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Disorder-tuned crossing of monopole and conventional pairing instabilities in multi-Weyl semimetals

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Abstract

We study how non-magnetic impurity scattering affects the emergence and possible coexistence of pairing instabilities in a two-node multi-Weyl semimetal. Within an explicitly specified projected impurity kernel—valley diagonal and momentum independent across each Fermi pocket, the leading behaviour of non-magnetic disorder in the small-pocket window $q_{\text{intra}}^{\text{max}}\xi_{\text{dis}} \ll 1 \ll |2\mathbf{Q}|\xi_{\text{dis}}$, treated at leading order in Born and Abrikosov–Gor’kov theory—quenched disorder tunes the leading pairing instability from a topologically nontrivial monopole channel to a conventional (s -wave) one, extending our earlier clean-system analysis into the disordered regime. A Born self-energy calculation in the chiral band basis then gives: (i) a band-isotropic conventional channel that is Anderson protected against intra-node scalar disorder ($\eta_s = 1$), introduced phenomenologically at the projected-band level; solving the competition for arbitrary η_s yields the crossing criterion $(1 - \eta_s)/(1 - \eta_m) < T_{c0}^{(s)}/T_{c0}^{(m)}$, so the mechanism tolerates substantial loss of conventional-channel protection; and (ii) a rank-one monopole sector fixing f_m as the exact eigenfunction with $\eta_m(J) = 1/(J + 2)$, exact given that kernel. The crossing location in $\Gamma_N/T_{c0}^{(m)}$ is set by $\eta_m(J)$, η_s , and $r = T_{c0}^{(s)}/T_{c0}^{(m)}$. From the clean projected BdG Hamiltonian the pure monopole nodes carry Berry charge $\pm J$, distinct from the gapped conventional solution; the clean nodal thermodynamics is charge-dependent, $N_{\text{SC}}(E) \propto E^{2/J}$ and $C \propto T^{1+2/J}$, and the residual density of states shows a threshold only for $J = 1$. The crossing lies in the moderately metallic regime, $\mu/\Gamma_N \simeq 11\text{--}14$ for the illustrative $T_{c0}^{(m)}/\mu = 0.133$ used in the figures ($\simeq 75\text{--}90$ at weak coupling $T_{c0}^{(m)}/\mu = 0.02$)

I. INTRODUCTION

Weyl and multi-Weyl semimetals are three-dimensional gapless topological phases whose band-touching points act as monopoles of Berry curvature, carrying an integer topological (monopole) charge J ; the linear ($J = 1$) Weyl node generalizes to anisotropic double- and triple-Weyl nodes with $J = 2, 3$ at which the dispersion is linear along one axis and grows as $k_{\perp}^J$ in the plane, thus violating the full Lorentz symmetry [1].

These phases are neither hypothetical nor exotic. The Weyl semimetal was first identified theoretically in the pyrochlore iridates [2] and in magnetically doped topological-insulator

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multilayers [3], and was then observed directly by angle-resolved photoemission in the non-centrosymmetric monpnictide TaAs [4, 5], following the material-specific predictions of Refs. [6, 7]. The defining consequences of the Berry monopole—the surface Fermi arcs, and the chiral anomaly inherited from the Adler–Bell–Jackiw physics of continuum Weyl fermions [8, 9]—are by now standard diagnostics of the phase. Crucially for the present work, crystalline point-group symmetry permits monopole charges larger than one: $C_{4,6}$ symmetry protects double-Weyl nodes with quadratic in-plane dispersion and C_6 protects triple-Weyl nodes with cubic dispersion [10], with the ferromagnet HgCr_2Se_4 [11] and SrSi_2 [12] identified as candidate double-Weyl materials. Higher monopole charge is thus a genuine material degree of freedom, and more broadly the classification of band degeneracies in crystals extends well beyond the Dirac and Weyl cases [13]. Throughout this work J enters quantitatively rather than as a label, through the anisotropy of the Fermi surface it induces.

Superconductivity in Weyl and Dirac systems has correspondingly become a subject in its own right. Proposals range from engineered superconductor/topological-insulator multilayers realizing Bogoliubov–Weyl nodes [14] and ferromagnetic superconductors with Majorana surface arcs [15], to intrinsic three-dimensional Dirac and Weyl superconductors [16], phonon-mediated pairing in doped Weyl metals [17], time-reversal-invariant topological pairing [18], and odd-parity channels [19]. What these treatments share is that the pairing must be assembled from Bloch states carrying Berry curvature, so the admissible gap functions are constrained by topology and not by point-group symmetry alone.

Doping such a semimetal and, at the same time, adding an attractive channel, produces a distinctive superconducting problem: because the normal-state Bloch eigenstates inherit the Berry monopole, the natural Cooper pair amplitudes (here the zero-momentum inter-node channel) are not ordinary spherical harmonics but *monopole harmonics*, carrying their own topological charge and an unavoidable nodal structure [20, 21]. A microscopic theory of the competition between such a topologically nontrivial “monopole” channel and a conventional, fully gapped s -wave channel has been previously developed by us in the *clean* limit [22–24], yielding explicit phase diagrams, critical temperatures, a “topological repulsion” between the two order parameters, and specific-heat signatures that can serve as a thermodynamic probe of the chirality index.

Real materials are disordered, and disorder affects the two channels very differently. Anderson’s theorem guarantees that the conventional BCS [25] s -wave state is insensitive to

weak non-magnetic disorder, while nodal and sign-changing order parameters are generically fragile, their transition temperature obeying an Abrikosov–Gor’kov pair-breaking law [26–28]. This contrast is what makes disorder a potential *tuning parameter* rather than merely a nuisance. Non-magnetic defects that would be almost invisible in a conventional s -wave material suppress T_c dramatically in an unconventional one, a sensitivity that has become one of the sharpest experimental discriminators of pairing symmetry [29, 30], and the corresponding pair-breaking theory is well developed for d -wave [31, 32], strongly anisotropic [33], and multiband $s_{\pm}$ gaps [34]. The last of these is the closest precedent for what follows: there, non-magnetic scattering alone can drive a transition between distinct pairing states, in a setting with no topology involved.

The dichotomy is nevertheless not the whole story: when the pairing carries internal (orbital, sublattice, or chiral) structure, a generalized Anderson theorem—formulated through the notion of superconducting fitness—can shield even a nodal gap from scattering far in excess of the gap scale, as established for topological-insulator-derived and doped-semimetal superconductors [35, 36]. Whether the monopole channel of a multi-Weyl superconductor is protected, pair-broken, or something in between is therefore a genuine question that must be settled by the chiral structure of its Bloch states.

A second, independent disorder effect complicates the picture. The multi-Weyl *normal* state is itself disorder-sensitive: beyond a critical strength it undergoes a non-Anderson, disorder-driven transition to a diffusive metal in which the zero-energy density of states turns on as an order parameter [37–39], and where disorder and interactions are strongly intertwined [40], a transition recently observed by ARPES through the disappearance of the topological Fermi arcs in NdAlSi [41]. Higher monopole charge makes this normal-state criticality *more* easily reached, for instance, a double-Weyl node already yields a finite residual density of states at weak disorder [37]. Any claim of a disorder-driven *pairing* transition must therefore demonstrate that it occurs while the host is still a good metal, well below the scale at which the underlying Weyl band-structure itself dissolves.

In this work, we pose the following question: *does scalar disorder select pairing topology* in a multi-Weyl superconductor, tuning a *crossing* of the monopole and conventional linearized pairing instabilities? Our answer turns out to be affirmative, as exposed by the arguments presented in this article. Here is a summary of the main results: (i) From a single Born self-energy calculation in the chiral band basis, we obtain the pair-breaking asymmetry: *con-*

ditional on the single-band anomalous-kernel assumption (Sec. II), the conventional channel is Anderson protected because the Bloch Berry phase that dresses the normal scattering rate cancels against the anomalous self-energy, while the monopole channel is pair-broken at a rate softened by the chiral suppression of backscattering, with a protection factor $\eta_m(J) = 1/(J+2)$ on the Fermi surface, fixed *exactly* by the rank-one $m = \pm J$ sector of the chiral coherence kernel (within the projected intra-node scalar model). (ii) A pair-breaking Abrikosov–Gor’kov (AG) solution, with a separately evaluated self-consistent Lorentzian spectral broadening of the density of states, locates the instability crossing, which moves to a smaller pair-breaking rate $\Gamma_N^*/T_{c0}^{(m)}$ with increasing monopole charge J , and supplies its thermodynamic fingerprints (specific-heat, residual density of states). (iii) The two competing pairings possess distinct nodal topology—the monopole solution carries BdG nodal charges $\pm J$, whereas the conventional solution is fully gapped. (iv) A disorder-symmetry classification shows that the selection is specific to intra-node potential disorder within the projected kernel; the operative requirement is differential pair-breaking (Eq. (28)) rather than the disorder label, and magnetic or chirality-mixing disorder requires a separate vertex calculation (Sec. VII). (v) A continuum Ginzburg–Landau estimate disfavors simple homogeneous coexistence, but does not fix the order of the ordered-state transition, which we leave open (App. H). (vi) Throughout, the transition sits in the moderately metallic regime, $\mu/\Gamma_N \simeq 11\text{--}14$ for the enhanced illustrative parameter set used in the figures and appreciably larger for weaker coupling (Sec. IX), separated from the Weyl disorder-criticality of the normal state. We deliberately quote *no* residual-resistivity or impurity-concentration values (Sec. XI); the only material-level target given is the conditional quantum lifetime $\tau_q = \hbar/(2\Gamma_N^*)$. The growing list of Weyl superconductors with resolved topological surface and pairing structure—most recently the superconducting Fermi arcs of PtBi₂ [42]—makes this an experimentally testable proof-of-principle prediction.

II. MODEL

We consider a pair of multi-Weyl nodes of opposite chirality at $\pm \mathbf{Q}$ with monopole charge $J \in \{1, 2, 3\}$. Measuring momenta from each node,

$$h_{\pm}(\mathbf{q}) = \pm v_z q_z \sigma_z + \alpha(q_-^J \sigma_+ + q_+^J \sigma_-), \quad q_{\pm} = q_x \pm i q_y, \quad (1)$$

with $E_{\pm}(\mathbf{q}) = \pm \sqrt{v_z^2 q_z^2 + \alpha^2 q_{\perp}^{2J}}$, $\sigma_{\pm} = (\sigma_x \pm i\sigma_y)/2$ (where σ_j , with $j = x, y, z$, are the Pauli matrices) and low-energy density of states (DOS) *per node and per pseudospin band*

$$N(\varepsilon) \propto |\varepsilon|^{2/J}, \quad N_0 \equiv N(\mu) = N_J |\mu|^{2/J}, \quad (2)$$

where the coefficient $N_J = \frac{\alpha^{-2/J}}{J v_z (2\pi)^2} B\left(\frac{1}{J}, \frac{1}{2}\right)$, for $B(x, y)$ the Beta function (see Appendix A for details).

Density-of-states multiplicities. Because several distinct densities of states could be meant here, we fix the convention once and use it throughout. $N(\varepsilon)$ in Eq. (2) is the DOS of a single node, for the single conduction pseudospin band, and $N_0 \equiv N(\mu)$ is its value at the Fermi level. This same N_0 is the one that enters (i) the dimensionless couplings $\lambda_a = g_a N_0 \langle |f_a|^2 \rangle$ of Eq. (13) and (ii) the Born rates Γ_b and Γ_N of Sec. III, since both the pairing and the intra-node impurity self-energy are evaluated on one node's conduction Fermi surface. The *total* normal-state DOS of the two-node system is $N_{\text{tot}} = 2N_0$ (times an additional factor of two if physical spin is retained as a degenerate spectator), and it is N_{tot} —not N_0 —that would enter a measured Sommerfeld coefficient. We use N_0 everywhere below and flag explicitly the one place, the normal-state specific heat C_n of App. E, where the multiplicity cancels in the ratio $\Delta C/C_n$ and is therefore immaterial.

