

Supplementary Information for “Convection-permitting scales break climate invariance in machine-learned precipitation downscaling”

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This Supplementary Information contains:

1. Supplementary Text: from the governing equations to the spectral energy budget
2. Supplementary Figures 1 to 4

Overview. The Supplementary Text derives the coupled kinetic and available-potential spectral energy budgets of the main text from a stated set of approximate governing equations. Supplementary Figure 1 compares the EnsCGP and GAN model performances in downscaling at different scale mappings. Figure 2 shows the idealized squall line initial conditions and reference states of surface warming experiments. Figure 3 shows that energy transfers are insensitive to the choice of vertical column. Figure 4 shows that energy transfers are insensitive to the convection schemes.

1 Supplementary Text: from the governing equations to the spectral energy budget

This text derives Eqs. (5)-(6) of the main text from the governing equations, with all symbols the same as in the main text.

2 Perturbed equations

With the perturbed buoyancy $b' = g \theta'_\rho / \bar{\theta}_\rho$, we have

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \varphi' + b' \hat{\mathbf{z}}, \quad (\text{S1})$$

$$\frac{\partial \theta'_\rho}{\partial t} + \mathbf{u}_h \cdot \nabla_h \theta'_\rho + w \frac{\partial \theta'_\rho}{\partial z} + w \frac{d\bar{\theta}_\rho}{dz} = \dot{\theta}'_{\text{cond}}. \quad (\text{S2})$$

The heating is resolved from the moisture budget: $\partial_t q_v + \mathbf{u}_h \cdot \nabla_h q_v + w \partial_z q_v = e - c$ (evaporation minus condensation). The net condensation rate is $c - e = -Dq_v/Dt$. Therefore, the potential-temperature source is

$$\dot{\theta}'_{\text{cond}} = \frac{\theta}{T} \frac{L_v}{c_p} \left(-\frac{Dq_v}{Dt} \right). \quad (\text{S3})$$

3 Energy tendency budgets

Multiplying Eq. (S2) by $\gamma \theta'_\rho$ and dotting Eq. (S1) with $\mathbf{u}$ gives, for $A \equiv \frac{\gamma}{2} \theta'^2_\rho = b'^2/(2N^2)$ and $K \equiv \frac{1}{2}(\mathbf{u}_h \cdot \mathbf{u}_h + w^2)$,

$$\frac{\partial A}{\partial t} + \mathbf{u}_h \cdot \nabla_h A + \frac{\gamma}{2} w \frac{\partial \theta'^2_\rho}{\partial z} + w b' = \gamma \theta'_\rho \dot{\theta}'_{\text{cond}}, \quad (\text{S4})$$

$$\frac{\partial K}{\partial t} + \mathbf{u}_h \cdot \nabla_h K + w \frac{\partial K}{\partial z} = -\nabla_h \cdot (\mathbf{u}_h \varphi') - \frac{\partial}{\partial z}(w \varphi') + \varphi' \nabla \cdot \mathbf{u} + w b', \quad (\text{S5})$$

where $\gamma (d\bar{\theta}_\rho/dz) \theta'_\rho = b'$ produced the $w b'$ term of Eq. (S4).

4 Fourier transform of the energy tendency budget

Every advection could be written as

$$\mathcal{A}(a) = \frac{1}{2} [(\mathbf{u}_h \cdot \nabla_h) a + \nabla_h \cdot (\mathbf{u}_h a)] = (\mathbf{u}_h \cdot \nabla_h) a + \frac{1}{2} a \nabla_h \cdot \mathbf{u}_h, \quad (\text{S6})$$

The thermodynamic equation then becomes:

$$\frac{\partial \hat{\theta}'_\rho(\mathbf{k}, z)}{\partial t} + \widehat{\mathcal{A}(\theta'_\rho)}(\mathbf{k}, z) + \int \hat{w}(\mathbf{p}, z) \frac{\partial \hat{\theta}'_\rho(\mathbf{k} - \mathbf{p}, z)}{\partial z} d\mathbf{p} + \frac{d\bar{\theta}_\rho}{dz} \hat{w}(\mathbf{k}, z) = \hat{\theta}'_{\text{cond}}(\mathbf{k}, z), \quad (\text{S7})$$

The shell-to-shell matrix is the band-filtered contraction of Eq. (S6): with a^P the field band-passed to giver shell P , the unfiltered anomaly wind as mediator, and the projection taken on receiver shell Q ,

$$T_X[Q, P](z) = -\langle a^Q \mathcal{A}(a^P) \rangle = -\langle a^Q (\mathbf{u}_h \cdot \nabla_h) a^P \rangle - \frac{1}{2} \langle (\nabla_h \cdot \mathbf{u}_h) a^Q a^P \rangle, \quad T_X[Q, P] = -T_X[P, Q], \quad (\text{S8})$$

where for E_K the field a is velocity, and for E_A , $a = \theta'_\rho$. Positive $T_X[Q, P]$ is transfer from giver P to receiver Q .

