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Uniform inf–sup stability of quartic and quintic Scott–Vogelius elements on Freudenthal meshes: a protected raw edge-star lifting

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Abstract

Let V_h^k be the continuous vector Lagrange space of degree k, with homogeneous Dirichlet trace, on the uniform three-dimensional Freudenthal triangulation, and let $Q_h^k = \operatorname{div} V_h^k$. Zhang proved a mesh-uniform right inverse of the divergence for $k \geq 6$. In his edge stage, however, the actual edge-trace functions already have degree $k \geq 4$; degree six is used only to repair elementwise divergence means. The missing point is to turn that observation into a genuinely local and order-independent lifting.

We give a separate protected raw edge-star lemma for $k = 4, 5$. The complete geometry consists of seven classes and thirty-seven oriented/boundary configurations; an independent census verifies all 117 boundary-decorated source envelopes. For every configuration we specify an exact rational local linear map. Its source trace space, target reproduction, and protection of all non-target edges are certified by integer matrix identities. Exact Bernstein mass and stiffness calculations give the uniform reference-patch bound $C_{\mathrm{ref}} < 385$. The protected maps have dependency depth zero, permit a 189-colour assembly, and have overlap at most 19. An exactly verified continuous piecewise-quartic two-cube macro-patch operator then repairs the element means without changing any edge trace. For $k = 4, 5$, combining this construction with Zhang's mean, vertex, and face stages and the low-degree vanishing of the final face-zero, cell-mean-zero residual yields mesh-independent inf–sup stability. Zhang's theorem supplies the cases $k \geq 6$, giving the result for every fixed $k \geq 4$.

1 Setting and main result

Let $\Omega = (0,1)^3$. Divide Ω into N^3 congruent cubes of side $h = N^{-1}$, and split each cube by the Freudenthal pattern. On the reference cube, with vertices

$$a_1 = (0,0,0), \quad a_2 = (1,0,0), \quad a_3 = (1,1,0), \quad a_4 = (1,1,1),$$
$$a_5 = (0,1,0), \quad a_6 = (1,0,1), \quad a_7 = (0,1,1), \quad a_8 = (0,0,1),$$

we use

$$K_1 = [a_1, a_2, a_3, a_4], \qquad K_2 = [a_1, a_3, a_4, a_5],$$
$$K_3 = [a_1, a_2, a_4, a_6], \qquad K_4 = [a_1, a_4, a_5, a_7],$$
$$K_5 = [a_1, a_4, a_6, a_8], \qquad K_6 = [a_1, a_4, a_7, a_8].$$

Write $\mathcal{T}_h$ for the resulting tetrahedral mesh and set

$$V_h^k = \{v_h \in H_0^1(\Omega;\mathbb{R}^3) : v_h|_K \in [\mathbb{P}_k(K)]^3 \quad (K \in \mathcal{T}_h)\},$$
$$Q_h^k = \operatorname{div} V_h^k \subset L_0^2(\Omega).$$

We write $|v|_1 = \|\nabla v\|_{L^2(\Omega)}$.

Theorem 1.1 (Freudenthal Scott–Vogelius stability)**.** *For every fixed integer $k \geq 4$ there exists $\beta_k > 0$, independent of N and $h = N^{-1}$, such that*

$$\sup_{0 \neq v_h \in V_h^k} \frac{(\operatorname{div} v_h, q_h)_\Omega}{|v_h|_1} \geq \beta_k \, \|q_h\|_{L^2(\Omega)} \qquad (q_h \in Q_h^k).$$

Equivalently, there is a linear right inverse $R_h : Q_h^k \to V_h^k$ with

$$\operatorname{div} R_h q_h = q_h, \qquad |R_h q_h|_1 \leq C_k \, \|q_h\|_0 \,,$$

where C_k is independent of h.

A bounded right inverse immediately implies the displayed inf–sup estimate by testing with $v_h = R_h q_h$. The cases $k \geq 6$ are already covered by Zhang's theorem [4]. The new work below treats $k = 4, 5$.

2 The precise gap in the existing construction

Zhang's right inverse is assembled in five stages: tetrahedral means, vertex values, edge traces, face traces, and element interiors. The stages other than the edge correction already have the degrees required here.

Proposition 2.1 (Published stages used below)**.** *The following operators have constants independent of h.*

(i) For $k \geq 3$, cell means are lifted by a field in V_h^3.

(ii) For $k \geq 3$, a zero-cell-mean residual has its elementwise vertex values lifted by a field in V_h^3 while preserving every cell mean.

(iii) For $k \geq 4$, a zero-cell-mean residual vanishing on all tetrahedral edges has its face traces lifted while preserving every cell mean.

(iv) For $k \geq 6$, a residual vanishing on every face and having zero cell mean has an element-bubble lifting. For $k = 4, 5$ such a residual is zero.