Impurity model: smooth, intra-node-dominated disorder. The disorder is a Gaussian, zero-mean, nonmagnetic random potential specified by its correlator in momentum space,

$$\overline{V(\mathbf{r})} = 0, \quad \overline{V_{\mathbf{q}} V_{\mathbf{q}'}} = (2\pi)^3 \delta(\mathbf{q} + \mathbf{q}') W(\mathbf{q}), \quad (3)$$

where $W(\mathbf{q}) = n_{\text{imp}} |V(\mathbf{q})|^2 \geq 0$ is the disorder power spectrum of scatterers of density n_{imp} , treated in the Born limit. We take the potential to be *smooth on the scale of the internodal separation*, i.e. of finite range ξ_{dis} with $\xi_{\text{dis}} |2\mathbf{Q}| \gg 1$, so that

$$\frac{W(2\mathbf{Q})}{W(0)} \ll 1. \quad (4)$$

Equation (4) suppresses inter-node scattering, but by itself it is *not* enough to justify the projected kernel used below. A correlator smooth enough to kill $2\mathbf{Q}$ processes is, generically, also forward-peaked *within* a single pocket, in which case the intra-node kernel would carry an extra factor $W(\mathbf{k}_F - \mathbf{k}'_F)/W(0)$, the normal eigenvalue would no longer be $\kappa_0 = \frac{1}{2}$, and f_m would not remain an exact eigenfunction. A second condition is therefore required: the correlation length must be short compared with the *intra-pocket* momentum transfers, so

that $W(\mathbf{k}_F - \mathbf{k}'_F) \simeq W(0)$ across one Fermi surface. The two requirements together define a *two-scale window*,

$$q_{\text{intra}}^{\text{max}} \xi_{\text{dis}} \ll 1 \ll |2\mathbf{Q}| \xi_{\text{dis}}, \quad (5)$$

where $q_{\text{intra}}^{\text{max}}$ is the largest momentum transfer between two states on the *same* pocket and $2\mathbf{Q}$ is the actual (vector) node separation. We take the nodes at $\pm Q\hat{\mathbf{z}}$, i.e. separated along the rotation axis, as in the standard multi-Weyl realizations; the condition is then purely geometric, $q_{\text{intra}}^{\text{max}} \ll 2Q$. Because the pocket is anisotropic there is no single k_F , and it is bounded by the two Fermi momenta

$$k_{F,z} = \frac{\mu}{v_z}, \quad k_{F,\perp} = \left(\frac{\mu}{\alpha}\right)^{1/J}, \quad q_{\text{intra}}^{\text{max}} \leq 2\sqrt{k_{F,z}^2 + k_{F,\perp}^2}, \quad (6)$$

which we use only as an order-of-magnitude scale, not as a sharp bound. The window Eq. (5) therefore requires simply

$$k_{F,z} \ll Q \quad \text{and} \quad k_{F,\perp} \ll Q, \quad (7)$$

i.e. Fermi pockets small compared with the node separation—a low-doping, small-pocket regime. We emphasize that Eq. (7) is a statement about *momenta*, not about energies: with $\mathbf{Q} \parallel \hat{\mathbf{z}}$ the combination αQ^J is not an inter-valley energy of the band structure. Which of the two conditions in Eq. (7) binds more tightly depends on the material parameters v_z , α and Q and on the doping. If instead $\mathbf{Q}$ points in a generic direction, the inequalities should be imposed component-wise on Q_z and $Q_\perp$, or directly as $q_{\text{intra}}^{\text{max}} \ll |2\mathbf{Q}|$.

Within this window the disorder acts, to leading order, as a *valley-diagonal, intra-pocket momentum-independent* random potential, and it is precisely this projected ensemble—not generic smooth disorder—for which the kernel Eq. (19) and the eigenvalue $\eta_m = 1/(J+2)$ are exact. Two departures are left uncomputed and are not claimed: generic short-range disorder ($W(2\mathbf{Q}) \sim W(0)$), which requires the full valley-space anomalous ladder (Sec. VII); and generic forward-peaked disorder ($q_{\text{intra}}^{\text{max}} \xi_{\text{dis}} \gtrsim 1$), which requires diagonalizing the finite-range kernel $W(\mathbf{k}_F - \mathbf{k}'_F) |O|^2 / W(0)$ and would change κ_0 , η_s and η_m as functions of ξ_{dis} . With Eq. (5) the intra-node Born rate is set by the zero-momentum weight, $\Gamma_b = \pi N_0 W(0)$; we write $W(0) \equiv n_{\text{imp}} |V|^2$ throughout, understanding $n_{\text{imp}} |V|^2$ as the constant intra-pocket value in this scale-separated limit, so that the rate conventions of Eq. (25) retain their familiar form.

A further distinction should be kept in mind, since the two are easily conflated: the *internal matrix structure* of the impurity potential (valley-even τ_0 , valley-odd τ_z , intervalley

$\tau_{x,y}$) and the *momentum transfer* it supplies are independent properties. A valley-even scalar potential $V(\mathbf{r})\tau_0$ has inter-node matrix elements whenever its Fourier support extends to $|\mathbf{q}| \simeq 2Q$; it is Eq. (4), restricting that support, and not the matrix structure alone, that suppresses inter-node processes here.

Superconductivity is introduced in the four-component Nambu spinor built from the *orbital-pseudospin* states of the two nodes, $\Psi_{\mathbf{k}} = (c_{+, \mathbf{k}1}, c_{+, \mathbf{k}2}, c_{-, -\mathbf{k}1}^\dagger, c_{-, -\mathbf{k}2}^\dagger)^\top$, where 1, 2 label the orbital (sublattice) pseudospin components of Eq. (1) and *not* physical spin, which is deliberately omitted for notational simplicity throughout the main text; it is reintroduced explicitly, with matrices s_i , only in App. G 7. This gives the BdG block

$$\mathcal{H}_{\text{BdG}}(\mathbf{k}) = \begin{pmatrix} h_+(\mathbf{k}) - \mu & \hat{\Delta}_a(\mathbf{k}) \\ \hat{\Delta}_a^\dagger(\mathbf{k}) & -h_-^*(-\mathbf{k}) + \mu \end{pmatrix}, \quad \hat{\Delta}_m(\mathbf{k}) = \Delta_m M_m^{(J)}. \quad (8)$$

The two pairing channels enter this block on a different footing, and we write them differently. For the *monopole* channel the gap is the constant, charge-parity-dependent pseudospin matrix $M_m^{(J)}$ of Eq. (9), whose conduction-band projection generates the form factor f_m ; this is a genuine matrix in the orbital space. For the *band-isotropic conventional* channel we do *not* posit a constant matrix $M_s^{(J)}$: as discussed below and in App. G 4, that channel is defined directly at the projected-band level by its form factor $f_s(\hat{\mathbf{k}}) = 1$, entering the linearized gap equation through $\Delta_s f_s(\hat{\mathbf{k}})$ rather than through Eq. (8). Where a gap matrix is needed for it, the momentum-dependent, gauge-covariant band-sewing construction $M_s(\mathbf{k}) = u_+(\mathbf{k})u_-^T(-\mathbf{k})$ of App. G 2 is used. The two-component structure of Eq. (1) is an orbital (sublattice) pseudospin, not physical spin. Because the Cooper pair is *inter-node* (node “+” at $\mathbf{k}$, with node “−” at $-\mathbf{k}$), the pairing lives off-diagonal in node space; in the full node $\otimes$ pseudospin space the gap is $\hat{\Delta}_m(\mathbf{k}) = \tau_+ \otimes \delta_m(\mathbf{k}) + \tau_- \otimes \delta'_m(\mathbf{k})$ with $\delta_m(\mathbf{k}) = \Delta_m g(\mathbf{k}) M_m^{(J)}$, and fermionic antisymmetry fixes $\delta'(\mathbf{k}) = -\delta^T(-\mathbf{k})$, i.e. $\hat{\Delta}_m(\mathbf{k}) = -\hat{\Delta}_m^T(-\mathbf{k})$ (see App. G). The pseudospin pairing matrix that yields each conduction channel is *charge-parity dependent*: projecting onto the conduction bands, $f_a(\hat{\mathbf{k}}) = u_+^\dagger(\mathbf{k}) M_a^{(J)} u_-^*(-\mathbf{k})$, one finds the monopole gap $f_m = \cos \psi e^{iJ\phi} \propto (k_x + ik_y)^J$ from

$$M_m^{(J)} = \begin{cases} i\sigma_y, & J \text{ odd,} \\ \sigma_x, & J \text{ even,} \end{cases} \quad (9)$$

whereas the complementary matrix projects to *zero*: σ_x gives zero for odd J , and $i\sigma_y$ gives zero for even J (App. G 1). This assignment is the one required by the antiunitary book-

keeping of Eq. (11): the antisymmetric $i\sigma_y$ pairs Θ_J -partners precisely when $\Theta_J^2 = -1$, i.e. for odd J . For the conventional channel, we take the isotropic nodeless conduction-band gap $f_s(\hat{\mathbf{k}}) = 1$ as an *effective-model input*, inserted at the projected level (in contrast to the monopole gap, which we derived microscopically above from $M_m^{(J)}$). We are explicit that this is an asymmetry of the construction: we do not claim the conventional channel arises from the same orbital-pseudospin projection. The single-band Abrikosov–Gor’kov treatment used throughout dresses both the normal and the anomalous self-energy with one coherence factor $|O|^2 = \frac{1}{2}(1 + \mathbf{d} \cdot \mathbf{d}')$. Within it, the results are exact: a *constant* $f_s = 1$ is the top eigenfunction with $\eta_s = 1$ identically (for $f_s = 1$ the normal- and anomalous-line averages coincide term by term), and the monopole f_m occupies the rank-one $m = \pm J$ sector with $\eta_m = 1/(J + 2)$.

Through out this work we assume that the conventional projected channel is Anderson protected, $\eta_s = 1$, i.e. that its anomalous impurity vertex equals the normal one (see App. G for details). Our assumption is precisely that the analogous node-space contraction reduces to $|O|^2$. We state once, here, the logical standing of each ingredient; *(i) Assumption-free.* Results that follow from the clean projected Hamiltonian alone: the angular moments, the nodal laws $N_{\text{SC}} \propto E^{2/J}$ and $C \propto T^{1+2/J}$, the clean specific-heat jumps, and the pure-state BdG nodal charges of magnitude J . *(ii) Exact given the projected kernel.* $\eta_m(J) = 1/(J + 2)$ is the exact rank-one eigenvalue of the valley-diagonal, intra-pocket momentum-independent kernel Eq. (19) in the sector $m = \pm J$. *(iii) Assumed.* The equality of the normal and anomalous coherence kernels, hence $\eta_s = 1$. App. G constructs a band-sewing gap matrix $M_s(\mathbf{k}) = u_+(\mathbf{k})u_-^T(-\mathbf{k})$ with $f_s \equiv 1$ and exact fermionic antisymmetry, and derives a *sufficient* condition—that the impurity potential preserves the anti-unitary sewing relation, $\hat{\mathcal{A}}\hat{V}_{\mathbf{k}\mathbf{k}'}\hat{\mathcal{A}}^{-1} = \hat{V}_{-\mathbf{k}, -\mathbf{k}'}$, which holds for valley-even scalar disorder and fails for valley-odd (τ_z) and intervalley ($\tau_{x,y}$) components. We do *not* claim a local orbital realization: no momentum-independent orbital sewing matrix exists for Eq. (1) (App. G 4), and a spectator spin singlet does not supply one (App. G 7). The band-isotropic channel is therefore a projected-band input, η_s is treated as a parameter, and the consequences of $\eta_s < 1$ are quantified by Eq. (28).

Let us now analyze the anti-unitarity. Writing the transverse terms as $\alpha \text{Re}(k_x + ik_y)^J \sigma_x +$

$\alpha \text{Im}(k_x + ik_y)^J \sigma_y$ and using $\mathcal{H}_0(\mathbf{k}) = \text{diag}(h_+(\mathbf{k}), h_-(\mathbf{k}))$, the operator $\Theta_J = \sigma_{a(J)} K$, for

$$a(J) = \begin{cases} y, & J \text{ odd} \\ x, & J \text{ even} \end{cases} \quad (10)$$

satisfies the properties

$$\Theta_J h_\chi(\mathbf{k}) \Theta_J^{-1} = h_\chi(-\mathbf{k}), \quad \Theta_J^2 = \begin{cases} -1, & J \text{ odd} \\ +1, & J \text{ even} \end{cases} \quad (11)$$

and hence is an exact anti-unitary symmetry of *each* node ($\chi = \pm$) for $J = 1, 2, 3$. (For an anti-unitary $\Theta = UK$, $\Theta^2 = UU^*$ is phase-convention independent, and $\sigma_y \sigma_y^* = -1$ while $\sigma_x \sigma_x^* = +1$.) The Kramers-like square for odd J is the origin of the parity dependence of $M_m^{(J)}$, since the antisymmetric $i\sigma_y$ pairs Θ -partners only when $\Theta^2 = -1$. We use Θ_J only to organize the disorder classes: intra-node scalar disorder $\tau_{0,z} \otimes \sigma_0$ preserves Θ_J , while chiral σ_z disorder does not. An important distinction must be drawn here, since Θ_J is a *within-node* pseudospin anti-unitary, whereas the operator that controls the conventional-channel protection is the *pairing* antiunitary $\mathcal{A}$ that exchanges the two paired nodes (App. G). These are not the same operator, and they classify disorder differently: the valley-even scalar potential $\tau_0 \otimes \sigma_0$ is invariant under both, but the valley-odd $\tau_z \otimes \sigma_0$, while it preserves Θ_J , obeys $\mathcal{A} \hat{V}_z \mathcal{A}^{-1} = -\hat{V}_z$ and therefore does *not* protect the conventional channel. Because it is $\mathcal{A}$ and not Θ_J that enters the anomalous vertex, we accordingly restrict the exact protection claim to *valley-even* intra-node scalar disorder $\tau_0 \otimes \sigma_0$ (Sec. III and App. G); for τ_z disorder one expects $\eta_s < 1$ and $\Gamma_s^{\text{pb}} > 0$. Two pairing channels, distinguished by the form factor f_a : a *band-isotropic conventional* channel $f_s(\hat{\mathbf{k}}) = 1$, $\langle f_s \rangle_{\text{FS}} = 1$, fully gapped; and a topologically nontrivial *monopole* inter-node channel with $\langle f_m \rangle_{\text{FS}} = 0$ and point nodes. For a genuine multi-Weyl pair the nodes carry Chern charges $C_\pm = \pm J$, so the zero-center-of-mass inter-node pairing bundle has monopole charge $q = (C_+ - C_-)/2 = J$, following the standard inter-node monopole-pairing construction [21, 22]; the consequence is that the pair amplitude is a section of a charge- J monopole bundle and is therefore expanded in monopole harmonics rather than ordinary spherical harmonics, which is why f_m cannot be a nonvanishing constant. We do not rely on this bundle-level bookkeeping anywhere below: the topological statement we actually use is the *local* BdG nodal charge, derived from the Bogoliubov Hamiltonian itself by a degree argument in Sec. VI, which is both rigorous and

