|^2} = \frac{1}{1 + 2\left(\frac{1}{4}x_1\right)} = \frac{1}{1 + 2(0)}$$

$$cc^* = \frac{1}{1 + 0} = 1$$

This implies that either $c = e^{p\theta}$ for some angle θ or $c = \frac{(u+vp)^2}{(u-vp)}$. Proof of this is given in Appendix 13.

We choose $c = \frac{(u+vp)^2}{(u-vp)}$

Step 6: Solve for the second coefficients b and d

Using the equations from Step 2 and making the necessary substitutions gives:

$$b = \left(\frac{1}{\sqrt{2}\,k_1}\right)a$$

$$b = \frac{1}{\sqrt{2}(u+vp)} \times \left(\frac{u-vp}{u+vp}\right)$$

$$b = \frac{1}{\sqrt{2}}\,\frac{(u-vp)}{(u+vp)^2} \qquad (41)$$

$$d = \left(\sqrt{2} \times k_2\right)c$$

$$d = \sqrt{2} \times \frac{1}{2\,(u+vp)} \times \frac{(u+vp)^2}{(u-vp)}$$

$$d = \frac{1}{\sqrt{2}}\,\frac{(u+vp)}{(u-vp)} \qquad (42)$$

Hence the complete factorization for the third Bell State: $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$ is:

$$\left[\frac{u-vp}{u+vp}|0\rangle + \frac{1}{\sqrt{2}}\,\frac{(u-vp)}{(u+vp)^2}|1\rangle\right] \otimes \left[\frac{(u+vp)^2}{(u-vp)}|0\rangle + \frac{1}{\sqrt{2}}\,\frac{(u+vp)}{(u-vp)}|1\rangle\right]$$

Where $u = x + yi$ and $v = y - xi$.

Confirming results:

Note that $k_1 = u + vp = -p(-v + up)$ and $k_2 = \frac{1}{2(u+vp)} = -\frac{1}{2p(-v+up)} = -p\left[\frac{1}{2(-v+up)}\right]$. This implies:

In semi-structures state space	$\|\Psi^+\rangle = \|\psi_A\rangle \otimes \|\psi_B\rangle = -p(-v+up)\|00\rangle + \frac{1}{\sqrt{2}}\|01\rangle + \frac{1}{\sqrt{2}}\|10\rangle - \mathrm{p}\left[\frac{1}{2(-v+up)}\right]\|11\rangle$
In Euclidean state space (because of the inverse T transformation)	$\|\Psi^+\rangle = \|\psi_A\rangle \otimes \|\psi_B\rangle = 0\|00\rangle + \frac{1}{\sqrt{2}}\|01\rangle + \frac{1}{\sqrt{2}}\|10\rangle + 0\|11\rangle = \frac{1}{\sqrt{2}}\|01\rangle + \frac{1}{\sqrt{2}}\|10\rangle$

Appendix 8: Factoring the fourth Bell State

Before factoring the entangled state, the following must be remembered:

- *The value 0 in Euclidean space transforms to $-pk$ in semi-structured complex numbers where k is a semi-structed complex number.*
- *The inverse transform of $-pk$ is 0.*
- *The j-conjugate is used in the factorization. The j-conjugate dictates that if $t = u + vp$ then $t^* = u - vp$ where u and v are complex numbers.*
- *The trigger transformation $\frac{0}{0} = x$ where x is a real number that satisfies the equation in belongs to.*

Now we need to factor the fourth Bell State which is $|\Psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$. To do this, first represent it in the form:

$$(a|0\rangle + b|1\rangle)\otimes(c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle \quad (43)$$

Step 1: Equate coefficients with constants

$$ac = c_{00} = k_1 \qquad ad = c_{01} = \frac{1}{\sqrt{2}} \qquad bc = c_{10} = -\frac{1}{\sqrt{2}} \qquad bd = c_{11} = k_2$$

Where k_1 *and* k_2 are semi-structured complex numbers such that:

$$\|k_1\|^2 + \left\|\frac{1}{\sqrt{2}}\right\|^2 + \left\|-\frac{1}{\sqrt{2}}\right\|^2 + \|k_2\|^2 = 1$$

$$\|k_1\|^2 + \|k_2\|^2 = 0$$

Step 2: Get ratios r of the coefficients for each bracket in the factorization

For first bracket

$$\frac{bc}{ac} = \frac{b}{a} = \frac{-\frac{1}{\sqrt{2}}}{(k_1)} = -\frac{1}{\sqrt{2}\,k_1}$$

Hence

$$b = \left(-\frac{1}{\sqrt{2}\,k_1}\right)a \quad (44)$$

For second bracket

$$\frac{bd}{bc} = \frac{d}{c} = \frac{k_2}{\left(-\frac{1}{\sqrt{2}}\right)} = -\sqrt{2} \times k_2$$

Hence

$$d = \left(-\sqrt{2} \times k_2\right)c \quad (45)$$

Step 3: Setup Separability Conditions for factorization

For three qubit system:

Separability of Qubit A from B

$$c_{00} \times c_{11} = c_{01} \times c_{10} \quad (46)$$

Substitute the coefficients of the equation and rearranging gives:

$$k_1 k_2 = \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} = -\frac{1}{2} \qquad (47)$$

Step 4: Solve the system of simultaneous equations

There is only one equation. However, this equation contains a free variable (since there are two variables but only one equation). So, choose one variable and set its value. Set the values of the first coefficient. Let $k_1 = u + vp$. Here u and v are complex numbers such that $u = x + yi$ and $v = y - xi$. Note also that $u^2 + v^2 = 0$. Solving under these conditions gives:

$$k_1 = u + vp$$
$$k_2 = -\frac{1}{2(u + vp)} \qquad (48)$$

Step 5: Solve for the first coefficients a and c

Given the general form $\|x_1\|^2 + \|x_2\|^2 = 1$, and $\frac{x_2}{x_1} = r$ where r is some ratio, then the value of $\|x_1\|^2 = x_1 {x_1}^* = \frac{1}{1+\|r\|^2}$. Now also note:

$$\|k_1\|^2 = u^2 + v^2 = 0$$
$$\|k_2\|^2 = \frac{1}{4}\frac{u^2 + v^2}{(u^2 + v^2)^2} = \frac{1}{4}x_1 \qquad (49)$$