Proof. These are Lemmas 3.1, 3.2, 3.4, and 3.5 of [4]. For the last sentence, a polynomial of degree at most $k-1$ that vanishes on all four faces is divisible by the quartic barycentric bubble. It is therefore zero for $k = 4$, while for $k = 5$ it is a constant multiple of the bubble and its zero mean makes that constant zero. □

The proof of Zhang's edge lemma explicitly constructs the edge-trace part for every $k \geq 4$. It then inserts degree-six bubbles solely to restore the mean of the divergence on individual tetrahedra without disturbing the edge traces. A direct quotation-free summary of the relevant logical structure is

$$\text{raw edge matching in } P_k \ (k \geq 4) \quad + \quad \text{edge-invisible } P_6 \text{ mean correction.}$$

The first summand must still be localized, protected from later edge lifts, and bounded independently of the number of cubes. The next four sections prove exactly that missing statement by finite exact matrices rather than by an informal reading of a sequential "borrow one degree of freedom" construction.

3 Edge traces and the finite star catalogue

For a mesh edge e, let $\omega_e = \mathrm{star}(e)$ be the union of tetrahedra containing e, and let P_e be the union of all Kuhn cubes containing e. Thus P_e contains at most four cubes. For an incident tetrahedron K, write the endpoints of e as local vertices a, b. Put $r = k - 1$. If

$$q|_K = \sum_{|\beta|=r} q_{K,\beta} B^r_\beta, \qquad B^r_\beta = \frac{r!}{\beta_0! \cdots \beta_3!} \lambda_0^{\beta_0} \cdots \lambda_3^{\beta_3},$$

then the interior edge-trace vector consists of

$$q_{K,\beta(j)}, \qquad \beta(j)_a = r - j, \quad \beta(j)_b = j, \quad j = 1, \ldots, k-2,$$

for every $K \supset e$. The two endpoint coefficients are omitted because our edge stage is applied after the vertex residual has been annihilated.

Every Freudenthal edge has a direction in $\{0,1\}^3 \setminus \{0\}$. Weight-three directions give one body-diagonal configuration. A weight-two direction has one constant coordinate and therefore gives, for each of three orientations, one interior and two boundary configurations. A weight-one direction has two constant coordinates; each can be low-boundary, interior, or high-boundary. Hence it gives $3^2 = 9$ configurations per orientation. The four codimension-two boundary positions split into two active and two inactive positions according to whether the two boundary signs agree. Consequently

$$1 + (3 + 6) + (3 + 12 + 6 + 6) = 37.$$

Lemma 3.1 (Complete edge-star catalogue)**.** *For every $N \geq 3$, up to translation by whole cubes, every mesh edge belongs to exactly one of the seven classes in Table 1. Across those geometries there are exactly* 117 *boundary-decorated source stars. Every actual endpoint-zero source trace lies in the admissible trace space of its corresponding protected template. For each canonical representative the indicated relation is necessary and sufficient, as certified by the exact source witnesses in Section 4. The class multiplicities sum to thirty-seven oriented/boundary configurations.*

Proof. An incident-cube patch occupies at most two consecutive cube layers in each coordinate. Consequently it cannot meet both opposite domain faces when $N \geq 3$. Translation-normalizing every edge of a $4 \times 4 \times 4$ box therefore enumerates every possible interior, face-boundary, and domain-edge geometry. The exact enumeration gives the thirty-seven rows in Appendix A. Boundary conditions can additionally suppress velocity coefficients at the target endpoints, on incident edges, and on faces containing the target edge; these are precisely the coefficients entering the one-sided divergence trace. Their exact enumeration gives 117 decorated signatures, independently for $N = 3, 4, 5, 6$. For each signature and each $k = 4, 5$, row reduction over $\mathbb{Q}$ verifies

$$H_{\tau,k} S_e(\ker E_e) = 0,$$

so its actual source trace lies in the certified template envelope. The source-space identities below establish equality for each canonical representative. The two exceptional mesh sizes $N = 1, 2$ are absorbed separately by finite dimensionality in the proof of Theorem 1.1. □

For SIFD, after ordering the four incident tetrahedra cyclically, the relation for every interior Bernstein mode is

$$y_1 - y_2 - y_3 + y_4 = 0. \tag{1}$$

For SBFD it is $y_1 - y_2 = 0$, and for DDEI the only admissible target is zero. These are exactly the singular-edge and boundary compatibility relations; ordinary classes have the full target space.

class	no.	tets	cubes	m	admissibility	d	s_0	C_4	C_5
OIA	3	6	4	$6(k-2)$	none	$6(k-2)$	1	167.571	384.271
SIFD	3	4	2	$4(k-2)$	checkerboard	$3(k-2)$	4	70.861	122.059
OIBD	1	6	1	$6(k-2)$	none	$6(k-2)$	1	90.485	205.928
OBFA	12	3	2	$3(k-2)$	none	$3(k-2)$	1	73.858	139.162
SBFD	6	2	1	$2(k-2)$	equal pair	$k-2$	2	24.237	44.123
ODEA	6	2	1	$2(k-2)$	none	$2(k-2)$	1	43.975	83.751
DDEI	6	1	1	$k-2$	zero	0	1	0	0

Table 1: The seven classes. Here m is the number of target trace coordinates, d their admissible dimension, s_0 is the coordinate scale in (6), and C_k is the largest certified reference-patch constant within the class. OIA: ordinary interior axis; SIFD: singular interior face diagonal; OIBD: ordinary interior body diagonal; OBFA: ordinary boundary-face axis; SBFD: singular boundary-face diagonal; ODEA: ordinary active domain edge; DDEI: inactive Dirichlet domain edge.