free of inter-node gauge conventions. The corresponding lowest inter-node monopole gap is

$$f_m(\hat{\mathbf{k}}) = \cos \psi e^{iJ\phi} = \frac{\alpha (k_x + ik_y)^J}{\mu} \quad (\rightarrow \sin \theta e^{i\phi} \text{ for the isotropic } J = 1 \text{ node}), \quad (12)$$

obtained by projecting the constant inter-node pairing with the *charge-parity-dependent* matrix $M_m^{(J)}$ of Eq. (9) onto the conduction band, $f_m(\mathbf{k}) \propto u_+^\dagger(\mathbf{k}) M_m^{(J)} u_-^*(-\mathbf{k})$. Explicitly (with conduction spinors $u_\pm$ of the two nodes): for odd J , $M_m^{(J)} = i\sigma_y$ gives $|u_+^\dagger(i\sigma_y)u_-^*| = \cos \psi$ with azimuthal winding J (while σ_x projects to *zero*); for even J , $M_m^{(J)} = \sigma_x$ gives the same $\cos \psi$ and winding J (and $i\sigma_y$ projects to zero). The explicit spinors and the evaluation are given in App. G 1. Two facts about this gap, both established below, are central. (i) One must distinguish the *pseudospin texture degree* from the *pairing line-bundle Chern number*. As a section of the Bloch bundle, $f_m = \cos \psi e^{iJ\phi}$ resembles a degree-one texture in the abstract $\hat{\mathbf{d}}$ frame, but the physical object is the pairing line bundle over the momentum Fermi surface, whose obstruction is set by the J -fold winding $f_m \propto (k_x + ik_y)^J$. Correspondingly, the Berry charge on a small sphere enclosing each BdG point node is $\pm J$: this *is* the genuine charge- J monopole gap, with $\mathcal{C}_{\text{BdG}} = J$ (Sec. VI). We write “correspondingly” rather than “equivalently” deliberately. The two are distinct invariants on distinct base spaces—the pairing-bundle Chern class is carried by the pair section over the Fermi surface, whereas $\mathcal{C}_{\text{BdG}}$ is a Chern number of Bogoliubov eigenstates over a small momentum sphere around one node—and they are related as two manifestations of the same J -fold winding obstruction rather than as the same object. Only the local BdG charge is used below, and it is established independently by the degree argument of Sec. VI. (ii) f_m is not a variational trial state but the *exact* eigenfunction of the projected impurity ladder in the rank-one azimuthal sector $m = \pm J$, with a closed-form pair-breaking eigenvalue $\eta_m(J) = 1/(J+2)$ (Sec. III). The remaining modeling input is the pairing symmetry class itself, an inter-node channel with the parity-dependent matrix $M_m^{(J)}$ and the antiunitary Θ_J of Eq. (11), classified in the full node $\otimes$ pseudospin space rather than by the pseudospin matrix alone, not the choice of harmonic. All Fermi-surface averages use the single measure $\langle \cdots \rangle_{\text{FS}} = [\int d\psi (\cos \psi)^{2/J-1} (\cdots)] / [\int d\psi (\cos \psi)^{2/J-1}]$ (see Appendix B), with $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We use the *unnormalized* convention $f_s = 1$, $f_m = \cos \psi e^{iJ\phi}$ throughout (so Δ_a is the gap amplitude, the monopole gap magnitude being $\Delta_m \cos \psi$), with $\langle |f_s|^2 \rangle = 1$, $\langle |f_m|^2 \rangle = 2/(J+2)$, and $\langle |f_m|^4 \rangle = (2/J)(2/J+2)/[(2/J+1)(2/J+3)]$ (see Appendix B for details); effective couplings $\lambda_a \equiv g_a N_0 \langle |f_a|^2 \rangle$ absorb this measure. This single convention is used in the gap, AG, heat-

capacity, and residual-DOS equations and in all plots. In the clean weak-coupling limit each channel obeys

$$T_{c0}^{(a)} = \frac{2e^{\gamma_E}}{\pi} \Lambda e^{-1/\lambda_a}, \quad \lambda_a = g_a N_0 \langle |f_a|^2 \rangle_{\text{FS}}, \quad (13)$$

so that the regime of interest,

$$T_{c0}^{(m)} > T_{c0}^{(s)} \iff \lambda_m > \lambda_s \iff g_m \langle |f_m|^2 \rangle > g_s \langle |f_s|^2 \rangle, \quad (14)$$

is a condition on the effective channel couplings λ_a , and the effective couplings $\lambda_a = g_a N_0 \langle |f_a|^2 \rangle$ differ from the bare attractions through the second moment $\langle |f_a|^2 \rangle$ (which is 1 for s -wave but $2/(J+2)$ for the monopole), so a given difference in λ is not the same difference in microscopic pairing strength. The asymmetry needed is modest. For the couplings *actually used* in the numerics: at the gap-equation cutoff $\Lambda = 10 \mu$, inverting Eq. (13) for the clean scales $T_{c0}^{(m)} = 0.133 \mu$ and $T_{c0}^{(s)} = 0.083 \mu$ gives

$$\lambda_m \simeq 0.225, \quad \lambda_s \simeq 0.203, \quad (15)$$

i.e. a $\simeq 11\%$ stronger monopole coupling, which yields $T_{c0}^{(m)}/T_{c0}^{(s)} = 1.60$ and hence $r = 0.625$. These are the values used in all figures and tables below. This is a *chosen, plausible* regime, whenever pairing is dominated by inter-node, large-momentum ($\sim 2\mathbf{Q}$) processes: the chiral Bloch structure projects such processes onto the charge- J monopole harmonic, while the conventional s -wave channel is additionally weakened by the Coulomb pseudo-potential μ^* , which is expected to suppress λ_s but not the higher- ℓ monopole channel, as in unconventional superconductors. This is the clean-limit regime established for monopole pairing in multi-Weyl systems by Refs. [22–24]; we take it as the starting point and ask how disorder resolves the competition.

III. DISORDER SELF-ENERGY IN THE CHIRAL BASIS: ANDERSON PROTECTION AND MONOPOLE PAIR-BREAKING

We treat both protection and pair-breaking on the same footing by computing the Born self-energy in the chiral (conduction-band) basis. For $\mu > 0$ each node contributes a conduction Fermi surface with Bloch spinor $|u(\hat{\mathbf{k}})\rangle$ (with pseudospin texture $\mathbf{d}(\hat{\mathbf{k}})$). Scalar disorder $V(\mathbf{r})\sigma_0$ scatters band states with the chiral coherence factor

$$O(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \langle u(\hat{\mathbf{k}}) | u(\hat{\mathbf{k}}') \rangle, \quad |O(\hat{\mathbf{k}}, \hat{\mathbf{k}}')|^2 = \frac{1}{2}(1 + \mathbf{d} \cdot \mathbf{d}'), \quad (16)$$

where $\mathbf{d}(\hat{\mathbf{k}})$ is the pseudospin (Anderson) unit vector, *not* the Euclidean momentum direction; $O \rightarrow 0$ at $\mathbf{d}' = -\mathbf{d}$ (a *pseudospin* reversal, which for the multi-Weyl texture need not coincide with physical momentum backscattering). Within this projected single-band kernel the eigenvalues below are exact and parity-independent. Both $\eta_s = 1$ and $\eta_m = 1/(J+2)$ are properties of the *same* kernel Eq. (19): the first is an input (Sec. II; a sufficient condition for it is derived in App. G), the second the exact rank-one eigenvalue of that input (for the valley-diagonal, intra-pocket momentum-independent kernel defined by the two-scale window Eq. (5)).

Self-consistent Born equations. We use the *normalized* quasiclassical propagator, $\check{g}^2 = \check{1}$, so that the density of states appears once and only once in the self-energy:

$$\check{g}(\hat{\mathbf{k}}', i\omega_n) = \frac{i\tilde{\omega}_n\tau_0 + \tilde{\Delta}(\hat{\mathbf{k}}')\tau_1}{\sqrt{\tilde{\omega}_n^2 + \tilde{\Delta}(\hat{\mathbf{k}}')^2}}, \quad (17)$$

(written in a *local gauge patch*; globally the complex gap occupies τ_1, τ_2 , with a momentum-dependent connection, which does not affect the angle-averaged rates, and the τ_3 term integrates to zero). With this normalization the disorder self-energy is

$$\check{\Sigma}(\hat{\mathbf{k}}, i\omega_n) = \pi N_0 n_{\text{imp}} |V|^2 \int d\mu_{\text{FS}}(\hat{\mathbf{k}}') \mathcal{C}(\hat{\mathbf{k}}, \hat{\mathbf{k}}') \check{g}(\hat{\mathbf{k}}', i\omega_n) = \Gamma_b \int d\mu_{\text{FS}}(\hat{\mathbf{k}}') \mathcal{C}(\hat{\mathbf{k}}, \hat{\mathbf{k}}') \check{g}(\hat{\mathbf{k}}', i\omega_n), \quad (18)$$

with $\Gamma_b = \pi N_0 W(0)$ the bare rate of Eq. (25), and $d\mu_{\text{FS}}(\hat{\mathbf{k}}')$ is the anisotropic density-of-states measure $\propto (\cos \psi')^{2/J-1} d\psi' d\phi'/2\pi$ (see Appendix B); it reduces to $d\hat{\mathbf{k}}'/4\pi$ only for the isotropic $J = 1$ node. We fix the normalization convention once here: $\check{g}$ is dimensionless and obeys $\check{g}^2 = \check{1}$, the density of states appears solely through the explicit prefactor πN_0 , and in the normal state $\check{g} \rightarrow i \text{sgn}(\omega_n)\tau_0$. Other common conventions absorb a factor $-i\pi N_0$ into $\check{g}$; the rates below are unchanged provided the factor is counted once.

The crucial point is the coherence structure $\mathcal{C}$. The normal (particle) line carries $O(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$ and the anomalous (Cooper) line pairs the Θ -conjugate partner, which carries $O^*(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$. *Under the single-band anomalous-kernel assumption* (stated in Sec. II: the conventional channel's anomalous vertex equals its normal vertex), both the τ_0 and τ_1 channels are weighted by the same real kernel

$$\mathcal{C}(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = |O(\hat{\mathbf{k}}, \hat{\mathbf{k}}')|^2 = \frac{1}{2}(1 + \mathbf{d}(\hat{\mathbf{k}}) \cdot \mathbf{d}(\hat{\mathbf{k}}')), \quad (19)$$

Spectral decomposition. Let us first consider the *isotropic* $J = 1$ node (the anisotropic $J > 1$ kernel is treated below), expanding the kernel in Legendre polynomials, $\mathcal{C}(\hat{\mathbf{k}}, \hat{\mathbf{k}}') =$

$\frac{1}{2}P_0 + \frac{1}{2}P_1$. Its eigenvalues κ_ℓ (kernel eigenvalues, distinct from the pairing couplings λ_a) on angular-momentum- ℓ form factors are $\kappa_\ell = c_\ell/(2\ell + 1)$, i.e.

$$\kappa_0 = \frac{1}{2}, \quad \kappa_1 = \frac{1}{6}, \quad \kappa_{\ell \geq 2} = 0, \quad (20)$$

so that $\int (d\hat{\mathbf{k}}'/4\pi) \mathcal{C}(\hat{\mathbf{k}}, \hat{\mathbf{k}}') f_\ell(\hat{\mathbf{k}}') = \kappa_\ell f_\ell(\hat{\mathbf{k}})$. The normal scattering rate is set by κ_0 , as follows

$$\frac{\hbar}{2\tau} = \pi N_0 n_{\text{imp}} |V|^2 \kappa_0 = \frac{1}{2} \pi N_0 n_{\text{imp}} |V|^2, \quad (21)$$

with the factor $\kappa_0 = \frac{1}{2}$ being the familiar chiral halving of the quantum lifetime. The pair-breaking rate for a gap of angular momentum ℓ_a is the *mismatch* between the normal (κ_0) and anomalous (κ_{ℓ_a}) renormalizations,

$$\boxed{\Gamma_a^{\text{pb}} = \pi N_0 n_{\text{imp}} |V|^2 (\kappa_0 - \kappa_{\ell_a}) = \frac{\hbar}{2\tau} (1 - \eta_a), \quad \eta_a = \frac{\kappa_{\ell_a}}{\kappa_0}.} \quad (22)$$

Consequences. For the *conventional* channel ($\ell_s = 0$) the gap is the leading eigenfunction, $\kappa_{\ell_s} = \kappa_0$, hence $\eta_s = 1$ and $\Gamma_s^{\text{pb}} = 0$: scalar disorder does not pair-break the conventional channel even in the Weyl semimetal. This is Anderson's theorem *as realized within the assumed kernel* (Sec. II), not a theorem derived for the full node-space vertex: the Berry phase that halves $1/\tau$ [Eq. (21)] cancels identically between numerator and denominator of the pairing renormalization and drops out. For the *monopole* channel ($\ell_m = 1$),

$$\eta_m = \frac{\kappa_1}{\kappa_0} = \frac{1/6}{1/2} = \frac{1}{3}, \quad \Gamma_m^{\text{pb}} = \frac{2}{3} \frac{\hbar}{2\tau}, \quad (23)$$

i.e. the chiral kernel reduces the pair-breaking rate to *two thirds* of the unprotected value $\Gamma^{\text{pb}} = \Gamma_N$ (the rate for a channel with vanishing anomalous eigenvalue, $\eta = 0$) at the same normal scattering rate—the chiral suppression of pseudospin reversal partially shields even the topological channel, but cannot protect it fully. (Eq. (23) is the isotropic value for the spherical $J = 1$ node; the Fermi-surface anisotropy of the $J = 2, 3$ nodes is treated below and gives $\eta_m(J) = 1/(J + 2)$.)

