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Min Son

Min.Son@unibw.de

Universität der Bundeswehr München

Lars Zigan

Universität der Bundeswehr München

Tobias Sander

Universität der Bundeswehr München

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Tomographic Reconstruction of Infrared Radiance from Axisymmetric Sprays in a Hot Environment Using Multi-Parameter Regularization

Min Son^{1,2*}, Lars Zigan^{1,2} and Tobias Sander^{1,2}

¹Institute of Thermodynamics, Universität der Bundeswehr München, Werner-Heisenberg-Weg 39, Neubiberg, 85577, Germany.

²Forschungszentrum MORE, Universität der Bundeswehr München, Werner-Heisenberg-Weg 39, Neubiberg, 85577, Germany.

*Corresponding author(s). E-mail(s): Min.Son@unibw.de;
Contributing authors: Lars.Zigan@unibw.de; Tobias.Sander@unibw.de;

Abstract

This study proposes a robust algorithm for tomographic reconstruction of infrared radiance from an axisymmetric fuel spray in a hot environment by directly solving the radiative transfer equation. The proposed algorithm combines differential infrared thermography (DIT) with multi-parameter regularized inversion. First, the DIT emissivity is used to reconstruct the local extinction coefficient, and the local radiative source term is then recovered by solving the attenuated line-of-sight integral. To stabilize the ill-posed inversion, a global smoothness penalty is combined with an additional center-axis constraint that imposes the zero-gradient condition required by axisymmetry. A numerical phantom study, including configurations relevant to evaporating cold sprays and flashing jets, is conducted to evaluate the reconstruction accuracy under well-defined conditions and noisy measurement scenarios. The results demonstrate that the proposed method extends conventional DIT beyond the equivalent-slab approximation by resolving the line-of-sight-integrated emissivity into a local extinction field and accounting for self-attenuation within the spatially resolved axisymmetric spray structure. This approach allows reconstruction of the local radiative source term and local radiance with good agreement with the ground truth in the main spray region. Consequently, this establishes an IR-tomographic framework for recovering the internal radiance and potentially the temperature field of axisymmetric sprays with appropriate radiometric calibration. An additional noise-suppression strategy is introduced to improve reconstruction robustness at higher noise levels.

Keywords: Tomographic Reconstruction, Infrared Radiance, Spray, Differential Thermography, Multi-Parameter Regularization, Phantom Study

1 Introduction

Modern combustion engines are designed to operate at high chamber pressures and temperatures to improve combustion efficiency. In modern liquid rocket and internal combustion engines, the chamber pressure can exceed the critical pressure of the fuel, while the fuel temperature often remains low or close to room temperature as it exits the injection nozzle. The injected fuel is subsequently heated by the hot combustion gases and surrounding chamber structures, and its temperature may exceed the critical temperature during the injection process. This regime is commonly referred to as transcritical injection (Segal and Polikhov 2009) and involves complex thermodynamic spray behavior, including phase transition (Crua et al. 2017) and phase separation (Traxinger et al. 2019).

At the opposite pressure extreme, propellant sprays in upper-stage rocket engines may undergo substantial phase change before ignition because the chamber pressure can be close to vacuum. Under such conditions, the injected propellant may become highly superheated relative to the local saturation temperature and undergo rapid vaporization, a phenomenon known as flash boiling (Luo and Haidn 2016). Similar to transcritical injection, the rapid phase change and highly transient spray structure make quantitative measurements under flash-boiling conditions particularly challenging (Rees et al. 2020).

Therefore, quantitative investigations of spray behavior under extreme thermodynamic conditions, including both transcritical injection and flash boiling, are particularly relevant to high-performance combustion systems, in which rapid mixing and stable combustion are directly linked to system performance and stability (Liu et al. 2017b; Oschwald et al. 2006; Son et al. 2025).

To investigate sprays in hot environments, optical and laser-optical diagnostics have been widely used. Light extinction measurements (LEM) or diffuse-background illumination (DBI), often called simply “shadowgraphy”, is one of the most common techniques for observing the overall spray structure (Linne et al. 2010; Knorsch et al. 2013), and the liquid phase can be distinguished by combining it with Mie-scattering signals (Baab et al. 2018). Several studies have attempted to quantify spray characteristics from optical images, such as spray angle, penetration (Jeyaseelan et al. 2023), and liquid-air volume fraction (Weiss et al. 2020; Lazzaro et al. 2024). However, optical diagnostics are limited in dense sprays because extinction and multiple light interactions make quantitative interpretation difficult. Optical extinction tomography has been applied to reconstruct the internal structure of dense fuel sprays, although large optical depths require careful treatment to suppress reconstruction artifacts (Lazzaro et al. 2024). Laser-induced fluorescence (LIF) has also been employed to quantify spray characteristics, such as vapor temperature (Trost et al. 2013) and droplet size (Mishra et al. 2020). Nevertheless, LIF usually requires fluorescent tracers, which are typically added to the fuel, and its calibration becomes complicated under extreme conditions

due to strong phase changes, pyrolysis and chemical reaction of tracers, and low LIF signals. Rayleigh scattering can partially provide gas-phase temperature information, but it is generally restricted to diluted regions to avoid contributions of elastic scattering from droplets (Allison et al. 2016), which excludes the dense spray region of primary interest.

Infrared (IR) imaging is a non-intrusive two-dimensional diagnostic technique that has been widely applied to quantitative measurements in thermo-fluid systems (Carlo-magno and Cardone 2010). IR thermography has been applied to characterize thermal distributions of sprays (Lamanna et al. 2013; Goltzke et al. 2016) and droplets (Aka-fuah et al. 2010; Saha et al. 2010). However, IR thermography for sprays and droplets has typically been used to obtain surface temperature or effective temperature, rather than tomographic temperature inside dense sprays. The fundamental challenge of IR tomography originates from the radiative transfer equation (RTE), which describes the coupling between local emission and attenuation, as expressed in Eq. 1.

$$I_{\text{meas}} = I_{\text{bg}} \cdot e^{-\delta_{\text{max}}} + \int_0^{S_{\text{max}}} j(s) \cdot e^{-\delta(s)} ds \quad (1)$$

I_{meas} is the observed radiance, I_{bg} is the background radiance, $j(s)$ is the local emission source term, and s denotes the coordinate along the ray path of total length S_{max} , with the background side located at $s = 0$ and the detector located at $s = S_{\text{max}}$. $\delta(s)$ represents the local optical depth (LOD), i.e., the cumulative attenuation from a local position s to the detector, while δ_{max} denotes the total optical depth (TOD) of the medium along the entire line of sight.

Solving the RTE includes two major challenges, self-attenuation of radiation within the spray and the ill-posed inversion of the integral source term. Traditional tomographic studies based on optical diagnostics often simplify the inversion by assuming a transparent medium (Klinner and Willert 2012; Weiss et al. 2020; Cai and Kamin-ski 2017). However, in dense sprays or soot-laden flames, signal attenuation cannot be neglected (Yu et al. 2019). Li et al. (2018) solved the RTE for sooting flame tomography using a multi-wavelength technique to estimate the attenuation. However, this method relies on iterative estimation loops that may accumulate errors and assumes that attenuation is caused exclusively by soot particles (nanometer-scale), which is not directly applicable to spray studies since sprays contain micrometer-scale droplets.

Differential infrared thermography (DIT) has been used in spray studies to avoid direct inversion of the ill-posed integral term by estimating the spray transmittance $e^{-\delta_{\text{max}}}$ from Eq. 1 (Lamanna et al. 2013; Goltzke et al. 2016). Although DIT can provide temperature fields of sprays, the DIT temperature cannot represent the local spray temperature because the 3-D spray structure is simplified as a thin slab layer rather than being spatially resolved along the line of sight. The DIT temperature instead represents an effective apparent temperature associated with the region where the LOD is of the order of unity according to the Eddington–Barbier approximation (Rutten 2003), which has also been applied in spray studies (Goltzke et al. 2016). Therefore, DIT alone is insufficient to recover the internal radiance or temperature distribution of optically thick sprays. In the present study, the integrated slab emissivity obtained from DIT is decomposed to estimate the local extinction coefficient, and the local radiative source term is subsequently reconstructed by directly solving the RTE.

The remaining difficulty is the ill-posedness of the inverse problem. Iterative techniques such as the algebraic reconstruction technique (ART) have historically been used because direct inversion is unstable. However, iterative methods have been reported to converge slowly as the solution approaches the ground truth (Wan et al. 2015). They are also affected by the semi-convergence phenomenon, in which the solution initially approaches the correct profile but later diverges as measurement noise is amplified (Cai and Kaminski 2017; Liu et al. 2017a). To stabilize direct inversion, regularization techniques such as Tikhonov regularization (Tikhonov 1975) and total variation regularization (Bachmayr and Burger 2009) have been employed. Tikhonov regularization promotes smooth solutions, whereas total variation regularization favors piecewise-constant solutions and can introduce stair-casing effects. However, a single global regularization term is not always sufficient for axisymmetric spray tomography, particularly near the symmetry axis where numerical artifacts can occur due to the Abel-type inversion. The reconstruction of axisymmetric fields from line-of-sight measurements is particularly sensitive to the choice of inversion algorithm and measurement noise, as already revealed for axisymmetric background-oriented-shadowgraphy measurements (Xiong et al. 2020).

To overcome the limitations of single-penalty regularization, multi-penalty models have been investigated in inverse problems (Bortolotti et al. 2025). Ito et al. (2011) showed that multi-parameter models are useful when the reconstructed solution has multiple distinct characteristics. Gazzola and Reichel (2016) demonstrated that combining regularization matrices with different null spaces can preserve various physical features of the target. Chen et al. (2008) further showed that individual penalties tailored to different scales can improve the stability of inversion. Based on this concept, we introduce a multi-parameter regularization framework that combines a global smoothness penalty with an additional center-axis constraint for axisymmetric IR tomography. This method provides a new regularization-based strategy for analyzing axisymmetric sprays.

This study presents a robust framework for tomographic reconstruction of IR radiance from an axisymmetric spray in a hot environment by directly solving the RTE. The DIT technique is used to obtain the integrated emissivity, from which the local attenuation information is reconstructed. The local source term is then recovered using a multi-parameter regularized inversion. A numerical phantom study is conducted to demonstrate that the proposed algorithm can reconstruct the radiative source term even in the presence of measurement noise. Finally, an additional noise-suppression strategy is introduced to improve robustness under severely noisy measurement conditions.

2 Problem Definition

2.1 Radiative Transfer Equation

For IR tomography of sprays, a single IR camera simultaneously captures the spray radiance and the background radiance, as shown in Figure 1, thereby simplifying the experimental setup, as illustrated in Figure 2. In practical measurements, the signal is also affected by the chamber and optical windows. However, these contributions can be

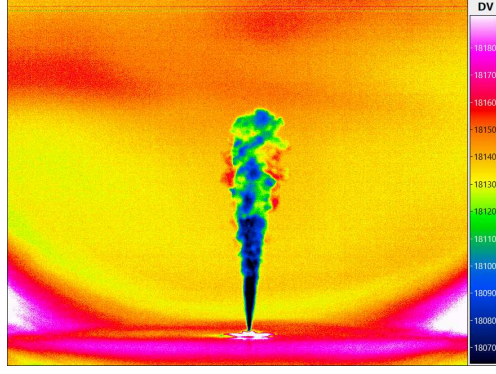


Fig. 1 Example IR-snapshot of a cyclopentane spray injected at room temperature into a nitrogen-filled chamber at 55 bar and 563 K. A gray-body radiator is in the background. The color bar represents the raw intensity values captured by the camera sensor through a bandpass filter ($3.410 \pm 0.040 \mu\text{m}$) for C-H stretching, before any postprocessing is applied.

