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Critical Witnesses and Square-Root Deadline Response in a Driven Toda Hamiltonian

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Abstract

A finite detector gate can make an analytic Hamiltonian flow induce a one-sided square-root edge in ensemble first-passage probability. We fix a forcing-strobed launch in a periodically driven Toda Hamiltonian and partition slow phases into forward passage, return, and deadline-unhit censoring. Under finite first-event charts and split Morse-witness hypotheses, the critical zero set defines a finite direct-image module. Its complete Fitting flag recovers the unordered cyclic-factor scan valuations. For a finite reduced factorisation met properly by the scan, each valuation is a sum of multiplicity-weighted component contacts; the cyclic module need not split componentwise. A weighted Morse formula supplies the response sign and coefficient. A two-witness angular calibration, obtained by normalising Toda's sudden energy-transfer formula, gives a nontrivial flag and an exact square-root law; it is not a full-flow approximation. In the full equations, fixed-momentum launches with $p_0 = 2.20$ and 1.80 give opposite deadline folds at $\mu \approx 0.188303$ and 0.194567 for $H = 3T/2$. Their critical trajectories lie on invariant symmetry fibres, where opposite effective-barrier shifts, not slow-mode energy exchange, interpret the numerically determined witness signs. Constant instantaneous-energy and phasewise baseline-energy-held preparations show that transverse curvature and response magnitude depend on launch protocol. A conditional time-to-deeper-winding audit separates delayed progression from censoring by an arbitrary T cutoff. Step, derivative, phase, target, deadline and recrossing audits are reported. The computation is sampled, not interval-certified, and defines neither a thermal rate nor a product yield.

Keywords: square-root response, first passage, Fitting ideal, barrier passage, driven reaction dynamics, Toda Hamiltonian

1 Introduction

Transition-state theory replaces configuration-space intuition by invariant phase-space structures near a reaction bottleneck. Normally hyperbolic invariant manifolds, their stable and unstable manifolds, and dividing surfaces provide the geometric basis of this construction [1–3]. Multisaddle transport may be organised as a phase-space reaction network, as in the three-dimensional HCN isomerisation problem [4]. In time-dependent systems, moving transition-state structures and ensemble propagation supply finite-time diagnostics and rate calculations [5–7]. Finite-time transmission and delay are therefore not introduced here [8]. Morse bifurcations of transition states are likewise established [14]. More generally, singular density and volume changes at critical values go back to van Hove and have precise differentiability theorems in the work of Hoveijn [12, 13]. MacKay and Strub relate this mechanism to transition-state flux through sublevel volumes [14]. The present object is instead a fixed-ensemble, trajectory-defined first-passage decision with an explicit deadline and competing return event.

The present question is narrower. Fix an initial ensemble, two competing observation sections and a deadline. As a parameter varies, which local data determine a potentially nonsmooth change of the observed phase fraction? A reduced discriminant locates an event, but it does not say whether the parameter path is tangent to the event, whether several critical phases have coalesced in the projection, or whether the critical point is a maximum or a minimum. Those distinctions are decisive for a response curve.

Fitting ideals encode multiple-point and discriminant structures of finite analytic maps [9]. Saito used a Fitting discriminant in the singularity and unfolding theory associated with a period mapping [10, §1.3, p. 1233]. Neither Fitting discriminants nor Morse-volume singularities are introduced here. We construct the finite direct-image module of *critical deadline witnesses*, pull its Fitting flag to a one-dimensional scan under split Morse hypotheses, and couple each reduced branch to its multiplicity, Hessian inertia and ensemble density. If one cyclic factor has a finite reduced factorisation met properly by the scan, its valuation is the sum of multiplicity-weighted component contacts; the module need not split into componentwise summands at their intersection. The factorisation

$$\text{critical witnesses} \longrightarrow \text{Fitting flag} \longrightarrow \text{branch contact} \longrightarrow \text{ensemble response}$$

for a trajectory-defined finite-horizon decision map is the mathematical contribution. An exact two-witness calibration displays the nontrivial flag. Its angular factor is the normalised form of Toda’s sudden energy-transfer approximation; the calibration is auxiliary and does not replace the full flow. The Toda folds themselves are separate one-witness applications. The zeroth Fitting ideal alone determines neither witness partition, reduced contact, inertia nor response coefficient.

The chemical calculation uses Toda’s periodically driven Hamiltonian [15]. We prescribe the launch section and uniform slow-phase measure before scanning the coupling μ . The observable is the fraction that reaches $q = \pi + 10^{-3}$ before a negative return to $q = 0$ and no later than the detector gate H . Its one-sided square-root edge transfers mass chiefly between forward passage and deadline-unhit censoring; it is not a forward–return channel exchange. We compare fixed momentum, constant instantaneous energy, and phasewise baseline-energy-held launches. A conditional time-to-deeper-winding distribution then distinguishes delayed progression from observation-window censoring. Finite-window recrossing is reported separately. None of these quantities is a canonical or microcanonical rate constant, lobe volume, committor, or molecular product yield [16, 18].

This paper addresses a finite-horizon first-passage question distinct from the held companion event-ledger study. The latter examines finite-window manifold incidences, a reversor-fixed trace and surrogate reconnection. No trace CSV, signed-offset ensemble, or surrogate event is used as numerical evidence below. The Hamiltonian and the elementary angular factor shared with the companion are model background; no text, figure, table or numerical record is reused. Earlier trace values were used only to choose broad exploratory brackets. The present paper uses new momenta $p_0 = 1.80, 2.20$, a trajectory-defined deadline boundary, new response data, and a separate critical-module theorem; fixed-momentum sampling itself is not claimed as new. Section 2 defines the experiment. Section 3 develops the mathematics. Sections 4 and 5 give the full flow computation and chemical interpretation. Proofs requiring local-ring or hitting-time preliminaries are included in the appendices.

2 Driven Toda model and fixed ensemble

2.1 Hamiltonian and equations of motion

We use Toda’s non-autonomous Hamiltonian [15, pp. 378–385, Eq. (93)]

$$H(q, p, \theta, I, t) = \frac{p^2}{2} - V \cos q + \frac{I^2}{2} - \mu(\cos q + 1) \cos \theta - \varepsilon \cos q \cos(\Omega t), \quad (1)$$

with $V = 1$, $\varepsilon = 0.1$ and $\Omega = 1.5$. Its forcing period is $T = 2\pi/\Omega = 4.188790204786$. The unwrapped reaction coordinate obeys

$$\dot{q} = p, \quad \dot{p} = -[V + \mu \cos \theta + \varepsilon \cos(\Omega t)] \sin q, \quad \dot{\theta} = I, \quad \dot{I} = -\mu(\cos q + 1) \sin \theta. \quad (2)$$

Proposition 2.1 (Slow-phase reflection) *The involution*

$$(q, p, \theta, I) \mapsto (q, p, -\theta, -I)$$

commutes with the flow in Eq. (2). Consequently the target and return times on $E(p_0)$ are invariant under $\theta \mapsto -\theta \pmod{2\pi}$; they are even about both fixed phases 0 and π .

Proof Cosine is even and sine is odd. Substitution in Eq. (2) leaves the q, p equations unchanged and changes both sides of the θ, I equations by the same sign. The observation sections depend only on q, p , so their first-hit times have the same symmetry. $\square$

Proposition 2.2 (Invariant symmetry fibres and barrier shift) *Let $s \in \{1, -1\}$ and set $\theta_s = 0$ for $s = 1$ and $\theta_s = \pi$ for $s = -1$. The fibre*

$$\mathcal{F}_s = \{(q, p, \theta, I) : \theta = \theta_s, I = 0\}$$

is invariant. Its reaction coordinate satisfies

$$\ddot{q} = -[V + s\mu + \varepsilon \cos(\Omega t)] \sin q. \quad (3)$$

The instantaneous potential difference from $q = 0$ to $q = \pi$ on this fibre is

$$\Delta U_s(t) = 2[V + s\mu + \varepsilon \cos(\Omega t)]. \quad (4)$$

Thus increasing μ raises the instantaneous barrier on $\mathcal{F}_+$ and lowers it on $\mathcal{F}_-$.

Proof At θ_s , one has $\sin \theta_s = 0$. If $I = 0$, then $\dot{\theta} = 0$ and $\dot{I} = 0$, so the fibre is invariant. Substitution of $\cos \theta_s = s$ in Eq. (2) gives Eq. (3). Evaluating the potential part of Eq. (1) at $q = \pi$ and $q = 0$ gives $V + \varepsilon \cos(\Omega t)$ and $-V - 2s\mu - \varepsilon \cos(\Omega t)$, respectively. Their difference is Eq. (4). $\square$

No slow-mode energy exchange occurs on either critical symmetry orbit: $I(t) = 0$ and $\theta(t) = \theta_s$ identically. Neighbouring nonsymmetric phases may exchange energy, but that is not the mechanism of the critical witness itself. Since the reduced equation is non-autonomous, the sign of the instantaneous barrier shift alone does not imply monotonicity of a first-hit time; the latter is determined by the full trajectory calculation.

The cylinder $q = \pi, p = 0$ is invariant, and its normal coefficient satisfies $1 + \mu \cos \theta + 0.1 \cos(1.5t) \geq 0.68$ for the reported $0 \leq \mu \leq 0.22$. No invariant-manifold approximation enters the definition of the observable.

Definition 2.3 (Conditioned launch ensemble) For a fixed $p_0 > 0$, let

$$E(p_0) = \{(q, p, \theta, I, t) = (0, p_0, \theta, 0, 0) : \theta \in \mathbb{T}\}$$

with probability measure $d\theta/(2\pi)$. This is a conditioned preparation with uncontrolled slow phase. It is not a canonical or microcanonical ensemble.

Example 2.4 (Positive: a declared reproducible ensemble) The two production ensembles are $E(2.20)$ and $E(1.80)$. Every initial coordinate, the phase density and the forcing strobe are fixed before the μ scan.

Example 2.5 (Negative: an unspecified energy shell) The phrase “initial conditions near the barrier” does not determine a probability. Changing the distribution in p, I or forcing phase changes the response and cannot be absorbed into a property of the flow.

At the launch strobe, the Hamiltonian on $E(p_0)$ is

$$H_0(\mu, \theta; p_0) = \frac{p_0^2}{2} - V - \varepsilon - 2\mu \cos \theta. \quad (5)$$

Consequently a fixed- p_0 coupling scan changes the initial-energy distribution as well as the subsequent dynamics.

Definition 2.6 (Constant instantaneous-energy preparation) Fix a real number E_0 such that the radicand below is positive on the parameter interval. At the forcing strobe $t = 0$, put

$$p_E(\mu, \theta) = \sqrt{2[E_0 + V + \varepsilon + 2\mu \cos \theta]} \quad (6)$$

and let

$$E^{\text{em}}(E_0; \mu) = \{(0, p_E(\mu, \theta), \theta, 0, 0) : \theta \in \mathbb{T}\}$$

carry $d\theta/(2\pi)$. Every launch then satisfies $H(0) = E_0$. This is an isoenergetic launch protocol, not an invariant energy shell of the non-autonomous Hamiltonian. It changes the whole launch curve and agrees with a fixed- p_0 ensemble only at a chosen critical witness.

Example 2.7 (Positive: an alternative isoenergetic preparation) Equation (6) removes the explicit $-2\mu \cos \theta$ variation in Eq. (5) while retaining the same slow-phase density and forcing strobe. Comparing its deadline response with the fixed- p_0 response quantifies sensitivity to this different preparation choice.

Example 2.8 (Negative: energy matching is not microcanonical sampling) The set $E^{\text{em}}(E_0; \mu)$ is a one-dimensional launch curve at one forcing phase. It is neither a flux-weighted dividing-surface ensemble nor a microcanonical measure in extended phase space.

Definition 2.9 (Phasewise baseline-energy-held comparator) Fix a reference witness (μ_*, p_0) and suppose the radicand below is positive on the parameter-phase domain. Put

$$p_{\text{hold}}(\mu, \theta) = \sqrt{p_0^2 + 4(\mu - \mu_*) \cos \theta}. \quad (7)$$

The launch curve

$$E^{\text{hold}}(\mu; \mu_*, p_0) = \{(0, p_{\text{hold}}(\mu, \theta), \theta, 0, 0) : \theta \in \mathbb{T}\}$$

carries $d\theta/(2\pi)$. It satisfies

$$H_0(\mu, \theta; p_{\text{hold}}) = H_0(\mu_*, \theta; p_0) \quad \text{for every } \theta, \quad (8)$$

and $p_{\text{hold}}(\mu_*, \theta) = p_0$ for every phase.