Monopole-harmonic check. Equation (23) uses the ordinary-harmonic addition theorem. One might worry that the genuine monopole-harmonic ($q = 1$) addition theorem, which governs the projection of $Y_{1,1,m}$ onto the $q = 0$ kernel Eq. (19), shifts the result. It does not: in this projected kernel the Berry-connection phase of $O(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$ cancels against that of its partner, so the monopole harmonic reproduces the value already obtained in closed form above. The cancellation is specific to the assumed Θ -compatible kernel; for a Θ -broken

(triplet/chiral) vertex the Berry phase survives and even the isotropic component is incompletely protected. The remaining shift to η_s comes from inter-node scattering (momentum transfer $2\mathbf{Q}$, with weight $W(2\mathbf{Q})/W(0)$, small by assumption in the smooth-disorder regime of Eq. (4) but not in the generic short-range case (Sec. VII)). The consequences of any resulting $\eta_s < 1$ are quantified by Eq. (28).

We therefore diagonalize the projected impurity ladder on the *actual* anisotropic Fermi surface. Parametrizing it by $v_z q_z = \mu \sin \psi$, $\alpha q_\perp^J = \mu \cos \psi$, the pseudospin is $\mathbf{d} = (\cos \psi \cos J\phi, \cos \psi \sin J\phi, \sin \psi)$, the chiral kernel is $|O|^2 = \frac{1}{2}(1 + \mathbf{d} \cdot \mathbf{d}')$, and the density-of-states measure is $w(\psi) \propto (\cos \psi)^{2/J-1}$ (Appendix A). The pair-breaking eigenproblem is $\langle |O(\hat{\mathbf{k}}, \hat{\mathbf{k}}')|^2 f(\hat{\mathbf{k}}') \rangle_{\hat{\mathbf{k}}'} = \lambda_f f(\hat{\mathbf{k}})$, with $\eta_f = \lambda_f / \langle |O|^2 \rangle_{\text{FS}} = 2\lambda_f$ and pair-breaking rate $\Gamma_f^{\text{pb}} = \Gamma_N(1 - \eta_f)$. Crucially, because $\mathbf{d} \cdot \mathbf{d}' = \sin \psi \sin \psi' + \cos \psi \cos \psi' \cos J(\phi - \phi')$ depends on azimuth only through $\cos J(\phi - \phi')$, the kernel is *block-diagonal in the azimuthal quantum number* m (eigenfunctions $\propto e^{im\phi} g(\psi)$), and only three sectors are populated:

- $m = 0$, $f = \text{const}$: $\kappa = \frac{1}{2}$, so $\eta_s = 1$ *exactly within the projected anomalous kernel of Eq. (19)* and, given that kernel, independently of Fermi-surface shape (Anderson protection of the conventional channel; the microscopic equality of normal and anomalous vertices remains the assumption of Sec. II);
- $m = 0$, $f = \sin \psi$: a *line-nodal* member (the gap vanishes on the $q_z = 0$ equator), with $\eta = \langle \sin^2 \psi \rangle_{\text{FS}} = J/(J+2)$ (Appendix B);
- $m = \pm J$, $f = \cos \psi e^{\pm iJ\phi}$: the *point-nodal* monopole gap. This sector is *rank one* [$K \propto \cos \psi \cos \psi'$], so f_m is the exact eigenfunction, with kernel eigenvalue $\kappa_m = \frac{1}{4} \langle \cos^2 \psi \rangle_{\text{FS}} = 1/(2(J+2))$ (Appendix B).

The two $m = 0$ entries and the $m = \pm J$ entry carry different numerical weights for a simple reason: All η 's are normalized to the constant mode, $\eta = \kappa/\kappa_0$ with $\kappa_0 = \frac{1}{2}$. In the $m = 0$ sector the azimuthal average of $\mathbf{d} \cdot \mathbf{d}'$ retains $\sin \psi \sin \psi'$ with weight $\frac{1}{2}$, giving $\kappa = \frac{1}{2} \langle \sin^2 \psi \rangle$ and hence $\eta = \langle \sin^2 \psi \rangle$. In the $m = \pm J$ sector the transverse term $\cos \psi \cos \psi' \cos J(\phi - \phi')$ contributes only half of its weight upon projection onto $e^{\pm iJ\phi}$, giving the extra factor $\frac{1}{2}$: $\kappa_m = \frac{1}{4} \langle \cos^2 \psi \rangle$ and $\eta_m = \frac{1}{2} \langle \cos^2 \psi \rangle$. With $\langle \cos^2 \psi \rangle = 2/(J+2)$ and $\langle \sin^2 \psi \rangle = J/(J+2)$ this yields $\eta_m = 1/(J+2)$ and $\eta_{\text{line}} = J/(J+2)$. For an isotropic ($J = 1$) node these three states are the degenerate $\ell = 1$ monopole triplet ($\eta = \frac{1}{3}$); the multi-Weyl anisotropy *splits* the

triplet, pushing the topological $m = \pm J$ member down to (see Appendix B)

$$\eta_m(J) = \frac{1}{2} \langle \cos^2 \psi \rangle_{\text{FS}} = \frac{1}{J+2}. \quad (24)$$

Therefore $\eta_m = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ for $J = 1, 2, 3$. This is an *exact* eigenvalue of the projected ladder for the physical point-nodal monopole gap $f_m = \cos \psi e^{iJ\phi}$ (Eq. (12)), not a trial-state expectation value: the rank-one structure of the $m = \pm J$ sector forces f_m to be the eigenfunction. Two consequences follow. First, $\eta_s = 1$ is exact within this projected scalar kernel. Second, the anisotropy *reduces* η_m with increasing J , i.e. strengthens the monopole pair-breaking $\Gamma_m^{\text{pb}} = (1 - \eta_m)\Gamma_N$; this sharpens, and does not weaken, the central trend that the crossing rate $\Gamma_N^*/T_{c0}^{(m)}$ decreases with J (Fig. 5).

Two-rate structure. A useful way to read Eqs. (21)–(23) is that scalar disorder enters the monopole channel through *two distinct rates*: the normal (frequency) renormalization is set by $\kappa_0 = \frac{1}{2}$ while the anomalous (gap) renormalization is set by $\kappa_1 = \frac{1}{6}$. The dimensionless disorder strength is $\gamma \equiv \frac{1}{2}n_{\text{imp}}|V|^2$, so that the bare rate is $\Gamma_b \equiv \pi N_0 n_{\text{imp}}|V|^2 = 2\pi\gamma N_0$, the normal rate is $\Gamma_N = \Gamma_b/2 = \pi\gamma N_0 = \hbar/2\tau$ (fixed self-consistently below by $N_0 \rightarrow N(\mu; \Gamma_N)$), the anomalous rate is $\Gamma_b\kappa_m = \Gamma_b/[2(J+2)]$ ($\Gamma_b/6$ at $J=1$), and their difference—the channel- m pair-breaking rate Γ_m^{pb} of Eq. (23), here abbreviated Γ_{pb} —is $\Gamma_{\text{pb}} = (1 - \eta_m)\Gamma_N = \Gamma_b(J+1)/[2(J+2)]$ ($\Gamma_b/3$ at $J=1$). Since several rates appear in this work, we define the complete set here, in the order in which they are constructed:

$$\begin{aligned} \Gamma_b &= \pi n_{\text{imp}}|V|^2 N_0 = 2\pi\gamma N_0 && \text{(bare Born rate),} \\ \Gamma_N &= \kappa_0 \Gamma_b = \frac{1}{2}\Gamma_b = \pi\gamma N_0 && \text{(normal/frequency rate),} \\ \Gamma_{\text{an},a} &= \kappa_a \Gamma_b && \text{(anomalous/gap rate),} \\ \Gamma_a^{\text{pb}} &= \Gamma_N - \Gamma_{\text{an},a} = (1 - \eta_a) \Gamma_N && \text{(pair breaking),} \\ \eta_a &= \kappa_a/\kappa_0 && \text{(protection factor).} \end{aligned} \quad (25)$$

Throughout, $n_{\text{imp}}|V|^2 \equiv W(0)$ denotes the constant intra-pocket value of the disorder power spectrum in the scale-separated limit Eq. (5); no separate pointlike-impurity assumption is made or needed. Here N_0 is the per-node, per-band DOS fixed in Sec. II, $\kappa_0 = \frac{1}{2}$ always, $\kappa_m = 1/[2(J+2)]$ for the monopole channel, and $\kappa_s = \kappa_0$ in the Anderson-protected limit $\eta_s = 1$. Two further rates appear later and are *not* members of this family: Γ_0 , the residual zero-energy broadening in the ordered state (App. F), and $\tau_q = \hbar/(2\Gamma_N)$, the quantum

lifetime used for material conversions (Sec. XI). This is the qualitative distinction from isotropic magnetic Abrikosov–Gor’kov, where the frequency and pair-breaking rates coincide. The self-consistent Born equations then carry an angle-independent $\tilde{\omega}_n = \omega_n + \Gamma_N \omega_n^{-1}$ -type renormalization while the gap retains its $\cos \psi e^{iJ\phi}$ winding under the anomalous rate $\Gamma_b/[2(J+2)]$, a separation we use in the thermodynamics below.

IV. GENERALIZED ABRIKOSOV–GOR’KOV GAP EQUATION

With Γ_a^{pb} from Eq. (22), each channel obeys (derived, with the rate conventions, in Appendix C, and discussed in Appendix D)

$$\ln \frac{T_{c0}^{(a)}}{T_c^{(a)}} = \psi\left(\frac{1}{2} + \frac{\Gamma_a^{\text{pb}}}{2\pi k_B T_c^{(a)}}\right) - \psi\left(\frac{1}{2}\right), \quad (26)$$

with $\psi(z)$ the Digamma function, and hence it is destroyed at $\Gamma_a^{\text{pb}} = \Gamma_c \simeq 0.882 k_B T_{c0}^{(a)}$. With $\Gamma_s^{\text{pb}} = 0$ the conventional channel is unaffected by pair-breaking *in the $\eta_s = 1$ projected-kernel limit* (for general η_s see Eq. (28)) and changes only through the disorder-dressed DOS $N(\gamma)$ entering $T_{c0}^{(s)}$; the monopole channel is suppressed per Eq. (26) with $\Gamma_m^{\text{pb}} = (1 - \eta_m)\Gamma_N = \frac{J+1}{J+2} \Gamma_N$.

Dimensionless form of the crossing. The AG crossing is most naturally stated in dimensionless variables. Since $\Gamma_m^{\text{pb}}/(2\pi k_B T_c^{(m)})$ depends on disorder only through $\Gamma_N/k_B T_{c0}^{(m)}$ and the charge only through $\eta_m(J)$, the crossing condition $T_c^{(m)}(\gamma) = T_c^{(s)}(\gamma)$ is governed by the two universal inputs $\{\Gamma_N/k_B T_{c0}^{(m)}, \eta_m(J)\}$ together with the clean ratio $T_{c0}^{(s)}/T_{c0}^{(m)}$. This is the dimensionless projected-model content (at fixed r and η_s) of the charge trend: at fixed clean ratio, larger J (smaller η_m) reaches the crossing at smaller $\Gamma_N/k_B T_{c0}^{(m)}$. The conversion to a bare disorder coordinate γ (and hence any bare-coordinate values) additionally carries the DOS normalization N_J , the cutoff, μ , and the fixed- μ vs. fixed-density choice, and is therefore *not* automatically comparable across J . All disorder strengths are therefore quoted as $\Gamma_N/k_B T_{c0}^{(m)}$ or Γ_N/μ .

AG crossing boundary for general η_s . Because we started with $\eta_s = 1$ as a working assumption (Sec. II), we now formulate the competition for *arbitrary* η_s and identify $\eta_s = 1$ as the Anderson-protected limit. Both channels obey the same digamma law with their own rates

$$\Gamma_s^{\text{pb}} = (1 - \eta_s)\Gamma_N, \quad \Gamma_m^{\text{pb}} = (1 - \eta_m)\Gamma_N, \quad (27)$$

and each is destroyed when its own Γ^{pb} reaches [27, 43] the critical limit $\Gamma_c = 0.882 k_B T_{c0}$. Hence, the monopole channel dies at $\Gamma_N^{(m0)} = 0.882 k_B T_{c0}^{(m)} / (1 - \eta_m)$ and the conventional one at $\Gamma_N^{(s0)} = 0.882 k_B T_{c0}^{(s)} / (1 - \eta_s)$, respectively. A crossing at *finite* T_c exists—necessary and sufficient given the verified monotonicity of the AG scaling function (see below)—precisely when the conventional channel outlives the monopole channel, $\Gamma_N^{(s0)} > \Gamma_N^{(m0)}$, i.e.

$$\boxed{\frac{1 - \eta_s}{1 - \eta_m} < r \equiv \frac{T_{c0}^{(s)}}{T_{c0}^{(m)}}} \iff \eta_s > \eta_s^c \equiv 1 - r(1 - \eta_m). \quad (28)$$

Two caveats on scope. Eq. (28) is derived at *fixed* clean ratio r ; if disorder-induced DOS feedback is retained, $r \rightarrow r(\Gamma_N)$ and the criterion becomes an implicit local condition [cf. Eq. (II)]. Moreover, at $\eta_s = 1$ the conventional curve is flat, so the crossing temperature is simply $T^* = T_{c0}^{(s)}$, i.e. $T^*/T_{c0}^{(m)} = r = 0.625$ for every J ; the charge dependence resides entirely in Γ_N^* . Three consequences matter for the scope of this paper. (i) The criterion is a statement about the *ratio* of pair-breaking rates, not about exact protection: the mechanism requires only that the conventional channel be pair-broken *sufficiently more slowly* than the monopole channel. (ii) With the clean ratio $r = 0.625$ used below and the derived $\eta_m = 1/(J + 2)$, the crossing survives for all $\eta_s > \eta_s^c = 0.583, 0.531, 0.500$ ($J = 1, 2, 3$), with Γ_N^* shifting upward continuously as η_s decreases toward η_s^c [Fig. 1(b)]. We quote these critical values directly rather than describing them as a fraction of “protection,” since η_s is a kernel eigenvalue ratio and not a linear measure of protection. (iii) Conversely, if the unresolved node-space vertex were to drive η_s below η_s^c , the crossing would disappear, and in this case Eq. (28) becomes a criterion to quantify how much protection is needed (as the AG boundary, conditional on the verified monotonicity and fixed r , see Appendix D). All crossing values quoted later are conditional in this precise sense.