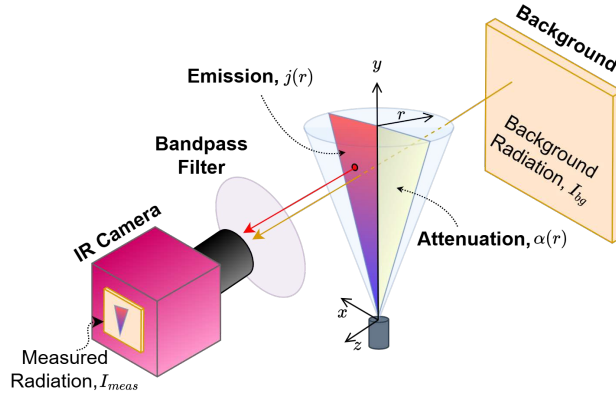


Fig. 2 Schematic of the experimental setup for axisymmetric evaporating/flashing jets.

corrected in practical measurements and are therefore neglected in the present study. To maximize the contribution of spray radiance, a bandpass filter is typically selected in a spectral region with strong absorption, such as the C–H stretching band for hydrocarbon fuels (Clark et al. 2009). In addition, the ambient gas should preferably be non-radiative and transparent within the selected spectral band. As a homonuclear diatomic molecule, nitrogen is a suitable choice due to the absence of strong dipole-active vibrational transitions in the mid-IR range, minimizing radiative interference (Bernath 2025). Additionally, nitrogen is chemically inert with respect to the fuel, avoiding additional radiative contributions from chemical reactions.

Based on the assumptions summarized in the previous section, the measured line-of-sight signal can be formulated as an axisymmetric radiative transfer problem. Radiative transfer through the optical system must account for local attenuation within the non-uniform spray structure (Chandrasekhar 2013). Since the fuel is

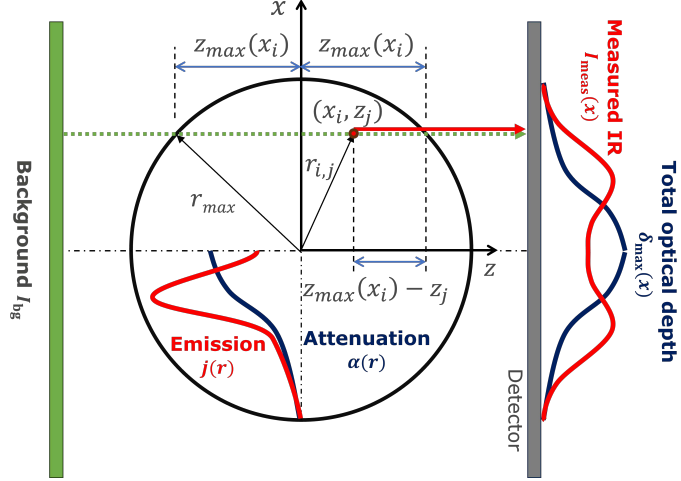


Fig. 3 Top-view schematic of the axisymmetric spray geometry and radiative transfer along the line of sight.

injected through a narrow nozzle and spreads into the combustion chamber, the spray core is typically denser, while the outer region becomes progressively diluted as an optically thin spray region. However, in a selected IR spectral range with strong fuel absorption, such as the C–H stretching band, attenuation is assumed to be dominated by absorption. Under this assumption, the same local absorption coefficient that determines the LOD also contributes to the local radiative source term through Kirchhoff’s law. In practice, the extinction coefficient profile may be more complex than a simple Gaussian distribution because the local absorption coefficient depends not only on fuel concentration but also on thermodynamic conditions such as temperature and pressure. For IR imaging in a narrow spectral band, the radiance described by the RTE can be related to the camera digital signal through radiometric calibration using a black-body calibrator (Golzke et al. 2016).

The inverse reconstruction proposed in this study starts from several assumptions that are required to reduce the radiative transfer problem to a tractable axisymmetric tomographic formulation. These assumptions define the range of validity of the reconstructed attenuation coefficient, source term, and local radiance.

First, the spray is assumed to be axisymmetric with respect to the injection axis, as shown in the top view in Figure 3. The line-of-sight coordinate is denoted by z , and the lateral image coordinate is denoted by x . The radial position is related to the image and line-of-sight coordinates as follows.

$$r = \sqrt{x^2 + z^2} \quad (2)$$

This assumption is reasonable for time-averaged spray images under steady-state conditions with a single-hole injector (Son et al. 2025). Therefore, the local radiative properties are treated as functions of the radial coordinate only, i.e., $\alpha = \alpha(r)$ and $j = j(r)$. Under this assumption, the line-of-sight measurement at a given axial position can be interpreted as a projection through a rotationally symmetric

spray cross-section. Any deviation from axisymmetry, e.g., spray bending, azimuthal non-uniformity, or lateral oscillation, would appear as a reconstruction error.

Second, scattering is not included explicitly in the inverse reconstruction. This approximation follows the practical treatment of DIT thermography, where the radiative signal is mainly interpreted in terms of emission and attenuation (Lamanna et al. 2013; Golzke et al. 2016). However, in dense liquid regions close to the nozzle, radiation originating from chamber structures (e.g. wall and windows) or heaters may be scattered by droplets and may contribute to the measured IR signal, as discussed by Golzke et al. Therefore, the reconstructed attenuation coefficient and source term should be interpreted as effective radiative quantities within the selected IR band. In practice, an appropriate bandpass filter, e.g., the C–H band for hydrocarbon fuels, should be used to enhance the relative contribution of spray emission and reduce background radiation from chamber structures or heaters, thereby reducing their potential scattering contribution to the measured signal.

Third, local thermodynamic equilibrium is assumed for the radiative emission of the spray. Under this assumption, the local emission can be directly related to the local temperature through a calibration-based black-body radiance in the selected wavelength band. In this study, the local radiance $I(r)$ does not represent an absolute physical spectral radiance, but rather an effective black-body radiometric signal obtained from the camera calibration. This assumption allows the reconstructed source term to be converted into an effective local radiance and, subsequently, into a temperature field using a calibration curve.

Assuming an axisymmetric spray structure and considering the top view shown in Figure 3, the measured camera signal $I_{\text{meas}}(x)$ at a given axial height and lateral position x can be expressed by Eq. 3.

$$I_{\text{meas}}(x) = I_{\text{bg}}(x) \cdot e^{-\delta_{\text{max}}(x)} + \int_{-z_{\text{max}}(x)}^{z_{\text{max}}(x)} j(r) \cdot e^{-\delta(x,z)} dz \quad (3)$$

In Eq. 3, the first term on the right-hand side represents the background radiation, considering its emissivity, after being attenuated by the two windows and the entire spray depth, and $\delta_{\text{max}}(x) = \delta(x, -z_{\text{max}}(x))$ is the TOD at a given x . Assuming an axisymmetric spray, the maximum optical path z_{max} is defined in Eq. 4. The latter integral term in Eq. 3 accounts for the local radiative source term $j(r)$ attenuated according to the LOD in the $+z$ -direction.

$$z_{\text{max}}(x) = \sqrt{r_{\text{max}}^2 - x^2} \quad (4)$$

The local radiative source term can be expressed in the general band-integrated form j_{tot} , consisting of thermal emission j_e and scattering j_s , as in Eq. 5, where Ω denotes the observation direction.

$$j_{\text{tot}}(r, \Omega) = j_e(r) + j_s(r, \Omega) \quad (5)$$

Under the assumption of local thermodynamic equilibrium, the emission term is defined in Eq. 6.

$$j_e(r) = \kappa(r) I(r) \quad (6)$$

Here, $\kappa(r)$ is the local absorption coefficient, and $I(r) = I(T(r))$ is the local radiance corresponding to the local temperature $T(r)$. It does not represent an absolute physical spectral radiance, but rather a camera-calibrated radiometric signal in arbitrary units within the selected wavelength band. The scattering term is given in Eq. 7, where $\sigma(r)$ is the local scattering coefficient, $p(\Omega', \Omega)$ is the phase function, and $I_{\text{inc}}(r, \Omega')$ denotes the incident radiance from direction Ω' .

$$j_s(r, \Omega) = \sigma(r) \int_{4\pi} p(\Omega', \Omega) I_{\text{inc}}(r, \Omega') d\Omega' \quad (7)$$

For practical application in the present study, scattering is neglected within the selected wavelength band (Sharkov 2003; Lamanna et al. 2013), such that $j(r) \approx j_e(r)$. This approach is well suited for vapor-phase flow and has been verified in the study of a flashing jet (Lamanna et al. 2013).

The local extinction coefficient is defined as the sum of the absorption and scattering coefficients.

$$\alpha(r) = \kappa(r) + \sigma(r) \quad (8)$$

The LOD $\delta(x_i, z_j)$ is then defined from the local extinction coefficient along the line of sight.

$$\delta(x_i, z_j) = \int_{z_j}^{z_{\max}(x_i)} \alpha \left(\sqrt{x_i^2 + \zeta^2} \right) d\zeta \quad (9)$$

Under negligible scattering, $\sigma \approx 0$ and $\alpha \approx \kappa$. Hence, Eq. 6 can be simplified as follows.

$$j(r) = \alpha(r) I(r) \quad (10)$$

The basic concept of this study is to solve the RTE to obtain the local radiance $I(r)$ and subsequently determine the local temperature $T(r)$ through radiometric calibration.

2.2 Reconstruction of Extinction Coefficient

The Differential Infrared Thermography (DIT) technique allows for the temperature measurement of a transmissive target with two different radiating backgrounds, assuming that the temperature or radiance of the target remains constant while the background changes (Sharkov 2003; Lamanna et al. 2013; Goltke et al. 2016). This means that the spray field remains statistically consistent between the two DIT measurements used to evaluate the effective slab emissivity. Therefore, the local attenuation coefficient and source term are considered identical for the two background conditions. If the spray structure or chamber radiation changes significantly between the two measurements, the estimated emissivity and the subsequent reconstruction contain additional uncertainty.

When the spray is represented as a thin slab instead of a three-dimensional volume, as illustrated in Figure 4, the integral term in Eq. 3 reduces to Eq. 11, where ε_{DIT} is the DIT emissivity and I_{DIT} denotes the DIT radiance.