Example 2.10 (Positive: a phasewise matched scan) Equation (8) removes the scan-induced change of each phase's baseline launch energy without flattening its phase distribution. It is therefore the matched comparator for the fixed- p_0 scan at $\mu = \mu_*$.

Example 2.11 (Negative: matching does not isolate a force mechanism) The held comparator still changes the initial momentum with μ and hence changes the launch embedding. A difference between its response and the fixed- p_0 response is preparation sensitivity, not a causal decomposition of barrier geometry and off-fibre energy transfer.

2.2 Deadline and recrossing observables

Put $\eta = 10^{-3}$. For $x \in E(p_0)$ define

$$\tau_F(x) = \inf\{t > 0 : q(t) = \pi + \eta, p(t) > 0\}, \quad (9)$$

$$\tau_R(x) = \inf\{t > 0 : q(t) = 0, p(t) < 0\}, \quad (10)$$

where the return event is armed only after the trajectory has entered $q > 0$. The infimum of the empty set is $+\infty$.

Definition 2.12 (Deadline response) For $H > 0$ and phase density w , the forward deadline response is

$$P_H^F(\mu, p_0; w) = \int_{\mathbb{T}} \mathbf{1}_{\{\tau_F \leq H, \tau_F < \tau_R\}} w(\theta) d\theta. \quad (11)$$

The production density is $w = (2\pi)^{-1}$. The primary deadline is $H = 3T/2 = 6.283185307180$; $5T/4$, $7T/4$ and $2T$ are sensitivity audits.

Definition 2.13 (Three-state deadline decision) Up to the null set of nontransverse or tied events, put

$$F_H = \{\tau_F \leq H, \tau_F < \tau_R\},$$

$$R_H = \{\tau_R \leq H, \tau_R < \tau_F\},$$

$$U_H = \{\tau_F > H, \tau_R > H\}.$$

Their weighted masses P_H^F, P_H^R, P_H^U satisfy

$$P_H^F + P_H^R + P_H^U = 1. \quad (12)$$

The label U_H means deadline-unhit, not an asymptotic dynamical state.

Example 2.14 (Positive: a censored decision simplex) The triples (P_H^F, P_H^R, P_H^U) retain whether lost forward mass has entered return or remains unobserved at the gate. The numerical CSV records all three counts and verifies Eq. (12) row by row.

Example 2.15 (Negative: a false two-channel interpretation) Replacing U_H by either forward or return forces a binary channel label onto censored trajectories. A square-root change in P_H^F can then be misreported as a reaction-channel switch although no second product channel has been observed.

The deadline is the open duration of an observation gate after the forcing-strobed launch; the periodic drive is not switched off at H . The dimensionless Toda calculation is not calibrated to a particular detector. Changing H/T changes the measured decision map and is therefore an observable-dependence audit, not a time-discretisation test.

For the control in Definition 2.6, write

$$P_H^{F,\text{em}}(\mu; E_0) = \frac{1}{2\pi} \int_{\mathbb{T}} \mathbf{1}_{\{\tau_F(\mu, \theta; p_E) \leq H, \tau_F < \tau_R\}} d\theta. \quad (13)$$

The launch map now depends analytically on μ wherever the radicand in Eq. (6) stays positive; the local response theorem therefore applies to this protocol under the same first-hit separation hypotheses.

For the comparator in Definition 2.9, write

$$P_H^{F,\text{hold}}(\mu; \mu_*, p_0) = \frac{1}{2\pi} \int_{\mathbb{T}} \mathbf{1}_{\{\tau_F(\mu, \theta; p_{\text{hold}}) \leq H, \tau_F < \tau_R\}} d\theta. \quad (14)$$

Its launch embedding is analytic wherever Eq. (7) is real and nonzero.

Definition 2.16 (Deeper-winding and finite-window recrossing response) Let τ_D be the first crossing of the deeper forward-winding section $q = 3\pi/2$ with $p > 0$, and, after such a crossing, let τ_{rec} be the first subsequent crossing of $q = \pi$ with $p < 0$. For an allowance $a > 0$, put

$$D_{H,a} = F_H \cap \{\tau_D - \tau_F \leq aT\}, \quad (15)$$

$$R_{H,a,\Delta} = D_{H,a} \cap \{\tau_{\text{rec}} \leq \tau_D + \Delta\}, \quad (16)$$

$$P_{H,a}^D = \int_{\mathbb{T}} \mathbf{1}_{D_{H,a}} w(\theta) d\theta, \quad (17)$$

$$P_{H,a,\Delta}^{\text{rec}} = \int_{\mathbb{T}} \mathbf{1}_{R_{H,a,\Delta}} w(\theta) d\theta, \quad (18)$$

$$S_D(a) = \frac{1}{P_H^F} \int_{\mathbb{T}} \mathbf{1}_{F_H} \mathbf{1}_{\{\tau_D - \tau_F > aT\}} w(\theta) d\theta, \quad P_H^F > 0. \quad (19)$$

The two P fractions use the full launch ensemble as denominator. The survival function instead uses the forward set as denominator and counts right-censored observations when no deeper hit is resolved within the largest allowance. It is undefined when $P_H^F = 0$; the corresponding machine-readable field is blank rather than zero. The audited allowances are $a = 0.5, 1, 1.25, 1.5, 2$; the production recrossing allowance is $a = 1.5$. The production post-deep window is $\Delta = 5T$, with $4T$ and $6T$ as censoring audits. Failure to reach the deeper section within one allowance is not absence of later deeper winding and is not non-recrossing.

Definition 2.17 (Changed-sector fate ledger) Fix a symmetric offset $\delta > 0$ and the phase grid $\Theta_N = \{2\pi j/N : 0 \leq j < N\}$. Let

$$L_{\pm}(\theta) = \mathbf{1}_{F_H(\mu_* \pm \delta)}(\theta), \quad S_{\delta} = \{\theta \in \Theta_N : L_{-}(\theta) \neq L_{+}(\theta)\}.$$

For $\theta \in S_{\delta}$, let $\sigma_F(\theta) \in \{-1, +1\}$ be the unique sign for which $L_{\sigma_F}(\theta) = 1$. The changed-sector fate count is

$$C_{\delta} = \sum_{\theta \in S_{\delta}} \mathbf{1}_{R_{H,a,\Delta}(\mu_* + \sigma_F(\theta)\delta)}(\theta).$$

For comparison, the plus-side complement count is

$$B_{\delta}^{+} = \sum_{\theta \in \Theta_N \setminus S_{\delta}} \mathbf{1}_{R_{H,a,\Delta}(\mu_* + \delta)}(\theta).$$

The count C_{δ} follows each changed phase on its forward side, whereas B_{δ}^{+} is evaluated only at the plus-side ensemble. They do not partition one ensemble. If C_{δ}^{+} denotes the first sum evaluated uniformly at $\mu_* + \delta$, then $C_{\delta}^{+} + B_{\delta}^{+}$ is the global plus-side recrossing count.

Example 2.18 (Positive: the disappearing sector is followed before loss) For a loss fold, $L_- = 1$ and $L_+ = 0$ on the changed sector. Definition 2.17 evaluates its later fate at $\mu_* - \delta$, where those trajectories are forward and hence eligible for deeper-winding and recrossing tests.

Example 2.19 (Negative: a structurally zero fate count) Evaluating every changed phase only at $\mu_* + \delta$ makes the loss sector ineligible for a post-forward observable. A zero obtained in that manner is a consequence of the conditioning, not evidence of dynamical commitment.

Example 2.20 (Positive: delayed deeper winding) If a forward trajectory satisfies $(\tau_D - \tau_F)/T = 1.2$, then it contributes to $S_D(1)$ but belongs to $D_{H,1.25}$. The change of its binary status reflects the observation allowance; the recorded delay itself is unchanged.

Example 2.21 (Negative: a zero denominator is not zero survival) If $P_H^F = 0$, no forward-conditioned population exists. Assigning $S_D(a) = 0$ would confuse an undefined conditional probability with evidence that every forward trajectory reached the deeper section.

Example 2.22 (Positive: a finite-time barrier-passage distinction) On the two critical symmetry fibres, Eq. (4) changes with opposite sign under the same increase of μ . The resulting deadline loss and birth are therefore compatible with an effective-barrier shift even though the coupling term itself vanishes at $q = \pi$.

Example 2.23 (Negative: a false slow-energy mechanism) It is incorrect to attribute either critical witness to energy transfer into the slow mode. Proposition 2.2 gives $I(t) = 0$ on the entire critical trajectory. Energy exchange for nearby nonsymmetric phases does not alter this exact statement.

Example 2.24 (Negative: finite non-recrossing is not commitment) A phase counted as non-recrossing up to $\tau_D + 5T$ may return later. A phase that fails to reach the deeper-winding section by $\tau_F + aT$ is censored earlier still. Changing a can therefore change the count without changing the trajectory. Hence neither subtraction defines an asymptotic product yield.

3 Critical-event modules and branchwise response

This section separates four data that are often conflated: the reduced parameter support of an event, the finite module carried by its critical witnesses, the order with which a parameter path meets each witness branch, and the real Hessian signature that governs the local response. All rings below are germs of real-analytic functions. The module and Fitting-ideal statements have complex-analytic analogues by complexification; the Hessian-sign and volume-response statements are intrinsically real. Neither Fitting ideals nor Morse-volume singularities are introduced here; the point is their branchwise use for a trajectory-defined finite-horizon decision map.

3.1 Finite critical modules and their Fitting flags

Let B be a real-analytic k -manifold, let X be a real-analytic m -manifold, and let

$$F : B \times X \longrightarrow \mathbb{R}$$

be real analytic. Fix $b_0 \in B$ and distinct points $x_1, \dots, x_N \in X$ satisfying

$$F(b_0, x_j) = 0, \quad D_x F(b_0, x_j) = 0, \quad \det D_{xx}^2 F(b_0, x_j) \neq 0. \quad (20)$$

Choose pairwise disjoint coordinate neighbourhoods U_j of the x_j . After shrinking them and a neighbourhood V of b_0 , assume that these are all the critical zeros of F in $V \times \bigcup_j U_j$. Write $A = \mathcal{O}_{B, b_0}$.

Theorem 3.1 (Finite critical module and scan Fitting flag) *Under the preceding hypotheses, there are unique real-analytic critical-point branches $\sigma_j : V \rightarrow U_j$ and critical-value germs*

$$h_j(b) = F(b, \sigma_j(b)), \quad h_j(b_0) = 0.$$

Let $\mathcal{C}_j$ be the analytic critical-zero germ defined in $V \times U_j$ by $F = 0$ and $D_x F = 0$, and let $\pi : \bigsqcup_j \mathcal{C}_j \rightarrow V$ be parameter projection. Then

$$M_{\text{crit}} := (\pi_* \mathcal{O}_{\bigsqcup_j \mathcal{C}_j})_{b_0} \simeq \bigoplus_{j=1}^N A/(h_j). \quad (21)$$

For $0 \leq i < N$ its Fitting ideals are

$$\text{Fitt}_i(M_{\text{crit}}) = \left\langle \prod_{j \in J} h_j : J \subset \{1, \dots, N\}, |J| = N - i \right\rangle, \quad (22)$$

and $\text{Fitt}_i(M_{\text{crit}}) = A$ for $i \geq N$. In particular,

$$\text{Fitt}_0(M_{\text{crit}}) = \left(\prod_{j=1}^N h_j \right), \quad \text{Supp } M_{\text{crit}} = \bigcup_{j=1}^N V(h_j). \quad (23)$$

Thus the scheme-theoretic reduced support has defining ideal $\sqrt{\text{Fitt}_0(M_{\text{crit}})}$. Over a real-analytic base this ordinary radical need not equal the ideal of the set of real points, which may require the real radical. Sign and direction inequalities selecting a physical real branch are further data and are not encoded by these Fitting ideals.