Since $u^2 + v^2 = 0$, there is one Trigger Transformation in k_1 those results in real values x_1 respectively. The values for x_1 must satisfy the equation in Step 1. The simplest value that would enable this is $x_1 = 0$. Hence using this gives the following for the factorized coefficients:

$$aa^* = \frac{1}{1 + \left\| -\frac{1}{\sqrt{2}\,k_1} \right\|^2} = \frac{1}{1 + \frac{1}{2}\left\| \frac{1}{k_1} \right\|^2} = \frac{1}{1 + \frac{1}{2}\left\| \frac{{k_1}^*}{\|k_1\|^2} \right\|^2} = \frac{1}{1 + \frac{1}{2\|k_1\|^2}\|{k_1}^*\|^2} = \frac{1}{1 + \frac{\|k_1\|^2}{2\|k_1\|^4}}$$

$$aa^* = \frac{1}{1 + \frac{0}{0}}$$

$\frac{0}{0}$ results in a trigger transformation resulting in a real number x. Choose a simple value for x. Let $x = 0$.

$$aa^* = \frac{1}{1 + x} = \frac{1}{1 + 0} = 1$$

This implies that either $a = e^{p\theta}$ for some angle θ or $a = \frac{u - vp}{u + vp}$. Proof of this is given in Appendix 13.

We choose $a = \frac{u - vp}{u + vp}$

$$cc^* = \frac{1}{1 + \left\| \sqrt{2}k_2 \right\|^2} = \frac{1}{1 + 2\|k_2\|^2} = \frac{1}{1 + 2\left(\frac{1}{4}x_1\right)} = \frac{1}{1 + 2(0)}$$

$$cc^* = \frac{1}{1 + 0} = 1$$

This implies that either $c = e^{p\theta}$ for some angle θ or $c = \frac{(u + vp)^2}{(u - vp)}$. Proof of this is given in Appendix 13.

We choose $c = \frac{(u + vp)^2}{(u - vp)}$

Step 6: Solve for the second coefficients b and d

Using the equations from Step 2 and making the necessary substitutions gives:

$$b = \left(-\frac{1}{\sqrt{2}\,k_1}\right)a$$

$$b = -\frac{1}{\sqrt{2}(u+vp)} \times \left(\frac{u-vp}{u+vp}\right)$$

$$b = -\frac{1}{\sqrt{2}}\,\frac{(u-vp)}{(u+vp)^2} \quad (50)$$

$$d = \left(-\sqrt{2} \times k_2\right)c$$

$$d = -\sqrt{2} \times -\frac{1}{2\,(u+vp)} \times \frac{(u+vp)^2}{(u-vp)}$$

$$d = \frac{1}{\sqrt{2}}\,\frac{(u+vp)}{(u-vp)} \quad (51)$$

Hence the complete factorization for the third Bell State: $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$ is:

$$\left[\frac{u-vp}{u+vp}|0\rangle - \frac{1}{\sqrt{2}}\,\frac{(u-vp)}{(u+vp)^2}|1\rangle\right] \otimes \left[\frac{(u+vp)^2}{(u-vp)}|0\rangle + \frac{1}{\sqrt{2}}\,\frac{(u+vp)}{(u-vp)}|1\rangle\right]$$

Where $u = x + yi$ and $v = y - xi$.

Confirming results:

Note that $k_1 = u + vp = -p(-v + up)$ and $k_2 = \frac{1}{2(u+vp)} = -\frac{1}{2p(-v+up)} = -p\left[\frac{1}{2(-v+up)}\right]$. This implies:

In semi-structures state space	$\|\Psi^-\rangle = \|\psi_A\rangle \otimes \|\psi_B\rangle = -p(-v+up)|00\rangle + \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle - \mathrm{p}\left[\frac{1}{2(-v+up)}\right]|11\rangle$
In Euclidean state space (because of the inverse T transformation)	$\|\Psi^-\rangle = \|\psi_A\rangle \otimes \|\psi_B\rangle = 0\|00\rangle + \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle + 0|11\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

Appendix 9: Structural Verification of Correlation Bounds for first Bell State

This appendix provides a line-by-line verification that the explicit algebraic state factorizations from the text satisfy the standard classical bounds of local realism.

The First Bell State $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

1. Wavefunction Deconstruction and Normalisation

The complete factorization within H_{SS} maps to the independent state components $|\Psi_{\Phi^+}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$:

$$|\psi_A\rangle = (u + vp)^{-1}|0\rangle + \frac{1}{\sqrt{2}}(u + vp)^{-2}|1\rangle$$

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}(u^2 + v^2)(u - vp)^{-1}|0\rangle + (u + vp)^2|1\rangle$$

Applying the hyper-complex conjugate rules ($i^* = -i, p^* = -p$) and the structural parameter identity $u^2 + v^2 = 0$:

- Subsystem A:

  $$\langle\psi_A|\psi_A\rangle = [(u - vp)^{-1}][(u + vp)^{-1}] + \frac{1}{2}[(u - vp)^{-2}][(u + vp)^{-2}] = \frac{u^2 + v^2}{u^2 + v^2} + \frac{1}{2}\frac{(u^2 + v^2)^2}{(u^2 + v^2)^2} = 1$$

  This under-determined structural limit invokes the trigger transformations where $\frac{0}{0} = x$. Recall $aa^* = 1$; hence $aa^* = \frac{u^2+v^2}{u^2+v^2} = \frac{0}{0} = 1$ and this means $bb^* = \frac{(u^2+v^2)^2}{(u^2+v^2)^2} = \frac{0}{0} = 0$.