The scattering rate is itself fixed self-consistently within a phenomenological Lorentzian spectral-broadening scheme, $\Gamma_N = \pi\gamma N(\mu; \Gamma_N)$. Because that convolution retains only the imaginary self-energy, is ultraviolet sensitive for $J = 1, 2$, and is *not* propagated into the phase diagrams presented here, we develop it separately in App. I, where we also verify that the instability crossing survives when the feedback is coupled to the linearized T_c equations. All results in the main text are at fixed μ with the feedback disabled.

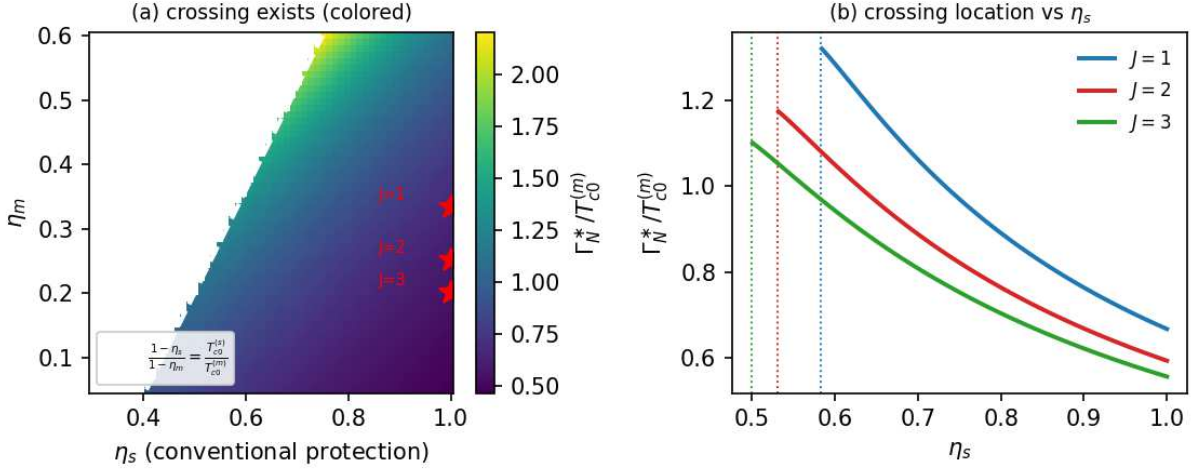


FIG. 1. Crossing criterion for general channel protection, clean ratio $r = T_{c0}^{(s)}/T_{c0}^{(m)} = 0.625$. (a) Crossing location $\Gamma_N^*/T_{c0}^{(m)}$ over the (η_s, η_m) plane; white region: no crossing at finite T_c , i.e. the strict inequality in Eq. (28) fails (the equality line itself belongs to the no-crossing side, where the two curves meet only at $T_c = 0$). Dashed line: the AG boundary, Eq. (28). Stars mark the assumed $\eta_s = 1$ with the derived $\eta_m = 1/(J+2)$. Since both protection factors are microscopically unresolved, the stars should be read as projected-kernel benchmark points, not as predictions for generic scalar disorder; the plane itself is the substantive content. (b) Crossing location versus η_s at fixed η_m ; dotted lines mark η_s^c . The crossing persists down to η_s^c and disappears below it. Grid: 60×55 in (η_s, η_m) , each point solving the two digamma equations by bisection to a precision 10^{-10} ; the dashed boundary is Eq. (28).

V. THERMODYNAMICS

Clean specific-heat jump. The gap magnitude follows from the gap equation

$$1 = g_a N_0 \int_0^\Lambda d\xi \langle |f_a|^2 \tanh(E/2T)/E \rangle_{\mathbf{k}}, \quad (29)$$

with $E = \sqrt{\xi^2 + \Delta_a^2 |f_a(\hat{\mathbf{k}})|^2}$. The jump at T_c is derived in Appendix E, and is given by the explicit expression

$$\Delta C/C_n = \frac{12}{7\zeta(3)} \langle |f_a|^2 \rangle^2 / \langle |f_a|^4 \rangle, \quad (30)$$

with C_n the specific heat of the normal state at the critical temperature. The gapped conventional channel gives the BCS value $\Delta C/C_n = 1.43$, while for the nodal monopole

channel we find 1.19, 0.95, 0.79 for $J = 1, 2, 3$, respectively. The more anisotropic gap at higher charge giving a smaller jump.

Low-energy spectrum: a charge-dependent power law. The monopole gap $|f_m| = \cos \psi$ has point nodes at the two Fermi-surface poles ($\cos \psi = 0$). There $|f_m|$ vanishes linearly in the polar angle $\vartheta \equiv \frac{\pi}{2} - |\psi|$, while the density-of-states measure scales as $(\cos \psi)^{2/J-1} \sim \vartheta^{2/J-1}$. The fraction of the Fermi surface with gap below E therefore scales as $\int_0^{E/\Delta} \vartheta^{2/J-1} d\vartheta \propto E^{2/J}$, so the quasiparticle density of states and the electronic specific heat obey

$$N_{\text{SC}}(E) \propto E^{2/J}, \quad C_{\text{es}}(T) \propto T^{1+2/J}, \quad (31)$$

i.e. $C_{\text{es}} \propto T^3, T^2, T^{5/3}$ for $J = 1, 2, 3$ [Fig. 2]. Only the isotropic ($J = 1$) node gives the familiar point-node law $C \sim T^3$; the higher-charge nodes are “softer,” and the charge-dependent exponent $1 + 2/J$ is itself a low-temperature fingerprint of the monopole charge.

Residual density of states under disorder: a charge-dependent threshold. The low-energy behavior is governed by the point node and is more subtle than a naive node-filling estimate. Scalar Born disorder generates a zero-energy scattering rate $\Gamma_0 = \text{Im} \tilde{\omega}^R(0)$. Retaining *both* the normal and anomalous self-energies in the real-frequency continuation of the angle-resolved equations of Appendix C, and using that the rank-one $m = \pm J$ sector preserves the $\cos \psi$ angular structure of the gap at a fixed effective spectral amplitude $\tilde{\Delta}$ (App. F), the $\omega \rightarrow 0$ limit reduces to

$$1 = \Gamma_N \langle (\tilde{\Delta}^2 \cos^2 \psi + \Gamma_0^2)^{-1/2} \rangle_{\text{FS}}, \quad \frac{N(0)}{N_0} = \frac{\Gamma_0}{\Gamma_N}, \quad (32)$$

whose $\Gamma_0 \rightarrow 0$ limit is controlled by $\langle 1/|f_m| \rangle = \langle 1/\cos \psi \rangle_{\text{FS}}$, i.e. by $\int_0 d\vartheta \vartheta^{2/J-2}$ near a pole. *The convergence of this integral, and hence the existence of a threshold, depends on the charge:*

- $J = 1$ ($2/J - 2 = 0$): $\langle 1/\cos \psi \rangle = \pi/2$ is finite, so $N(0) = 0$ *exactly* below a threshold $\Gamma_N^c = \frac{2\tilde{\Delta}}{\pi}$ (equivalently $\Gamma_c^{\text{pb}} = (1 - \eta_1) \frac{2\tilde{\Delta}}{\pi} = \frac{4\tilde{\Delta}}{3\pi} \simeq 0.42 \tilde{\Delta}$). The clean point-node thermodynamics survives up to this rate.
- $J = 2$ ($2/J - 2 = -1$): $\langle 1/\cos \psi \rangle$ is *logarithmically divergent*, so there is *no* threshold; arbitrarily weak disorder generates an exponentially small residual scale, with the *complete* asymptotic $\Gamma_0 = 4\tilde{\Delta} e^{-\pi\tilde{\Delta}/(2\Gamma_N)}$ (prefactor and exponent both analytic; see Appendix F).

- $J = 3$ ($2/J - 2 = -4/3$): $\langle 1/\cos\psi \rangle$ is *power divergent*, so again no threshold, with the algebraic residual $\Gamma_0 = 8\Gamma_N^3/\tilde{\Delta}^2$ (coefficient exact; App. F).

All three cases are solved from Eq. (32) (see Fig. 3). Thus a strictly vanishing residual DOS over a finite disorder window is special to $J = 1$; for $J \geq 2$ a residual Sommerfeld term $C_{\text{es}}/T \rightarrow \text{const}$ is present at any nonzero disorder *within the fixed- $\tilde{\Delta}$ Born theory used here*, though it is exponentially ($J = 2$) or algebraically ($J = 3$) small at weak disorder. That qualifier is essential rather than formal: at the exponentially (or algebraically) small scales involved, rare-event and localization physics, finite-size effects, and the full self-consistent solution of the gap equation may all modify the experimentally accessible asymptotics, and none of these is contained in Eq. (32). We stress that the threshold is *not* universal across J : using the spherical $J = 1$ average $\langle 1/\sin\theta \rangle = \pi/2$ for all J would incorrectly predict a finite threshold in every case, whereas the correct multi-Weyl measure makes $\langle 1/|f_m| \rangle$ divergent for $J \geq 2$. The threshold/exponential/algebraic *classification* is fixed by the convergence of $\int_0 d\vartheta \vartheta^{2/J-2}$ near the node and is therefore robust within the momentum-independent projected Born kernel used here—not for arbitrary finite-range Born kernels, for which the angular average $\langle 1/|f_m| \rangle$ is reweighted; only the numerical prefactors (threshold coefficient, exponential constant c , algebraic prefactor) inherit the disorder-renormalized amplitude $\tilde{\Delta}$ through Eq. (32). Table I collects the charge-dependent thermodynamic signatures together with their differing logical status.

Disorder dependence of the gap and jump. Away from T_c the anisotropic nodal state is *not* described by the isotropic Abrikosov–Gor’kov spectral formulas (there is no hard BCS gap edge to be driven downward); the full $\Delta(T; \gamma)$ would require the angle-resolved nonlinear Green’s function, which we do *not* solve; the residual-DOS results below are obtained at a *fixed* renormalized amplitude $\tilde{\Delta}$ and presented as functions of $\Gamma_N/\tilde{\Delta}$. The rigorous, disorder-independent anchors are the clean jumps (0.95 for the $J = 2$ monopole, and 1.43 for the conventional channel, see Appendix E) and the linearized digamma law $T_c(\Gamma^{\text{pb}})$ of Eq. (26). Under regular thermodynamic behavior the *absolute* heat-capacity discontinuity vanishes as $T_c^{(m)} \rightarrow 0$; we assert no monotonic trend for the *normalized* jump $\Delta C/C_n$, which would require the disorder-dressed quartic coefficient as defined in Eq. (30). Anderson protection removes *pair-breaking* from the conventional channel ($\Gamma_s^{\text{pb}} = 0$), so its clean jump structure is disorder-independent; its T_c , however, still shifts through the normal-state DOS $N(\mu; \Gamma_N)$

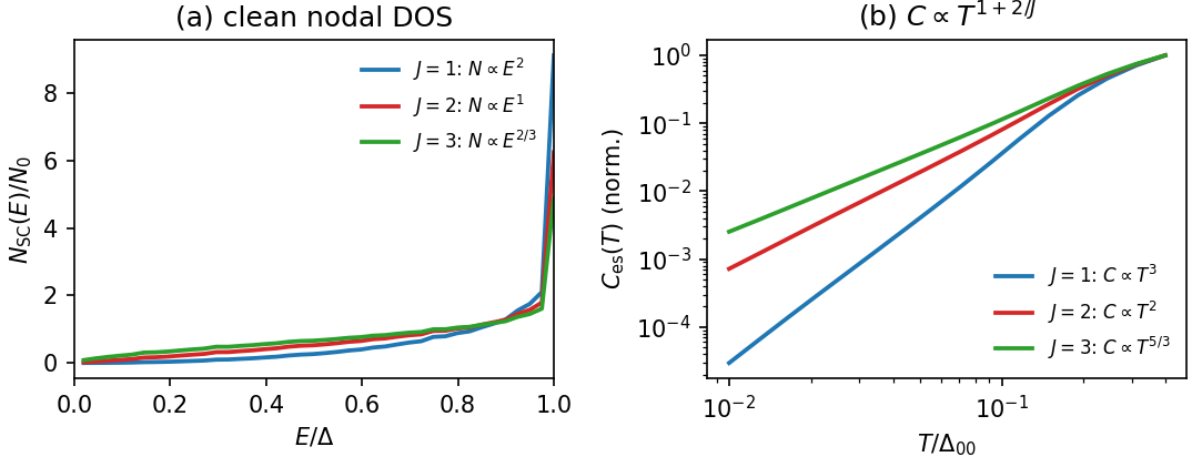


FIG. 2. Clean nodal thermodynamics of the monopole channel. (a) Quasiparticle density of states $N_{\text{SC}}(E)/N_0$; the point node gives $N_{\text{SC}} \propto E^{2/J}$ ($E^2, E^1, E^{2/3}$ for $J = 1, 2, 3$)—not a hard gap. (b) Electronic specific heat $C_{\text{es}} \propto T^{1+2/J}$ ($T^3, T^2, T^{5/3}$), a charge-sensitive low-temperature fingerprint. Curves computed from the anisotropic gap $|f_m| = \cos \psi$ with the multi-Weyl measure.

entering $T_{\text{cd}}^{(s)}$ (Sec. IX), and is not “pinned” in that sense.