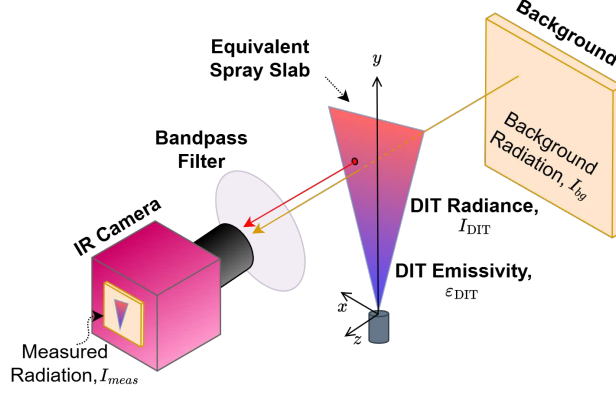


Fig. 4 Schematic of the slab approximation of a three-dimensional spray.

$$\int_{-z_{\max}(x)}^{z_{\max}(x)} j(r) \cdot e^{-\delta(x,z)} dz = \varepsilon_{\text{DIT}}(x) \cdot I_{\text{DIT}}(x) \quad (11)$$

By neglecting the scattering effects produced by the dispersed sprays, the reduction of the background radiation as it traverses the spray can be described in terms of the spray transmittance τ_{DIT} via the Beer–Lambert law, as given in Eq. 12a. Under this assumption, $\delta_{\max}$ is then determined using Eq. 12b.

$$\tau_{\text{DIT}}(x) = 1 - \varepsilon_{\text{DIT}}(x) = e^{-\delta_{\max}(x)} \quad (12a)$$

$$\delta_{\max}(x) = -\ln(1 - \varepsilon_{\text{DIT}}(x)) \quad (12b)$$

Under this approximation, the raw signals of two IR images, $I_{\text{meas},1}$ and $I_{\text{meas},2}$, obtained with two different background conditions, $I_{\text{bg},1}$ and $I_{\text{bg},2}$, can be written as follows.

$$I_{\text{meas},1}(x) = I_{\text{bg},1} (1 - \varepsilon_{\text{DIT}}(x)) + \varepsilon_{\text{DIT}}(x) I_{\text{DIT}}(x) \quad (13a)$$

$$I_{\text{meas},2}(x) = I_{\text{bg},2} (1 - \varepsilon_{\text{DIT}}(x)) + \varepsilon_{\text{DIT}}(x) I_{\text{DIT}}(x) \quad (13b)$$

By combining Eqs. 13a–13b, the integrated spray emissivity can be obtained from the two spray images and the corresponding background images without spray, as given in Eq. 14. This is used to obtain ε_{DIT} , written directly in terms of the four measured images and the two chamber-noise contributions.

$$\varepsilon_{\text{DIT}} = 1 - \frac{I_{\text{meas},2} - I_{\text{meas},1}}{I_{\text{bg},2} - I_{\text{bg},1}} \quad (14)$$

Consequently, I_{DIT} can be obtained from Eq. 15, and the corresponding temperature field can then be determined using the calibration curve.

$$I_{\text{DIT}} = \frac{I_{\text{meas},1} - I_{\text{bg},1} (1 - \varepsilon_{\text{DIT}})}{\varepsilon_{\text{DIT}}} \quad (15)$$

Note that I_{DIT} represents an integrated spray radiance and is therefore treated as the radiative contribution of an equivalent transmissive slab. This assumption has

previously been successfully used to determine the axial temperature of hot vapor sprays in a cold environment (Lamanna et al. 2013). However, for optically thick cold vapor sprays in a warm environment, the DIT temperature should be interpreted as an effective temperature associated mainly with the outer spray region where the LOD is of the order of unity (Golzke et al. 2016), which is also the case in the present study. For this reason, tomographic reconstruction is required to recover the cross-sectional radiance and temperature distributions within the spray.

Under the assumption of axisymmetry, the total optical depth $\delta_{\max}(x)$ is the Abel transform of the local extinction-coefficient profile $\alpha(r)$, as described in Eq. 16.

$$\delta_{\max}(x) = \int_{-z_{\max}(x)}^{z_{\max}(x)} \alpha\left(\sqrt{x^2 + \zeta^2}\right) d\zeta = 2 \int_x^{r_{\max}} \frac{\alpha(r) r}{\sqrt{r^2 - x^2}} dr \quad (16)$$

The spray cross-section is discretized into concentric rings. The continuous integral is approximated by a finite sum, which can be written in linear algebraic form.

$$\mathbf{\Delta} = \mathbf{K}\mathbf{A}, \quad (17)$$

$\mathbf{\Delta}$ is the known vector of TOD derived from Eq. 12b, $\mathbf{A}$ is the unknown vector of the radial extinction coefficient $\alpha(r)$, and $\mathbf{K}$ is the geometric kernel matrix describing the path length of each line of sight through each concentric ring. The element K_{ij} corresponds to the path length of the i -th ray passing through the j -th ring.

2.3 Reconstruction of Local Source Term

Once the local attenuation coefficient $\alpha(r)$ is determined, the local source term $j(r)$ can be reconstructed. By combining Eqs. 3 and 12b, the RTE can be divided into a known term and an unknown integral term.

$$\underbrace{I_{\text{meas}}(x) - I_{\text{bg}} \cdot (1 - \varepsilon_{\text{DIT}}(x))}_{\text{Known}} = \underbrace{\int_{-z_{\max}(x)}^{z_{\max}(x)} j(r(x, \zeta)) \cdot \exp\left[-\int_z^{z_{\max}(x)} \alpha(r(x, \zeta)) d\zeta\right] dz}_{\text{Unknown}} \quad (18)$$

The unknown term can be approximated by a discretized sum, as in Eq. 19, where $\mathbf{H}$ is the attenuation-weighted geometric projection matrix constructed from the corresponding LODs, and $\mathbf{J}$ is the vector of the local radiative source term $j(r)$.

$$\mathbf{G} = \mathbf{H}\mathbf{J}. \quad (19)$$

This equation represents a linear mapping from the local source term $j(r)$ to the known $g_{\text{true}}(x)$. To reconstruct $j(r)$, an optimization approach similar to that in the previous section can be adopted.

Consequently, the radiative source term of the axisymmetric spray is successfully reconstructed. From the relation between the radiative source term and the local radiance, as in Eq. 10, the local radiance of the spray was reconstructed using Eq. 20.

$$I(r) = \frac{j(r)}{\alpha(r)} \quad (20)$$

3 Phantom study

3.1 Phantom configuration

A numerical phantom study is conducted to validate the proposed two-step reconstruction procedure under well-defined conditions, where the exact radial distributions are known. The purpose of this study is to evaluate whether the inverse algorithm can recover the local extinction coefficient $\alpha(r)$ and the local radiative source term $j(r)$ from synthetic line-of-sight measurements affected by measurement noise. Since the ground-truth fields are prescribed analytically, the reconstructed results can be compared directly with the known reference profiles.

The phantom represents an axisymmetric spray cross-section. Therefore, all physical quantities are functions of the radial coordinate r only. Then, the computational domain is defined by the maximum radius.

$$r_{\max} = 2.0 \quad (21)$$

The radial grid is discretized using $N_r = 300$ points from 0 to $r_{\max}$. The line-of-sight integration is evaluated using $N_z = 300$ points. The same radial grid is used for both the image coordinate x and the radial coordinate r .

$$x_i = r_i, \quad i = 1, \dots, N_r \quad (22)$$

The forward calculation generates synthetic projection data from the prescribed radial fields. These projection data are then used as artificial measurements for the inverse reconstruction. In this way, the numerical phantom provides a controlled benchmark for assessing the stability and accuracy of the reconstruction algorithm.

3.2 Definition of the ground-truth phantom

Since we focus on the scenario of a cold vapor jet injected into a hot environment, the ground truth is generated assuming similar conditions. The extinction coefficient is assumed to be high at the spray center and to gradually decrease toward the outer boundary, representing a dense core and a relatively dilute periphery. Accordingly, the radial distribution of the extinction coefficient is represented by a Gaussian-like profile, as defined in Eq. 23, where $\alpha_0 = 3.0$ and $C_\alpha = 0.6$.

$$\alpha_{\text{true}}(r) = \alpha_0 \exp \left[- \left(\frac{r}{C_\alpha} \right)^2 \right] \quad (23)$$

Here, the maximum extinction coefficient $\alpha_0 = 3.0$ simulates very dense sprays near the nozzle, for which the corresponding TOD is approximately 3.19 and the transmittance is around 4%. It has been reported that the TOD often exceeds 10 for fuel sprays measured by shadowgraphy using a visible light source (Lehnert et al. 2023; Sonawane et al. 2026). However, TOD values in the range of 0.5–3.0 have been observed in spray studies in the mid-infrared range (Lamanna et al. 2013; Golzke et al. 2016). Therefore, $\alpha_0 = 3.0$ is applied to represent a challenging optical condition in this study. The profile parameter C_α , which governs the spray width, is selected accordingly.

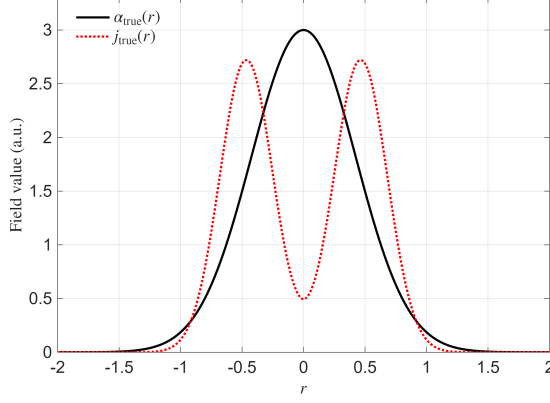


Fig. 5 Ground-truth radial profiles of the extinction coefficient α_{true} and radiative source j_{true} .

The ground-truth local source term is based on a Gaussian-type distribution. To generate a non-trivial, spatially displaced source distribution that mimics a hollow-cone structure, the base profile is symmetrically shifted and superposed. Although implemented step-by-step via discrete array manipulations in the numerical code, the final ground-truth radial source profile $j_{\text{true}}(r)$ is mathematically equivalent to the following closed analytical form, where r_{shift} represents the radial displacement of the source peaks from the symmetry axis, which is the number of 140 grid points in this study.

$$j_{\text{true}}(r) = J_0 \exp \left[1 - \left(\frac{r - r_{\text{shift}}}{C_J} \right)^2 \right] + J_0 \exp \left[1 - \left(\frac{r + r_{\text{shift}}}{C_J} \right)^2 \right] \quad (24)$$

The parameters governing the amplitude and shape of the emission zones are appropriately set to $J_0 = 1.0$ and $C_J = 0.3$ to simulate an extreme case of an evaporating spray injected into a hot environment; the jet core remains cold, while its outer edge is hot. Because of the strong heat transfer at the spray boundary, the radiative source term exhibits a maximum at a specific location near the spray edge. As the surrounding gas is considered non-radiative, the radiation intensity then decreases again toward the outer edge of the spray. This formulation produces a smooth but spatially displaced source distribution with distinct off-axis peaks, as shown in Figure 5, which is fundamentally more challenging to reconstruct than a simple centered Gaussian profile.

The corresponding ground-truth local radiance of the phantom can be expressed using the relation defined in Eq. (20).

$$I_{\text{true}}(r) = \frac{j_{\text{true}}(r)}{\alpha_{\text{true}}(r)} \quad (25)$$

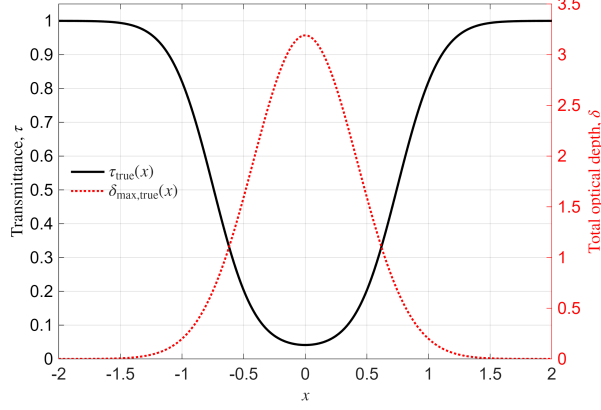


Fig. 6 Ground-truth lateral profiles of the transmittance τ_{true} and total optical depth $\delta_{\text{max,true}}$.