Let $\gamma : (\mathbb{R}, 0) \rightarrow (B, b_0)$ be a real-analytic scan such that no $h_j \circ \gamma$ is identically zero. Put

$$r_j = \text{ord}_0(h_j \circ \gamma) \in \mathbb{N}, \quad R = \mathbb{R}\{t\}.$$

We call r_j the Fitting valuation of the j th witness along the scan. Suppose, for a given j , that the critical-value germ has the finite reduced factorisation

$$h_j = u_j \prod_{\alpha=1}^{s_j} q_{j\alpha}^{m_{j\alpha}}, \quad u_j(b_0) \neq 0, \quad (24)$$

where the $q_{j\alpha}$ are pairwise nonassociated reduced germs and $m_{j\alpha} \geq 1$. Assume that the scan is not contained in any component. The integer $m_{j\alpha}$ is its branch multiplicity, while

$$c_{j\alpha} := \text{ord}_0(q_{j\alpha} \circ \gamma) \quad (25)$$

is its reduced geometric contact. If $q_{j\alpha} = 0$ is smooth, we also write $c_{j\alpha} = \text{TI}_{j\alpha}(\gamma)$; it equals one precisely for a transverse scan. These data obey

$$r_j = \sum_{\alpha=1}^{s_j} m_{j\alpha} c_{j\alpha}. \quad (26)$$

When $s_j = 1$ we suppress α and write $r_j = m_j c_j$. Factorisation of the divisor does not in general split its cyclic module: at a point where two components meet,

$$A/(q_{j1}^{m_{j1}} q_{j2}^{m_{j2}}) \not\cong A/(q_{j1}^{m_{j1}}) \oplus A/(q_{j2}^{m_{j2}}) \quad (27)$$

in general. The direct sum in Eq. (21) separates distinct phase witnesses x_j , not the reduced components of one critical-value divisor.

Then

$$\gamma^* M_{\text{crit}} \simeq \bigoplus_{j=1}^N R/(t^{r_j}), \quad \text{length}_R(\gamma^* M_{\text{crit}}) = \sum_{j=1}^N r_j = \text{ord}_0 \text{Fitt}_0(\gamma^* M_{\text{crit}}). \quad (28)$$

If $r_1 \leq \dots \leq r_N$ and $v_i = \text{ord}_0 \text{Fitt}_i(\gamma^* M_{\text{crit}})$ with $v_N = 0$, then

$$v_i = \sum_{j=1}^{N-i} r_j, \quad r_{N-i} = v_i - v_{i+1} \quad (0 \leq i < N). \quad (29)$$

Hence the complete Fitting flag on a one-dimensional scan recovers the unordered Fitting-valuation multiset $\{r_j\}$. Without independent knowledge of the factorisation in Eq. (24), it does not recover the component pairs $(m_{j\alpha}, c_{j\alpha})$. Nor does it recover the witness labels, the real Hessian inertias, or an observation density.

Proof At (b_0, x_j) the derivative with respect to x of the map $D_x F$ is the nonsingular Hessian in Eq. (20). The analytic implicit-function theorem therefore gives a unique analytic solution $x = \sigma_j(b)$ of $D_x F = 0$. In the local analytic ring of $B \times X$, the components of $D_x F$ are local coordinates transverse to this graph: the analytic inverse-function theorem identifies their ideal with the ideal generated by the components of $x - \sigma_j(b)$. Quotienting first by that ideal and then by F gives

$$\mathcal{O}_{B \times X, (b_0, x_j)} / (F, D_x F) \simeq A/(F(b, \sigma_j(b))) = A/(h_j).$$

The chosen neighbourhoods are disjoint, so their direct image is the direct sum in Eq. (21).

A presentation of this module is

$$A^N \xrightarrow{\text{diag}(h_1, \dots, h_N)} A^N \longrightarrow M_{\text{crit}} \longrightarrow 0.$$

By definition, Fitt_i is generated by the $(N-i) \times (N-i)$ minors of a presentation matrix. The nonzero minors of this diagonal matrix are precisely the products in Eq. (22). The support of a finite module is the zero set of its zeroth Fitting ideal, proving Eqs. (22) and (23).

For the scan, analyticity and finite order give $h_j \circ \gamma = \tilde{u}_j(t) t^{r_j}$ with $\tilde{u}_j(0) \neq 0$. Multiplication by a unit does not change a cyclic quotient, so base change to R gives the first isomorphism in Eq. (28). The residue classes of $1, t, \dots, t^{r_j-1}$ form an $\mathbb{R}$ -basis of $R/(t^{r_j})$; hence its length is r_j . Length is additive on finite direct sums. Finally, in the discrete valuation ring R , an ideal generated by several powers of t is generated by the one with the smallest exponent. Among products of $N-i$ diagonal entries, the smallest valuation is $r_1 + \dots + r_{N-i}$. This

proves Eqs. (28) and (29). Under Eq. (24), additivity of order under products and invariance under multiplication by a unit give

$$\text{ord}_0(h_j \circ \gamma) = \sum_{\alpha=1}^{s_j} m_{j\alpha} \text{ord}_0(q_{j\alpha} \circ \gamma) = \sum_{\alpha=1}^{s_j} m_{j\alpha} c_{j\alpha},$$

which proves Eq. (26).

For the non-splitting assertion, it suffices to exhibit its obstruction. If $a = q_{j_1}^{m_{j_1}}$ and $b = q_{j_2}^{m_{j_2}}$ satisfy $(a) \cap (b) = (ab)$, as do powers of distinct coordinate branches in a regular analytic local ring, there is an exact sequence

$$0 \longrightarrow A/(ab) \longrightarrow A/(a) \oplus A/(b) \longrightarrow A/(a, b) \longrightarrow 0, \quad (30)$$

where the last arrow sends $(\bar{f}, \bar{g})$ to $\overline{f - g}$. At a common zero, $A/(a, b) \neq 0$; hence the first arrow is not surjective. More intrinsically, $\text{Fitt}_1(A/(ab)) = A$, whereas $\text{Fitt}_1(A/(a) \oplus A/(b)) = (a, b) \neq A$ at the common zero. The two modules are therefore not isomorphic. Thus component factorisation records a valuation sum but does not supply a componentwise direct-sum decomposition. $\square$

Remark 3.2 (Distinct multiplicities and contacts) The length of the phase critical algebra $\mathcal{O}_{X, x_j}/(D_x F(b_0, \cdot))$, the length of a special critical fibre, the branch multiplicities $m_{j\alpha}$, the reduced scan contacts $c_{j\alpha}$, the Fitting valuation $r_j = \sum_{\alpha} m_{j\alpha} c_{j\alpha}$, and the intersection length of distinct parameter hypersurfaces are different invariants. For a Morse witness the phase critical-algebra length is one. Its parameter divisor can nevertheless be nonreduced, and several Morse witnesses can give a special fibre of length greater than one.

Theorem 3.3 (Reduce–separate–then–test) *Let a local event equation have a factorisation*

$$H = u \prod_{\alpha=1}^s q_{\alpha}^{m_{\alpha}}, \quad u(b_0) \neq 0, \quad q_{\alpha}(b_0) = 0, \quad m_{\alpha} \geq 1,$$

where the q_{α} are distinct reduced branch germs. If $\sum_{\alpha} m_{\alpha} \geq 2$, then $dH(b_0) = 0$. This remains true when every branch is smooth and a scan γ is transverse to every branch.

For a smooth branch $q_{\alpha} = 0$ and a scan not contained in that branch,

$$\text{ord}_0(q_{\alpha} \circ \gamma) = 1 \iff dq_{\alpha}(b_0)\dot{\gamma}(0) \neq 0. \quad (31)$$

Consequently, a transversality index must be evaluated after removing nilpotent multiplicities and separating the reduced branches, or on the corresponding critical-witness branches. Differentiating a product generator is not a branchwise transversality test. Radicalisation alone is insufficient at an intersection of two distinct reduced branches.

Proof The product rule gives

$$dH = u \sum_{\alpha} m_{\alpha} q_{\alpha}^{m_{\alpha}-1} \left(\prod_{\beta \neq \alpha} q_{\beta}^{m_{\beta}} \right) dq_{\alpha} + \left(\prod_{\alpha} q_{\alpha}^{m_{\alpha}} \right) du.$$

Every coefficient on the right vanishes at b_0 when the total number of incident factors, counted with multiplicity, is at least two. This proves the first assertion. For Eq. (31), Taylor's formula gives $q_{\alpha}(\gamma(t)) = t dq_{\alpha}(b_0)\dot{\gamma}(0) + O(t^2)$. The coefficient of t is nonzero exactly when the order is one. If two distinct reduced branches meet, their square-free product still has zero differential at the meeting point, which proves the final assertion. $\square$

3.2 Hessian signature and fixed-ensemble response

Let dV be a fixed smooth positive volume density on X , and let $w \geq 0$ be a fixed smooth ensemble density. In a local chart at x_j , write $w dV = \varpi_j(x) dx$. The ensemble is fixed: neither w nor dV changes with the scanned parameter.

Theorem 3.4 (Branchwise definite-Morse response) *Assume the hypotheses of Theorem 3.1. Suppose in addition that every $D_{xx}^2 F(b_0, x_j)$ is definite and $\varpi_j(x_j) > 0$. Put*

$$\epsilon_j = \begin{cases} +1, & D_{xx}^2 F(b_0, x_j) \text{ is negative definite,} \\ -1, & D_{xx}^2 F(b_0, x_j) \text{ is positive definite,} \end{cases}$$

and

$$\kappa_j = \frac{\omega_m 2^{m/2} \varpi_j(x_j)}{\sqrt{|\det D_{xx}^2 F(b_0, x_j)|}}, \quad (32)$$

where ω_m is the volume of the Euclidean unit ball in $\mathbb{R}^m$ and $[u]_+ = \max\{u, 0\}$. For disjoint sufficiently small witness neighbourhoods, the weighted superlevel response

$$P(b) = \int_{\bigcup_j U_j} \mathbf{1}_{\{F(b,x) > 0\}} w(x) dV(x)$$

is the sum of a term smooth across the event and one potentially nonsmooth event contribution from each witness.

More precisely, let γ be a one-dimensional analytic scan and write

$$h_j(\gamma(t)) = a_j t^{r_j} + o(t^{r_j}), \quad a_j \neq 0. \quad (33)$$

Then there is a function P_{reg} , smooth at $t = 0$, such that the j th local contribution satisfies

$$P_j(\gamma(t)) - P_{j,\text{reg}}(t) = \epsilon_j \kappa_j [\epsilon_j a_j t^{r_j}]_+^{m/2} + o(|t|^{r_j m/2}). \quad (34)$$

The potentially nonsmooth event part of $P - P_{\text{reg}}$ is the sum of these expressions. Thus a one-dimensional phase fold has exponent

$$\frac{r_j}{2} = \frac{1}{2} \sum_{\alpha=1}^{s_j} m_{j\alpha} c_{j\alpha} \quad (35)$$

whenever Eq. (24) applies. A reduced smooth branch met transversely has $m_j = c_j = 1$ and hence a one-sided square-root edge. This is not a Whitney cusp or a cusp catastrophe. The single number $\sum_j r_j$ in Eq. (28) does not determine the response exponent or its sign.

Proof Fix j and use coordinates centred at x_j . The critical-point branch can first be translated to the origin. The parameterised Morse lemma [11] then gives a parameter-dependent local diffeomorphism under which

$$F(b, x) = h_j(b) + Q_j(y),$$

where Q_j is a fixed definite quadratic form with the same inertia as the Hessian. Equivalently, the same leading term follows directly from

$$F(b, x) = h_j(b) + \frac{1}{2} x^T H_j x + o(|x|^2), \quad H_j = D_{xx}^2 F(b_0, x_j).$$

Regular zero-set pieces outside a smaller neighbourhood move by the implicit function theorem and contribute a smooth term. It remains to calculate the small ellipsoid created or removed at the critical point.