4 The exact protected local maps

Fix one of the thirty-seven configurations τ and $k \in \{4, 5\}$. All bases and matrices below are determined from the tetrahedral coordinates and the Bernstein divergence formula

$$d_{K,\beta} = k \sum_{i=0}^{3} \nabla\lambda_i \cdot v_{\beta+e_i}, \qquad |\beta| = k-1. \tag{2}$$

Coefficients at the same physical Bernstein node are shared across neighbouring tetrahedra. Hence continuity is built into the unknown vector.

4.1 Exact characterization of the source trace space

Let $x \in \mathbb{R}^{n_s}$ be the local velocity coefficients on the incident star, with all physical Dirichlet face coefficients removed. Let

$$E_{\tau,k}x \in \mathbb{R}^{2n_t}, \qquad S_{\tau,k}x \in \mathbb{R}^m$$

be, respectively, the endpoint and interior coefficients of $\operatorname{div} x$ on the target edge, where n_t is the number of incident tetrahedra. Define

$$Y_{\tau,k} = S_{\tau,k}(\ker E_{\tau,k}).$$

The certificate stores integer matrices

$$H \in \mathbb{Z}^{(m-d)\times m}, \quad B \in \mathbb{Z}^{m\times d}, \quad X \in \mathbb{Z}^{n_s\times d}, \quad A \in \mathbb{Z}^{(m-d)\times 2n_t},$$

and coordinate matrices

$$D \in \mathbb{Z}^{d\times m}, \qquad R \in \mathbb{Z}^{(m-d)\times m}.$$

They satisfy, exactly over $\mathbb{Z}$,

$$HB = 0, \tag{3}$$

$$EX = 0, \qquad SX = s_X B, \tag{4}$$

$$AE = s_A HS, \tag{5}$$

$$[B\ H^T]\begin{bmatrix} D \\ R \end{bmatrix} = s_0 I_m, \qquad DB = s_0 I_d, \qquad RB = 0. \tag{6}$$

Here $s_X = k$ for every active case, $s_A = 1$ in all seventy-four records, and s_0 is listed in Table 1. Since s_A is nonzero, equation (5) gives $S(\ker E) \subset \ker H$. Equation (6) makes $[B\ H^T]$ invertible and gives $\ker H = \operatorname{range} B$. Finally, (4) gives the reverse inclusion in the source image. Thus

$$Y_{\tau,k} = \ker H = \operatorname{range} B. \tag{7}$$

This proves, rather than assumes, that the listed checkerboard, equal-pair, and zero relations are the complete relations.

4.2 The protected output operator

Let $z \in \mathbb{R}^{n_v}$ be the continuous degree-k velocity coefficients on P_e, with every coefficient on ∂P_e set to zero. Let $T_{\tau,k}z \in \mathbb{R}^m$ be the target interior edge trace. Let $P_{\tau,k}z$ collect

(a) both endpoint coefficients on the target edge, and

(b) every divergence-trace Bernstein coefficient on every other tetrahedral edge in P_e.

The certificate stores $C \in \mathbb{Z}^{n_v \times d}$ satisfying

$$P_{\tau,k}C = 0, \qquad T_{\tau,k}C = s_C B, \tag{8}$$

where $s_C = k$ in every active case. For $y \in Y_{\tau,k}$ define the exact local linear map

$$\widehat{L}_{\tau,k}y = \frac{1}{s_C s_0}\, CDy. \tag{9}$$

Because $y = B\alpha$ and $DB = s_0 I$, equations (8) give

$$P_{\tau,k}\widehat{L}_{\tau,k}y = 0, \qquad T_{\tau,k}\widehat{L}_{\tau,k}y = y. \tag{10}$$

The first identity is the *protected-trace property*: the lift changes no non-target edge and does not recreate either target endpoint. Since the field itself vanishes on ∂P_e, it extends by zero to a member of V_h^k.

Lemma 4.1 (Exact local protected lifting)**.** *For every one of the thirty-seven configurations and each $k \in \{4, 5\}$, the map* (9) *is defined on the certified template space* $\ker H_{\tau,k}$*. Every actual endpoint-zero target trace of* $\operatorname{div} V_h^k$ *lies in this space. The map reproduces that target trace, protects all other edge traces, and is supported on the union of the cubes incident to the target edge.*

Proof. There are seventy-four records, named `cfgXX-kY-class`. The exact verifier reconstructs E, S, P, T directly from the tetrahedral coordinates using (2); it does not trust stored copies of those matrices. It then verifies (3)–(6) and (8) over the integers. The boundary-decoration verifier checks the required containment in $\ker H_{\tau,k}$ for all 117 decorated source stars and both degrees. The algebra above converts those finite identities into the asserted map. Appendix A lists all thirty-seven geometries and points to the corresponding exact records. □

Remark 4.2. The matrix entries are supplied as sparse integer pairs, not rounded decimal coefficients. Printing several hundred pages of zeros would make the paper less auditable, not more. The JSON record, its reconstruction verifier, and the generated Lean constant together are the exact specification of each linear map.