3.3 Synthetic forward projection

Using the prescribed ground-truth fields, the noiseless line-of-sight signal was calculated from the radiative transfer model. For each lateral position x_i , the projected source contribution $I_{\text{proj,true}}$ can be defined.

$$I_{\text{proj,true}}(x_i) = \int_{-z_{\text{max}}(x_i)}^{z_{\text{max}}(x_i)} j_{\text{true}} \left(\sqrt{x_i^2 + z^2} \right) \cdot \exp \left[- \int_z^{z_{\text{max}}(x_i)} \alpha_{\text{true}} \left(\sqrt{x_i^2 + \zeta^2} \right) d\zeta \right] dz \quad (26)$$

Then, the true TOD $\delta_{\text{max,true}}$ and the overall transmittance of the phantom obtained from the TOD can be calculated as shown in Figure 6.

To apply DIT, two different black-body background radiances are used for the two background temperatures, as $I_{\text{bg},1} = 2.0$ and $I_{\text{bg},2} = 4.0$. Hence, the corresponding measured signals are defined as Eqs. 27a and 27b as shown in Figure 7.

$$I_{\text{meas},1}(x_i) = I_{\text{proj,true}}(x_i) + \tau_{\text{true}}(x_i) I_{\text{bg},1} \quad (27a)$$

$$I_{\text{meas},2}(x_i) = I_{\text{proj,true}}(x_i) + \tau_{\text{true}}(x_i) I_{\text{bg},2} \quad (27b)$$

Combining Eqs. 14 and 15, the DIT emissivity ε_{DIT} and DIT radiance I_{DIT} can be derived, as shown in Figure 8. It is clear that the radiance predicted by DIT, I_{DIT} , is lower than the ground truth I_{true} at the peak and higher in the central region, increasing the discrepancy along the centerline where the spray is optically thick. In addition, the peak position of I_{DIT} is shifted relative to the centerline compared to the ground truth. However, I_{DIT} relatively agrees with the ground truth at the outer spray boundary, suggesting that the diluted spray boundary temperature estimated by the DIT could be acceptable. Nevertheless, this comparison indicates that the DIT alone is inadequate to evaluate the spray temperature under these harsh conditions.

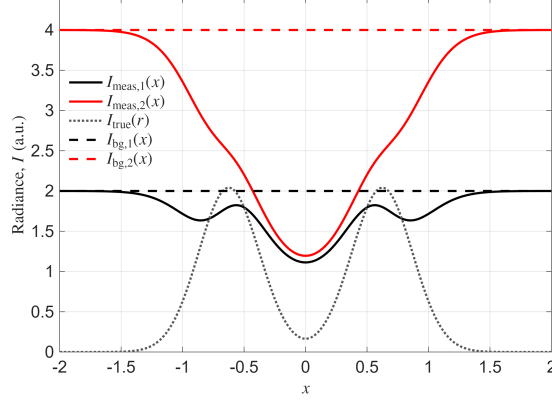


Fig. 7 Lateral profiles of the measured signals with two different backgrounds.

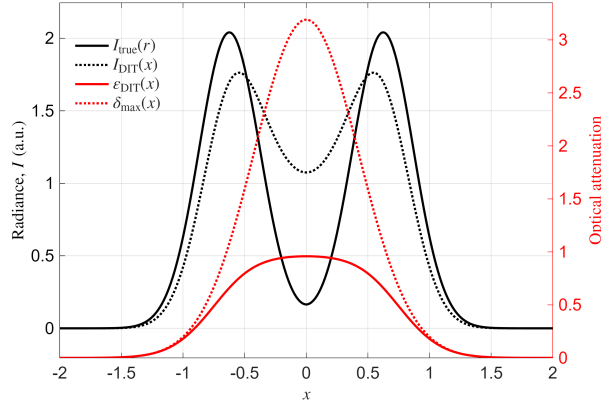


Fig. 8 Lateral profiles of the DIT emissivity ε_{DIT} and DIT radiance I_{DIT} .

3.4 Noise Modeling

In real applications, however, the measured signals unavoidably contain noise caused by various sources, such as pixel bias of the image sensor and strong turbulent flow. Therefore, it is necessary to evaluate the robustness of algorithms using noisy datasets. To investigate this issue, Gaussian random noise is generated and added to the measured intensity I_{meas} . The standard deviation of the noise s_{noise} is prescribed relative to the maximum value of the clean profile as defined by Eq. 28, where ϵ_{noise} is the relative noise level.

$$s_{\text{noise}} = \epsilon_{\text{noise}} \max(I_{\text{meas}}) \quad (28)$$

Then, the noisy measurement data are generated by the following equation, where ξ is an independent random variable following a standard normal distribution. In the numerical implementation, a fixed random seed is used to ensure reproducibility.

$$I_{\text{meas,noise}}(x) = I_{\text{meas}}(x) + s_{\text{noise}}\xi(x) \quad (29)$$

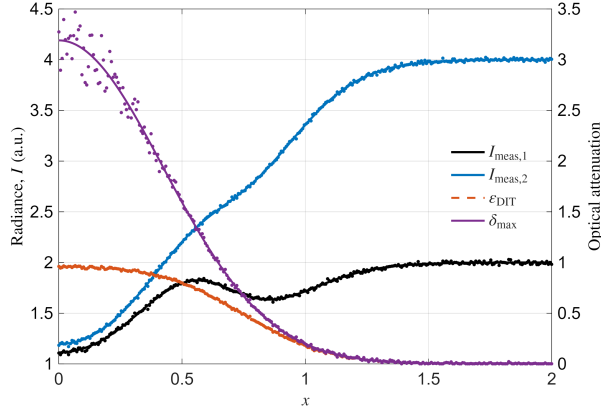


Fig. 9 Phantom data with 0.3% noise. Lines represent the clean phantom data, while markers indicate the corresponding noisy data.

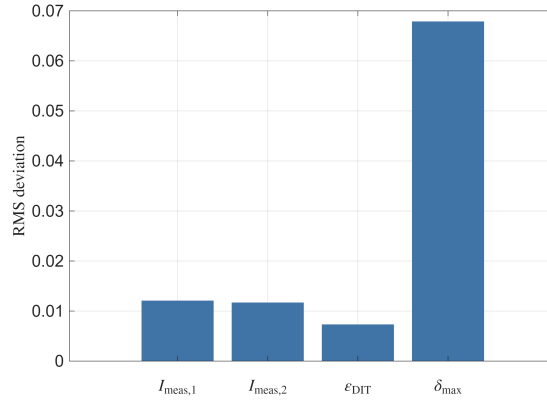


Fig. 10 Comparison of RMS values for the phantom data with 0.3% noise.

First, small perturbations corresponding to a noise level of 0.3%, i.e., $\epsilon_{\text{noise}} = 0.003$, are applied to the two measured radiances, $I_{\text{meas},1}$ and $I_{\text{meas},2}$, as shown in Figure 9. The effect of the noise level is further discussed in Section 6. A comparison of the root-mean-square (RMS) deviations in Figure 10 clearly indicates that noise propagation into the DIT emissivity ϵ_{DIT} is mitigated, whereas the noise in δ_{max} is markedly amplified. This behavior arises from the definition of the TOD in Eq. 12b; δ_{max} is related to ϵ_{DIT} through the logarithmic relation $\delta_{\text{max}} = -\ln(1 - \epsilon_{\text{DIT}})$, making δ_{max} increasingly sensitive to small variations in ϵ_{DIT} as the emissivity increases. Consequently, a stable inversion algorithm that is robust against noisy data is necessary.

4 Inversion Problem

To reconstruct the spray radiance profile, two inverse problems have to be solved; the local extinction coefficient $\alpha(r)$ as in Eq. 17, and the local source term $j(r)$ as in Eq. 19. Since both inverse problems share the same linear structure, those can be

written in the integrated form of Eq. 30, where $\mathbf{y}$ represents the measured data vector, $\mathbf{F}$ is the forward operator and $\mathbf{x}$ is the unknown radial field to be reconstructed.

$$\mathbf{y} = \mathbf{F}\mathbf{x} \quad (30)$$

For the two cases considered in this study, the variables are defined as follows, corresponding to Eqs. 17 and 19.

- Extinction coefficient: $\mathbf{y} = \mathbf{\Delta}$, $\mathbf{F} = \mathbf{K}$, $\mathbf{x} = \mathbf{A}$
- Source term: $\mathbf{y} = \mathbf{G}$, $\mathbf{F} = \mathbf{H}$, $\mathbf{x} = \mathbf{J}$

We can then apply numerical algorithms to address the inverse problem and obtain $\mathbf{x}$ by inverting the forward operator $\mathbf{F}$. In the following sections, we mainly compare two approaches (i.e. direct inversion and iterative inversion) and afterward discuss the limitations of these methods under noisy measurement conditions.

4.1 Direct Inversion

As the most straightforward approach, Eq. 30 can be directly inverted for both cases as follows.

$$\mathbf{x} = \mathbf{F}^{-1}\mathbf{y} \quad (31)$$

To keep numerical stability for inversion, Cholesky factorization (Anderson et al. 1999) is applied. Since Cholesky factorization requires symmetric positive definiteness, the transposed forward operator is additionally multiplied before factorization.

$$\mathbf{F}^\top \mathbf{y} = \mathbf{F}^\top \mathbf{F} \mathbf{x} = \mathbf{B} \mathbf{x} \quad (32)$$

Then the symmetric matrix $\mathbf{B}$ can be factorized using a low triangle matrix $\mathbf{L}_c$ as follows.

$$\mathbf{B} = \mathbf{L}_c \mathbf{L}_c^\top \quad (33)$$

$$\mathbf{L}_c = \begin{bmatrix} \ell_{11} & 0 & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & 0 & \cdots & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & \ell_{nn} \end{bmatrix} \quad (34)$$

Consequently, the unknown $\mathbf{x}$ can be stably reconstructed.

$$\mathbf{x} = (\mathbf{L}_c^\top)^{-1} (\mathbf{L}_c)^{-1} \mathbf{F}^\top \mathbf{y} \quad (35)$$

Using the direct inversion approach, the radial distributions of the extinction coefficient α and the radiative source term j are reconstructed, as shown in Figure 11. To exclude additional interference from the extinction coefficient, the local source term is reconstructed using α_{true} .