If H_j is negative definite and $h = h_j(b) > 0$, rescale $x = h^{1/2}z$. The rescaled superlevel set converges to $\{z : z^T(-H_j)z < 2\}$, while the rescaled density converges uniformly to $\varpi_j(x_j)$. Definite quadratic bounds provide a fixed integrable dominating indicator. Dominated convergence therefore gives

$$\int \mathbf{1}_{\{F>0\}} \varpi_j(x) dx = \frac{\omega_m 2^{m/2} \varpi_j(x_j)}{\sqrt{|\det H_j|}} h^{m/2} + o(h^{m/2}).$$

For $h < 0$ the corresponding small superlevel component is absent. If H_j is positive definite, the same calculation applies to the ellipsoidal complement when $h < 0$ and enters with a minus sign relative to the smooth background. This proves the formula with ϵ_j . Substitution of Eq. (33) proves Eq. (34); disjointness makes the contributions additive. $\square$

3.3 Finite-horizon first-passage specialisation

Let $X_\lambda(t, z)$ be a real-analytic parameter-dependent vector field, possibly depending on time, and let $\Phi_\lambda(t, 0; z)$ be its solution map from the fixed initial time 0. Write $\Phi_\lambda^t(z)$ for this map. The same notation is obtained by adjoining the clock equation $\dot{s} = 1$ and using the resulting autonomous flow. Let $\iota_\lambda(\theta, p)$ be a real-analytic parameterised family of launch embeddings, where $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ and p is a scalar launch coordinate. A parameter-independent launch embedding is the special case in which ι_λ is constant in λ . Let S_F and S_R be disjoint relatively open directed strata of analytic hypersurfaces. Denote their first positive directed hitting times by τ_F and τ_R , with value $+\infty$ when no hit occurs. A direction is specified by a local defining function r_α and the inequality $\partial_t(r_\alpha \circ \Phi) > 0$; replacing r_α by a positive analytic unit does not alter it. The supporting hypersurface is denoted by Σ_α .

Corollary 3.5 (Finite-horizon first-passage response) *Fix a deadline $H > 0$ and write $\bar{z}(t) = \Phi_{\lambda_0}^t(\iota_{\lambda_0}(\theta_0, p_0))$. Assume that the flow exists on $[0, H + \eta]$ for some $\eta > 0$ and that the following finite crossing ledger is admissible.*

First, $t = H$ is an isolated transverse directed hit in the interior of S_F . It is the first directed target hit, and no directed return hit precedes it. Second, every intersection of $\bar{z}((0, H + \eta])$ with $\bar{S}_F \cup \bar{S}_R$ is one of finitely many transverse intersections at strictly ordered times

$$0 < t_1 < \dots < t_{\ell-1} < t_\ell = H.$$

Each lies in the interior of its stratum. Small pairwise disjoint time intervals J_i about these crossings can be chosen so that, for an auxiliary Riemannian distance and some $d_0 > 0$,

$$\text{dist}(\bar{z}(K), \bar{S}_F \cup \bar{S}_R) \geq d_0, \quad K = [\delta, H + \eta] \setminus \bigcup_{i=1}^{\ell} J_i, \quad (36)$$

where $\delta = 0$ unless the launch lies on a supporting hypersurface; in the latter case $\delta > 0$ is chosen inside the uniform departure interval below. All crossings before H have their recorded reference directions, none of which is a directed target or return hit. The strict reference ordering has a positive time gap: with $t_0 = \delta$ and $t_{\ell+1} = H + \eta$,

$$g_0 := \min_{0 \leq i \leq \ell} (t_{i+1} - t_i) > 0. \quad (37)$$

Finally, if the launch lies on the return hypersurface, initial departure is uniformly transverse: for a defining function r_{out} oriented towards the designated outgoing side there are a neighbourhood $\mathcal{N}$ and $\kappa_0 > 0$ such that

$$\partial_t r_{\text{out}}(\Phi_\lambda^t(\iota_\lambda(\theta, p))) \Big|_{t=0} \geq \kappa_0 \quad \text{for } (\lambda, \theta, p) \in \mathcal{N}. \quad (38)$$

Then the continued terminal root is the first target time and is analytic on a neighbourhood of the reference initial condition. For fixed p_0 and a smooth phase density w with $w(\theta_0) > 0$, put

$$C_H(\lambda, \theta) = H - \tau_F(\lambda, \theta, p_0), \quad P_H(\lambda, p_0) = \int_0^{2\pi} \mathbf{1}_{\{\tau_F \leq H, \tau_F < \tau_R\}} w(\theta) d\theta.$$

Choose an open arc U in this neighbourhood and $\chi \in C_c^\infty(U)$ with $0 \leq \chi \leq 1$ and $\chi = 1$ near θ_0 . Define the localised response

$$P_{H,U}(\lambda, p_0) = \int_0^{2\pi} \chi(\theta) \mathbf{1}_{\{\tau_F \leq H, \tau_F < \tau_R\}} w(\theta) d\theta.$$

If

$$C_H(\lambda_0, \theta_0) = 0, \quad \partial_\theta C_H(\lambda_0, \theta_0) = 0, \quad \partial_{\theta\theta}^2 C_H(\lambda_0, \theta_0) \neq 0, \quad (39)$$

then $(C_H, \partial_\theta C_H)$ defines a finite critical-event module as in Theorem 3.1. Along an analytic scan $\lambda = \lambda(s)$, let $\theta_c(s)$ be the continued phase-critical branch. If

$$\left. \frac{d}{ds} C_H(\lambda(s), \theta_c(s)) \right|_{s=0} \neq 0,$$

then the scan order is one. There is a function $P_{H,U,\text{reg}}$, smooth at $s = 0$, such that

$$P_{H,U}(\lambda(s), p_0) - P_{H,U,\text{reg}}(s) = \epsilon \frac{2\sqrt{2} w(\theta_0)}{\sqrt{|\partial_{\theta\theta}^2 C_H(\lambda_0, \theta_0)|}} [\epsilon a s]_+^{1/2} + o(|s|^{1/2}), \quad (40)$$

where a is the displayed nonzero scan derivative and $\epsilon = -\text{sign}(\partial_{\theta\theta}^2 C_H)$.

For a global conclusion, put $K_\chi = \text{supp}(1 - \chi)$ and fix a compact scan interval I about $s = 0$. Suppose $I \times K_\chi$ has a finite cover by first-event charts with the following uniform properties. On each chart, every trajectory has a finite ledger of all target and return hypersurface intersections up to $H + \eta$; the intersections lie in stratum interiors, have analytic times $t_{a1} < \dots < t_{a\ell_a}$. Between fixed crossing neighbourhoods the orbit is at distance at least $d_* > 0$ from the stratum closures, and initial departure satisfies Eq. (38) with a common positive bound wherever it is required. For the local defining functions fixed on the charts, every crossing derivative has absolute value at least $\kappa_* > 0$. Distinct continued roots have a uniform order gap

$$t_{a,i+1} - t_{a,i} \geq g_*, \quad |\tau_F - \tau_R| \geq g_* \quad \text{whenever both first times are finite,} \quad (41)$$

for some $g_* > 0$. Thus target–return ties, tangencies, stratum-boundary hits, and positive roots accumulating at $t = 0$ are absent.

Assume further that the only remaining decision boundary in $I \times K_\chi$ is a finite family of pairwise disjoint analytic graphs $\theta = \beta_k(s)$ of deadline roots. On these graphs the first target precedes return by at least g_* and

$$|\partial_\theta C_H(\lambda(s), \beta_k(s))| \geq d_*, \quad \text{dist}_\mathbb{T}(\beta_k(s), \beta_l(s)) \geq g_* \quad (k \neq l). \quad (42)$$

Away from fixed neighbourhoods of these graphs, all inequalities determining the first-event label are separated from equality by at least d_* . Then there is neither root accumulation nor an uncharted competing event on the complement. The complementary response $P_H - P_{H,U}$ is smooth along the scan, and Eq. (40) gives the potentially nonsmooth event term of the global response after subtraction of a smooth function.

Proof Let r_F define the analytic hypersurface containing S_F near the terminal hit. There

$$\partial_{tr_F}(\Phi_{\lambda_0}^t(\iota_{\lambda_0}(\theta_0, p_0)))|_{t=H} = dr_F(X_{\lambda_0}(H, \Phi_{\lambda_0}^H(\iota_{\lambda_0}(\theta_0, p_0)))) \neq 0.$$

The analytic implicit-function theorem applied to $r_F(\Phi_\lambda^t(\iota_\lambda(\theta, p))) = 0$ gives an analytic hitting-time function near H : the argument is analytic because both the finite-time solution map and the launch family are analytic. The same argument at each earlier transverse crossing gives a unique analytic continuation of that crossing. Nonzero crossing derivatives preserve their signs, and the strict gaps between the finitely many reference times, quantified by Eq. (37), preserve their order after the neighbourhood is shrunk. Openness keeps every continued crossing in the interior of its stratum. Thus direction and order preservation are consequences of the reference transversality and time gaps, not additional hypotheses.

It remains to exclude an unrecorded root pair. On the compact set K in Eq. (36), continuous dependence of the flow on parameters and initial data preserves a positive distance from both stratum closures. The uniqueness clauses of the implicit-function theorem exclude additional roots inside the J_i . If the launch lies on Σ_R , Eq. (38) and continuity give a common $0 < \delta_1 \leq \delta$ on which every nearby orbit remains on the outgoing side and has no positive return. Thus no grazing pair can be born between the recorded crossings, their directions and order remain fixed, and the continued terminal root is the first directed target root and precedes any directed return. Hence the deadline event on U is exactly $C_H \geq 0$. Equations (39) are the one-dimensional Morse-witness hypotheses for $F = C_H$. In the parameterised Morse chart the zero set of C_H has one-dimensional phase measure zero. Therefore the indicators of $C_H \geq 0$ and $C_H > 0$ differ only on a null set, so the superlevel response theorem applies without changing the integral. Apply Theorem 3.4 with the density χw , $m = 1$ and $\omega_1 = 2$; since $\chi(\theta_0) = 1$, this gives Eq. (40) for $P_{H,U}$. Under the additional global hypotheses, each finite first-event chart has fixed crossing order by Eq. (41); uniform separation excludes new roots between its recorded crossings. The stated graph condition makes the complementary decision set a finite union of circular intervals whose only moving endpoints are among the $\beta_k(s)$; constant-label charts determine which intervals are selected. The implicit-function theorem and Eq. (42) make these endpoints smooth and uniformly separated. Integration of the fixed smooth density $(1 - \chi)w$ between them shows that $P_H - P_{H,U}$ is smooth along the scan. $\square$

Proposition 3.6 (Integrated positive-margin law) *Under the hypotheses of Theorem 3.4, replace the indicator response by the local positive margin*

$$Q(b) = \int_{\bigcup_j U_j} [F(b, x)]_+ w(x) dV(x), \quad [u]_+ = \max\{u, 0\}.$$

After subtraction of a smooth term, the j th definite-Morse contribution along the scan in Eq. (33) is

$$d_j [\epsilon_j a_j t^{r_j}]_+^{m/2+1} + o(|t|^{r_j(m/2+1)}), \quad d_j = \frac{2^{m/2+1} \omega_m \varpi_j(x_j)}{(m+2) \sqrt{|\det D_{xx}^2 F(b_0, x_j)|}}. \quad (43)$$

For one phase dimension the exponent is $3r_j/2$.

Proof In the Morse coordinates used in Theorem 3.4, a negative-definite witness contributes the positive quadratic cap when $h_j > 0$. At a positive-definite witness, the identity $[z]_+ = z + [-z]_+$ separates a smooth integral from the positive cap of $-F$ when $h_j < 0$. Thus both cases give the cap selected by $[\epsilon_j h_j]_+$. Put

$$H_j = D_{xx}^2 F(b_0, x_j), \quad A_j = -\epsilon_j H_j.$$

Then A_j is positive definite. On the active side, with $h = \epsilon_j h_j(b) > 0$ and $x = (2h)^{1/2} A_j^{-1/2} z$, the cap equals

$$\frac{(2h)^{m/2} h \varpi_j(x_j)}{\sqrt{\det A_j}} \int_{|z| < 1} (1 - |z|^2) dz + o(h^{m/2+1}).$$

Since $\det A_j = |\det H_j|$, this has the coefficient stated above. Polar integration gives $\int_{|z| < 1} (1 - |z|^2) dz = 2\omega_m/(m+2)$. Equation (43) follows on substituting $h_j(\gamma(t)) = a_j t^{r_j} + o(t^{r_j})$. $\square$

Proposition 3.7 (A finite deadline is not an exact stable-boundary audit) *Suppose the uncensored passage time at signed distance d from a hyperbolic stable boundary satisfies*

$$\tau(d) = -\lambda^{-1} \log |d| + O(1), \quad \lambda > 0.$$

For every finite H there is a punctured neighbourhood of the boundary in which $\tau(d) > H$. In particular, a fixed-deadline computation cannot uniformly certify the uncensored side label in any full neighbourhood of the stable boundary. The deadline boundary $\tau_F = H$ and the invariant stable boundary are distinct objects.