5 Reference-patch norm estimate

The exact maps also carry a uniform norm certificate. Let $\psi_1, \dots, \psi_d$ be the piecewise-polynomial fields represented by the columns of C/s_C, and form the exact rational Gram matrix

$$(G_C)_{ab} = (\nabla\psi_a, \nabla\psi_b)_{P_e}.$$

In target coordinates, the map (9) has Gram matrix

$$G_Y = \left(\frac{D}{s_0}\right)^T G_C \left(\frac{D}{s_0}\right). \tag{11}$$

Let $\lambda_{\tau,k}$ be the maximum absolute row sum of G_Y. Then

$$\left|\widehat{L}_{\tau,k} y\right|_1^2 \le \lambda_{\tau,k} \|y\|_{\ell^2}^2 .$$

For a degree-r Bernstein polynomial on a tetrahedron, let M_r be the exact mass matrix. The squared dual norm of coefficient extraction at β is $(M_r^{-1})_{\beta\beta}$. Summing these coefficient estimates over the interior target-edge modes and the incident tetrahedra gives an exact rational number $\theta_{\tau,k}$ such that

$$\|y\|_{\ell^2}^2 \le \theta_{\tau,k} \|q\|_{L^2(\omega_e)}^2 .$$

Therefore

$$\left|\widehat{L}_{\tau,k} q\right|_1^2 \le \lambda_{\tau,k}\theta_{\tau,k} \|q\|_{L^2(\omega_e)}^2 . \tag{12}$$

Every integral in M_r and G_C is a factorial expression in barycentric multi-indices and is evaluated over $\mathbb{Q}$. Among all seventy-four records the exact maximum is

$$\max_{\tau,k} \lambda_{\tau,k}\theta_{\tau,k} = \frac{8121511}{55} < 385^2, \tag{13}$$

attained by configuration 21 at $k = 5$. Hence

$$\left|\widehat{L}_{\tau,k} q\right|_{H^1(P_e)} \le 385 \|q\|_{L^2(\omega_e)} . \tag{14}$$

Under the physical scaling $x = x_0 + h\widehat{x}$, set $v(x) = h\widehat{v}(\widehat{x})$. Then the divergence trace is unchanged, while both $|v|_{H^1(P_e)}$ and $\|q\|_{L^2(\omega_e)}$ acquire the factor $h^{3/2}$. Thus the same constant 385 holds on every physical edge star and is independent of h.

6 Dependency depth, colouring, and overlap

For every mesh edge e, let L_e be the translated and scaled map from Lemma 4.1, applied to the trace of a residual r_h that vanishes at all tetrahedral vertices. Extend $L_e r_h$ by zero outside P_e.

Lemma 6.1 (Depth-zero assembly)**.** *All edge lifts can be applied simultaneously. For every mesh edge f,*

$$\operatorname{div} \sum_e L_e r_h \Big|_f = r_h|_f.$$

The dependency depth is zero.

Proof. The target identity in (10) supplies the summand $e = f$. The protected identity makes every summand $e \neq f$ vanish on f. No ordering, borrowing chain, or induction across the mesh remains. □

For an edge, orient its direction as a nonzero binary vector $\delta \in \{0,1\}^3$ and let $a \in \mathbb{Z}^3$ be its lower endpoint. Colour it by

$$\operatorname{col}(e) = \big(\delta, a_1 \bmod 3, a_2 \bmod 3, a_3 \bmod 3\big). \tag{15}$$

There are $7 \cdot 3^3 = 189$ colours. This is an explicit schedule, not merely an existence argument. A smaller existence bound also follows from the conflict graph: a support uses at most four cubes and each cube contains nineteen mesh edges, so its conflict degree is at most $4(19-1) = 72$ and greedy colouring uses at most 73 colours. If two distinct edges of the same colour had a common incident cube, then in every coordinate their anchors would differ by at most one, while being congruent modulo three; hence the anchors would be equal. Therefore same-coloured support patches are disjoint.

A Kuhn cube contains exactly twelve coordinate edges, six selected face diagonals, and one body diagonal, hence nineteen mesh edges. Consequently, on the interior of every tetrahedron at most nineteen raw support patches are active; the support multiplicity is therefore at most nineteen almost everywhere. Moreover, each tetrahedron belongs to the source stars of exactly its six edges. Combining these facts with (14) gives

$$\left|\sum_e L_e r_h\right|_1^2 \leq 19 \sum_e |L_e r_h|_1^2 \tag{16}$$

$$\leq 19 \cdot 385^2 \sum_e \|r_h\|_{0,\omega_e}^2 \tag{17}$$

$$= 19 \cdot 6 \cdot 385^2 \, \|r_h\|_0^2 . \tag{18}$$

The same estimate can alternatively be summed colour by colour. The explicit squared constant in (18) is 16897650.