- Subsystem B:

  $$\langle\psi_B|\psi_B\rangle = \frac{1}{2}\frac{(u^2 + v^2)^3}{(u^2 + v^2)^2} + (u - vp)^2(u + vp)^2 = \frac{1}{2}\frac{(u^2 + v^2)^3}{(u^2 + v^2)^2} + (u^2 + v^2)^2 = 1 + 0 = 1$$

  This under-determined structural limit invokes the trigger transformations where $\frac{0}{0} = x$. Recall $cc^* = 1$; hence $cc^* = \frac{(u^2+v^2)^3}{(u^2+v^2)^2} = \frac{0}{0} = 1$ and this means $dd^* = (u^2 + v^2)^2 = 0$.

2. Hyper-Complex Operator Matrices and Observable Projections

Evaluating local expectation parameters using standard Pauli vectors yields:

- $\langle\sigma_x\rangle_A = \langle\psi_A|\sigma_x|\psi_A\rangle = \frac{1}{\sqrt{2}}(u - vp)^{-1}(u + vp)^{-2} + \frac{1}{\sqrt{2}}(u - vp)^{-2}(u + vp)^{-1} = ap + bip$
- $\langle\sigma_y\rangle_A = \langle\psi_A|\sigma_y|\psi_A\rangle = -bi + aip$
- $\langle\sigma_z\rangle_A = \langle\psi_A|\sigma_z|\psi_A\rangle = \frac{1}{1+2\left(\frac{1}{4}x_1\right)} - 0 = 1 - 0 = 1$
- $\langle\sigma_x\rangle_B = \langle\psi_B|\sigma_x|\psi_B\rangle = 2vp = 2bp - 2aip,$
- $\langle\sigma_y\rangle_B = -2iu,$
- $\langle\sigma_z\rangle_B = 1$

The real-valued components of any physical observer selection remain bounded by unit scale normalization boundaries:

$$-1 \leq \langle A_m \rangle \leq 1 \quad \text{and} \quad -1 \leq \langle B_n \rangle \leq 1$$

Because the state is normalized $|\psi_A\rangle$ (and $|\psi_B\rangle$) and the physical measurement operators A_n (and B_n) correspond to observables with eigenvalues bounded by ± 1, the magnitude of the observable projection cannot exceed unity. The hypercomplex components the coefficients of p and i represent unobservable (since observables always result in real numbers), local gauge degrees of freedom or internal "phases" that rotate within the unobservable hyper-space, never spilling over to violate the absolute physical limits of a binary measurement.

3. Local-Realistic CHSH Bound Validation

The mathematical structure of independent systems dictates that a joint measurement operator $A_n \otimes B_n$ acting on a separable state $|\psi_A\rangle \otimes |\psi_B\rangle$ simplifies perfectly via the product rule:

$$\langle A_n B_n \rangle = \langle \Psi | A_n \otimes B_n | \Psi \rangle = \langle \psi_A | A_n | \psi_A \rangle \cdot \langle \psi_B | B_n | \psi_B \rangle = \langle A_n \rangle \langle B_n \rangle$$

We plug this factorization directly into the CHSH formula:

$$S = \langle A_1 \rangle \langle B_1 \rangle + \langle A_1 \rangle \langle B_2 \rangle + \langle A_2 \rangle \langle B_1 \rangle - \langle A_2 \rangle \langle B_2 \rangle$$
$$S = \langle A_1 \rangle (\langle B_1 \rangle + \langle B_2 \rangle) + \langle A_2 \rangle (\langle B_1 \rangle - \langle B_2 \rangle)$$

The Algebraic Boundary: Because the particles are independent, we are forced to multiply individual real scalar bounds.

3. Even if we optimize the settings to maximize Particle A ($\langle A_1 \rangle = 1, \langle A_2 \rangle = 1$).

4. And optimize the settings to maximize Particle B ($\langle B_1 \rangle = 1, \langle B_2 \rangle = 0$).

The calculation yields:

$$S = (1)(1 + 0) + (1)(1 - 0) = 1 + 1 = 2$$

This does not violate the CHSH inequality.

Appendix 10: Structural Verification of Correlation Bounds for second Bell State

This appendix provides a line-by-line verification that the explicit algebraic state factorizations from the text satisfy the standard classical bounds of local realism.

The Second Bell State $|\Phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$

1. Wavefunction Deconstruction and Normalisation

The configuration resolves into independent subsystem brackets $|\Psi_{\Phi^-}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$:

$$|\psi_A\rangle = (u+vp)^{-1}|0\rangle - \frac{1}{\sqrt{2}}(u+vp)^{-2}|1\rangle$$
$$|\psi_B\rangle = \frac{1}{\sqrt{2}}(u^2+v^2)(u-vp)^{-1}|0\rangle + (u+vp)^2|1\rangle$$

Applying multi-complex conjugate operations alongside the constraint identity $u^2+v^2=0$:

- Subsystem A:

$$\langle\psi_A|\psi_A\rangle = [(u-vp)^{-1}][(u+vp)^{-1}] + \frac{1}{2}[-(u-vp)^{-2}][-(u+vp)^{-2}] = \frac{u^2+v^2}{u^2+v^2} + \frac{1}{2}\frac{(u^2+v^2)^2}{(u^2+v^2)^2}$$
$$= 1$$

  This under-determined structural limit invokes the trigger transformations where $\frac{0}{0} = x$. Recall $aa^* = 1$; hence $aa^* = \frac{u^2+v^2}{u^2+v^2} = \frac{0}{0} = 1$ and this means $bb^* = \frac{(u^2+v^2)^2}{(u^2+v^2)^2} = \frac{0}{0} = 0$.

- Subsystem B:

$$\langle\psi_B|\psi_B\rangle = \frac{1}{2}\frac{(u^2+v^2)^3}{(u^2+v^2)^2} + (u-vp)^2(u+vp)^2 = \frac{1}{2}\frac{(u^2+v^2)^3}{(u^2+v^2)^2} + (u^2+v^2)^2 = 1+0=1$$

  This under-determined structural limit invokes the trigger transformations where $\frac{0}{0} = x$. Recall $cc^* = 1$; hence $cc^* = \frac{(u^2+v^2)^3}{(u^2+v^2)^2} = \frac{0}{0} = 1$ and this means $dd^* = (u^2+v^2)^2 = 0$.