Proof There are $C > 0$ and $d_0 > 0$ such that $\tau(d) \geq -\lambda^{-1} \log |d| - C$ for $0 < |d| < d_0$. Hence $0 < |d| < \min\{d_0, e^{-\lambda(H+C)}\}$ implies $\tau(d) > H$. $\square$

3.4 A two-witness split module

Proposition 3.8 (Exact Fitting flag of a transverse split) *Let $B_0 = \mathbb{R}\{\nu, \kappa\}$ and consider the finite module*

$$M_{\text{split}} = B_0/(\nu + \kappa) \oplus B_0/(\nu - \kappa). \quad (44)$$

Its Fitting flag is

$$\text{Fitt}_0(M_{\text{split}}) = (\nu^2 - \kappa^2), \quad \text{Fitt}_1(M_{\text{split}}) = (\nu, \kappa), \quad \text{Fitt}_2(M_{\text{split}}) = B_0. \quad (45)$$

The two parameter branches meet transversely and their intersection length is

$$\text{length}_{B_0} B_0/(\nu + \kappa, \nu - \kappa) = 1.$$

After restriction to the diagonal scan $\kappa = 0$, with $A_0 = \mathbb{R}\{\nu\}$,

$$M_{\text{split}}|_{\kappa=0} \simeq A_0/(\nu) \oplus A_0/(\nu), \quad \text{Fitt}_0 = (\nu^2), \quad \text{Fitt}_1 = (\nu).$$

Thus transverse boundary-overlap length one, two simple critical witnesses, and scan Fitting order two coexist and must not be identified.

Proof The diagonal presentation with entries $\nu + \kappa$ and $\nu - \kappa$ gives Eq. (45); since 2 is a unit, $(\nu + \kappa, \nu - \kappa) = (\nu, \kappa)$ and the quotient has length one. Tensoring with $B_0/(\kappa)$ gives the displayed scan restriction. $\square$

Proposition 3.9 (Toda sudden-transfer normalisation) *Let $V > 0$, $\mu \neq 0$, and let $E_{\text{exc}} > 0$ be the reaction-coordinate excess above a declared passage threshold before a sudden transfer. Suppose the Toda sudden-transfer approximation assigns the retained reaction-coordinate energy change*

$$\Delta H_s(\theta_0) = -\frac{8\mu^2}{V} \sin^2 \theta_0. \quad (46)$$

Declare the surrogate passage margin to be $M_s = E_{\text{exc}} + \Delta H_s$. Its normalisation by the positive threshold excess is

$$\frac{M_s(\theta_0)}{E_{\text{exc}}} = 1 - \rho \sin^2 \theta_0, \quad \rho = \frac{8\mu^2}{V E_{\text{exc}}}. \quad (47)$$

Thus $\rho = 1$, equivalently $|\mu| = \sqrt{V E_{\text{exc}}/8}$, is the two-witness threshold at $\theta_0 = \pi/2, 3\pi/2$.

Equation (47) is an auxiliary energy-threshold calibration derived from the Toda sudden-transfer formula. It freezes the phase during transfer and replaces a trajectory-defined first-event decision by the sign of an energy margin. It is therefore not a full-flow hitting-time map and does not imply a finite-time passage fraction, rate, or product yield without the separate first-passage hypotheses of Corollary 3.5.

Proof The retained excess is

$$M_s(\theta_0) = E_{\text{exc}} - \frac{8\mu^2}{V} \sin^2 \theta_0.$$

Division by $E_{\text{exc}} > 0$ gives Eq. (47). At $\rho = 1$ the normalised margin vanishes exactly when $\sin^2 \theta_0 = 1$, which on the phase circle gives the two stated phases. Solving $8\mu^2/(V E_{\text{exc}}) = 1$ gives the displayed threshold. $\square$

Proposition 3.10 (Exact two-witness angular calibration) *Let*

$$C_\rho(\theta) = 1 - \rho \sin^2 \theta, \quad \theta \in \mathbb{R}/(2\pi\mathbb{Z}), \quad (48)$$

and set $\nu = \rho - 1$ and $A_\rho = \mathbb{R}\{\nu\}$. By Proposition 3.9, this is the normalised Toda sudden-transfer margin; it may also be read as an abstract angular calibration. It is not the Toda full-flow first-hitting-time map. At $\rho = 1$ it has precisely two critical zeros, $\theta = \pi/2$ and $3\pi/2$, in neighbourhoods of those points. Their critical-value germs are both $-\nu$, and hence

$$\begin{aligned} M_{\text{cal}} &\simeq A_\rho/(\nu) \oplus A_\rho/(\nu), \\ \text{Fitt}_0(M_{\text{cal}}) &= (\nu^2), \quad \text{Fitt}_1(M_{\text{cal}}) = (\nu), \quad \text{Fitt}_2(M_{\text{cal}}) = A_\rho. \end{aligned} \quad (49)$$

For the scan $\nu = t$, each witness has branch multiplicity $m_j = 1$, reduced contact $c_j = 1$, and Fitting valuation $r_j = m_j c_j = 1$.

With normalised uniform phase measure, define

$$P_{\text{cal}}(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{1}_{\{C_\rho(\theta) > 0\}} d\theta.$$

For ρ near one,

$$P_{\text{cal}}(\rho) = \begin{cases} 1, & \rho \leq 1, \\ \frac{2}{\pi} \arcsin(\rho^{-1/2}) = 1 - \frac{2}{\pi} \arctan \sqrt{\rho - 1}, & \rho > 1, \end{cases} \quad (50)$$

and therefore

$$P_{\text{cal}}(1 + \nu) = 1 - \frac{2}{\pi} \nu^{1/2} + \frac{2}{3\pi} \nu^{3/2} + O(\nu^{5/2}) \quad (\nu \downarrow 0). \quad (51)$$

Thus the first Fitting ideal distinguishes two simple witnesses from one nonreduced cyclic summand having the same zeroth Fitting ideal. The calculation verifies the algebra-response bookkeeping in closed form; it does not establish any full-flow Toda passage or chemical rate.

Proof Differentiation gives

$$\partial_\theta C_\rho = -\rho \sin(2\theta), \quad \partial_{\theta\theta}^2 C_\rho = -2\rho \cos(2\theta).$$

At $\rho = 1$, the only critical points at which C_ρ also vanishes are $\pi/2$ and $3\pi/2$; the Hessian equals 2 at each. Their critical branches are constant and their critical values are $1 - \rho = -\nu$. The diagonal presentation $\text{diag}(-\nu, -\nu)$ proves Eq. (49), and Eqs. (24)–(26) give $m_j = c_j = r_j = 1$.

If $\rho \leq 1$ and is positive, $C_\rho \geq 0$ on the circle, with only a measure-zero equality set at $\rho = 1$. If $\rho > 1$, the condition $C_\rho > 0$ is $|\sin \theta| < \rho^{-1/2}$ and has total angular length $4 \arcsin(\rho^{-1/2})$. This proves the first equality in Eq. (50). For $\nu > 0$, the elementary identity

$$\arcsin((1 + \nu)^{-1/2}) = \frac{\pi}{2} - \arctan \sqrt{\nu}$$

proves the second. Expanding $\arctan x = x - x^3/3 + O(x^5)$ proves Eq. (51). Equivalently, each positive Hessian witness has $\epsilon_j = -1$ and $\kappa_j = 1/\pi$ in Theorem 3.4, so the two local terms sum to $-(2/\pi)\nu^{1/2}$. $\square$

3.5 Sharp positive and negative examples

Example 3.11 (Positive: an admissible first-hit fold) Take $H > 2$, let $\dot{q} = 1 + \lambda/H$, and use the parameter-independent launch map

$$q(0) = -H - (1 - \cos \theta).$$

Declare $q = 0$ with positive velocity to be the target; a return section may be placed strictly behind every nearby launch. Then

$$\tau_F(\lambda, \theta) = \frac{H + 1 - \cos \theta}{1 + \lambda/H}, \quad C_H(\lambda, \theta) = H - \frac{H + 1 - \cos \theta}{1 + \lambda/H}.$$

At $(\lambda, \theta) = (0, 0)$ the terminal root is isolated and transverse, the reference segment has positive separation from all other observation strata, and the crossing ledger consists only of the terminal hit. The phase fold is a nondegenerate maximum. For normalised uniform phase measure and $0 < \lambda < 2$,

$$P_H(\lambda) = \frac{1}{\pi} \arccos(1 - \lambda) = \frac{\sqrt{2}}{\pi} \lambda^{1/2} + O(\lambda^{3/2}),$$

whereas $P_H(\lambda) = 0$ for $\lambda < 0$ near zero. This is precisely Eq. (40), since $\partial_{\theta\theta}^2 C_H(0, 0) = -1$ and $w(0) = 1/(2\pi)$. If U is a small arc about zero, then $1 - \cos \theta$ has a positive minimum on $\text{supp}(1 - \chi)$. On a smaller scan interval the complement is uniformly deadline-unhit and is covered by finitely many constant-label first-event charts. Thus the same square-root edge is also the global event term in this example.

Example 3.12 (Negative: an earlier grazing pair destroys first-hit analyticity) Fix $0 < a < H$ and consider the analytic scalar trajectory

$$y_\lambda(t) = (t - H)((t - a)^2 - \lambda t).$$

It is realised from the parameter-independent initial value $y_\lambda(0) = -Ha^2$ by the analytic nonautonomous equation $\dot{y} = \partial_t y_\lambda(t)$. Declare $y = 0$ with $\dot{y} > 0$ to be the target. At $\lambda = 0$, the root $t = H$ is transverse and directed, while $t = a$ is a nondirected double root. Thus the terminal root continues analytically as H . For small $\lambda < 0$ it remains the first directed root. For small $\lambda > 0$, however, the first directed root is

$$t_-(\lambda) = a + \frac{\lambda}{2} - \frac{1}{2} \sqrt{\lambda(4a + \lambda)}.$$

Consequently τ_F is not even continuous at $\lambda = 0$. The example violates Eq. (36) at the grazing time a and shows why an analytic terminal root alone does not define an analytic first-hit branch.

Example 3.13 (Negative: tangential initial departure creates an immediate return) Launch on the return section $y = 0$ and let $y_\lambda(t) = t(t - \lambda)$. Take $y < 0$ as the outgoing side and a crossing with $\dot{y} > 0$ as return. For every $\lambda > 0$ the trajectory initially leaves into $y < 0$ and returns at $t = \lambda$; these return times accumulate at zero. At $\lambda = 0$ the initial derivative vanishes. Hence no common positive interval is free of return roots. Uniform initial transversality in Eq. (38) is therefore essential.

Example 3.14 (Negative: an uncharted first-event tie creates another edge) Consider two independent coordinates with $\dot{q} = \dot{r} = 1$ and the parameterised launch

$$q(0) = \lambda - \theta^2, \quad r(0) = 0.$$

Declare $q = 1$ a forward target and $r = 1$ a return. Their directed first times are

$$\tau_F = 1 - \lambda + \theta^2, \quad \tau_R = 1.$$

At $(\lambda, \theta) = (0, 0)$ they tie and the tie function is tangent in phase. For $\lambda > 0$, the forward-first phase set is $|\theta| < \sqrt{\lambda}$ and has local length $2\sqrt{\lambda}$; for $\lambda < 0$ it is empty. If this point were left in the purported smooth complement, that complement would carry a second one-sided square-root edge. The target–return gap and tie-exclusion clauses in Eq. (41) cannot be omitted.

Example 3.15 (Positive: a multicomponent valuation is the weighted contact sum) Let $B = (\mathbb{R}^2, 0)$ with coordinates (ν, κ) and set $F((\nu, \kappa), x) = \nu\kappa^2 - x^2$. The unique phase-critical branch is $x = 0$, its Hessian is -2 , and its critical value is $h = \nu\kappa^2$. Along the scan $\gamma(t) = (t^2, t^3)$ the two reduced contacts are $c_\nu = 2$ and $c_\kappa = 3$, while the multiplicities are $m_\nu = 1$ and $m_\kappa = 2$. Hence

$$r = m_\nu c_\nu + m_\kappa c_\kappa = 2 + 6 = 8, \quad \gamma^* M_{\text{crit}} \simeq \mathbb{R}\{t\}/(t^8).$$

The local superlevel set is $|x| < |t|^4$ and has length $2|t|^4$, in agreement with the one-dimensional exponent $r/2 = 4$.