VI. NODAL CHARGE OF THE PURE-STATE SOLUTIONS

The strongest topological statement is a property of the two *clean* pure-state solutions, and we present it as such. The monopole state is *nodal* (point nodes on the Fermi surface), so it has no global three-dimensional gapped BdG invariant; what is well defined is the Berry (Chern) charge on a small sphere enclosing each BdG point node, computed directly from the Hermitian clean projected BdG Hamiltonian $h(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}$. For the disorder statement, we add the observation that, within this broadening scheme, a scalar identity self-energy $\text{Re } \Sigma(0) \propto \sigma_0$ shifts μ and the gap magnitude, but does not rotate the eigenvectors carrying the Berry charge, so the pure-state nodal charges are unchanged by such disorder so long as the nodes remain isolated (the shift does not merge or gap them) and the projected two-band description remains valid, i.e. $\Gamma_N \ll \mu$.

Projected onto the chiral Fermi surface, $h = \mathbf{n}(\mathbf{k}) \cdot \boldsymbol{\tau}$ with $\mathbf{n} = (\text{Re } \Delta, -\text{Im } \Delta, \xi)$ and $\xi = v(|\mathbf{k}| - k_F)$ (we write $\mathbf{n}$ for the *BdG* vector to distinguish it from the normal-state pseudospin texture $\mathbf{d}$ of Sec. III). A BdG point node occurs where $\xi = 0$ and $\Delta(\hat{\mathbf{k}}) = 0$.

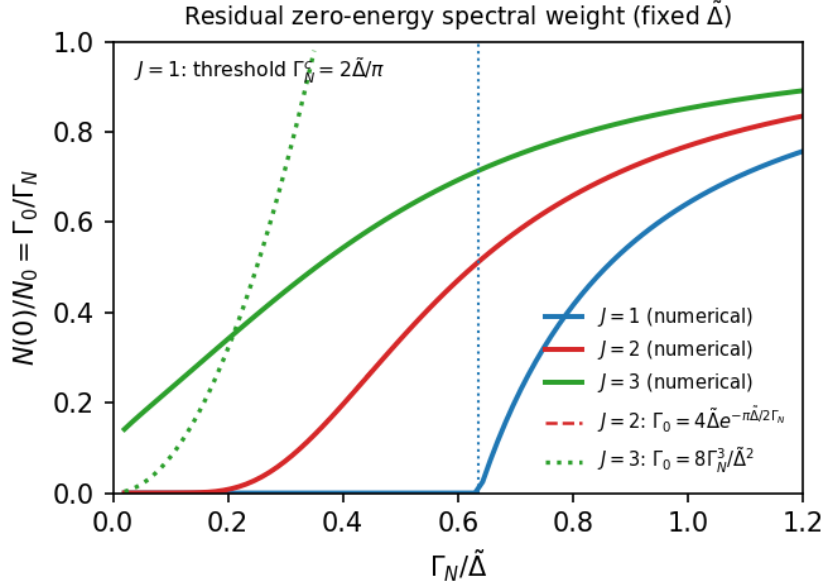


FIG. 3. Residual zero-energy DOS $N(0)/N_0 = \Gamma_0/\Gamma_N$ from the angle-resolved Born self-consistency Eq. (32), versus normal scattering rate; $\tilde{\Delta}$ is a *fixed effective spectral amplitude* (no nonlinear gap equation is solved in Appendix F). The point-node threshold $\Gamma_N^c = \frac{2\tilde{\Delta}}{\pi}$ exists *only* for $J = 1$ (dotted line; $N(0) = 0$ strictly below). For $J = 2$ the onset is exponential, $\Gamma_0 = 4\tilde{\Delta}e^{-\pi\tilde{\Delta}/(2\Gamma_N)}$, and for $J = 3$ algebraic, $\Gamma_0 = 8\Gamma_N^3/\tilde{\Delta}^2$ (dashed and dotted overlays; exact prefactors derived in Appendix F); both have *no* threshold. A residual Sommerfeld term therefore accompanies any disorder for $J \geq 2$ *within this fixed- $\tilde{\Delta}$ Born theory* (see main text for the scope of that qualifier). Curves are the full numerical solutions of Eq. (32); the analytic forms are shown only as asymptotic checks.

The conventional gap $\Delta = \Delta_s$ never vanishes on the Fermi surface, so the spectrum is fully gapped and there are no BdG nodes (trivial nodal content, which we label $\mathcal{C}_{\text{BdG}} = 0$). The monopole gap $f_m = \cos \psi e^{iJ\phi} \propto (k_x + ik_y)^J$ [Eq. (12)] vanishes at the two Fermi-surface poles, where the BdG spectrum hosts a pair of point nodes. The Berry charge on a small sphere enclosing each node is $\pm J$, which we now establish by a local-degree argument.

Consider the north node at $\mathbf{k}_0 = (0, 0, k_F)$ and expand in the local displacement $\delta\mathbf{k} = \mathbf{k} - \mathbf{k}_0$. Writing the clean projected BdG Hamiltonian as $h = \hat{\mathbf{n}} \cdot \boldsymbol{\tau}$ with $\mathbf{n} = (\text{Re } \Delta, -\text{Im } \Delta, \xi)$, the two transverse components inherit the gap winding while the longitudinal component is

TABLE I. Charge-dependent summary. The columns have *different* logical status. $\Delta C/C_n$, $N_{\text{SC}}(E)$ and $C_{\text{es}}(T)$ are clean projected-Hamiltonian results (assumption-free). η_m is the exact eigenvalue of the *assumed* valley-diagonal, intra-pocket momentum-independent projected impurity kernel Eq. (19). The residual-DOS onset is a zero-energy Born asymptotic at fixed $\tilde{\Delta}$ on the reflection-symmetric branch (Appendix F). Order of limits: for $J \geq 2$ the residual DOS is nonzero at any disorder within the fixed- $\tilde{\Delta}$ Born theory, so at fixed $\Gamma_N \neq 0$ the Born-theory $T \rightarrow 0$ law is $C/T \rightarrow \text{const}$; the clean power law $C \propto T^{1+2/J}$ is observable only in the intermediate window above the (exponentially or algebraically small) residual scale.

J	projected kernel	clean projected Hamiltonian			Born, fixed $\tilde{\Delta}$
	$\eta_m = \frac{1}{J+2}$	$\Delta C/C_n$	$N_{\text{SC}}(E)$	$C_{\text{es}}(T)$	residual-DOS onset
1	1/3	1.19	E^2	T^3	threshold $\Gamma_N^c = \frac{2\tilde{\Delta}}{\pi}$
2	1/4	0.95	E^1	T^2	none; $\Gamma_0 = 4\tilde{\Delta}e^{-\pi\tilde{\Delta}/(2\Gamma_N)}$
3	1/5	0.79	$E^{2/3}$	$T^{5/3}$	none; $\Gamma_0 = 8\Gamma_N^3/\tilde{\Delta}^2$

the normal dispersion,

$$n_x + i n_y \propto (\delta k_x + i \delta k_y)^J, \quad n_z = \xi \simeq v_{\parallel} \delta k_{\parallel}, \quad (33)$$

so that the node is a multi-Weyl point of azimuthal degree J . Parameterize a small enclosing sphere of radius R by $\delta \mathbf{k} = R(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Along the equator the transverse part gives $\arg(n_x + i n_y) = J\phi$, i.e. the image winds J times as ϕ runs once around; meanwhile $n_z \propto \cos \theta$ changes sign exactly once between the north ($\theta = 0$) and south ($\theta = \pi$) poles of the enclosing sphere, so the image of $\hat{\mathbf{n}}$ sweeps from one pole of the target sphere to the other. The map $\hat{\mathbf{n}} : S^2 \rightarrow S^2$ therefore has degree J (up to orientation) and the Chern number is $\mathcal{C} = +J$ at the north node, with the orientation of the enclosing sphere fixed by the outward normal in $\delta \mathbf{k}$.

We fix the convention by working in the *same* global Cartesian frame at both poles, so that the transverse coordinates—and hence the winding $\arg(n_x + i n_y) = +J\phi$ —are unchanged. The only difference is then the sign of the radial dispersion, $\xi \simeq \pm v_{\parallel} \delta k_z$ at the north/south pole, since the outward Fermi-surface normal is $+\hat{z}$ at one and $-\hat{z}$ at the other. Reversing n_z at fixed transverse winding composes $\hat{\mathbf{n}}$ with a reflection, so the degree changes sign and $\mathcal{C}_{\text{north}} = -\mathcal{C}_{\text{south}} = J$, consistent with vanishing total Berry flux over the closed Fermi surface.

The J -fold covering thus resides in the *image* of the map, not in the enclosing surface, which is an ordinary sphere in physical momentum traversed once; no rescaled coordinates and no branch or covering ambiguity are involved. The two competing *pure-state* solutions therefore have distinct nodal topology: the monopole solution carries BdG nodal charges $\pm J$, while the conventional solution is fully gapped (Fig. 4),

$$|\mathcal{C}_{\text{BdG}}| = \begin{cases} J \text{ per node,} & \text{pure monopole solution (point-nodal),} \\ 0, & \text{pure conventional solution (gapped),} \end{cases} \quad (34)$$

where $\mathcal{C}_{\text{BdG}}$ is the Berry charge of *one* node of the clean projected 2×2 nodal Hamiltonian. We stress the scope. First, this is a property of the two pure states, computed from the clean projected nodal Hamiltonian; the topological-Hamiltonian construction above shows only that a static scalar $\text{Re } \Sigma(0) \propto \sigma_0$ does not change the local eigenvector map while the enclosing surface remains gapped. Second, and importantly, we do *not* claim a sharp “jump” of a bulk invariant along the physical disorder path: at a mere crossing of linearized instabilities the ordered-state gap need not change discontinuously, and whether the nodes reconstruct, split, or gap—and in what order relative to the linearized crossing—is part of the unresolved nonlinear phase diagram (Sec. VIII), in particular whether an $s + m$ coexistence window intervenes. What is rigorous is the contrast between the two pure-state nodal structures ($\pm J$ vs. gapped); the topology along the actual ordered-state boundary depends on that nonlinear analysis. This is the disorder-driven analog of the clean topological-to-trivial pairing changes of our previous works Refs. [22, 24].

VII. BEYOND INTRA-NODE SCALAR DISORDER: SCOPE

The disorder calculation treated in this paper concerns *intra-node scalar* disorder within the projected kernel of Eq. (19) (Table II), for which the projected kernel gives $\eta_m = 1/(J+2)$ and, under the assumption of Sec. II, $\eta_s = 1$. Two other classes matter physically and we delimit them rather than model them.