The results indicate that both α and j are accurately reconstructed, except for a minor disturbance in α near the centerline. Based solely on these results, the direct

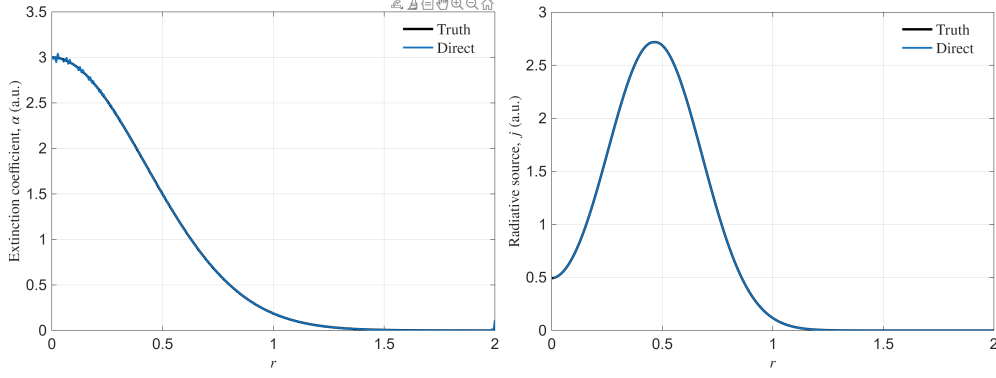


Fig. 11 Reconstructed radial distribution by using direct inversion; (left) the extinction coefficient α and (right) the radiative source term j with the true extinction coefficient α_{true} .

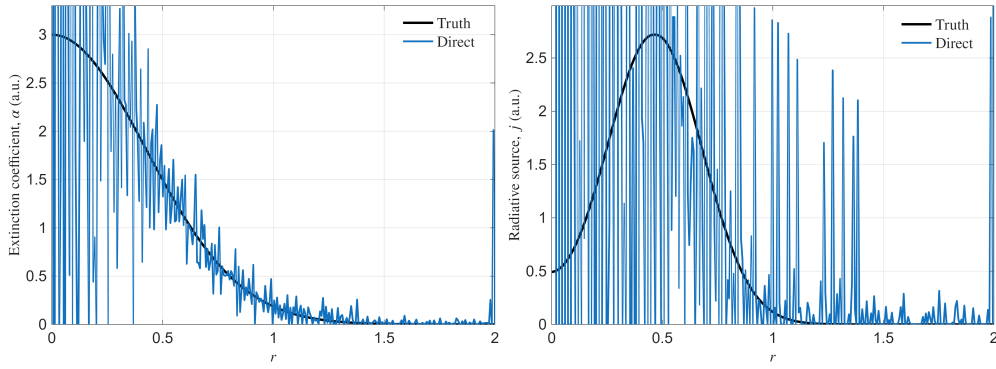


Fig. 12 Reconstructed radial distribution by using direct inversion (0.3% noise); (left) the extinction coefficient α and (right) the radiative source term j with the true extinction coefficient α_{true} .

inversion method appears to be suitable. However, as expected, direct inversion does not successfully reconstruct noisy phantom data with a noise level of 0.3%, as shown in Figure 12. It is obvious that the direction inversion algorithm is highly vulnerable to even small perturbations in the measured data.

4.2 Iterative Inversion

To overcome failure of the direct inversion approach, iterative inversion algorithms can be employed, such as Simultaneous Iterative Reconstruction Techniques (SIRT), which have been successfully used in tomography studies (Grauer et al. 2023). These methods are developed to reduce the high computational cost of performing a line by-line inversion by updating multiple lines simultaneously. SIRT solvers adopt a generalized update rule for the unknown vector $\mathbf{x}$. For the forward model $\mathbf{y} = \mathbf{F}\mathbf{x}$, the estimate at the $(k + 1)$ -th iteration is obtained according to Eq. 36, where ω is the additional relaxation parameter, $\mathbf{M}$ and $\mathbf{T}$ are diagonal weighting matrices.

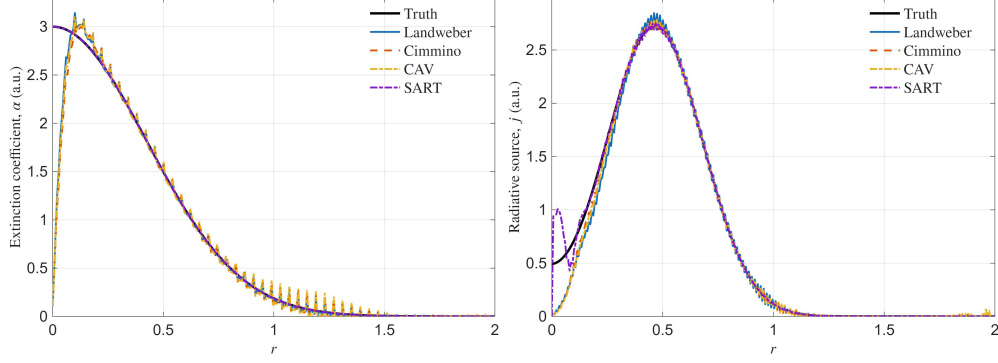


Fig. 13 Radial distributions reconstructed using the SIRT algorithm: (left) extinction coefficient α and (right) radiative source term j reconstructed using the ground-truth extinction coefficient α_{true} .

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \omega \mathbf{T} \mathbf{F}^T \mathbf{M} (\mathbf{y} - \mathbf{F} \mathbf{x}^{(k)}) \quad (36)$$

In this study, four different algorithms are investigated, which can be determined by the choice of weighting matrices $\mathbf{M}$ and $\mathbf{T}$, as summarized in Table 1. $\mathbf{M}$ controls the residuals along the line-of-sight, while $\mathbf{T}$ controls the back-projection residuals.

The Landweber method (Landweber 1951) represents the steepest descent in the least-squares functional. In the Cimmino method (Cimmino 1938), m denotes the number of measurements related to the spray layer on the y-axis of Figure 2 in this study. In the component averaging method, so called CAV (Censor et al. 2008), $\mathbf{S}_M$ represents a diagonal sparsity scaling matrix where each j -th diagonal element is defined as $\sqrt{s_j}$, with s_j being the number of non-zero elements in the j -th column of the forward operator $\mathbf{F}$. Consequently, the modified row norm becomes $\|\mathbf{S}_M \mathbf{F}_{i,*}\|_2^2 = \sum_j s_j F_{i,j}^2$. This term acts as a sparsity-oriented weighting factor derived from generalized oblique projections, effectively scaling the updates based on the intersection density of the line-of-sight rays. The Simultaneous Algebraic Reconstruction Technique (SART) (Andersen and Kak 1984) has been widely used in tomographic applications, normalizing the updates by the L_1 -norms of the rows and columns to improve convergence speed and robustness in the presence of noise.

The subsequent phantom study, these iterative methods are executed for $N = 1000$ iterations to ensure convergence. Although these iterative approaches are computationally more stable than direct inversion, they often show semi-convergence even without noise, as compared in Figure 13 (left). The α reconstructed with Landweber,

Table 1 Diagonal weighting matrices for the SIRT algorithms.

Method	$\mathbf{T}_{j,j}$	$\mathbf{M}_{i,i}$
Landweber (Landweber 1951)	1	1
Cimmino (Cimmino 1938)	1	$m^{-1} \ \mathbf{F}_{i,*}\ _2^{-2}$
CAV (Censor et al. 2008)	1	$\ \mathbf{S}_M \mathbf{F}_{i,*}\ _2^{-2}$
SART (Andersen and Kak 1984)	$\ \mathbf{F}_{*,j}\ _1^{-1}$	$\ \mathbf{F}_{i,*}\ _1^{-1}$

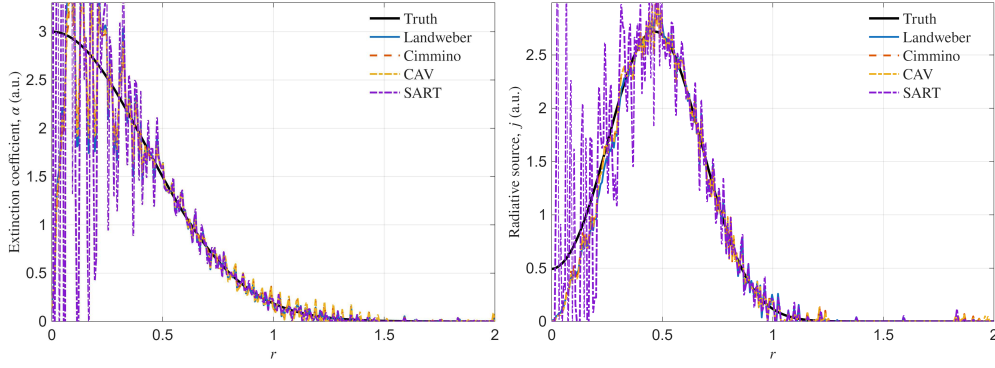


Fig. 14 Reconstructed radial distribution by using SIRT algorithms (0.3% noise): (left) extinction coefficient α and (right) radiative source term j reconstructed using the ground-truth extinction coefficient α_{true} .

Cimmino and CAV presents overall oscillations and a sharp decay near the centerline, i.e., at $r = 0$, whereas the SART method follows the original profile more smoothly. These features are also observed in the source-term reconstruction when using the true α , as shown in Figure 13 (right). SART obviously performs better than the other methods, but it still fails to accurately reconstruct near the centerline.

When 0.3% noise is introduced into the measurement dataset, the reconstruction method achieves better overall performance than direct inversion. However, as shown in Figure 14, iterative methods remain highly sensitive to noise. Under noisy conditions, the other methods surpass SART, in contrast to the clean-data case; in particular, the radiative source term j can be reconstructed with fairly acceptable accuracy. Despite this improvement, predicting the centerline remains an inherent vulnerability of the iterative approach. Consequently, an alternative method is required to address this issue.

5 Regularized Inversion

5.1 Single-Parameter Tikhonov Regularization and its Limitations

To mitigate the noise amplification observed in the direct and iterative inversions, a standard approach is to introduce Tikhonov regularization in the second-order form (Tikhonov 1975; Daun et al. 2006). The objective function including regularization is defined as Eq. 37.

$$\Phi(\mathbf{x}) = \underbrace{\|\mathbf{F}\mathbf{x} - \mathbf{y}\|_2^2}_{\text{Data fidelity}} + \underbrace{\mu_2 \|\mathbf{L}_2 \mathbf{x}\|_2^2}_{\text{Tikhonov regularization}} \quad (37)$$

The first term comes from the direct inversion defined as Eq. 30, where $\mathbf{F}$ is the forward operator, $\mathbf{y}$ is the measured data vector, $\mathbf{x}$ is the unknown radial field. The second term denotes the Tikhonov regularization, where $\mathbf{L}_2$ is the second-order

finite-difference operator and μ_2 is the regularization weight. The first term induces agreement of the data fidelity with the measured projection data, whereas the second term suppresses excessive curvature and high-frequency oscillations from noise in the reconstructed radial profile.