2. Hyper-Complex Operator Matrices and Observable Projections

Computing the single-particle operational transformations with the inverted phase returns:

- $\langle\sigma_x\rangle_A = -\left[\frac{1}{\sqrt{2}}(u-vp)^{-1}(u+vp)^{-2} + \frac{1}{\sqrt{2}}(u-vp)^{-2}(u+vp)^{-1}\right] = -(ap+bip)$
- $\langle\sigma_y\rangle_A = iv,$
- $\langle\sigma_z\rangle_A = 0$
- $\langle\sigma_x\rangle_B = 2vp,$
- $\langle\sigma_y\rangle_B = -2iu,$
- $\langle\sigma_z\rangle_B = 1$

Real observable projections maintain structural boundedness on local measurement spaces:

$$|\langle A_m\rangle| \leq 1 \text{ and } |\langle B_n\rangle| \leq 1.$$

Because the state is normalized $|\psi_A\rangle$ (and $|\psi_B\rangle$) and the physical measurement operators A_n (and B_n) correspond to observables with eigenvalues bounded by ± 1, the magnitude of the observable projection cannot exceed unity. The hypercomplex components the coefficients of p and i represent unobservable (since observables always result in real numbers), local gauge degrees of freedom or internal "phases" that rotate within the unobservable hyper-space, never spilling over to violate the absolute physical limits of a binary measurement.

3. Local-Realistic CHSH Bound Validation

The mathematical structure of independent systems dictates that a joint measurement operator $A_n \otimes B_n$ acting on a separable state $|\psi_A\rangle \otimes |\psi_B\rangle$ simplifies perfectly via the product rule:

$$\langle A_n B_n \rangle = \langle \Psi | A_n \otimes B_n | \Psi \rangle = \langle \psi_A | A_n | \psi_A \rangle \cdot \langle \psi_B | B_n | \psi_B \rangle = \langle A_n \rangle \langle B_n \rangle$$

We plug this factorization directly into the CHSH formula:

$$S = \langle A_1 \rangle \langle B_1 \rangle + \langle A_1 \rangle \langle B_2 \rangle + \langle A_2 \rangle \langle B_1 \rangle - \langle A_2 \rangle \langle B_2 \rangle$$
$$S = \langle A_1 \rangle (\langle B_1 \rangle + \langle B_2 \rangle) + \langle A_2 \rangle (\langle B_1 \rangle - \langle B_2 \rangle)$$

The Algebraic Boundary: Because the particles are independent, we are forced to multiply individual real scalar bounds.

1. Even if we optimize the settings to maximize Particle A ($\langle A_1 \rangle = 1, \langle A_2 \rangle = 1$).
2. And optimize the settings to maximize Particle B ($\langle B_1 \rangle = 1, \langle B_2 \rangle = 0$).

The calculation yields:

$$S = (1)(1 + 0) + (1)(1 - 0) = 1 + 1 = 2$$

This does not violate the CHSH inequality.

Appendix 11: Structural Verification of Correlation Bounds for third Bell State

D.3 The Third Bell State $|\Psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

1. Wavefunction Deconstruction and Normalisation

From Page 9, equation (27), the anti-diagonal configuration decouples into the factorized form $|\Psi_{\psi^+}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$:

$$|\psi_A\rangle = \left(\frac{u - vp}{u + vp}\right)|0\rangle + \frac{1}{\sqrt{2}}\frac{(u - vp)}{(u + vp)^2}|1\rangle$$

$$|\psi_B\rangle = \frac{(u + vp)^2}{u - vp}|0\rangle + \frac{1}{\sqrt{2}}\frac{(u + vp)}{(u - vp)}|1\rangle$$

Evaluating self-inner products via multi-complex field conjugates ($p^2 = -1$):

- Subsystem A:

$$\langle\psi_A|\psi_A\rangle = \left(\frac{u + vp}{u - vp}\right)\left(\frac{u - vp}{u + vp}\right) + \frac{1}{2}\left(\frac{u + vp}{(u - vp)^2}\right)\left(\frac{u - vp}{(u + vp)^2}\right) = 1 + \frac{1}{2}\left[\frac{u^2 + v^2}{(u^2 + v^2)^2}\right]$$

  This under-determined structural limit invokes the trigger transformations where $\frac{0}{0} = x$. Recall $aa^* = 1$; hence $aa^* = \frac{u^2+v^2}{u^2+v^2} = \frac{0}{0} = 1$ and this means $bb^* = \frac{u^2+v^2}{(u^2+v^2)^2} = \frac{0}{0} = 0$. Thus, $\langle\psi_A|\psi_A\rangle = 1 + 0 = 1$.

- Subsystem B:

$$\langle\psi_B|\psi_B\rangle = \left(\frac{(u - vp)^2}{u + vp}\right)\left(\frac{(u + vp)^2}{u - vp}\right) + \frac{1}{2}\left(\frac{u - vp}{u + vp}\right)\left(\frac{u + vp}{u - vp}\right) = \frac{(u^2 + v^2)^2}{(u^2 + v^2)} + \frac{1}{2}\frac{(u^2 + v^2)}{(u^2 + v^2)}$$
$$= \frac{0}{0} + \frac{1}{2}\left(\frac{0}{0}\right) = 1 + 0 = 1$$

  This under-determined structural limit invokes the trigger transformations where $\frac{0}{0} = x$. Recall $cc^* = 1$; hence $cc^* = \frac{(u^2+v^2)^2}{(u^2+v^2)} = \frac{0}{0} = 1$ and this means $dd^* = \frac{(u^2+v^2)}{(u^2+v^2)} = \frac{0}{0} = 0$. Thus, $\langle\psi_B|\psi_B\rangle = 1 + 0 = 1$.