Inter-node ($2\mathbf{Q}$) scattering. Its strength relative to intra-node scattering is set parametrically by the Fourier content of the impurity potential,

$$\frac{\Gamma_{2\mathbf{Q}}}{\Gamma_0} \sim \frac{W(2\mathbf{Q})}{W(0)}, \quad (35)$$

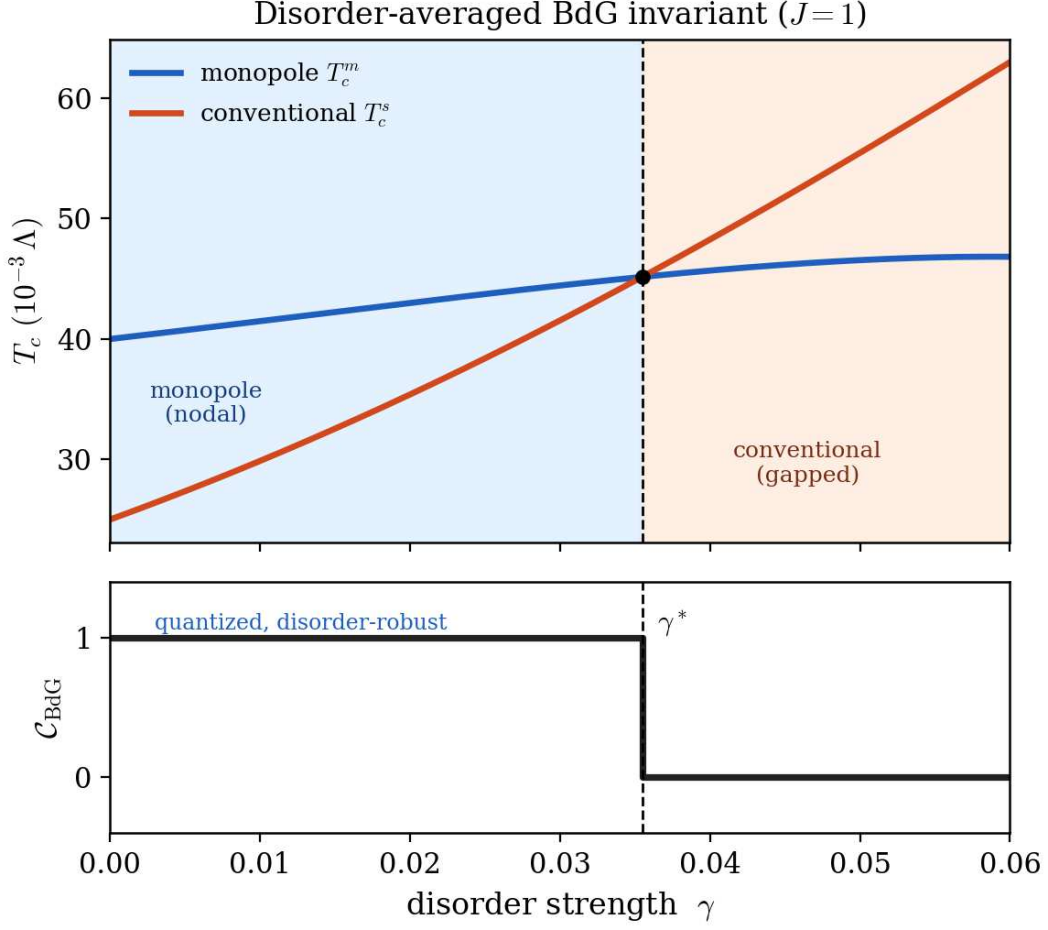


FIG. 4. The two *pure-state* gap structures, shown side by side with *no disorder axis* so that no jump is implied. (a) The monopole solution has point nodes on the Fermi surface, each carrying BdG Berry charge $\mathcal{C}_{\text{BdG}} = \pm J$ (Sec. VI). (b) The conventional solution is fully gapped, $\mathcal{C}_{\text{BdG}} = 0$. This is a comparison of two clean solutions; the topology along any physical disorder path depends on the unresolved nonlinear phase diagram (Sec. VIII).

which is the same ratio that defines the smooth-disorder condition Eq. (4): the regime studied in this paper is $W(2\mathbf{Q})/W(0) \ll 1$, whereas short-ranged (in the limit, pointlike) disorder has $W(2\mathbf{Q}) \simeq W(0)$ and mixes the valleys at leading order. We do *not* compute the inter-node channel eigenvalues: that requires diagonalizing the full valley-space impurity ladder with the anisotropic measure, including the possibility that inter-node scattering *also* pair-breaks the conventional channel ($\eta_s < 1$). The reason is structural: for zero-center-of-mass inter-node pairing the anomalous ladder contains one impurity matrix element from *each* paired node, $K^{\text{an}}(\mathbf{k}, \mathbf{k}') \sim \overline{U_+(\mathbf{k}, \mathbf{k}')U_-^*(-\mathbf{k}, -\mathbf{k}')}$, which is not generically equal to

TABLE II. Pair-breaking rates for *intra-node scalar* disorder—the disorder class analyzed in the projected kernel Eq. (19). Rates are quoted separately rather than as a ratio (the conventional rate vanishes under the assumption). $\eta_s = 1$ is the assumption of Sec. II; $\eta_m = 1/(J + 2)$ is exact within the projected kernel. For general η_s see Eq. (28).

disorder	Θ_J preserved?	η_s	$\Gamma_s^{\text{pb}}/\Gamma_N$	$\Gamma_m^{\text{pb}}/\Gamma_N$
scalar intra-node	preserved	1 (assumed)	0	$\frac{J+1}{J+2}$

$|U_+(\mathbf{k}, \mathbf{k}')|^2$. Replacing it by the normal kernel $|O|^2$ requires the sewing identity of App. G 2 and is *not* a consequence of the disorder being scalar; a full node-resolved treatment could therefore shift *both* η_s away from unity and η_m away from $1/(J + 2)$. This is the central technical problem left open here, and we leave it for future work. Equation (28) *parameterizes the consequences* of an effective $\eta_s < 1$; it does not replace computing the inter-node vertex.

Magnetic / chirality-mixing disorder. This breaks the antiunitary Θ_J and is expected to pair-break both channels. The actual requirement is not strictly a nonmagnetic host but the inequality Eq. (28): the conventional channel must be pair-broken sufficiently more slowly than the monopole channel. Magnetic disorder is disfavored because it is expected to break both comparably, but the criterion, not the disorder class, is the operative condition.

VIII. NATURE OF THE TRANSITION: CROSSING OF PAIRING INSTABILITIES

At the crossing rate Γ_N^* the two *linearized* instabilities cross and the leading pairing symmetry of the normal state changes. A crossing of two $T_c(\Gamma_N)$ curves is well defined within the projected model, but it is *not* the ordered-state phase boundary, and we do not treat it as one. Whether the physical boundary is first order, or whether an $s + m$ coexistence or an $s + im$ time-reversal-breaking window intervenes, requires the two-component Ginzburg–Landau functional with the disorder-dressed quartic vertices *and the actual crystalline point group*, which we do not compute. A continuum (C_∞) estimate, given in App. H, yields a biquadratic ratio $R_{\text{clean}} = 2 \langle |f_m|^2 \rangle / \sqrt{\langle |f_m|^4 \rangle} = 1.83, 1.63, 1.48$ for $J = 1, 2, 3$, all exceeding unity, so that simple homogeneous coexistence is disfavored within that estimate. It does not determine the transition order: the continuum expansion drops the phase-locking term

$\propto e^{2iJ\phi}$ restored by a discrete C_n whenever $n \mid 2J$, and disorder renormalizes all quartic coefficients. We therefore leave the order undetermined and make no claims of hysteresis, latent heat, or rare-region rounding.

IX. VALIDITY OF THE PROJECTED BORN TREATMENT NEAR WEYL DISORDER-CRITICALITY

A doped Weyl semimetal undergoes a disorder-driven semimetal-to-diffusive-metal transition where the Born expansion becomes uncontrolled. It is therefore essential that the superconducting transition occurs at weaker disorder. The Born parameter is $\Gamma_N/\mu = \hbar/(2\tau\mu) = \pi\gamma N(\mu)/\mu$ (for the isotropic $J = 1$ node this equals $1/(k_F\ell)$). The monopole channel is *destroyed* when $\Gamma_m^{\text{pb}} = (1 - \eta_m)\Gamma_N$ reaches $\Gamma_c = 0.882 k_B T_{c0}^{(m)}$, i.e. at $\Gamma_N^{(m0)} = \Gamma_c/(1 - \eta_m)$. This is an *upper* scale, distinct from the earlier instability crossing $\Gamma_N^* < \Gamma_N^{(m0)}$. We do not use a rough estimate here: both scales are taken directly from the AG solution and tabulated below, giving $\Gamma_N^*/\mu = 0.089, 0.079, 0.074$ for $J = 1, 2, 3$. The entire phase diagram therefore lives in a moderately metallic regime, separated from the disorder-criticality (which requires $\Gamma_N \sim \mu$, i.e. $\mu/\Gamma_N \sim 1$, or the node physics at $\mu = 0$).

Separation of scales for $J = 1, 2, 3$. We make the separation quantitative by comparing the crossing rate Γ_N^* (of order $T_{c0}^{(m)}$) with the rate at which the broadening reaches the Fermi energy, $\Gamma_N = \mu$, beyond which the Fermi surface dissolves and the chiral-basis Born treatment loses control. We do not quote a corresponding statement in the bare disorder coordinate γ : as discussed in Sec. X, γ carries the density-of-states normalization N_J , the cutoff and the ensemble choice, and is not itself observable.

J	η_m	$\Gamma_N^*/T_{c0}^{(m)}$	$\Gamma_N^{(m0)}/T_{c0}^{(m)}$	Γ_N^*/μ
1	1/3	0.667	1.32	0.089
2	1/4	0.593	1.18	0.079
3	1/5	0.556	1.10	0.074

(the crossing temperature is $T^*/T_{c0}^{(m)} = r$ for every J in the $\eta_s = 1$ limit).

Numerical parameters and cutoff. All curves use fixed μ and the same effective-model cutoff; the crossing is set by the *pair-breaking* asymmetry ($\eta_s = 1$ vs. $\eta_m = 1/(J+2)$), which is cutoff-independent *because the cutoff-sensitive DOS feedback is omitted* (a property of the

decomposition, not of the full model); so the tabulated $\Gamma_N^*/T_{c0}^{(m)}$ and Γ_N^*/μ do not depend on the cutoff ratio. The cutoff enters only the separately reported DOS feedback $N(\mu; \Gamma_N)$, which we now treat as UV-sensitive for $J = 1$ (Sec. IX); the disorder-feedback data are computed at $\Lambda/\mu = 10$, and their $J = 1$ values carry the cutoff caveat. Fixing the carrier density rather than μ shifts $\mu(\gamma)$ and would modify the $J = 1$ numbers most; we quote the fixed- μ ensemble throughout. The instability crossing sits at $\Gamma_N^*/\mu \simeq 0.07\text{--}0.13$ for all J , and the margin *widens* with J : although the broadened DOS generates a finite residual weight at the node already at weak disorder for $J = 2, 3$ (the Bera–Sau–Roy non-Anderson rounding at $\varepsilon = 0$), this is parametrically suppressed for the pairing, which lives on the Fermi surface at $\varepsilon = \mu$, by powers of $\Gamma_N/\mu \ll 1$ (rather than being strictly immaterial). As long as $\Gamma_N \ll \mu$ the Fermi surface remains a sharp metallic sheet with well-defined chiral Bloch states, so the node rounding (a separate phenomenon at $\varepsilon = 0$) has an influence on the finite- μ Fermi-surface pairing calculation that is parametrically suppressed by Γ_N/μ , and correspondingly on the Abrikosov–Gor’kov suppression; the relevant control parameter is $\Gamma_N^*/\mu \simeq 0.07\text{--}0.13$, not the node-DOS. The separation is supported by the weak-coupling hierarchy $T_{c0}^{(m)}/\mu \ll 1$. We acknowledge how strong that hierarchy actually is: the numerical value $T_{c0}^{(m)} = 0.133\mu$ corresponds in BCS weak coupling to $\Delta_{00}/\mu = 1.764 T_{c0}^{(m)}/\mu \simeq 0.24$, i.e. a gap of order a quarter of the Fermi energy. This is *not* parametrically small, but the value is chosen so that the crossing and the destruction rate are graphically resolvable on the same axes; it should be read as an illustrative, deliberately enhanced pairing scale, not as a quantitatively weak-coupling parameter set. At $\Delta_{00}/\mu \sim 0.24$ one expects appreciable corrections from the energy dependence of the density of states, valence-band and interband pairing components, and particle–hole asymmetry, none of which we retain. The dimensionless results that do not reference μ —the kernel eigenvalues, the crossing criterion Eq. (28), and the rates in units of $T_{c0}^{(m)}$ —are unaffected by this choice; only the conversions to μ (Sec. XI) inherit it, and they would improve for a genuinely weak-coupling $T_{c0}/\mu \lesssim 0.02$.

Consequently the projected Born treatment is a leading-order treatment in a moderately metallic regime, but only marginally so: the relative nominal expansion parameter Γ_N/μ is $\simeq 7\text{--}9\%$ across $J = 1, 2, 3$ (power counting, not an uncertainty bound) (largest, $\simeq 9\%$, for $J = 1$)—comparable to the $\simeq 11\%$ clean coupling asymmetry, so the quantitative crossing values should be read with a commensurate uncertainty. We also caution that μ/Γ_N conflates the quantum and transport lifetimes and, for $J > 1$, an anisotropic velocity tensor with no

single scalar k_F ; these enter the prefactors of any material estimate (Sec. XI). Beyond-Born effects, including rare-region states that round the nominal semimetal-to-diffusive-metal transition into an avoided quantum-critical crossover [44], and quantum-interference and interaction corrections [45], are expected to be subleading at finite μ , being suppressed by the finite Fermi-surface density of states or by Γ_N/μ . We have not calculated these effects at finite doping, however, so this is a power-counting expectation rather than a demonstrated bound; a full accounting of vertex and crossed diagrams, rare regions, and fixed-density $\mu(\gamma)$ shifts is left for future work. The one quantity that probes stronger disorder is the DOS *enhancement* $N(\gamma)/N_0$ relevant to the $J = 1$ channel, which reaches $\mu/\Gamma_N \sim 3\text{--}4$ at the largest disorder shown; its magnitude is therefore broadening-scheme-level (qualitative), whereas the Abrikosov–Gor’kov suppression that drives the transition is the leading pair-breaking result.