Example 3.16 (Negative: divisor components do not split one witness module) Let $A = \mathbb{R}\{\nu, \kappa\}$ and $F((\nu, \kappa), x) = \nu\kappa - x^2$. There is one phase witness, so

$$M_{\text{crit}} = A/(\nu\kappa), \quad \text{Fitt}_0(M_{\text{crit}}) = (\nu\kappa), \quad \text{Fitt}_1(M_{\text{crit}}) = A.$$

At the component intersection, the exact sequence

$$0 \longrightarrow A/(\nu\kappa) \longrightarrow A/(\nu) \oplus A/(\kappa) \longrightarrow A/(\nu, \kappa) \longrightarrow 0$$

has nonzero final term. Thus the first arrow is not an isomorphism. Indeed, the incorrectly split module would have first Fitting ideal (ν, κ) rather than A . The two divisor components describe two parameter directions of one critical-value germ, not two phase witnesses.

Example 3.17 (Negative: a Fitting valuation does not determine its factors) Compare $F_A(\nu, x) = \nu - x^2$ along $\nu = t^2$ with $F_B(\nu, x) = \nu^2 - x^2$ along $\nu = t$. In the first case $(m, c, r) = (1, 2, 2)$; in the second, $(m, c, r) = (2, 1, 2)$. Both pulled-back critical modules are $\mathbb{R}\{t\}/(t^2)$, and both local superlevel responses have length $2|t|$. The pulled-back Fitting flag therefore determines r but cannot decide whether its order came from a nonreduced branch or a tangent scan.

Example 3.18 (Positive: two transverse witnesses) Let $F(t, \theta) = t - \sin^2 \theta$ on the phase circle. At $t = 0$ the critical zeros $\theta = 0, \pi$ are nondegenerate maxima, each with critical value t . Hence

$$M_{\text{crit}} = \mathbb{R}\{t\}/(t) \oplus \mathbb{R}\{t\}/(t), \quad \text{Fitt}_0 = (t^2), \quad \text{Fitt}_1 = (t).$$

For the unnormalised uniform phase measure, $\{\sin^2 \theta < t\}$ has total length $4 \arcsin \sqrt{t} = 4\sqrt{t} + O(t^{3/2})$. Two transverse witnesses therefore double a square-root coefficient; they do not create a contact of order two.

Example 3.19 (Negative: one grazing witness has the same zeroth order) Let a smooth event branch be $\nu = 0$ and pull it back by the tangent scan $\nu = t^2$. The local family $F(t, x) = t^2 - x^2$ then has

$$\gamma^* M_{\text{crit}} = \mathbb{R}\{t\}/(t^2), \quad \text{Fitt}_0 = (t^2), \quad \text{Fitt}_1 = \mathbb{R}\{t\}.$$

Its uniform local response is the length of $|x| < |t|$, namely $2|t|$. This has the same zeroth Fitting order as Example 3.18, but a different complete Fitting flag and a different response exponent.

Example 3.20 (Negative: support and multiplicity do not determine surgery) The germs

$$F_{\text{def}}(\nu, x, y) = \nu + x^2 + y^2, \quad F_{\text{ind}}(\nu, x, y) = \nu + x^2 - y^2$$

have the same reduced event ideal (ν) , the same critical module $\mathbb{R}\{\nu\}/(\nu)$, the same phase critical-algebra length one, and the same transverse scan order. The first has a definite birth or death of a small circle, whereas the second has an indefinite local reconnection. The Hessian inertia, and for global conclusions the attaching data, cannot be recovered from the reduced event ideal or the Fitting module.

Example 3.21 (Negative: a vanishing density changes the exponent) For $F(t, x) = t - x^2$ and density $w(x) = x^2$,

$$\int_{-\sqrt{t}}^{\sqrt{t}} x^2 dx = \frac{2}{3} t^{3/2} \quad (t > 0).$$

The event ideal and scan order are unchanged, but the square-root term vanishes because $w(0) = 0$. Positivity of the ensemble density at the witness is an essential hypothesis of Theorem 3.4.

Example 3.22 (Negative: indefinite signature changes the response class) On $[-1, 1]^2$, the area of $\{(x, y) : xy \leq t\}$ for $0 < t < 1$ exceeds its value at $t = 0$ by

$$2 \int_0^1 \min\{1, t/x\} dx = 2t(1 - \log t).$$

Thus an indefinite two-dimensional Morse point has a $t \log |t|$ singular part for this cutoff sublevel observable. By contrast, the area of $\{x^2 + y^2 \leq t\}$ is πt for $t > 0$. Equal event support and contact order do not determine the response class without the Hessian signature and a specified observable.

Example 3.23 (Negative: a product gradient rejects transverse witnesses) In the split module restricted to $\kappa = 0$, both critical witnesses have event equation $\nu = 0$ and are transverse to the scan $\nu = t$. The zeroth Fitting generator is nevertheless ν^2 , whose differential vanishes at $\nu = 0$. A transversality index computed from $d(\nu^2)$ would label both witnesses as grazing. Branchwise testing gives the correct two orders $r_1 = r_2 = 1$.

4 Numerical audit

4.1 Reproducible calculation

Equation (2) is integrated by classical fourth-order Runge–Kutta. Directed section crossings are bracketed within one step and located by bisection to a time interval of width at most 10^{-11} . Reported fold roots, derivatives and boundary continuations use the finest step $h = 0.0025$; $h = 0.01$ and $h = 0.005$ form their step audit. Direct first-event labels use 128 and 256 phases at $h = 0.005$. The deeper-winding calculation uses 128, 256, 512 and 1024 phases at $h = 0.005$; its 1024-phase values are reported only as descriptive finite-window diagnostics. Deep allowances $0.5T, T, 1.25T, 1.5T, 2T$ are evaluated from the same forward events, and target locations $\eta = 0, 10^{-4}, 10^{-3}, 10^{-2}$ form a section-sensitivity audit. The phase response is computed in two ways. First, direct labels are evaluated on the two uniform grids. Secondly, the exact reflection of Proposition 2.1, a bisection-isolated phase boundary and a 256-subinterval half-circle sign audit give a symmetry-reduced boundary estimate of the uniform-phase fraction. Agreement of these constructions is a numerical diagnostic, not an interval proof that excludes every unsampled root.

At a symmetry phase θ_* the event equation is

$$C_H(\mu, \theta; p_0) = H - \tau_F(\mu, \theta; p_0) = 0, \quad \partial_\theta C_H = 0. \quad (52)$$

The first condition is solved by bracketed bisection in μ . Symmetry gives the second exactly. Centred two-point first differences and three-point second differences estimate C_μ and $C_{\theta\theta}$ at two nested increments. A reported value is accepted only when the signs are unchanged across the nested-resolution audit. For the fixed-momentum scan, first- and second-order variational equations are also integrated with the base flow to obtain the endpoint derivatives $q_\mu(H)$ and $q_{\theta\theta}(H)$ independently of parameter differencing. Every event orbit is checked for a positive target crossing, absence of an earlier return, and absence of an earlier target hit.

The calculation is not an interval-arithmetic computer-assisted proof. The term *numerically audited* below means that all declared residual, bracketing, transversality, step-halving, phase-doubling and censoring gates pass. A rigorous enclosure would require an outward-rounded flow and global root isolation. Rigorous a posteriori manifold calculations provide a benchmark for that stronger standard [19].

4.2 Algebraic verification

The distributed exact-rational program constructs diagonal presentations for the examples in Section 3, computes their minors and checks the stated Fitting ideals. It also verifies that the two modules $A/(t) \oplus A/(t)$ and $A/(t^2)$ have the same zeroth Fitting ideal but different first Fitting ideals, and checks the nontrivial flag of the exact two-witness angular calibration in Proposition 3.10. These calculations audit the examples; the proofs do not depend on software.

Table 1 Predeclared numerical gates. Each comparison is regenerated by the distributed program.

Object	Production	Audit
Fold and boundary step	0.0025	0.01, 0.005
Direct first-event grids	$h = 0.005$, 256 phases	128 phases
Deeper-winding grids	$h = 0.005$, 1024 phases	128, 256, 512 phases
Deep allowance	$1.5T$	$0.5T, T, 1.25T, 2T$
Target section	$\eta = 10^{-3}$	$0, 10^{-4}, 10^{-2}$
Event root	bracket $\leq 10^{-11}$	three-step root spread
Phase fraction	boundary bisection	sign audit; phase grids
Derivative signs	centred differences	nested increments; variational ODE
Primary deadline	$3T/2$	$5T/4, 7T/4, 2T$
Recrossing window	$5T$	$4T, 6T$
Event semantics	target before return	directed crossing and one-hit audit

5 Square-root deadline response and protocol controls

5.1 Two opposite finite-time fold candidates

For $E(2.20)$, reflection forces the phase derivative to vanish at $\theta_* = 0$. The declared sampled half-circle audit finds no additional sign change on the reported branch. The numerically resolved deadline-fold candidate at $H = 3T/2$ lies at

$$\mu_U = 0.188302777751, \quad C_\mu = -19053.1, \quad C_{\theta\theta} = 2639.32.$$

The target crossing velocity is $0.010404257 > 0$. Thus increasing μ through μ_U produces a loss of sampled forward phases. For $E(1.80)$, reflection forces stationarity at $\theta_* = \pi$; again the sampled audit finds no additional reported sign change. The oppositely signed candidate lies at

$$\mu_L = 0.194566664355, \quad C_\mu = 3281.61, \quad C_{\theta\theta} = -1066.16,$$

with crossing velocity $0.026791662 > 0$. Increasing μ now produces a gain of sampled forward phases. The signs agree with the branchwise response theorem and distinguish loss from gain; the reduced support alone would not do so.

Proposition 2.2 gives the critical-orbit mechanism. Increasing μ raises the effective barrier on the upper $\theta_* = 0$ fibre and lowers it on the lower $\theta_* = \pi$ fibre. It does not transfer energy to the slow mode on either witness. The sampled labels also identify the decision states separated by each fold edge. At $\mu - \mu_U = 8 \times 10^{-4}$, the upper 256-phase grid contains 247 forward, zero return and nine deadline-unhit phases. For the lower grid, $\mu - \mu_L = -4 \times 10^{-4}$ gives zero forward, 217 return and 39 deadline-unhit phases, while $\mu - \mu_L = 8 \times 10^{-4}$ gives five forward, 215 return and 36 deadline-unhit phases. The primary potentially nonsmooth observable is therefore deadline barrier-passage

Table 2 Primary $H = 3T/2$ deadline-fold candidates. Derivatives use the time-margin normalisation $C_H = H - \tau_F$; $K > 0$ is the magnitude in $|\Delta P| \sim K[\mu - \mu_*]_+^{1/2}$, with loss and gain signs shown by the branch labels. Displayed digits identify the reproducible finest-step run, not certified accuracy.

Branch	p_0	θ_*	μ_*	C_μ	$C_{\theta\theta}$	K
Loss	2.20	0	0.188302777751	-19053.1	2639.32	1.209488
Gain	1.80	π	0.194566664355	3281.61	-1066.16	0.789765

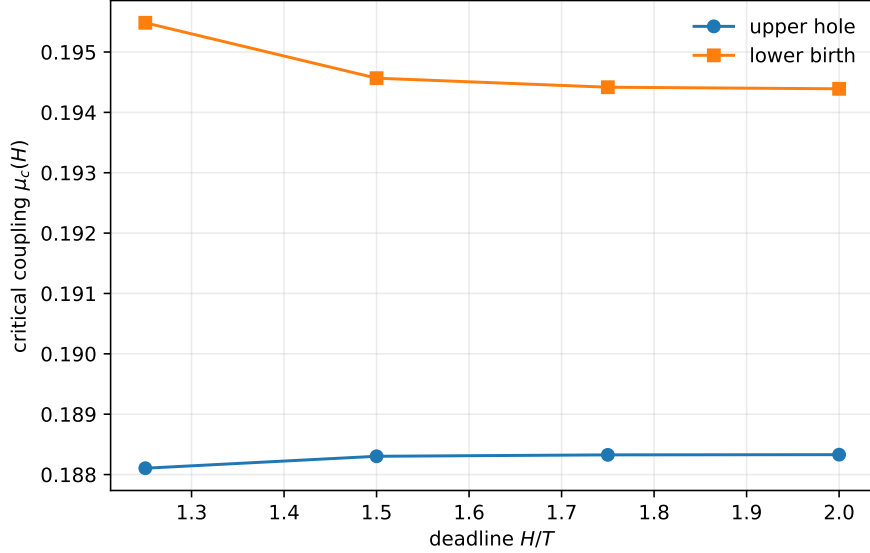


Fig. 1 Deadline dependence of the two critical-coupling candidates. Each point is a numerical approximation to the trajectory-defined root $\tau_F = H$ at the declared symmetry phase; no finite- H point is identified with an invariant stable boundary.

probability, with censoring explicit; it is not a binary exchange of forward and return channels.