Lemma 6.2 (Raw edge-star lifting)**.** *Let $N \geq 3$, let $k \in \{4,5\}$, and let $r_h \in Q_h^k$ vanish at every vertex of every tetrahedron. There is a linear field*

$$W_h r_h = \sum_{e \in \mathcal{E}_h} L_e(r_h|_{\omega_e}) \in V_h^k$$

such that

$$(\operatorname{div} W_h r_h)|_e = r_h|_e \qquad \text{for every tetrahedral edge } e,$$

$$|W_h r_h|_1 \leq 385\sqrt{114}\, \|r_h\|_0 .$$

Every local support contains at most four cubes, all non-target edge traces are protected, the dependency depth is zero, a 189*-colour disjoint-patch assembly is available, and the elementwise (hence almost-everywhere) support overlap is at most* 19*.*

Proof. Lemma 4.1 supplies every local map; the source-space identity (7) applies because $r_h \in \operatorname{div} V_h^k$ and its endpoint traces vanish. Lemma 6.1 gives the edge identity, and (18) gives the norm bound. □

7 Quartic repair of the cell means

The raw field need not preserve the zero mean of the input residual on each tetrahedron. We recall the exact degree-four repair that replaces Zhang's P_6 bubbles.

Let $\widehat{P} = [0,2] \times [0,1]^2$ be two adjacent Kuhn cubes, containing twelve tetrahedra, and define

$$\mathcal{W}_4(\widehat{P}) = \{b \in H^1_0(\widehat{P}; \mathbb{R}^3) : b|_K \in [\mathbb{P}_4(K)]^3, (\operatorname{div} b)|_e = 0 \text{ on every tetrahedral edge}\}.$$

Let $\mathcal{M}(b) = (⨍_K \operatorname{div} b)_{K=1}^{12}$.

Lemma 7.1 (Two-cube quartic mean repair)**.**

$$\mathcal{M}(\mathcal{W}_4(\widehat{P})) = \{m \in \mathbb{R}^{12} : \sum_{K=1}^{12} m_K = 0\}.$$

This map has a bounded linear right inverse on the fixed reference patch.

Proof. A continuous quartic vector field vanishing on $\partial\widehat{P}$ has 63 interior physical Bernstein nodes and hence 189 scalar unknowns. Using (2), let $E_{\mathrm{mp}} \in \mathbb{Z}^{192\times 189}$ impose zero divergence on every tetrahedral edge, and let $S_{\mathrm{mp}} \in \mathbb{Z}^{12\times 189}$ be the unnormalised sums of the twenty cubic Bernstein coefficients on each tetrahedron. The supplied sparse integer matrix $C_{\mathrm{mp}} \in \mathbb{Z}^{189\times 11}$ has 208 nonzero entries and satisfies

$$E_{\mathrm{mp}} C_{\mathrm{mp}} = 0, \qquad S_{\mathrm{mp}} C_{\mathrm{mp}} = 24[e_1 - e_{12}, \dots, e_{11} - e_{12}]. \tag{19}$$

The average of a cubic Bernstein polynomial is one twentieth of the sum of its coefficients. Hence the fields represented by $(5/6)C_{\mathrm{mp}}$ have mean vectors $e_j - e_{12}$ and span the zero-sum hyperplane. The reverse inclusion follows from the divergence theorem. Boundedness follows by finite-dimensional norm equivalence on the fixed patch. The primary verifier reconstructs both matrices and checks (19); an independent symbolic script additionally checks continuity, zero boundary trace, every edge restriction, and every cell average. □

The Freudenthal tetrahedron set is invariant under coordinate permutations; the velocity components are permuted in the same way. Thus the certified pair may be placed across a face normal to any coordinate axis. The usual translation and scaling $b(x) = h\widehat{b}((x - x_0)/h)$ preserves divergence means and scales the H^1 seminorm by $h^{3/2}$. A spanning-tree combination of two-cube repairs therefore yields the following bounded-patch version.

Lemma 7.2 (Bounded-patch mean repair)**.** *For every fixed M there is C_M such that the following holds. Assume the global mesh contains at least two cubes. If $w_h \in V_h^k$ is supported on a face-connected union ω_h of at most M cubes, then there is a face-connected union ω_h^+ of at most $M+1$ cubes containing ω_h and a field $b_h \in V_h^4$, supported in $\overline{\omega_h^+}$, with*

$$\begin{aligned} &(\operatorname{div} b_h)|_e = 0 && \text{on every tetrahedral edge,}\\ &⨍_K \operatorname{div} b_h = ⨍_K \operatorname{div} w_h && \text{for every } K,\\ &|b_h|_1 \le C_M\, |w_h|_1 . \end{aligned}$$

The constant is independent of h and the patch location.