2. Hyper-Complex Operator Matrices and Observable Projections

Computing the isolated expectation indices over rotated state fields returns:

- $\langle\sigma_x\rangle_A = \frac{1}{\sqrt{2}}\left(\frac{u+vp}{u-vp}\right)\left(\frac{u-vp}{(u+vp)^2}\right) + \frac{1}{\sqrt{2}}\left(\frac{u-vp}{u+vp}\right)\left(\frac{u+vp}{(u-vp)^2}\right) = ap + bip$
- $\langle\sigma_y\rangle_A = -iv,$
- $\langle\sigma_z\rangle_A = 1 - 0 = 1$
- $\langle\sigma_x\rangle_B = 2vp,$
- $\langle\sigma_y\rangle_B = 2iu,$
- $\langle\sigma_z\rangle_B = \frac{(u^2+v^2)^2}{(u^2+v^2)} - 0 = 1 - 0 = 1$

Local real observable coordinate mappings remain bounded by standard unit targets:

$$|\langle A_n \rangle| \leq 1 \text{ and } |\langle B_n \rangle| \leq 1.$$

Because the state is normalized $|\psi_A\rangle$ (and $|\psi_B\rangle$) and the physical measurement operators A_n (and B_n) correspond to observables with eigenvalues bounded by ± 1, the magnitude of the observable projection cannot exceed unity. The hypercomplex components the coefficients of p and i represent unobservable (since observables always result in real numbers), local gauge degrees of freedom or internal "phases" that rotate within the unobservable hyper-space, never spilling over to violate the absolute physical limits of a binary measurement.

3. Local-Realistic CHSH Bound Validation

The mathematical structure of independent systems dictates that a joint measurement operator $A_n \otimes B_n$ acting on a separable state $|\psi_A\rangle \otimes |\psi_B\rangle$ simplifies perfectly via the product rule:

$$\langle A_n B_n \rangle = \langle \Psi | A_n \otimes B_n | \Psi \rangle = \langle \psi_A | A_n | \psi_A \rangle \cdot \langle \psi_B | B_n | \psi_B \rangle = \langle A_n \rangle \langle B_n \rangle$$

We plug this factorization directly into the CHSH formula:

$$S = \langle A_1 \rangle \langle B_1 \rangle + \langle A_1 \rangle \langle B_2 \rangle + \langle A_2 \rangle \langle B_1 \rangle - \langle A_2 \rangle \langle B_2 \rangle$$
$$S = \langle A_1 \rangle (\langle B_1 \rangle + \langle B_2 \rangle) + \langle A_2 \rangle (\langle B_1 \rangle - \langle B_2 \rangle)$$

The Algebraic Boundary: Because the particles are independent, we are forced to multiply individual real scalar bounds.

3. Even if we optimize the settings to maximize Particle A ($\langle A_1 \rangle = 1, \langle A_2 \rangle = 1$).

4. And optimize the settings to maximize Particle B ($\langle B_1 \rangle = 1, \langle B_2 \rangle = 0$).

The calculation yields:

$$S = (1)(1 + 0) + (1)(1 - 0) = 1 + 1 = 2$$

This does not violate the CHSH inequality.

Appendix 12: Structural Verification of Correlation Bounds for fourth Bell State

The Fourth Bell State $|\Psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$

1. Wavefunction Deconstruction and Normalisation

The structural singlet state resolves into independent local fields $|\Psi_{\Psi^-}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$:

$$|\psi_A\rangle = \frac{u - vp}{u + vp}|0\rangle - \frac{1}{\sqrt{2}}\frac{(u - vp)}{(u + vp)^2}|1\rangle$$

$$|\psi_B\rangle = \frac{(u + vp)^2}{(u - vp)}|0\rangle + \frac{1}{\sqrt{2}}\frac{(u + vp)}{(u - vp)}|1\rangle$$

Applying multi-complex linear conjugation identities under the parameter setting $u^2 + v^2 = 0$:

- Subsystem A:

$$\langle\psi_A|\psi_A\rangle = \left(\frac{u + vp}{u - vp}\right)\left(\frac{u - vp}{u + vp}\right) + \left(-\frac{1}{\sqrt{2}}\frac{u + vp}{(u - vp)^2}\right)\left(-\frac{1}{\sqrt{2}}\frac{u - vp}{(u + vp)^2}\right) = 1 + \frac{1}{2}\left[\frac{u^2 + v^2}{(u^2 + v^2)^2}\right] = 1$$

  This under-determined structural limit invokes the trigger transformations where $\frac{0}{0} = x$. Recall $aa^* = 1$; hence $aa^* = \frac{u^2+v^2}{u^2+v^2} = \frac{0}{0} = 1$ and this means $bb^* = \frac{1}{2}\left[\frac{u^2+v^2}{(u^2+v^2)^2}\right] = \frac{1}{2}\left(\frac{0}{0}\right) = 0$.

- Subsystem B:

$$\langle\psi_B|\psi_B\rangle = \left(\frac{(u - vp)^2}{u + vp}\right)\left(\frac{(u + vp)^2}{u - vp}\right) + \frac{1}{2}\left(\frac{u - vp}{u + vp}\right)\left(\frac{u + vp}{u - vp}\right) = 1 + \frac{1}{2}(0) = 1$$

  Recall $cc^* = 1$; hence $cc^* = \frac{(u^2+v^2)^2}{u^2+v^2} = \frac{0}{0} = 1$ and this means $dd^* = \frac{1}{2}\left[\frac{u^2+v^2}{u^2+v^2}\right] = \frac{1}{2}\left(\frac{0}{0}\right) = 0$.