Figure 1 records the deadline dependence of the two roots, and Figure 2 compares the local boundary measure with the quadratic square-root prediction.

5.2 Deadline-gate dependence and conditioning

The event roots at $h = 0.005$ differ from those at $h = 0.0025$ by no more than $8.558e - 11$. Nested finite differences preserve the signs of C_μ and $C_{\theta\theta}$. Against the variational equations, the maximum relative differences of the finite-difference endpoint derivatives are $1.908e - 06$ for q_μ and $6.964e - 07$ for $q_{\theta\theta}$. True 128-to-256 grid doubling changes a directly sampled fixed-momentum fraction by at most 0.0039062; the separate maximum disagreement between a direct grid and a symmetry-reduced boundary

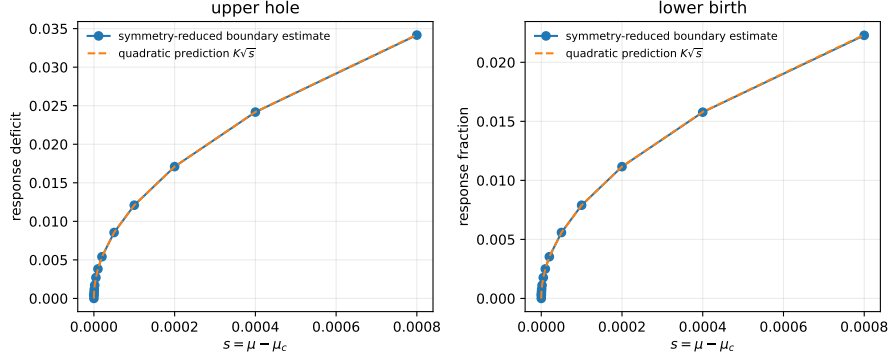


Fig. 2 Uniform-phase deadline responses near the two events. Marked curves are symmetry-reduced boundary estimates; dashed curves use the quadratic coefficient in Table 2. Direct phase grids provide the separate audit reported in the text.

fraction is 0.0078125. Boundary estimates carry their own bisection brackets and are used for the square-root rescaling. Over the declared small opening-side windows, the maximum absolute remainder in the rescaled fixed-momentum coefficient is $2.325e-05$. The 256-subinterval half-circle audit has 0 sampled sign-change failures across the 38 reported branch–offset cases. The sampled gate does not exclude even tangencies, root pairs between nodes, or other unsampled roots. Figure 3 reports the full resolution table rather than a single derivative estimate.

Moving the forward section through $\eta = 0, 10^{-4}, 10^{-3}, 10^{-2}$ gives coefficient ranges $[1.209481, 1.209551]$ for the upper branch and $[0.789701, 0.789772]$ for the lower branch. The coefficient is stable on this audit, whereas the critical coupling remains section-dependent; the calculation therefore does not identify a section-free transition parameter.

The endpoint normalisation in Proposition A.1 divides endpoint derivatives by the crossing velocity. The primary upper and lower velocities, approximately 0.0104 and 0.0268, are positive but small. The large values of C_μ and $C_{\theta\theta}$ therefore record a transverse but sensitively conditioned event, not an interval-certified count of significant digits. Full-precision values are retained in the CSV files; the text reports only digits stable under the declared audit.

The horizon audit has a structural meaning. Near a hyperbolic barrier, a trajectory at signed distance d from a stable boundary has delay of order $-\lambda^{-1} \log |d|$. Consequently every finite H leaves a strip of width $O(e^{-\lambda H})$ unresolved with respect to the infinite-time boundary. Our event is instead the defining finite-time surface $\tau_F = H$. Changing H is therefore a change of observable, not merely a discretisation error. On the finest reported step, changing H/T from 1.25 to 2 moves the upper candidate from 0.1881050 to 0.1883293 and the lower candidate from 0.1954841 to 0.1943906. This opposite drift is a protocol-sensitivity result of the deadline decision itself; it is not included in the RK4 error estimate.

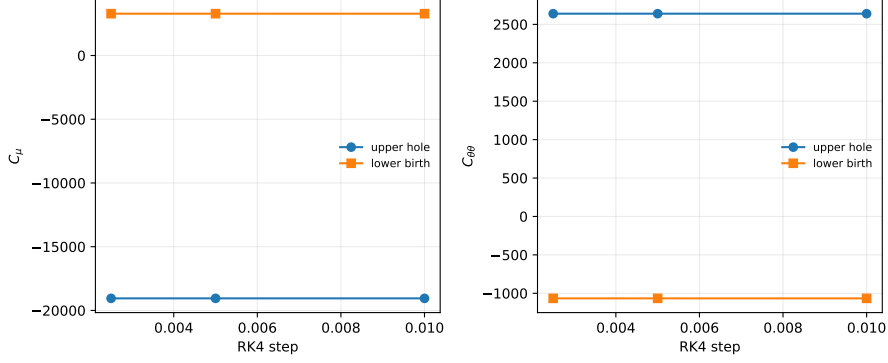


Fig. 3 Primary-deadline derivative estimates under RK4 step refinement. The nested finite-difference audit is tabulated in the distributed CSV; the nonzero signs are unchanged at every displayed step.

Table 3 Local response under three launch protocols at $H = 3T/2$. The constant-energy and phasewise-held roots reproduce the reference witness by construction. Opening side is the side on which the local change occurs. The quantities are numerical, not interval-certified.

Protocol	branch	side	C_μ	$C_{\theta\theta}$	K
Fixed p_0	upper	$\mu > \mu_*$	-19053.1	2639.32	1.209488
Fixed p_0	lower	$\mu > \mu_*$	3281.61	-1066.16	0.789765
Constant E_0	upper	$\mu < \mu_*$	-696.481	-817.277	0.415561
Constant E_0	lower	$\mu > \mu_*$	161.14	-459.018	0.266718
Phasewise held	upper	$\mu > \mu_*$	-696.481	2639.32	0.231246
Phasewise held	lower	$\mu > \mu_*$	161.14	-1066.16	0.175007

5.3 Three preparation protocols

For each branch, the control energy was chosen from the fixed-momentum witness itself:

$$E_U = 0.943394444, \quad E_L = 0.909133329.$$

Consequently $p_E(\mu_*, \theta_*) = p_0$ by construction. The two constant-energy roots therefore reproduce the same witness trajectories; their agreement with the fixed-momentum roots is not an independent root discovery. What changes is the off-witness launch family. By contrast, Eq. (8) makes the held comparator coincide with the fixed-momentum launch at μ_* for every phase, not only at the witness.

The constant-energy preparation reverses the upper opening to $\mu < \mu_*$ and changes its transverse curvature. The phasewise-held comparator instead retains the fixed-launch curvature at the reference coupling and opens on $\mu > \mu_*$, with coefficients 0.231246 and 0.175007. Thus the constant-energy reversal is not explained by removal of scan-induced energy drift alone: it also reflects replacement of the surrounding launch curve. The largest rescaled-coefficient remainders are $2.164e - 05$ for constant energy

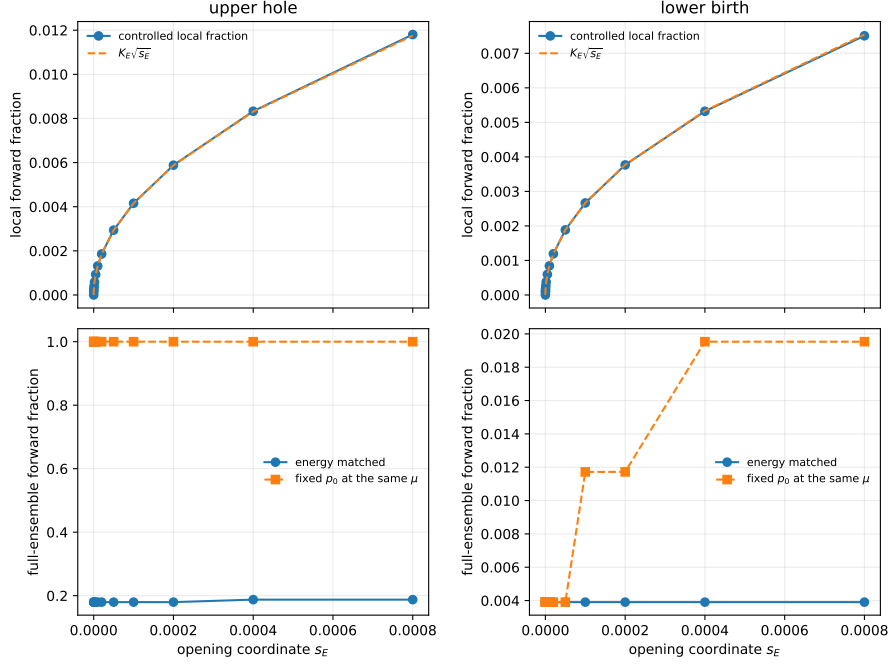


Fig. 4 Top: the symmetry-adjacent local constant-energy square-root edge and its quadratic prediction. Bottom: full-circle direct fractions for constant-energy and fixed- p_0 launches at the same coupling. Each control is anchored at the corresponding fixed-momentum witness, so the comparison does not provide independent agreement of the critical root.

and $4.452e - 06$ for phasewise holding. Direct 128-to-256 grid changes are at most 0.0078125 and 0.0039062, respectively. These are preparation-sensitivity comparisons, not causal subtractions.

Figure 4 isolates the constant-energy alternative, whereas Figure 5 places all three local responses on the same scale.

5.4 Deeper-winding and finite-window recrossing

The recrossing calculation uses the same fixed- p_0 ensemble and requires the same forward deadline event. Figure 6 reports the $a = 1.5$ deeper-winding and post-deep recrossing fractions. The conditional survival curves in Figure 7 retain the time-to-deep information that a single binary gate discards. Across the reported offsets, the largest observed delays are $1.189074T$ for the upper branch and $1.243079T$ for the lower branch. The number of sampled forward trajectories unresolved by $1.5T$ is 0. The lower branch therefore exhibits delayed deeper winding under this protocol; it is not confined to shallow passage.

The maximum 256-to-512 and 512-to-1024 recrossing-grid changes are 0.0078125 and 0.0039062. Their decrease under refinement is insufficient to establish convergence of a chemical quantity. Reporting a subtraction alone would also conceal both censoring layers.

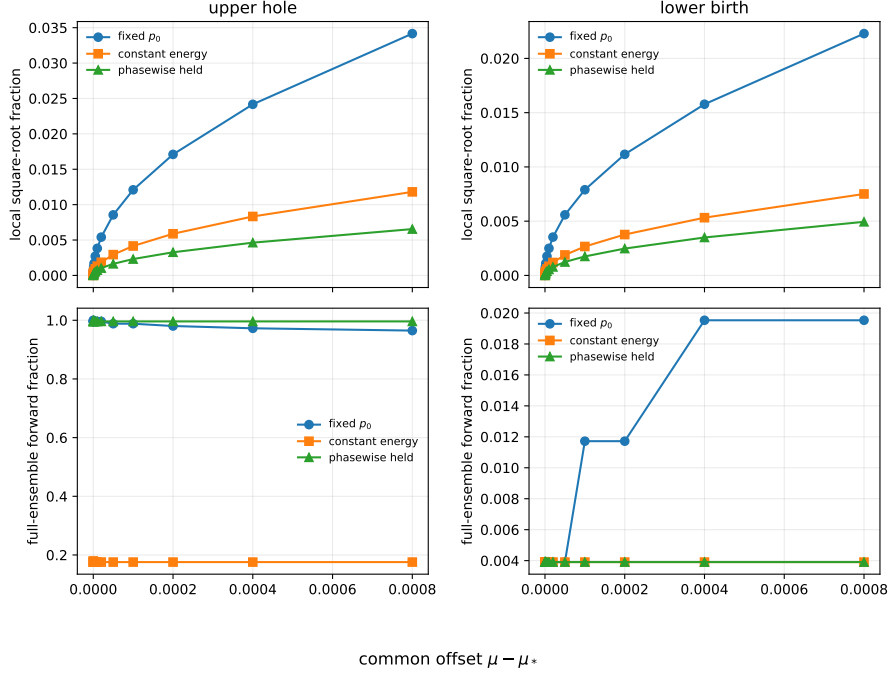


Fig. 5 Local deadline response under fixed momentum, constant instantaneous energy and phasewise baseline-energy holding. The third protocol matches the fixed launch at the reference coupling for every phase; the second does not.