Proof. If ω_h consists of one cube, adjoin any face neighbour; otherwise put $\omega_h^+ = \omega_h$. Number the tetrahedra in each cube C as $K_{C,1}, \dots, K_{C,6}$ and set

$$m_{C,i} = ⨍_{K_{C,i}} \operatorname{div} w_h.$$

The zero extension of w_h has zero trace on $\partial\omega_h^+$. Since all tetrahedra have the same volume, the divergence theorem gives

$$\sum_C \sum_{i=1}^{6} m_{C,i} = 0. \tag{20}$$

The transported two-cube operator of Lemma 7.1 realizes every zero-sum mean vector on a pair of adjacent cubes, has zero divergence trace on every edge, and has H^1 seminorm at most $Ch^{3/2}$ times the Euclidean norm of that mean vector. In each cube, for $i = 1, \dots, 5$, use such a pair corrector with mean vector $e_{C,i} - e_{C,6}$ and coefficient $m_{C,i}$. After these corrections the remaining target is concentrated on the reference tetrahedra $K_{C,6}$, with cube charges

$$s_C = \sum_{i=1}^{6} m_{C,i}, \qquad \sum_C s_C = 0.$$

Choose a rooted spanning tree of the cube face-adjacency graph. The added neighbour and the rooted tree are fixed once for each of the finitely many translated patch types, so the resulting operator is linear. For every nonroot cube C, let

$$S_C = \sum_{D \text{ in the subtree rooted at } C} s_D.$$

On the pair $C \cup p(C)$ use a corrector with mean vector $e_{C,6} - e_{p(C),6}$ and coefficient S_C. Comparing the coefficient at each tree vertex gives

$$\sum_{C \neq C_0} S_C (e_{C,6} - e_{p(C),6}) = \sum_C s_C e_{C,6}.$$

Thus the sum of the within-cube and tree correctors has exactly the required cell means; every summand has zero edge trace.

There are at most $6(M+1)$ correctors and the tree has bounded size. Hence

$$\sum_{C, i \leq 5} |m_{C,i}| + \sum_{C \neq C_0} |S_C| \leq C_M |m|_{\ell^2}.$$

Scaling, the triangle inequality, and Cauchy–Schwarz on each tetrahedron give

$$|b_h|_1 \leq C_M h^{3/2} |m|_{\ell^2} \leq C_M \|\operatorname{div} w_h\|_0 \leq C_M \sqrt{3}\, |w_h|_1 .$$

□

8 Mean-preserving edge lifting and the main theorem

Lemma 8.1 (Mean-preserving edge stage for $k = 4, 5$)**.** *Assume $N \geq 3$. If $r_h \in Q_h^k$, $k \in \{4, 5\}$, has zero mean on every tetrahedron and vanishes at every tetrahedral vertex, then there is $v_3 \in V_h^k$ such that*

$$\begin{aligned} &(\operatorname{div} v_3)|_e = r_h|_e && \text{on every tetrahedral edge,} \\ &\int_K \operatorname{div} v_3 = 0 && \text{for every } K \in \mathcal{T}_h, \\ &\quad |v_3|_1 \leq C_k \|r_h\|_0 , \end{aligned}$$

where C_k is independent of h.

Proof. Write the raw lifting as $W_h r_h = \sum_e w_e$, where $w_e = L_e(r_h|_{\omega_e})$. The local estimate and the fact that every tetrahedron has six edges give the component square-sum bound

$$\sum_e |w_e|_1^2 \le 385^2 \sum_e \|r_h\|_{0,\omega_e}^2 = 6 \cdot 385^2 \, \|r_h\|_0^2 . \tag{21}$$

Each w_e is supported on at most four face-connected cubes. Apply Lemma 7.2, with $M = 4$, to obtain b_e with the same cell means and zero divergence trace on every edge, and set

$$v_3 = \sum_e (w_e - b_e).$$

Term by term, the cell means vanish and the protected edge trace of w_e is unchanged. Each corrected support lies in a one-face enlargement of the raw support. A fixed cube is therefore reached through itself or one of its six face neighbours, each containing nineteen target edges; the corrected overlap is at most $7 \cdot 19 = 133$. Lemma 7.2, bounded overlap, and (21) yield

$$|v_3|_1^2 \le 133 \sum_e |w_e - b_e|_1^2 \le C_k \sum_e |w_e|_1^2 \le C_k \, \|r_h\|_0^2 . \qquad \square$$

Proof of Theorem 1.1. For $k \ge 6$, invoke [4]. Let $k \in \{4, 5\}$. For each of the two fixed meshes $N = 1, 2$, the divergence map between V_h^k and its image Q_h^k has a bounded linear right inverse by finite dimensionality. The two resulting constants are absorbed into C_k. Hence assume $N \ge 3$ and let $q_h \in Q_h^k$. Proposition 2.1(i) gives v_1 matching all cell means. Proposition 2.1(ii) gives v_2 matching all vertex values while preserving those means. Apply Lemma 8.1 to the remaining residual, obtaining v_3 that matches every edge trace and preserves every mean. Proposition 2.1(iii) gives v_4 matching all face traces and preserving the means. The last residual vanishes on all faces and has zero mean, hence is zero by Proposition 2.1(iv). Thus

$$R_h q_h = v_1 + v_2 + v_3 + v_4$$

has divergence q_h. The stability estimates at each stage, together with $\|\operatorname{div} v\|_0 \le \sqrt{3}\, |v|_1$, bound every residual and every lift by $C_k \, \|q_h\|_0$. Hence $|R_h q_h|_1 \le C_k \, \|q_h\|_0$ independently of h, proving the right-inverse and inf–sup statements. $\square$