The spectral energy budget terms

Multiplying Eq. (S7) by $\gamma \hat{\theta}'^*(\mathbf{k})$ and adding the conjugate gives the budget of $E_A(\mathbf{k}, z) = \frac{\gamma}{2} |\hat{\theta}'_\rho|^2$; the same operation on the transformed momentum equation with $\hat{\mathbf{u}}'^*(\mathbf{k})$ gives the budget of $E_K(\mathbf{k}, z) = \frac{1}{2} |\hat{\mathbf{u}}'|^2$:

$$\begin{aligned} \frac{\partial E_A}{\partial t} = & \underbrace{-\gamma \operatorname{Re} \left[\hat{\theta}'^*(\mathbf{k}) \widehat{\mathcal{A}(\theta'_\rho)}(\mathbf{k}) \right]}_{\rightarrow T_A(\mathbf{k}, z)} - \underbrace{\gamma \operatorname{Re} \left[\hat{\theta}'^*(\mathbf{k}) \int \hat{w}(\mathbf{p}) \partial_z \hat{\theta}'_\rho(\mathbf{k} - \mathbf{p}) d\mathbf{p} \right]}_{\rightarrow -\partial_z F_{A\uparrow} + J} \\ & - \underbrace{\frac{g}{\theta_\rho} \operatorname{Re} [\hat{\theta}'^*_\rho \hat{w}]}_{-C} + \underbrace{\gamma \operatorname{Re} [\hat{\theta}'^*_\rho \hat{\theta}'_{\text{cond}}]}_{+G}, \end{aligned} \quad (\text{S9})$$

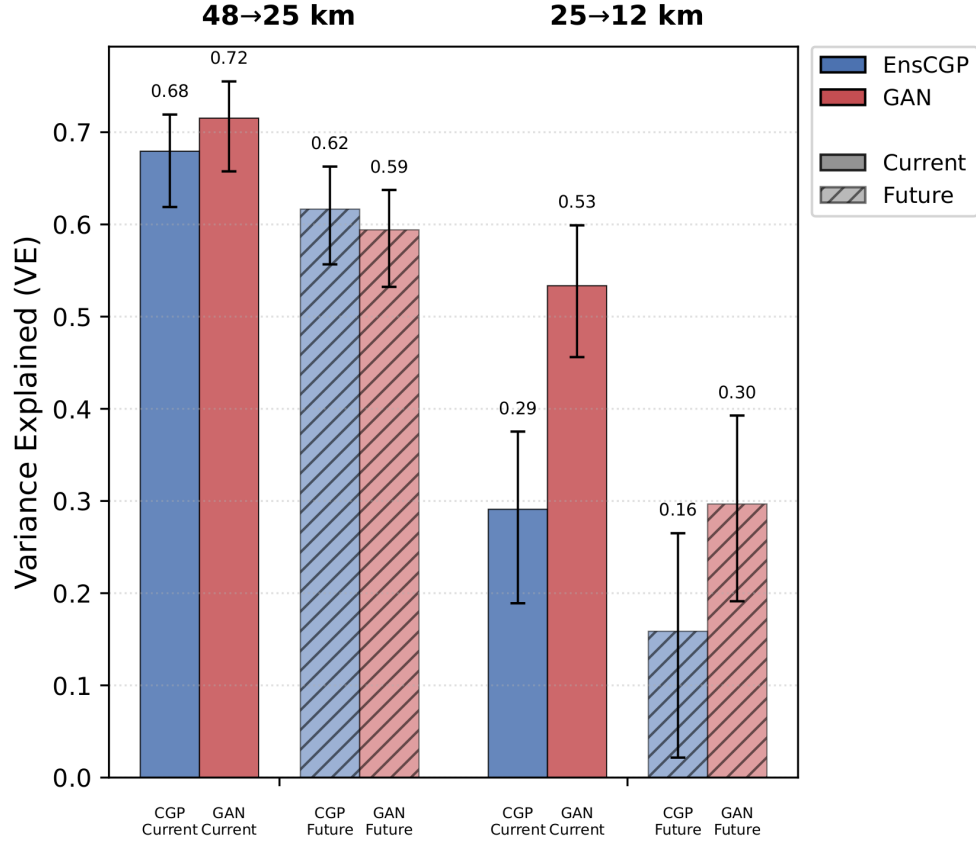
$$\begin{aligned} \frac{\partial E_K}{\partial t} = & \underbrace{-\operatorname{Re} \left[\hat{\mathbf{u}}'^*(\mathbf{k}) \cdot \widehat{\mathcal{A}(\mathbf{u}')(\mathbf{k})} \right]}_{\rightarrow T_K(\mathbf{k}, z)} - \underbrace{\operatorname{Re} \left[\hat{\mathbf{u}}'^*(\mathbf{k}) \cdot \int \hat{w}(\mathbf{p}) \partial_z \hat{\mathbf{u}}'(\mathbf{k} - \mathbf{p}) d\mathbf{p} \right]}_{\rightarrow -\partial_z \left[\frac{1}{2} \langle w' |\mathbf{u}'|^2 \rangle_{\mathbf{k}} \right]} \\ & - \underbrace{\operatorname{Re} [i\mathbf{k} \cdot \hat{\mathbf{u}}'_h(\mathbf{k}) \hat{\varphi}'(\mathbf{k})]}_{\sum_{\mathbf{k}} = 0 \rightarrow D_K} - \underbrace{\partial_z \langle w' \varphi' \rangle_{\mathbf{k}}}_{\rightarrow -\partial_z \langle w' \varphi' \rangle_{\mathbf{k}}} + \underbrace{\langle \varphi' \nabla \cdot \mathbf{u}' \rangle_{\mathbf{k}}}_{\rightarrow D_K} + \underbrace{\frac{g}{\theta_\rho} \operatorname{Re} [\hat{w}^* \hat{\theta}'_\rho]}_{+C}, \end{aligned} \quad (\text{S10})$$