6 Chemical interpretation and limits

The upper and lower launch momenta probe different sides of the unperturbed pendulum threshold $p_0 = 2\sqrt{V} = 2$. At $p_0 = 2.20$, the coupled slow degree does not exchange energy on the critical $\theta = 0$ orbit; increasing μ raises its effective barrier and removes a narrow set of phases from the deadline-passage set. At $p_0 = 1.80$, the $\theta = \pi$ critical orbit likewise has no slow-energy exchange, but increasing μ lowers its effective barrier and creates a passage window. These exact signs do not by themselves prove monotonic first-hit times in a driven system. They coexist with energy exchange on nonsymmetric trajectories and with the fixed- p_0 preparation drift in Eq. (5).

The protocol comparison resolves part of that ambiguity without claiming a causal decomposition. Phasewise holding removes the scan-induced change of each phase's launch energy while preserving the entire fixed- p_0 launch at μ_* . Both held folds remain on the $\mu > \mu_*$ side. Constant instantaneous energy instead replaces the phase profile already at μ_* ; its reversal of the upper opening is therefore a property of that alternative preparation, not the isolated effect of energy-drift removal. The barrier shift fixes the symmetry-witness sign interpretation; transverse curvature and ensemble response remain protocol dependent.

The deeper-winding audit changes the earlier chemical interpretation. A $+T$ cutoff leaves delayed trajectories unresolved, but every sampled forward trajectory in the

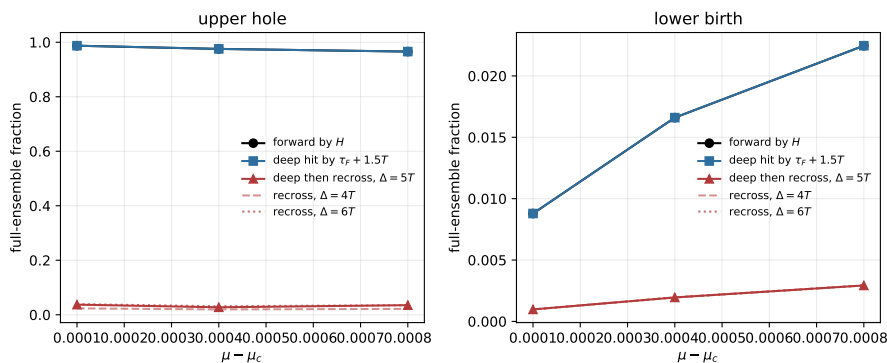


Fig. 6 Forward, deeper-winding-by-1.5T, and finite-window recrossing fractions for the fixed ensembles. A larger post-winding window can only increase the observed recrossing fraction. Failure to reach the deeper section by one allowance and absence of an observed return are distinct censored statements.

reported offset set reaches $q = 3\pi/2$ by $+1.5T$. The lower fold therefore connects to delayed deeper winding under the declared launch, rather than to a distinct shallow-only class. To test whether aggregate recrossing is the fate of the local fold mass, we use the binary forward-membership ledger in Definition 2.17, not equality of the three-state labels. At $N = 1024$, $a = 1.5$, $\Delta = 5T$ and $\delta = 8 \times 10^{-4}$, the upper sector has $|S_\delta| = 35$, with forward-side fate count $C_\delta = 0$. Its plus-side changed count is $C_\delta^+ = 0$, while $B_\delta^+ = 36$ on the unchanged plus-side complement. The corresponding lower counts are 23, 3, 3 and 0. The forward-side and plus-complement counts are intentionally not added; only $C_\delta^+ + B_\delta^+$ partitions the global plus-side recrossing count. Thus aggregate recrossing cannot in general be assigned to the square-root sector without this phasewise ledger. Because q is periodic, the unwrapped section records progress towards the equivalent well at $q = 2\pi$, not formation of a distinct molecular species; all return counts remain finite-window censored observations.

The interpretation is conditional. Uniform phase is a Haar preparation on the slow angle, not an equilibrium distribution. The Toda model is a mechanistic Hamiltonian, not a fitted molecular potential. The forward section records passage beyond a barrier neighbourhood; it does not identify a chemical product basin. The recrossing window tests one finite-depth failure mode. Rates, branching ratios and yields require a flux-weighted reactant ensemble, a product definition, and observation long enough to control recrossing. These limits separate the present result from lobe-rate calculations and roaming analyses [17, 18].

7 Conclusion

A finite-horizon response edge carries four distinct layers. Its reduced support locates the event. Under split hypotheses, the direct-image module and complete Fitting flag recover the unordered cyclic-factor valuations r_j . For one reduced component, $r_j = m_j \text{TI}_j$; with several components, $r_j = \sum_\alpha m_{j\alpha} c_{j\alpha}$, and the cyclic module need not split at their intersection. The flag does not recover the factors. Hessian inertia and scan orientation determine whether phase mass is gained or lost, while launch

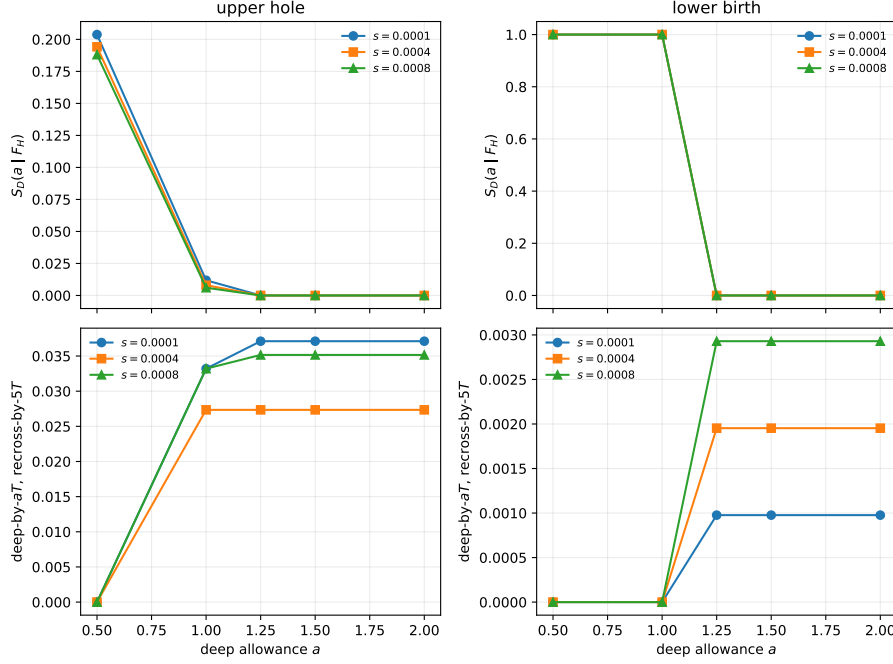


Fig. 7 Conditional survival $S_D(a) = \Pr(\tau_D - \tau_F > aT \mid F_H)$ on the 1024-phase audit grid. The decline between T and $1.5T$ identifies observation-window delay rather than a separate shallow-passage class.

density sets the coefficient. The Toda-normalised angular calibration shows why a zeroth Fitting ideal cannot replace this ledger.

Each Toda candidate is instead a one-witness application to the unreduced non-autonomous equations. The invariant-fibre calculation identifies its critical mechanism as an effective-barrier shift. The three-state labels show that the measured square-root edge is a deadline barrier-passage response with explicit censoring, not an exchange between two product channels. The three launch protocols establish preparation dependence of transverse curvature, opening side and coefficient. Deadline variation establishes observation-window dependence of the critical coupling, while conditional time-to-deep distinguishes delayed winding from an arbitrary $+T$ cutoff. Phasewise changed-sector accounting further separates local fold mass from a global recrossing background.

The theorem is local. The full-circle computation supplies sampled global consistency, not interval root exclusion. Higher-resolution deeper-winding and recrossing fractions remain conditioned finite-window diagnostics. The result therefore supports an auditable, protocol-dependent first-passage statement, not an invariant sheet, lobe volume, thermal rate, committor, asymptotic product yield or molecular validation.

A Endpoint and hitting-time derivatives

The code differentiates an endpoint equation rather than repeatedly differentiating a root-finding routine. The following identity gives the normalisation used in Table 2.

Proposition A.1 (Endpoint normalisation) *Let z denote any parameter or initial coordinate, and suppose the target is $q = q_F$. If the reference orbit reaches it for the first time at H with $v = \dot{q}(H) > 0$, put*

$$G(z) = q(H; z) - q_F, \quad C(z) = H - \tau_F(z).$$

Then $C_z = G_z/v$. If $z = (\mu, \theta)$ and $G_\theta = 0$ at the reference point, then also $C_{\theta\theta} = G_{\theta\theta}/v$.

Proof The local target equation is $q(\tau_F(z); z) - q_F = 0$. Differentiation gives $v\tau_{F,z} + q_z(H; z) = 0$. Hence $C_z = -\tau_{F,z} = G_z/v$. A second phase derivative gives $v\tau_{F,\theta\theta} + q_{\theta\theta} + 2q_{t\theta}\tau_{F,\theta} + q_{tt}\tau_{F,\theta}^2 = 0$. At a phase-critical point, $G_\theta = 0$ and therefore $\tau_{F,\theta} = 0$; the stated second-derivative identity follows. $\square$

Example A.2 (Positive: a transverse endpoint equation) For $q(t; z) = t + z$ and target $q_F = H$, one has $v = 1$, $G(z) = z$, $\tau_F(z) = H - z$, and $C(z) = z$. The endpoint and time-margin derivatives agree.

Example A.3 (Negative: tangency destroys the normalisation) For $q(t; z) = (t - H)^2 + z$ and target $q_F = 0$, the reference hit has $v = 0$. For $z < 0$ there are two nearby roots, while for $z > 0$ there is none. The hitting time is not an analytic single-valued function through $z = 0$.

B Computational records and audit semantics

The file `deadline_fold_roots.csv` contains every combination of two ensembles, four deadlines and three RK4 steps. The file `phase_response_boundaries.csv` records the bisection-resolved, symmetry-reduced phase-boundary estimate, response fraction, square-root rescaling and local fit. `direct_phase_grid_audit.csv` contains separately evaluated 128- and 256-phase labels. The three `energy_matched_*.csv` files record the constant-energy roots, local boundary response and direct first-event grids; the three `phasewise_energy_held_*.csv` files give the matched comparator. The file `sampled_half_circle_sign_change_audit.csv` records sampled sign changes and cannot exclude even tangencies between nodes. `target_section_sensitivity.csv` gives the η audit. `deep_time_survival_summary.csv` records conditional survival and delay quantiles. The two recrossing CSV files retain individual phase records, binary plus/minus forward membership, the unique forward-side fate record on each changed sector, and 128-, 256-, 512- and 1024-phase aggregates for all deep allowances and the $4T$, $5T$ and $6T$ post-deep windows. The JSON summary contains the maximum residual or spread entering each declared gate. Blank deeper-winding-hit or recrossing fields are censored observations; they are not zeros inferred beyond the observation window.

Supplementary information

Online Resource 1 contains the source, code, data, generated figures, validation report and machine-readable manifest needed to reproduce every numerical statement.

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Declarations

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Competing interests

The author declares no competing interests.

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Author contributions

Takashi Matsuhisa conceived the study, developed the analysis, prepared the computational audit and wrote the manuscript.

Data availability

No experimental data are reported. All trajectory labels, event roots, derivative audits and response summaries are supplied as CSV and JSON files and are regenerated by the distributed code.

Code availability

The dependency-free numerical kernel and exact algebra audit are included in Online Resource 1. Matplotlib is used only to draw the supplied vector-PDF figures.

Use of generative artificial intelligence

OpenAI Codex was used during mathematical drafting, LaTeX preparation and code development. The author reviewed the definitions, proofs, calculations, citations and final text and assumes full responsibility for the work.

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