9 Exact verification and Lean formalization

The reproducibility package contains six independent finite checks. The two principal data files are content-addressed by

$$\text{SHA256}(\texttt{all_edge_star_certificates.json}) = \texttt{f14caddc60c75542...f40bdfd2958931f3},$$
$$\text{SHA256}(\texttt{all_edge_star_norm_bounds.json}) = \texttt{af169660100ffcb3...20c6c5c1637a6327e}.$$

The full hashes are included in the package manifest.

(1) `verify_complete_edge_star_catalogue.py` enumerates every edge in a $4 \times 4 \times 4$ Freudenthal box, translation-normalizes its full star, obtains exactly thirty-seven keys, checks that the same catalogue is obtained for $N = 3, 5, 6$, and compares every key with the certificate metadata for both degrees.

(2) `verify_boundary_decorated_source_envelopes.py` enumerates the 117 endpoint/edge/face boundary decorations for each of $N = 3, 4, 5, 6$ and checks $H_{\tau,k} S_e(\ker E_e) = 0$ exactly for all 234 decorated/degree pairs.

(3) `verify_all_edge_star_certificates.py` and
`verify_all_edge_star_certificates_independent.py`
independently reconstruct every source and output trace matrix from the tetrahedral geometry, verify the one-sided target-row and velocity-node ordering, and check the protected identities for all seventy-four records. A mutation test confirms that the independent backend rejects a self-consistent counterfeit produced by corrupting the shared generator/primary-verifier geometry constructor. The primary verifier additionally checks the exact admissible-space dimension, source reachability and necessity, $HB = 0$, and the $[B, H^T]$ kernel decomposition and coordinate identities over $\mathbb{Q}$.

(4) `verify_all_edge_star_norm_bounds.py` and
`verify_all_edge_star_norm_bounds_independent.py`
reconstruct all Bernstein mass and H^1 Gram matrices over $\mathbb{Q}$ using separate implementations and check (13) entry by entry. A mutation test confirms that the independent backend rejects a self-consistent counterfeit accepted by a deliberately corrupted shared generator/verifier backend.

(5) `verify_p4_macro_patch.py` and
`independent_symbolic_check.py`
verify the independent degree-four mean-repair certificate.

(6) The accompanying Lake project, `freudenthal_k4_edge_star_lean`, encodes the seventy-four source/protection/target certificates, all seventy-four exact norm factorizations and rational bounds, the thirty-seven-case catalogue, a generic conditional depth-zero assembly lemma, the 189-colour disjointness theorem, the nineteen-edge cube count, and the factor $19 \cdot 6 \cdot 385^2$.

The generated Lean data contain no `sorry`, `admit`, or user-declared axioms. The theorem
`raw_edge_star_finite_audit`
checks the audit tuple

$$(37, 74, 189, 73, 19, 133, 0, 385),$$

the complete local-certificate table, the exact norm-factorization and rational-bound tables, and the quartic macro-patch certificate. The generic theorem `assembleProtected_eq` formalizes simultaneous depth-zero assembly conditional on target reproduction and off-target protection. The Python verification suite is also run with optimization enabled, and same-shape mutations of the catalogue, local maps, norm data, provenance hashes, and macro-patch witness are required to be rejected.

The accompanying Lake project pins Lean to version 4.30.0 and Mathlib to tag `v4.30.0`. A clean `lake build` followed by `lake exe edgeStarAudit` completed on the release sources. All load-bearing finite Boolean table evaluations use `decide +kernel`; aggregate and generic theorems use ordinary kernel-checked tactics. The seventy-four local records are split into eight bounded chunks to control kernel-evaluation memory. The expanded axiom report contains only Lean's standard `propext`, `Classical.choice`, and `Quot.sound` dependencies where applicable, with no project-declared axiom and no `native_decide` trust boundary.

The Lean layer checks the stored finite algebra and separately proves generic conditional assembly and colouring lemmas. It does not derive the semantic hypotheses of `assembleProtected_eq` from the seventy-four records, instantiate that lemma on a formal global Freudenthal mesh, or derive dependency depth from `checkProtected`. The reconstruction of the matrices from tetrahedral geometry, including the 117 boundary-decorated source-envelope checks, is performed independently by the exact Python verifiers and is not represented as an end-to-end Mathlib finite-element development. Affine Sobolev scaling and Zhang's published analytic stages remain mathematical arguments in this manuscript.

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The author thanks L. Ridgway Scott for formulating the prize problem and Shangyou Zhang for the constructive framework on which the proof builds. The finite template census and admissible-trace dimensions were cross-checked against the public exact-certificate archive of Kias Henry [2]; the protected local witnesses and their norm certificates in this work were generated and verified independently. The author used ChatGPT 5.6 Sol Pro and ChatGPT 5.6 Sol for proof-structure exploration, exact-arithmetic code generation, Lean formalization, and manuscript editing. Responsibility for all mathematical statements, computations, and submission decisions remains with the author.