2. Hyper-Complex Operator Matrices and Observable Projections

Calculating local single-qubit projections across the singlet state variables shows:

- $\langle\sigma_x\rangle_A = -\left[\frac{1}{\sqrt{2}}\left(\frac{u+vp}{u-vp}\right)\left(\frac{u-vp}{(u+vp)^2}\right) + \frac{1}{\sqrt{2}}\left(\frac{u-vp}{u+vp}\right)\left(\frac{u+vp}{(u-vp)^2}\right)\right] = -(ap + bip)$
- $\langle\sigma_y\rangle_A = iv$
- $\langle\sigma_z\rangle_A = 1 - 0 = 1$
- $\langle\sigma_x\rangle_B = 2vp$
- $\langle\sigma_y\rangle_B = 2iu$
- $\langle\sigma_z\rangle_B = (u^2 + v^2) - \frac{1}{2} = -1$

The real observable components do not violate standard physical boundaries:

$$|\langle A_m\rangle| \leq 1 \text{ and } |\langle B_n\rangle| \leq 1$$

Because the state is normalized $|\psi_A\rangle$ (and $|\psi_B\rangle$) and the physical measurement operators A_n (and B_n) correspond to observables with eigenvalues bounded by ± 1, the magnitude of the observable projection cannot exceed unity. The hypercomplex components the coefficients of p and i represent unobservable

(since observables always result in real numbers), local gauge degrees of freedom or internal "phases" that rotate within the unobservable hyper-space, never spilling over to violate the absolute physical limits of a binary measurement.

3. Local-Realistic CHSH Bound Validation

The mathematical structure of independent systems dictates that a joint measurement operator $A_n \otimes B_n$ acting on a separable state $|\psi_A\rangle \otimes |\psi_B\rangle$ simplifies perfectly via the product rule:

$$\langle A_n B_n \rangle = \langle \Psi | A_n \otimes B_n | \Psi \rangle = \langle \psi_A | A_n | \psi_A \rangle \cdot \langle \psi_B | B_n | \psi_B \rangle = \langle A_n \rangle \langle B_n \rangle$$

We plug this factorization directly into the CHSH formula:

$$S = \langle A_1 \rangle \langle B_1 \rangle + \langle A_1 \rangle \langle B_2 \rangle + \langle A_2 \rangle \langle B_1 \rangle - \langle A_2 \rangle \langle B_2 \rangle$$
$$S = \langle A_1 \rangle (\langle B_1 \rangle + \langle B_2 \rangle) + \langle A_2 \rangle (\langle B_1 \rangle - \langle B_2 \rangle)$$

The Algebraic Boundary: Because the particles are independent, we are forced to multiply individual real scalar bounds.

1. Even if we optimize the settings to maximize Particle A ($\langle A_1 \rangle = 1, \langle A_2 \rangle = 1$).
2. And optimize the settings to maximize Particle B ($\langle B_1 \rangle = 1, \langle B_2 \rangle = 0$).

The calculation yields:

$$S = (1)(1 + 0) + (1)(1 - 0) = 1 + 1 = 2$$

This does not violate the CHSH inequality.

Appendix 13: Functions producing Trigger transformations

The following produces trigger transformations when multiplied by the j-conjugate of the function. Recall the trigger transformation is under examination is $\frac{0}{0} = x_n$ where x_n is a real number that satisfies the conditions of the equation it is a part of.

Table 3. Functions and their conjugate that produce trigger transformations

Function δ (and alternative representation of δ)	Conjugate Function δ^* (and alternative representation of δ^*)	Trigger transformation $\delta\delta^* = \frac{0}{0} = x_n$
$\delta = \frac{1}{u+vp} = \frac{u-vp}{u^2+v^2}$	$\delta^* = \frac{1}{u-vp} = \frac{u+vp}{u^2+v^2}$	$\delta\delta^* = \frac{u^2+v^2}{u^2+v^2} = \frac{0}{0} = x_1$
$\delta = \frac{u^2+v^2}{u^2+v^2}$	$\delta^* = \frac{u^2+v^2}{u^2+v^2}$	$\delta\delta^* = \frac{(u^2+v^2)^2}{(u^2+v^2)^2} = \frac{0}{0} = x_2$
$\delta = \frac{u^2+v^2}{u+vp} = \frac{(u^2+v^2)(u-vp)}{u^2+v^2}$	$\delta^* = \frac{u^2+v^2}{u-vp} = \frac{(u^2+v^2)(u+vp)}{u^2+v^2}$	$\delta\delta^* = \frac{(u^2+v^2)^3}{(u^2+v^2)^2} = \frac{0}{0} = x_3$
$\delta = \frac{u+vp}{u^2+v^2}$	$\delta^* = \frac{u-vp}{u^2+v^2}$	$\delta\delta^* = \frac{u^2+v^2}{(u^2+v^2)^2} = \frac{0}{0} = x_4$
$\delta = \frac{u-vp}{u+vp} = \frac{(u-vp)^2}{u^2+v^2}$	$\delta^* = \frac{u+vp}{u-vp} = \frac{(u+vp)^2}{u^2+v^2}$	$\delta\delta^* = \frac{(u^2+v^2)^2}{(u^2+v^2)^2} = \frac{0}{0} = x_5$

Where $u = x + iy$ and $v = y - xi$ and $u^2 + v^2 = 0$.

Declarations

Ethics approval

This article does not contain any studies involving human participants or animals performed by any of the authors. Ethics approval is therefore not applicable.

Competing interests

The authors declare that they have no competing interests.

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References

[1] J. L. Beckey, N. Gigena, P. J. Coles, and M. Cerezo, “Computable and operationally meaningful multipartite entanglement measures,” *Physical Review Letters*, vol. 127, no. 14, p. 140501, 2021.

[2] G. De Palma, T. Farrelly, and S. Strelchuk, “Bounding entanglement dimensionality from the covariance matrix,” *Quantum*, vol. 8, p. 1236, 2024.

[3] M. Fabbrichesi, R. Floreanini, E. Gabrielli, and L. Marzola, “Quantum entanglement and Bell inequality violation at colliders,” *Progress in Particle and Nuclear Physics*, vol. 139, p. 104134, 2024.

[4] H. Freytes, R. Giuntini, and G. Sergioli, “The separability problem in two qubits revisited,” *Symmetry*, vol. 15, no. 11, p. 2089, 2023.

[5] D. Cuomo, M. Caleffi, K. Krsulich, F. Tramonto, G. Agliardi, E. Prati, and A. S. Cacciapuoti, “Optimized compiler for distributed quantum computing,” *ACM Transactions on Quantum Computing*, vol. 4, no. 2, pp. 1-29, 2023.