The two vertical fluxes $F_{K\uparrow}$ and $F_{A\uparrow}$ of Eq. (8) of the main text assemble from the vertical-advection and pressure terms. For the kinetic energy,

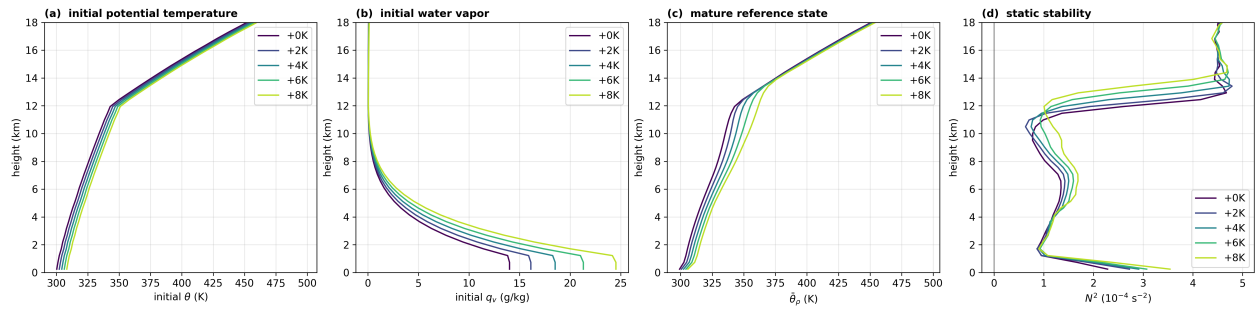
$$-\operatorname{Re} \left[\hat{\mathbf{u}}'^* \cdot \widehat{w \partial_z \mathbf{u}'} \right] = -\frac{\partial}{\partial z} \left[\frac{1}{2} \langle w' |\mathbf{u}'|^2 \rangle_{\mathbf{k}} \right] + \frac{1}{2} \langle (\partial_z w') |\mathbf{u}'|^2 \rangle_{\mathbf{k}}, \quad (\text{S11})$$

whose first term is the turbulent part of $F_{K\uparrow} = \langle w' \varphi' \rangle_{\mathbf{k}} + \frac{1}{2} \langle w' |\mathbf{u}'|^2 \rangle_{\mathbf{k}}$. For the available potential energy,