The second-order regularization operator $\mathbf{L}_2$ is defined with spacing Δr by the finite-difference approximation, as described in Eq. 38.

$$\mathbf{L}_2 = \frac{1}{\Delta r^2} \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \end{bmatrix}. \quad (38)$$

Minimizing the objective function Φ means that the derivative of Φ should be zero as $\frac{\partial \Phi}{\partial \mathbf{x}}$. To obtain the derivative, each term of Φ can be expanded as follows.

$$\|\mathbf{F}\mathbf{x} - \mathbf{y}\|_2^2 = (\mathbf{F}\mathbf{x} - \mathbf{y})^\top (\mathbf{F}\mathbf{x} - \mathbf{y}) = \mathbf{x}^\top \mathbf{F}^\top \mathbf{F} \mathbf{x} - 2\mathbf{x}^\top \mathbf{F}^\top \mathbf{y} + \mathbf{y}^\top \mathbf{y} \quad (39a)$$

$$\mu_2 \|\mathbf{L}_2 \mathbf{x}\|_2^2 = \mu_2 \mathbf{x}^\top \mathbf{L}_2^\top \mathbf{L}_2 \mathbf{x} \quad (39b)$$

Therefore, the full objective function becomes Eq. 40.

$$\Phi(\mathbf{x}) = \mathbf{x}^\top \mathbf{F}^\top \mathbf{F} \mathbf{x} - 2\mathbf{x}^\top \mathbf{F}^\top \mathbf{y} + \mathbf{y}^\top \mathbf{y} + \mu_2 \mathbf{x}^\top \mathbf{L}_2^\top \mathbf{L}_2 \mathbf{x} \quad (40)$$

The derivative with respect to $\mathbf{x}$ is then derived as Eq. 41, setting to zero yields.

$$\frac{\partial \Phi}{\partial \mathbf{x}} = 2\mathbf{F}^\top \mathbf{F} \mathbf{x} - 2\mathbf{F}^\top \mathbf{y} + 2\mu_2 \mathbf{L}_2^\top \mathbf{L}_2 \mathbf{x} = 0 \quad (41)$$

The regularized normal equation is consequently obtained as follows.

$$(\mathbf{F}^\top \mathbf{F} + \mu_2 \mathbf{L}_2^\top \mathbf{L}_2) \mathbf{x} = \mathbf{F}^\top \mathbf{y} \quad (42)$$

$$\mathbf{B}_T \mathbf{x} = \mathbf{F}^\top \mathbf{y} \quad (43)$$

In Eq. 43, $\mathbf{B}_T$ is decomposed using Cholesky factorization, in the same manner as in the direct inversion approach described by Eqs. 33 and 34.

The quality of the obtained solution depends on the regularization weight μ_2 , i.e., an appropriate value of μ_2 has to be determined. For this purpose, the L-curve has been widely adopted (Regińska 1996; Hansen 2001) and has been used successfully in the tomography of axisymmetric flame (Åkesson and Daun 2008). The L-curve is constructed on a logarithmic scale using two parameters; the residual norm $\log_{10} \|\mathbf{F}\mathbf{x}_k - \mathbf{y}\|_2$ on the x-axis and the second-derivative regularization norm $\log_{10} \|\mathbf{L}_2 \mathbf{x}_k\|_2$ on the y-axis, evaluated for each candidate value $\mu_{2,k}$ within the scanning range, where $k = 1, \dots, N_k$. Here, $\mu_{2,k}$ is scanned from 10^{-10} to 10^{-2} , when $N_k = 201$.

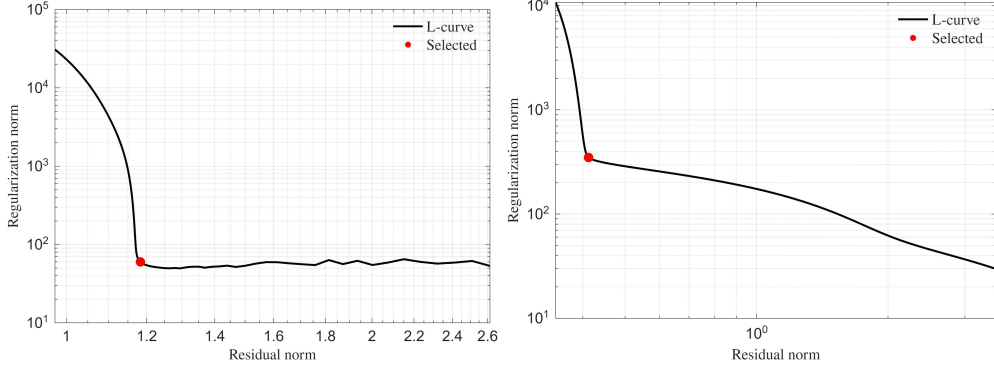


Fig. 15 L-curves and selected corner points using Tikhonov regularization (0.3% noise): (left) the extinction coefficient α and (right) the radiative source term j reconstructed using the ground-truth extinction coefficient α_{true} .

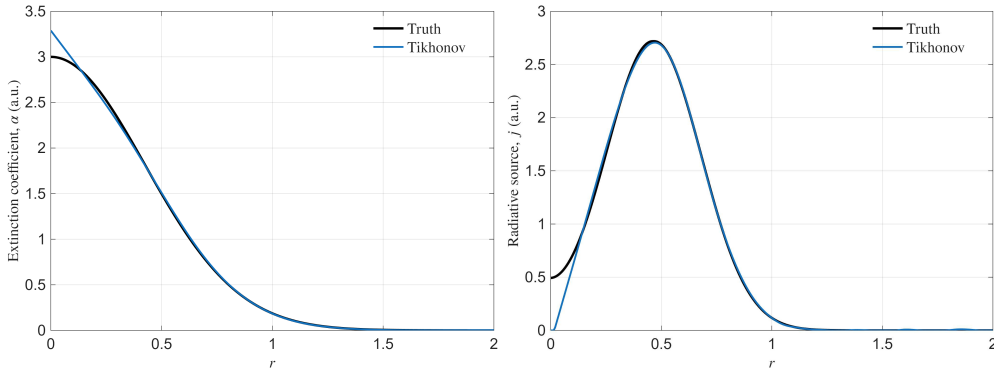


Fig. 16 Radial distributions reconstructed using Tikhonov regularization (0.3% noise): (left) the extinction coefficient α and (right) the radiative source term j reconstructed using the ground-truth extinction coefficient α_{true} .

The optimal point is known as the corner of the L-curve, as shown in Figure 15. This corner represents the point where a further reduction of the residual norm would require an excessive increase in the curvature of the reconstructed solution. Therefore, the selected μ_2 at the corner provides a compromise between projection fidelity and suppression of noise-induced oscillations (Hansen 2001).

To determine the optimal value of μ_2 , the curvature of the L-curve has been extensively employed (Åkesson and Daun 2008; Hansen 2001), and the minimum distance criterion can also be used as an approximation (Belge et al. 2002). To address failures in searching for the global corner in a very smooth L-curve (Hansen and O’Leary 1993), alternative parameters instead of the norms, e.g., μ_2 and $\mathbf{x}_k$, have been proposed (Rezghi and Hosseini 2009). Since optimizing μ_2 is beyond the scope of this study, the corner points were chosen manually; the sensitivity to small variations in μ_2 was negligible.

Using the selected μ_2 , the extinction coefficient α , and the radiative source term j with the true extinction coefficient are stably reconstructed even with 0.3% noise as shown in Figure 16. Although the Tikhonov formulation improves the stability of the inversion, it has an important limitation in axisymmetric tomography. The second-order operator $\mathbf{L}_2$ acts globally on the radial profile and does not explicitly impose the physical center-axis condition; the radial gradient should vanish at the symmetry axis for an axisymmetric scalar field, i.e. $\frac{\partial x}{\partial r}|_{r=0} = 0$. This limitation encourages an additional regularization term that can impose center-axis consistency.

5.2 Multi-regularized Inversion

As discussed above, Tikhonov regularization effectively stabilizes the reconstruction. However, the key requirement for the axisymmetric formulation, i.e. a zero gradient at the center, is not satisfied. To address this limitation, we introduce an additional regularization term in the objective function $\Phi(\mathbf{x})$, as given in Eq. 44, where μ_0 is the weighting factor for the softening constraint at the axis.

$$\Phi(\mathbf{x}) = \underbrace{\|\mathbf{F}\mathbf{x} - \mathbf{y}\|_2^2}_{\text{Datafidelity}} + \underbrace{\mu_2 \|\mathbf{L}_2 \mathbf{x}\|_2^2}_{\text{Tikhonov regularization}} + \underbrace{\mu_0 \|\mathbf{L}_0 \mathbf{x}\|_2^2}_{\text{Center-axis constraint}} \quad (44)$$

The center-axis can be smoothed by penalizing steep gradients close to the symmetry axis, as expressed in Eq. 45.

$$\mathbf{L}_0 = \frac{1}{\Delta r} [-1 \ 1 \ 0 \ \cdots \ 0] \quad (45)$$

The operator $\mathbf{L}_0$ effectively constrains the gradient of the reconstructed radial profile near the axis by enforcing $x_2 - x_1 \approx 0$, which damps oscillatory behavior while maintaining the overall trend near the axis. This strategy is reasonable when the x-grid at the center is sufficiently refined, i.e., when the image resolution is fine enough. This term is also useful for removing numerical artifacts near the axis by suppressing the non-physical jump in the reconstructed profile at the first two radial nodes.

To differentiate the new objective function, the additional constraint term can be expanded as follows.

$$\mu_0 \|\mathbf{L}_0 \mathbf{x}\|_2^2 = \mu_0 \mathbf{x}^T \mathbf{L}_0^T \mathbf{L}_0 \mathbf{x} \quad (46)$$

Therefore, the fully expanded objective function becomes Eq. 47.

$$\Phi(\mathbf{x}) = \mathbf{x}^T \mathbf{F}^T \mathbf{F} \mathbf{x} - 2\mathbf{x}^T \mathbf{F}^T \mathbf{y} + \mathbf{y}^T \mathbf{y} + \mu_2 \mathbf{x}^T \mathbf{L}_2^T \mathbf{L}_2 \mathbf{x} + \mu_0 \mathbf{x}^T \mathbf{L}_0^T \mathbf{L}_0 \mathbf{x} \quad (47)$$

Then, we can take the derivative with respect to $\mathbf{x}$ and set it to zero yields as in Eq. 48.

$$\frac{\partial \Phi}{\partial \mathbf{x}} = 2\mathbf{F}^T \mathbf{F} \mathbf{x} - 2\mathbf{F}^T \mathbf{y} + 2\mu_2 \mathbf{L}_2^T \mathbf{L}_2 \mathbf{x} + 2\mu_0 \mathbf{L}_0^T \mathbf{L}_0 \mathbf{x} = 0 \quad (48)$$

Consequently, the regularized equation takes the same form as $\mathbf{B}_M \mathbf{x}$, and can be solved using factorization as described in the previous section.

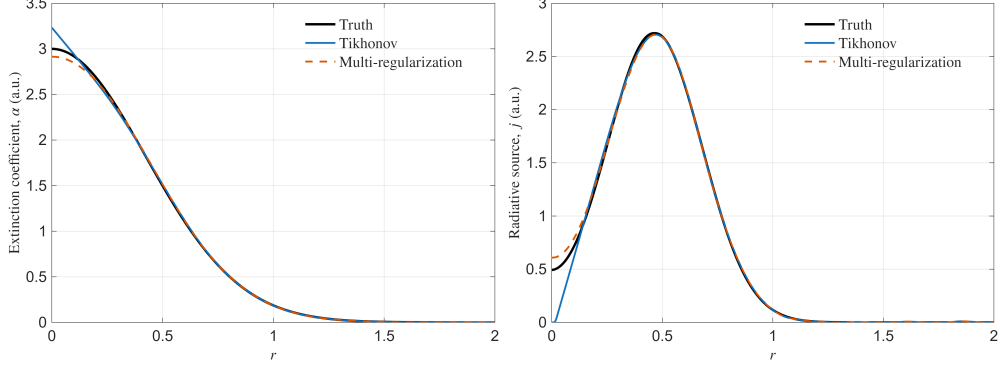


Fig. 17 Comparison of radial distributions reconstructed using single- and multi-regularization approaches (0.3% noise): (left) the extinction coefficient α and (right) the radiative source term j reconstructed using the ground-truth extinction coefficient α_{true} .