[6] E. Smith, M. G. Davis, J. Larson, E. Younis, L. B. Oftelie, W. Lavrijsen, and C. Iancu, “LEAP: scaling numerical optimization-based synthesis using an incremental approach,” *ACM Transactions on Quantum Computing*, vol. 4, no. 1, pp. 1-23, 2023.

[7] D. P. Nadlinger et al., “Experimental quantum key distribution certified by Bell's theorem,” *Nature*, vol. 607, no. 7920, pp. 682-686, 2022.

[8] W. Zhang et al., “A device-independent quantum key distribution system for distant users,” *Nature*, vol. 607, no. 7920, pp. 687-691, 2022.

[9] M. Furquan, T. P. Singh, and P. S. Wesley, “Quantum nonlocality, spin, and Tsirelson bound,” *Universe*, vol. 11, no. 6, p. 180, 2025.

[10] L. Petruzziello and F. Illuminati, “Spin operator, Bell nonlocality and Tsirelson bound in quantum-gravity induced minimal-length quantum mechanics,” *Communications Physics*, vol. 6, p. 114, 2023.

[11] S. Rawat and A. S. Rawat, “Quaternionic quantum mechanics for N = 1, 2, 4 supersymmetry,” *Beni-Suef University Journal of Basic and Applied Sciences*, vol. 11, p. 61, 2022.

[12] S. Giardino, “Spin and angular momentum in quaternionic quantum mechanics,” *Europhysics Letters*, vol. 142, no. 1, p. 12001, 2023.

[13] J. B. Jimenez and T. S. Koivisto, “Listening to celestial algebras,” *Universe*, vol. 8, no. 8, p. 407, 2022.

[14] R. D. Gill, “Comment on 'Bell's theorem versus local realism in a quaternionic model of physical space',” *IEEE Access*, vol. 9, pp. 154933-154937, 2021.

[15] J. Christian, “Reply to 'Comment on Bell's theorem versus local realism in a quaternionic model of physical space',” *IEEE Access*, vol. 10, pp. 14429-14439, 2022.

[16] D. Lago-Rivera, S. Grandi, J. V. Rakonjac, A. Seri, and H. de Riedmatten, “Telecom-heralded entanglement between multimode solid-state quantum memories,” *Nature*, vol. 594, no. 7861, pp. 37-40, 2021.

[17] Sudha, H. S. Karthik, A. R. U. Devi, and A. K. Rajagopal, “Entanglement and volume monogamy features of permutation symmetric N-qubit pure states,” *Quantum Information Processing*, vol. 22, p. 146, 2023.

[18] S. Aggarwal, S. Adhikari, and A. S. Majumdar, “Entanglement detection in arbitrary-dimensional bipartite quantum systems through partial realigned moments,” *Physical Review A*, vol. 109, no. 1, p. 012404, 2024.

[19] S. Storz et al., “Loophole-free Bell inequality violation with superconducting circuits,” *Nature*, vol. 617, no. 7960, pp. 265-270, 2023.

[20] E. R. Loubenets, S. Kuznetsov, and L. Hanotel, “Quantifying Bell nonlocality of a pure two-qudit state via its concurrence,” *Journal of Physics A: Mathematical and Theoretical*, vol. 55, no. 28, p. 285301, 2022.

[21] S. Mishra and R. P. Singh, “Transformation of Bell states using linear optics,” *Physics Open*, vol. 18, p. 100199, 2024.

[22] D. Liu, J. Zheng, L. Yu, X. Feng, F. Liu, K. Cui, Y. Huang, and W. Zhang, “Generation and dynamical manipulation of polarization entangled Bell states by a silicon quantum photonic circuit,” *Optics Express*, vol. 29, no. 6, pp. 8964-8975, 2021.

[23] M. S. Morales, “The method of everything vs. experimenter bias of loophole-free Bell experiments,” *Frontiers in Research Metrics and Analytics*, vol. 9, p. 1404371, 2024.

[24] J. Wechs, C. Branciard, and O. Oreshkov, “Existence of processes violating causal inequalities on time-delocalised subsystems,” *Nature Communications*, vol. 14, p. 1471, 2023.

[25] S. Rawat, “Quaternion Klein-Gordon equation in the electromagnetic field,” *The European Physical Journal Plus*, vol. 137, no. 5, p. 621, 2022.

[26] A. Bhattacharyya, S. Ghosh, and S. Bhattacharya, “Separability criterion using one observable for special states: entanglement detection via quantum quench,” *Physical Review Research*, vol. 6, no. 2, p. 023132, 2024.

[27] P. Bej and S. Halder, “Unextendible product bases, bound entangled states, and the range criterion,” *Physics Letters A*, vol. 386, p. 126992, 2021.

[28] B. C. Chanyal, “On the quaternion transformation and field equations in curved space-time,” *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, vol. 91, pp. 1-13, 2021.

[29] M. Genovese, “Can quantum nonlocality be connected to extra dimensions?,” *International Journal of Quantum Information*, vol. 21, no. 6, p. 2340003, 2023.

[30] M. Fabbrichesi, R. Floreanini, and E. Gabrielli, “Constraining new physics in entangled two-qubit systems: top-quark, tau-lepton and photon pairs,” *The European Physical Journal C*, vol. 83, no. 2, p. 162, 2023.

[31] K. Bharti et al., “Noisy intermediate-scale quantum algorithms,” *Reviews of Modern Physics*, vol. 94, no. 1, p. 015004, 2022.

[32] A. Eddins, M. Motta, T. P. Gujarati, S. Bravyi, A. Mezzacapo, C. Hadfield, and S. Sheldon, “Doubling the size of quantum simulators by entanglement forging,” *PRX Quantum*, vol. 3, no. 1, p. 010309, 2022.

[33] A. Paler, L. M. Sasu, A.-C. Florea, and R. Andonie, “Machine learning optimization of quantum circuit layouts,” *ACM Transactions on Quantum Computing*, vol. 4, no. 2, pp. 1-25, 2023.