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[1] P. E. Farrell, L. Mitchell, and L. R. Scott, Two conjectures on the Stokes complex in three dimensions on Freudenthal meshes, *SIAM J. Sci. Comput.* 46 (2024), A629–A644, `https://doi.org/10.1137/22M1533943`.

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A The thirty-seven exact edge configurations

Coordinates in the third column are normalized to one incident unit cube. Boundary labels refer to the physical $N \times N \times N$ cube lattice. The last two columns contain

$$\text{target-row count/admissible dimension/}s_C$$

for $k = 4$ and $k = 5$. The exact map for row `XX` and degree k is record `cfgXX-kY-class` in `all_edge_star_certificates.json` and in the generated Lean file `Freudenthal/EdgeStarData.lean`.

ID	class	normalized edge	physical boundary	tets	cubes	$k = 4$	$k = 5$
00	ODEA	`000--001`	$x = 0$, $y = 0$	2	1	4/4/4	6/6/5
01	ODEA	`000--010`	$x = 0$, $z = 0$	2	1	4/4/4	6/6/5
02	SBFD	`000--011`	$x = 0$	2	1	4/2/4	6/3/5
03	ODEA	`000--100`	$y = 0$, $z = 0$	2	1	4/4/4	6/6/5
04	SBFD	`000--101`	$y = 0$	2	1	4/2/4	6/3/5
05	SBFD	`000--110`	$z = 0$	2	1	4/2/4	6/3/5
06	OIBD	`000--111`	–	6	1	12/12/4	18/18/5
07	DDEI	`001--011`	$x = 0$, $z = N$	1	1	2/0/1	3/0/1
08	OBFA	`001--011`	$x = 0$	3	2	6/6/4	9/9/5
09	DDEI	`001--101`	$y = 0$, $z = N$	1	1	2/0/1	3/0/1
10	OBFA	`001--101`	$y = 0$	3	2	6/6/4	9/9/5
11	SBFD	`001--111`	$z = N$	2	1	4/2/4	6/3/5
12	SIFD	`001--111`	–	4	2	8/6/4	12/9/5
13	DDEI	`010--011`	$x = 0$, $y = N$	1	1	2/0/1	3/0/1
14	OBFA	`010--011`	$x = 0$	3	2	6/6/4	9/9/5
15	DDEI	`010--110`	$y = N$, $z = 0$	1	1	2/0/1	3/0/1
16	OBFA	`010--110`	$z = 0$	3	2	6/6/4	9/9/5
17	SBFD	`010--111`	$y = N$	2	1	4/2/4	6/3/5
18	SIFD	`010--111`	–	4	2	8/6/4	12/9/5
19	ODEA	`011--111`	$y = N$, $z = N$	2	1	4/4/4	6/6/5
20	OBFA	`011--111`	$y = N$	3	2	6/6/4	9/9/5
21	OIA	`011--111`	–	6	4	12/12/4	18/18/5
22	OBFA	`011--111`	$z = N$	3	2	6/6/4	9/9/5
23	DDEI	`100--101`	$x = N$, $y = 0$	1	1	2/0/1	3/0/1
24	OBFA	`100--101`	$y = 0$	3	2	6/6/4	9/9/5
25	DDEI	`100--110`	$x = N$, $z = 0$	1	1	2/0/1	3/0/1
26	OBFA	`100--110`	$z = 0$	3	2	6/6/4	9/9/5
27	SBFD	`100--111`	$x = N$	2	1	4/2/4	6/3/5
28	SIFD	`100--111`	–	4	2	8/6/4	12/9/5
29	ODEA	`101--111`	$x = N$, $z = N$	2	1	4/4/4	6/6/5
30	OBFA	`101--111`	$x = N$	3	2	6/6/4	9/9/5
31	OIA	`101--111`	–	6	4	12/12/4	18/18/5
32	OBFA	`101--111`	$z = N$	3	2	6/6/4	9/9/5
33	ODEA	`110--111`	$x = N$, $y = N$	2	1	4/4/4	6/6/5
34	OBFA	`110--111`	$x = N$	3	2	6/6/4	9/9/5
35	OIA	`110--111`	–	6	4	12/12/4	18/18/5
36	OBFA	`110--111`	$y = N$	3	2	6/6/4	9/9/5

B Certificate file schema

For each record, the fields are grouped as follows:

- `source_endpoint_rows` and `source_target_rows` store E and S;
- `admissibility_constraints` and `B_columns` store H and B;
- the source witnesses store X and A, and the coordinate fields store D, H^T, R;
- `P_rows`, `T_rows`, and `C_columns` store P, T, C.

Every sparse row or column is a list of zero-based integer index/value pairs. The node ordering and every patch tetrahedron are included, so the matrices can be reconstructed without trusting the stored rows.

Supplementary Files

This is a list of supplementary files associated with this preprint. Click to download.

- freudenthalk4completereproducibleproof.zip