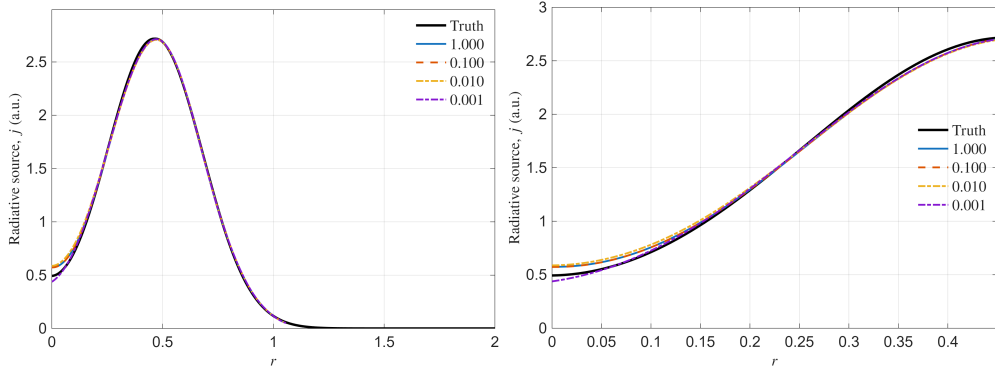


Fig. 18 Effect of the center-axis constraint weighting factor μ_0 on the reconstructed radiative source term j using the ground-truth extinction coefficient α_{true} : (left) original scale and (right) close-up near the axis.

$$(\mathbf{F}^T \mathbf{F} + \mu_2 \mathbf{L}_2^T \mathbf{L}_2 + \mu_0 \mathbf{L}_0^T \mathbf{L}_0) \mathbf{x} = \mathbf{F}^T \mathbf{y} \quad (49)$$

$$\mathbf{B}_M \mathbf{x} = \mathbf{F}^T \mathbf{y} \quad (50)$$

The radiation profiles of α and j reconstructed using the proposed multi-regularized inversion are accurately corrected near the axis, as illustrated in Figure 17, although the 0.3% noise influences the central amplitude. Here, $\mu_0 = 1.0$ is applied. This clearly indicates that the additional regularization term effectively satisfies the symmetry. Moreover, as shown in Figure 18, varying μ_0 from 1.0 to 0.001 has a negligible impact on the results.

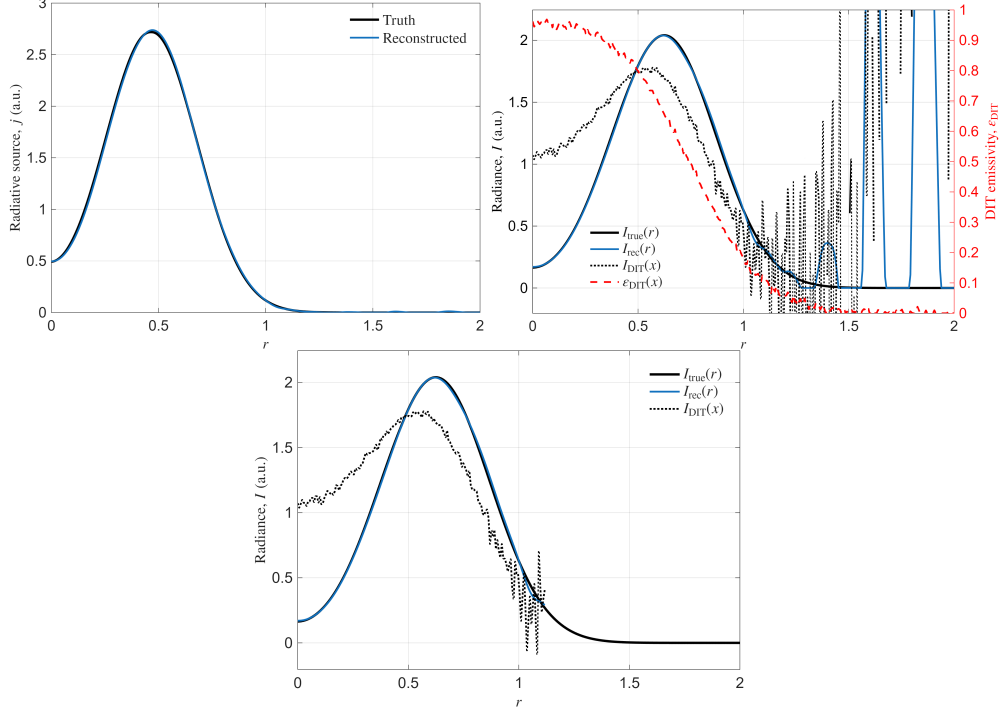


Fig. 19 Fully reconstructed results using the extinction coefficient α obtained by multi-regularization inversion (0.3 % noise): (left) the radiative source term j , (right) the reconstructed local radiance I_{rec} compared with the DIT radiance I_{DIT} , and (bottom) the masked local radiance I_{rec} .

6 Full Reconstruction Results and Discussion

Based on 0.3 % noisy measurement data, the radiative source term j_{rec} is reconstructed together with the extinction coefficient α_{rec} , as shown in Figure 19 (left). Using j_{rec} and α_{rec} , the reconstructed local radiance I_{rec} is then calculated using Eq. 25, and the result is shown in Figure 19 (right). The profiles of j_{rec} and I_{rec} agree well with the ground truth, even in the presence of noise, within the range where r or x equals 1.0. Near the outer boundary, because the radiance is defined as j_{rec} divided by α , small perturbations in low j_{rec} cause large fluctuations that cannot be avoided. Wei et al. (2021) successfully reduced the boundary fluctuation through external masking or thresholding (Wei et al. 2021). Similar to the approach used by Wei et al., the low-intensity region can be removed by applying a mask with a threshold derived from the ε_{DIT} profile, thereby excluding areas with low emission, as shown in Figure 19 (bottom); radiance values for which $\varepsilon_{\text{DIT}} < 0.05$ are omitted in this work.

When we compare I_{rec} with the radiance obtained by the DIT method as defined in Eq. 15, I_{DIT} approximately captures the outer region, but its profile is displaced toward the centerline, and the DIT radiance is overestimated near the axis. Therefore, it is clearly demonstrated that the proposed multi-regularized inversion approach significantly outperforms the direct DIT method for reconstructing axisymmetric radiance profiles of evaporating sprays from infrared thermography data.

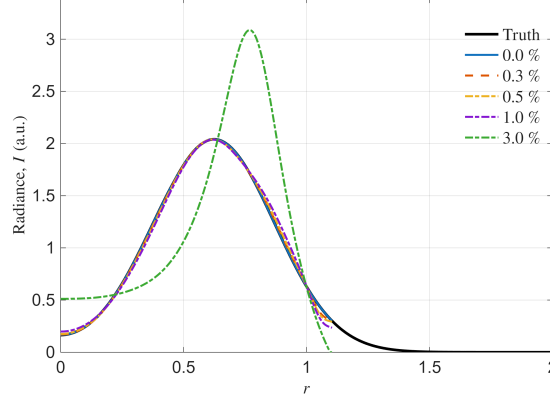


Fig. 20 Comparison of reconstructed local radiance I_{rec} at various noise levels.

It should be emphasized that the proposed method does not always yield a perfect reconstruction of the local radiance, as shown in Figure 19. To understand this, we return to Figure 17. Comparing the reconstructed j and α in Figure 17, we observe that α is slightly underestimated, while j is slightly overestimated near the axis. These deviations appear to be associated with the different shapes of the underlying radial profiles in the present phantom case. The Gaussian-like α profile, with its maximum at the axis, is slightly underestimated near the centerline, whereas the bimodal j profile shows a slight overestimation in the same region. Since the local radiance is obtained from the ratio $I_{\text{rec}} = j_{\text{rec}}/\alpha_{\text{rec}}$, both reconstruction biases propagate into the resulting radiance profile. Nevertheless, I_{rec} shows good agreement with the ground truth for the present phantom configuration.

Although more accurate estimates could, in principle, be obtained by carefully tuning μ_2 , this is only feasible when the ground truth is already known. In practice, however, the ground truth is unknown and μ_2 has to be chosen according to a consistent criterion, such as selecting the corner of the L-curve. Consequently, the observed tendency toward underestimation and overestimation near the axis represents an inherent limitation of the proposed method.

Another important aspect is the sensitivity to noise. In this study, a noise level of 0.3 % is primarily used because the other inversion methods are already unstable at this level. To further assess the robustness of the proposed method, noise levels up to 3.0 % are added to the measured phantom data, as illustrated in Figure 20. For noise levels up to 1.0 %, the spray radiance profiles are reconstructed in a stable manner, whereas the reconstruction at 3.0 % noise is unacceptably degraded. As anticipated from Figure 9 and Eq.12b, the TOD is highly sensitive to measurement noise due to its logarithmic nature. Consequently, at 3.0 % noise, δ_{max} becomes extremely unstable, as shown in Figure 21.

To reduce noise propagation, three smoothing strategies are evaluated. In the first strategy, the original ε profile is kept unchanged while δ_{max} is directly smoothed. In the second approach, ε is smoothed first, and δ_{max} is then implicitly smoothed via the relation $\delta_{\text{max}}(x) = -\ln[1 - \varepsilon(x)]$. In the third method, the measured data $I_{\text{meas},1}$ and $I_{\text{meas},2}$ are initially smoothed, after which ε and δ_{max} are successively obtained and

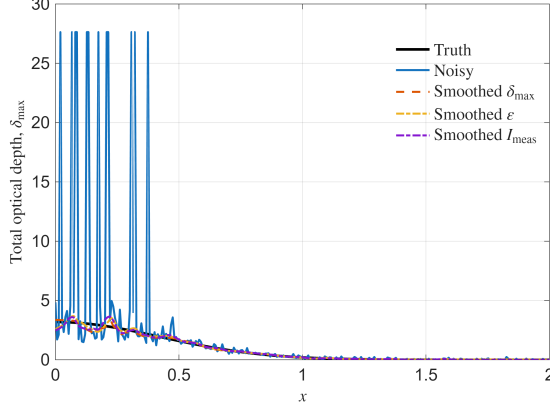


Fig. 21 Total optical depth $\delta_{\max}$ obtained from various smoothing approaches with 3.0 % noise.

smoothed using Eq. 14. As shown in Fig. 21, all three strategies effectively suppress the sharp noise spikes, though minor oscillations remain near the axis. For the smoothing procedure, a moving-average algorithm with a 15-point window is applied. While the smoothing quality is sensitive to the algorithm, a single representative method is employed here for all cases, as this aspect lies outside the main scope of the present work.