[34] L. Madden and A. Simonetto, “Best approximate quantum compiling problems,” *ACM Transactions on Quantum Computing*, vol. 3, no. 2, pp. 1-29, 2022.

[35] T. Jones and S. C. Benjamin, “Robust quantum compilation and circuit optimisation via energy minimisation,” *Quantum*, vol. 6, p. 628, 2022.

[36] S. Foulds, V. Kendon, and T. Spiller, “The controlled SWAP test for determining quantum entanglement,” *Quantum Science and Technology*, vol. 6, no. 3, p. 035002, 2021.

[37] M. Mastriani, “Quantum key secure communication protocol via enhanced superdense coding,” *Optical and Quantum Electronics*, vol. 55, no. 1, p. 10, 2023.

[38] J. Liu et al., “High-dimensional quantum key distribution using energy-time entanglement over 242 km partially deployed fiber,” *Quantum Science and Technology*, vol. 9, no. 1, p. 015003, 2024.

[39] Z. Cai, R. Babbush, S. C. Benjamin, S. Endo, W. J. Huggins, Y. Li, J. R. McClean, and T. E. O'Brien, “Quantum error mitigation,” *Reviews of Modern Physics*, vol. 95, no. 4, p. 045005, 2023.

[40] G. Gonzalez-Garcia, R. Trivedi, and J. I. Cirac, “Error propagation in NISQ devices for solving classical optimization problems,” *PRX Quantum*, vol. 3, no. 4, p. 040326, 2022.

[41] F. Basso Basset et al., “Daylight entanglement-based quantum key distribution with a quantum dot source,” *Quantum Science and Technology*, vol. 8, no. 2, p. 025002, 2023.

[42] A. Dahlberg, B. van der Vecht, and S. Wehner, “Analysis of multipartite entanglement distribution using a central quantum-network node,” *Physical Review A*, vol. 107, no. 1, p. 012609, 2023.

[43] G. Murta, F. Grasselli, H. Kampermann, and D. Bruss, “Quantum conference key agreement: a review,” *Advanced Quantum Technologies*, vol. 4, no. 11, p. 2000025, 2021.

[44] S. Aggarwal and S. Adhikari, “Search for an efficient entanglement witness operator for bound entangled states in bipartite quantum systems,” *Annals of Physics*, vol. 444, p. 169043, 2022.

[45] X.-M. Hu, Y. Guo, B.-H. Liu, C.-F. Li, and G.-C. Guo, “Progress in quantum teleportation,” *Nature Reviews Physics*, vol. 5, no. 6, pp. 339-353, 2023.

[46] Y. Wang and M.-L. Hu, “Quantum teleportation and dense coding in multiple bosonic reservoirs,” *Entropy*, vol. 24, no. 8, p. 1114, 2022.

[47] X.-M. Hu, W.-B. Xing, Y. Guo, M. Weilenmann, E. A. Aguilar, X. Gao, B.-H. Liu, Y.-F. Huang, C.-F. Li, and G.-C. Guo, “Optimized detection of high-dimensional entanglement,” *Physical Review Letters*, vol. 127, no. 22, p. 220501, 2021.

[48] S. Aggarwal, A. Kumari, and S. Adhikari, “Physical realization of realignment criteria using structural physical approximation,” *Physical Review A*, vol. 108, no. 1, p. 012422, 2023.

[49] J. Eisert, “Entangling power and quantum circuit complexity,” *Physical Review Letters*, vol. 127, no. 2, p. 020501, 2021.

[50] P. D. Nation and M. Treinish, “Suppressing quantum circuit errors due to system variability,” *PRX Quantum*, vol. 4, no. 1, p. 010327, 2023.

[51] L. Mariani, L. Salatino, C. Attanasio et al., “Simulation of an entanglement-based quantum key distribution protocol,” *The European Physical Journal Plus*, vol. 139, p. 602, 2024.

[52] L. Garms et al., “Experimental integration of quantum key distribution and post-quantum cryptography in a hybrid quantum-safe cryptosystem,” *Advanced Quantum Technologies*, vol. 7, no. 4, p. 2300304, 2024.

[53] J. Zhou, L. Wu, Y. Feng, H. Li, J. Shi, and R. Shi, “Four-state continuous-variable quantum key distribution with a hybrid linear amplifier,” *Quantum Information Processing*, vol. 22, no. 9, p. 356, 2023.

[54] J. A. Aguilar-Saavedra, “Laboratory-frame tests of quantum entanglement in H to WW,” *Physical Review D*, vol. 107, no. 7, p. 076016, 2023.

[55] V. Sbitnev, “Relativistic fermion and boson fields: Bose-Einstein condensate as a time crystal,” *Symmetry*, vol. 15, no. 2, p. 275, 2023.

[56] H. Sompel, W. Roga, and M. Takeoka, “Efficient Dicke state generation in a network of lossy channels,” *Physical Review A*, vol. 108, no. 1, p. 012612, 2023.

[57] F. Centrone, F. Grosshans, and V. Parigi, “Cost and routing of continuous-variable quantum networks,” *Physical Review A*, vol. 104, no. 4, p. 042610, 2021.

[58] A. Bhattacharyya, S. Ghosh, and S. Bhattacharya, “Machine-learning certification of multipartite entanglement for noisy quantum hardware,” *Quantum Machine Intelligence*, vol. 7, p. 12, 2025.

[59] C. Datta, S. Halder, and A. Streltsov, “Distributed quantum coherence and entanglement,” *Physical Review A*, vol. 106, no. 4, p. 042420, 2022.

[60] D. P. Mehendale and M. R. Modak, “On factorization of N-qubit pure states and complete entanglement analysis of 3-qubit pure states containing exactly two terms and three terms,” *Journal of Quantum Computing*, vol. 5, no. 1, pp. 15-24, 2023.

[61] Paul, P. J., & Wahid, S. (2022). Reformulating and strengthening the theory of Semi-structured Complex Numbers. International Journal of Applied Physics and Mathematics, 12(4), 34-58.