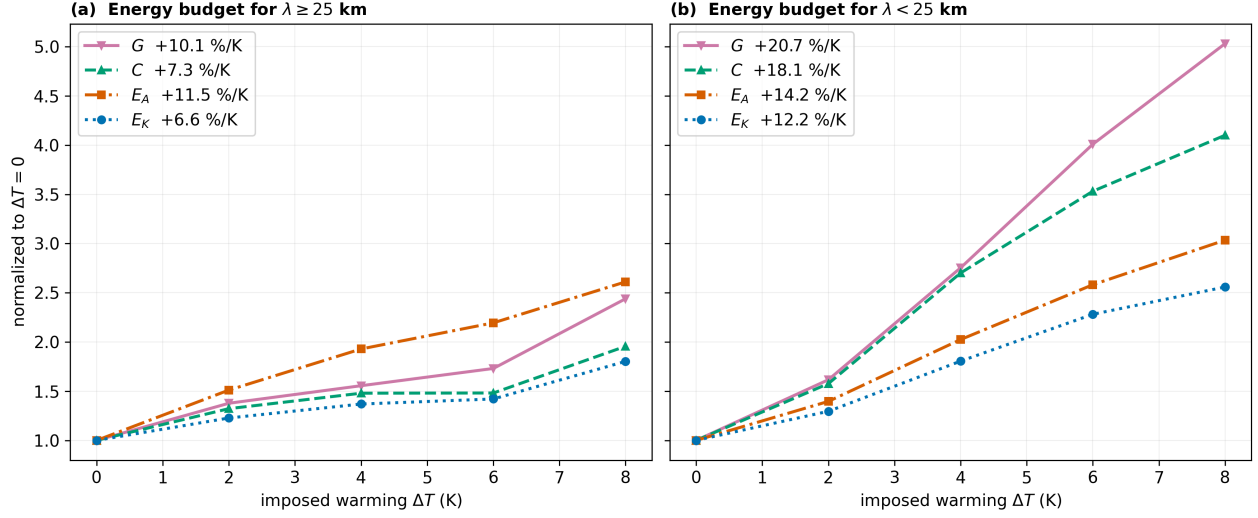
$$\gamma \operatorname{Re} \left[\hat{\theta}'^*_\rho \int \hat{w} \partial_z \hat{\theta}'_\rho \right] = \frac{\partial}{\partial z} \underbrace{\left[\frac{\gamma}{2} \langle w' \theta'^2_\rho \rangle_{\mathbf{k}} \right]}_{F_{A\uparrow}} - \underbrace{\frac{1}{2} \frac{d\gamma}{dz} \langle w' \theta'^2_\rho \rangle_{\mathbf{k}}}_{J} - \frac{\gamma}{2} \langle \theta'^2_\rho \partial_z w' \rangle_{\mathbf{k}}. \quad (\text{S12})$$



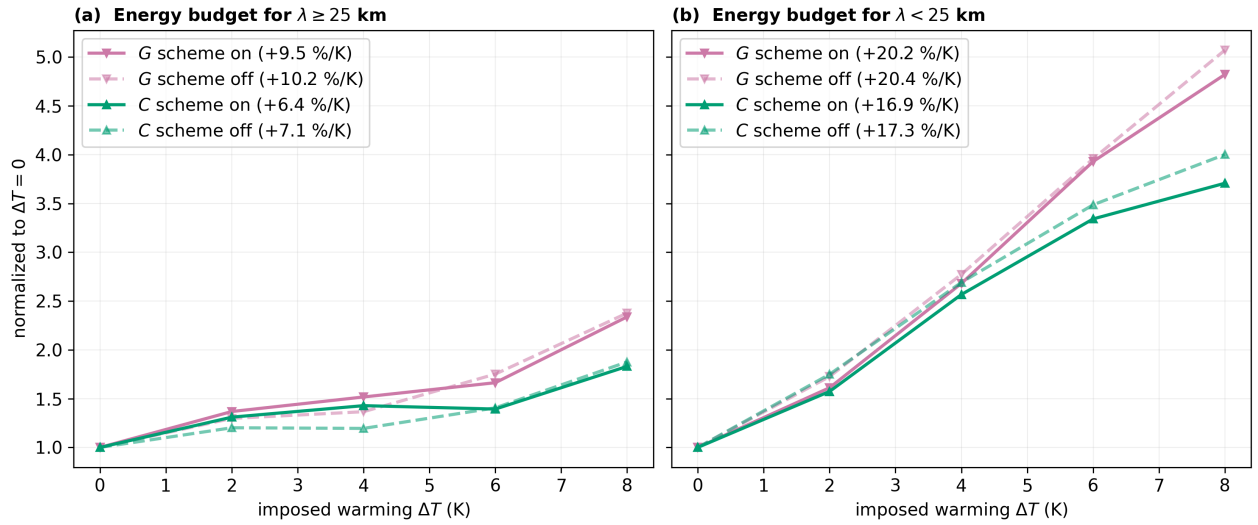
Supplementary Figure S1: Skills of EnsCGP and GAN operators at the coarse 48-25 and 25-12 km mappings. The variance explained (VE) shown here is a total-variance ratio: the sum of all eigenvalues of the covariance of the downscaled output divided by that of the target. Values near one indicate agreement in total variance.



Supplementary Figure S2: Reference state for the five surface warming levels in the 3 km idealized squall line runs. (a) Initial-condition potential temperature $\theta(z)$, and (b) water vapor $q_v(z)$ across $\Delta T = 0$ to $+8$ K. (c) The spun-up reference density potential temperature $\bar{\theta}_\rho(z)$, and (d) the static stability $N^2(z)$.



Supplementary Figure S3: Same as Figure 7 in the main text, but vertically integrated over the entire atmospheric column.



Supplementary Figure S4: Same as Figure 7 in the main text, but without the E_A E_K terms. Repeated for turning off the Grell-Freitas scheme in MPAS simulations.