Using three different strategies, the key profiles are reconstructed and compared, as shown in Figure 22. As expected, all three approaches yield results that are sufficiently consistent with the ground truth and substantially improved over the unsmoothed noisy data, regardless of the specific strategy used. The RMS errors, shown in Figure 23, confirm that smoothing significantly enhances the results. More specifically, applying smoothing directly to the measured data leads to slightly poorer performance, which may originate from noise propagation through the extended calculation chain leading to $\delta_{\max}$. Practically, in certain cases, strong noise in ε can significantly distort the values after the logarithm is taken, making it challenging to accurately determine $\delta_{\max}$ even with smoothing. Therefore, smoothing ε seems to be a reasonable approach, as it preserves most of the information in the ε domain while still generating a result that closely matches the ground truth.

6.1 Reconstruction for a Flashing Spray Configuration

To further evaluate the applicability of the proposed algorithm to different radial source distributions, an additional phantom configuration is considered by setting $r_{\text{shift}} = 0$ in Eq. 24. This results in a center-peaked Gaussian j distribution, representing a radiative structure similar to that observed for a flashing jet propagating into a colder surrounding medium (Lamanna et al. 2013). As shown in Figure 24, both the extinction coefficient α and the radiative source term j follow a single Gaussian profile that characterizes the strong radiation from the centerline, similar to a flashing jet. Consequently, both are underestimated. Since the underestimated value of α is

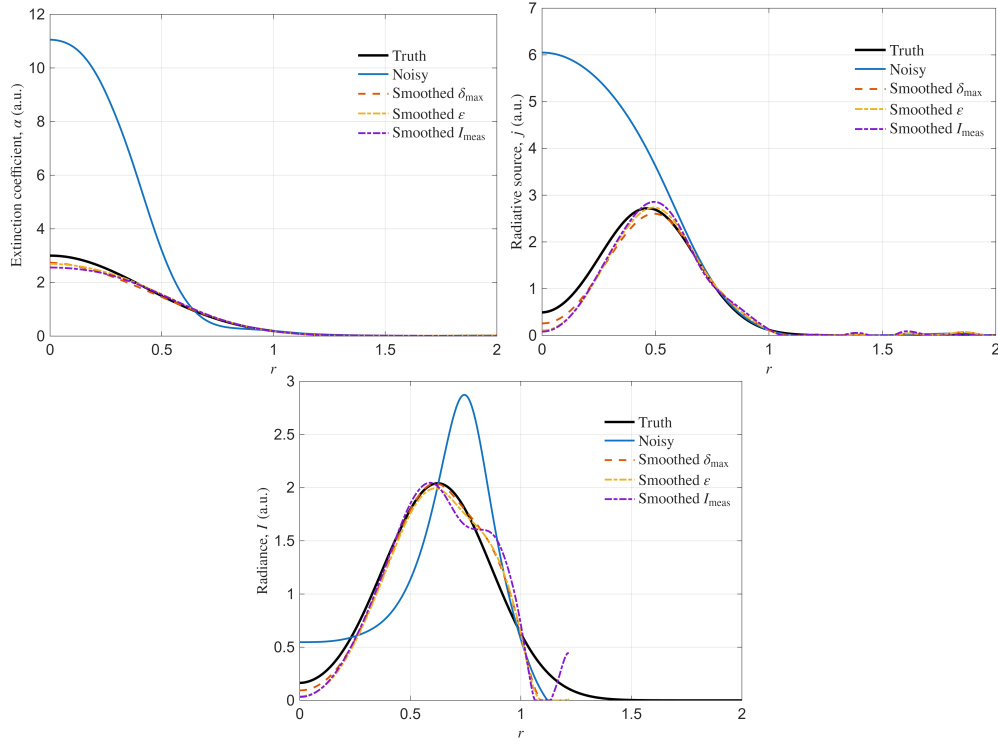


Fig. 22 Reconstruction results obtained from various approaches with 3.0 % noise: (left) the extinction coefficient α_{rec} , (right) the radiative source term j_{rec} , and (bottom) the local radiance I_{rec} .

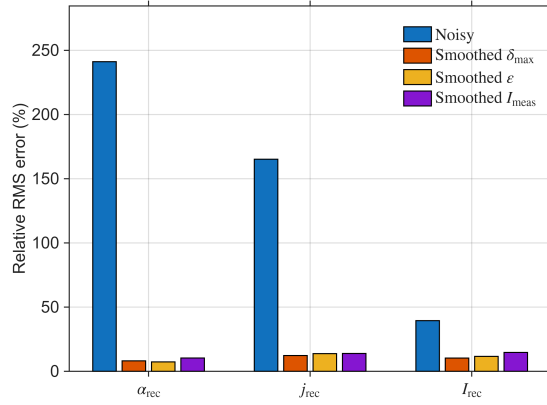


Fig. 23 RMS errors of the reconstructed extinction coefficient α_{rec} , radiative source term j_{rec} , and local radiance I_{rec} obtained using the three smoothing strategies.

subsequently used to evaluate j , the resulting estimate of j is even further underestimated. Despite this systematic underestimation, the proposed method remains more accurate than the DIT approach, as compared in Figure 24 (bottom).

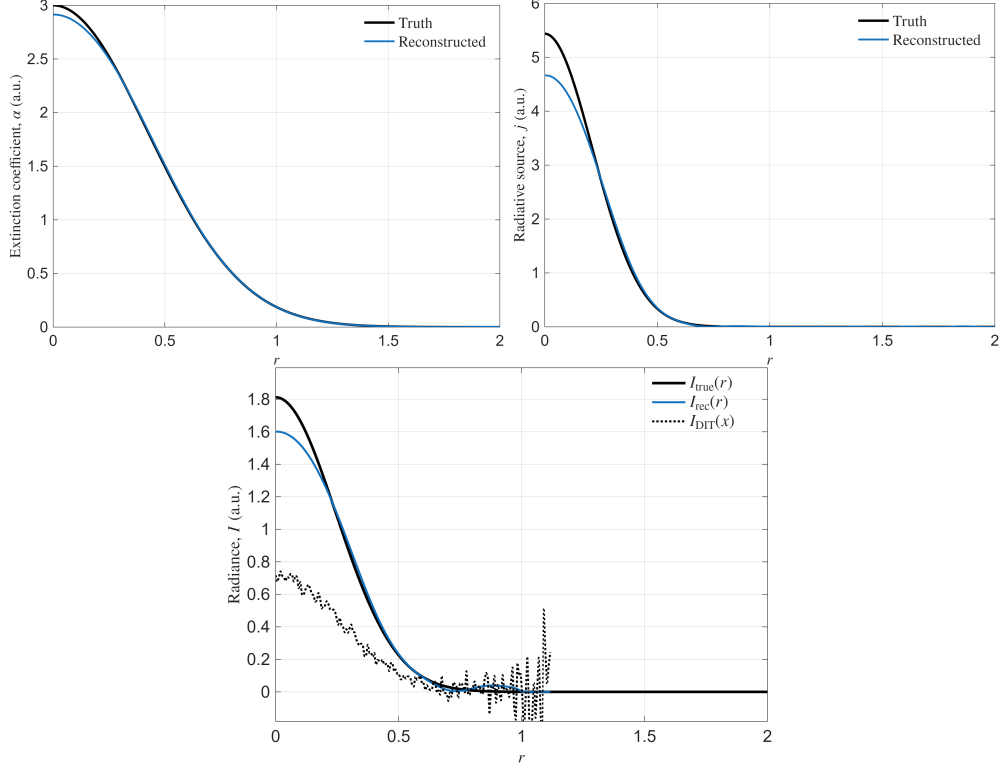


Fig. 24 Fully reconstructed results for the flashing jet scenario where $r_{\text{shift}} = 0$ (0.3% noise); (left) the extinction coefficient α , (right) the radiative source term j , and (bottom) comparison of the reconstructed local radiance I_{rec} with the DIT radiance I_{DIT} .

7 Conclusion

In this study, we introduced a robust framework for tomographic reconstruction of infrared (IR) radiance from an axisymmetric spray in a hot environment by directly solving the radiative transfer equation (RTE). The main novelty of the proposed approach is that the effective slab spray emissivity obtained from differential infrared thermography (DIT) is not used solely for apparent thermometry, as in conventional DIT, but is further decomposed to reconstruct the local extinction coefficient. The reconstructed attenuation field is then incorporated into the attenuated line-of-sight integral to reconstruct the local radiative source term. In this way, the proposed method extends conventional DIT from an apparent thermography technique to a tomographic reconstruction framework for optically attenuating sprays.

A numerical phantom study was conducted to validate the proposed two-step reconstruction procedure under predefined conditions, primarily representing an evaporating cold spray in a hot environment. An additional source configuration was also considered to investigate applicability of the suggested method to a flashing-jet-like radiative distribution. The results showed that conventional DIT can evaluate the outer spray region relatively well but cannot recover the internal radiance of

an optically thick spray. In particular, the DIT radiance was displaced relative to the ground-truth profile and showed significant deviations in the dense spray region near the axis. This confirms that the DIT radiance should be interpreted as an effective apparent quantity rather than a local tomographic radiance when strong self-attenuation is present.

The inverse reconstruction of both the extinction coefficient and the local radiative source term was found to be highly sensitive to measurement noise. Direct inversion accurately reproduced the ground-truth profiles in the noise-free case but became unstable even under a noise level of 0.3%. Iterative algorithms improved numerical stability compared with direct inversion but still showed high sensitivity to noise and non-physical fluctuations near the axis. These results indicate that a stable regularized inversion is essential for IR tomography of dense sprays in hot environments.

To address this issue, multi-parameter regularization was introduced. The proposed formulation combines a global smoothness penalty based on second-order Tikhonov regularization with an additional center-axis constraint that imposes the zero-gradient condition required by axisymmetry. Tikhonov regularization suppressed high-frequency oscillations caused by measurement noise, while the center-axis constraint enforced a zero gradient at the axis. Compared with single Tikhonov regularization, the multi-regularized inversion improved the reconstruction near the axis without significantly influencing the overall radial distribution. This shows that the proposed center-axis constraint effectively suppresses non-physical artifacts near the symmetry axis while preserving the overall radial distribution.

As a result, the local radiance was successfully recovered in the phantom study. The reconstructed radiance I_{rec} agreed well with the ground truth within the main spray region. Remaining fluctuations appeared near the outer boundary, where the extinction coefficient becomes small, because the local radiance is obtained from the ratio of the radiative source term to the extinction coefficient. These fluctuations can be reduced by applying a mask based on the DIT emissivity ε_{DIT} , excluding weakly radiating regions below a prescribed threshold. For higher noise levels similar to practical applications, additional smoothing techniques further improved the robustness of the reconstruction by suppressing noise amplification in the total optical depth.

Consequently, the proposed method provides a promising framework for quantitative IR tomography of axisymmetric sprays under high-pressure and high-temperature conditions. The key contribution is the combination of DIT-derived attenuation, RTE-based source-term reconstruction, and multi-parameter regularization with a physics-informed constraint. Although the reconstruction is demonstrated in this study only for spray thermography, the suggested framework may also be extended to other axisymmetric radiating and attenuating media, such as flames, as long as the assumptions behind the radiative transfer model remain valid. Future work will focus on applying the proposed framework to experimental spray measurements with precise calibration.

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Statements and Declarations

The authors declare that there are no conflicts of interest regarding the publication of this article.

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