X. PROOF-OF-CONCEPT RESULTS

We solve the Abrikosov–Gor’kov gap equation (with the DOS-feedback broadening treated as a *separate*, decoupled correction—the pair-breaking phase diagram and the DOS-feedback panel are not one joint self-consistent solution) using the *derived* FS-anisotropy values $\eta_m(J) = 1/(J + 2)$ ($\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ for $J = 1, 2, 3$). We quote energies in units of μ , with cutoff $\Lambda = 10\mu$ (consistent with the disorder-feedback data of Sec. IX) and clean $T_{c0}^{(m)} = 0.133\mu > T_{c0}^{(s)} = 0.083\mu$. *The phase-diagram crossing is computed with the DOS feedback switched off* (pair-breaking only), so $\Gamma_N^*/T_{c0}^{(m)}$ and Γ_N^*/μ are independent of the cutoff; the cutoff enters only the separately reported $N(\mu; \Gamma_N)$ feedback, which is not propagated into T_c . The complete parameter set used in every figure is: fixed μ , gap-equation cutoff $\Lambda = 10\mu$, couplings $\lambda_m = 0.225$ and $\lambda_s = 0.203$ [Eq. (15)], and DOS feedback disabled except where explicitly stated.

Charge dependence in the universal rate. In the universal variable the crossings sit at $\Gamma_N^*/T_{c0}^{(m)} = 0.667, 0.593, 0.556$ for $J = 1, 2, 3$, with the monopole channel destroyed at $\Gamma_N^{(m0)}/T_{c0}^{(m)} = 0.882/(1 - \eta_m) = 1.32, 1.18, 1.10$. Smaller η_m (larger J) reaches the crossing at a smaller rate, $\eta_m = 1/(J + 2)$ being the exact eigenvalue of the momentum-independent projected kernel Eq. (19).

We regard Fig. 5 as the primary presentation of the crossing: it uses the universal rate

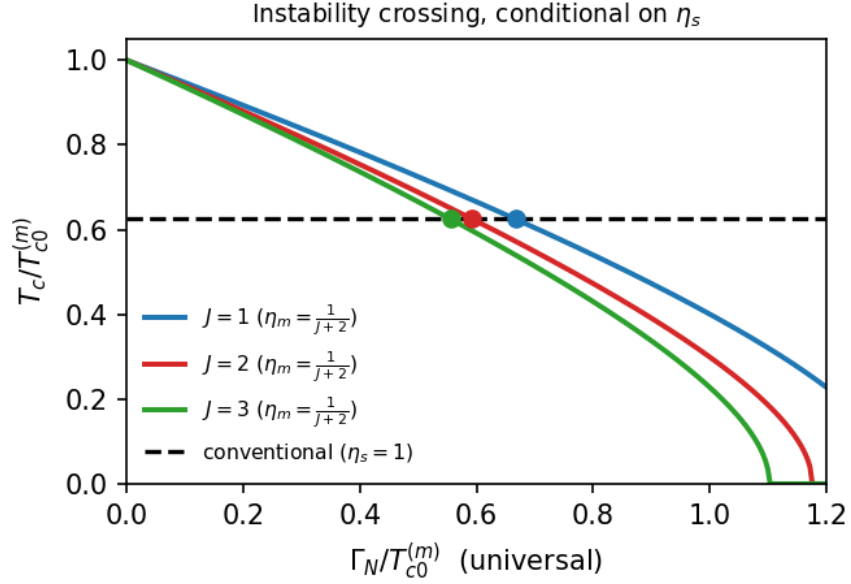


FIG. 5. Instability crossing in the *universal* variable $\Gamma_N/T_{c0}^{(m)}$ (primary presentation). Solid: monopole T_c with the derived $\eta_m = 1/(J+2)$; dashed: conventional channel at the assumed $\eta_s = 1$ (clean ratio $r = 0.625$). Markers: crossings at $\Gamma_N^*/T_{c0}^{(m)} = 0.667, 0.593, 0.556$ for $J = 1, 2, 3$. In this limit the conventional curve is flat, so the crossing temperature is $T^*/T_{c0}^{(m)} = r = 0.625$ for every J —the charge dependence lies entirely in Γ_N^* . Pair-breaking only; DOS feedback disabled.

variable, is comparable across J , and involves no DOS-feedback or cutoff input. We do not present phase diagrams in the bare coordinate γ , which is not directly observable. All disorder strengths are therefore quoted as the normal-state rate Γ_N , in units of either $T_{c0}^{(m)}$ or μ . Alongside Γ_N^* we quote the crossing temperature: in the $\eta_s = 1$ limit $T^*/T_{c0}^{(m)} = r = 0.625$ independently of J , so an experiment must resolve the change of pairing symmetry at a temperature $\sim 0.6 T_{c0}^{(m)}$, not merely a T_c suppression.

Specific heat. The thermodynamic signatures were given quantitatively in Figs. 2–3: the clean jumps are 1.19, 0.95, 0.79 (monopole) and 1.43 (conventional), and the low-energy laws are $N_{\text{SC}} \propto E^{2/J}$, $C \propto T^{1+2/J}$. A residual C/T distinguishes the nodal (monopole) from the gapped (conventional) state; for $J \geq 2$ it is nonzero at any disorder *within the fixed- $\tilde{\Delta}$ Born theory* [Eq. (32)], but the $J = 2$ scale is exponentially small, $\Gamma_0 = 4\tilde{\Delta}e^{-\pi\tilde{\Delta}/2\Gamma_N}$. Concretely, at $\Gamma_N/\tilde{\Delta} = 0.3$ one finds $\Gamma_0 \simeq 2 \times 10^{-2}\tilde{\Delta}$, already marginal, while at $\Gamma_N/\tilde{\Delta} = 0.2$ it is $\sim 2 \times 10^{-3}\tilde{\Delta}$, i.e. $\Gamma_0/k_B \simeq 20$ mK for a spectral gap $\tilde{\Delta}/k_B = 10$ K (we convert the

gap amplitude, not T_c)—far below any realistic base temperature. The $J = 2$ residual Sommerfeld term is therefore *masked by finite-temperature effects* in any practical experiment except very close to $\Gamma_N^{(m0)}$; the residual term is realistically resolvable only near the monopole-destruction rate $\Gamma_N^{(m0)}$ —the same class of observable proposed as a probe of topological quantum criticality in the clean theory [22, 23].

Robustness in both channel parameters. The relevant robustness question is not how the crossing rate moves when η_m alone is varied, but whether the crossing survives when the assumed conventional-channel protection is relaxed. That is answered within two-channel AG theory at fixed r , given the verified monotonicity, by Eq. (28) and Fig. 1: the crossing exists iff $(1 - \eta_s)/(1 - \eta_m) < r$, i.e. for $\eta_s > 1 - r(1 - \eta_m)$, which with $r = 0.625$ and $\eta_m = 1/(J + 2)$ means $\eta_s > 0.583, 0.531, 0.500$ for $J = 1, 2, 3$. Varying η_m at fixed $\eta_s = 1$ (the former one-parameter scan) is a strict subset of this two-parameter statement and is not shown separately.

XI. MODEL-LEVEL SCATTERING-RATE TARGETS

The projected model defines an *illustrative quantum-rate target*. The bare γ^* is not itself a laboratory observable, and the rate below depends on r , η_s , the assumed kernel, fixed μ , and disabled DOS feedback. With those caveats, the broadening scheme fixes the dimensionless rate at the *instability crossing* (where $T_c^{(m)} = T_c^{(s)}$, not at the later monopole-destruction rate $\Gamma_N^{(m0)}$),

$$\frac{\Gamma_N^*}{\mu} \simeq 0.089, 0.079, 0.074 \quad (J = 1, 2, 3), \quad (36)$$

so $\mu/\Gamma_N^* \simeq 11.3, 12.7, 13.5$ at the enhanced illustrative value $T_{c0}^{(m)}/\mu = 0.133$: the crossing lies in a moderately metallic regime for every charge, though $\mu/\Gamma_N^* \sim 10$ is moderate rather than parametrically large. Since Γ_N^*/μ is linear in $T_{c0}^{(m)}/\mu$, a weak-coupling choice improves this substantially, to $\mu/\Gamma_N^* \simeq 75\text{--}90$ at $T_{c0}^{(m)}/\mu = 0.02$. These figures refer to the decoupled pair-breaking calculation; the flagged $J = 2$, $\Lambda/\mu = 20$ entry of Table IV in App. I, with $\mu/\Gamma_N^* \simeq 5.9$, is excluded from such statements. (We write μ/Γ_N rather than $k_F\ell$, which is not a scalar for $J > 1$ because the velocity tensor is anisotropic; for the isotropic $J = 1$ node the identification $\mu/\Gamma_N = k_F\ell$ may be made parenthetically.) The one *conditional model-level* laboratory target (conditional on η_s , the clean ratio r , the AG form, the projected kernel, fixed μ , and disabled DOS feedback) is the *quantum scattering rate* at the crossing, Γ_N^*/μ ,

or equivalently the quantum lifetime $\tau_q = \hbar/(2\Gamma_N^*)$. For an illustrative doping range $\mu \simeq 20\text{--}50$ meV one has $\Gamma_N^*/\mu \simeq 0.089, 0.079, 0.074$ ($J = 1, 2, 3$), i.e. $\tau_q = \hbar/(2\Gamma_N^*) \simeq 0.074\text{--}0.185$ ps ($J = 1$), $0.083\text{--}0.209$ ps ($J = 2$), and $0.089\text{--}0.223$ ps ($J = 3$) [Table III], the smaller η_m at larger J reaching the crossing at a smaller rate and hence a longer lifetime. These lifetimes inherit the deliberately enhanced pairing scale $T_{c0}^{(m)}/\mu = 0.133$ discussed in Sec. IX and are therefore not a robust material target: because Γ_N^*/μ is linear in $T_{c0}^{(m)}/\mu$, a genuinely weak-coupling choice $T_{c0}^{(m)}/\mu = 0.02$ gives $\Gamma_N^*/\mu \simeq 0.012$ for $J = 2$ rather than 0.079, and required lifetimes about 6.6 times longer (lower block of Table III). What is comparatively robust is the *ordering* $\tau_q(J=1) < \tau_q(J=2) < \tau_q(J=3)$ and the ratios between charges, rather than the absolute picoseconds—and even that ordering holds only at fixed r , fixed $T_{c0}^{(m)}/\mu$, and the benchmark kernel $\eta_m = 1/(J+2)$. We deliberately stop here: converting τ_q to a residual resistivity or impurity concentration would require a consistent single-regime T -matrix and an anisotropic Boltzmann-vertex calculation for the J -fold node (there is no single scalar k_F or transport time for $J > 1$), which we do not carry out. The operative requirement is Eq. (28) rather than the disorder label—magnetic or chirality-mixing disorder requires a separate vertex calculation and is expected on symmetry grounds to reduce the conventional-channel protection—so a candidate should have disorder for which the conventional channel is pair-broken sufficiently more slowly than the monopole channel [Eq. (28)]; we make no compound-specific claim, as identifying one would need independent evidence of the node multiplicity, chemical potential, nonmagnetic disorder, and pairing channels.

XII. DISCUSSION AND CONCLUSIONS

As stated in the Introduction, this work extends the clean monopole-pairing analysis of Refs. [22–24] into the disordered regime within a Matsubara/Born and generalized Abrikosov–Gor’kov framework. A family of clean projected-Hamiltonian results—the angular moments, the specific-heat jumps, the nodal laws $N_{\text{SC}} \propto E^{2/J}$ and $C \propto T^{1+2/J}$, and the pure-state BdG nodal charges $\pm J$ —follows without any disorder input, while $\eta_m(J) = 1/(J+2)$ is the exact eigenvalue of the valley-diagonal, intra-pocket momentum-independent projected kernel Eq. (19). From these we derived the crossing of linearized instabilities: its existence, its location $\Gamma_N^*/T_{c0}^{(m)}$, and the charge trend (Sec. X), with the arbitrary- η_s boundary Eq. (28) requiring $\eta_s > 1 - r(1 - \eta_m)$, i.e. $\eta_s \gtrsim 0.5\text{--}0.58$ here. We verified the survival of

TABLE III. Conditional model-level targets at the crossing: the normal-state (quantum) scattering rate and the corresponding quantum lifetime $\tau_q = \hbar/(2\Gamma_N^*)$ for an illustrative doping range $\mu = 20$ –50 meV. We deliberately quote no residual resistivity or impurity concentration (Sec. XI). Values are conditional on $\eta_s = 1$ [Eq. (28)] and computed with the DOS feedback disabled. These numbers are *not* universal outputs of the crossing mechanism: since $\Gamma_N^*/\mu = (\Gamma_N^*/T_{c0}^{(m)}) \times (T_{c0}^{(m)}/\mu)$, they scale linearly with the assumed $T_{c0}^{(m)}/\mu$, which is stated explicitly in the first column. The upper block uses the deliberately enhanced, graphically convenient value $T_{c0}^{(m)}/\mu = 0.133$ used throughout the figures; the lower block gives a genuinely weak-coupling benchmark, $T_{c0}^{(m)}/\mu = 0.02$, for which the required lifetimes are 6.6 times longer.

$T_{c0}^{(m)}/\mu$	J	Γ_N^*/μ	τ_q (ps), $\mu = 20$ –50 meV
0.133 (enhanced)	1	0.089	0.074–0.185
0.133 (enhanced)	2	0.079	0.083–0.209
0.133 (enhanced)	3	0.074	0.089–0.223
0.02 (weak coupling)	1	0.0133	0.49–1.23
0.02 (weak coupling)	2	0.0119	0.55–1.39
0.02 (weak coupling)	3	0.0111	0.59–1.48

the crossing under the coupled Lorentzian-DOS-plus-AG scheme for $J = 2, 3$ (App. I, Table IV), and obtained the charge-dependent residual-DOS classification (threshold for $J = 1$, exponential for $J = 2$, algebraic for $J = 3$; App. F). The precise logical status of each of these results is set out below.

Our treatment is leading order in a *moderately* metallic regime, $\mu/\Gamma_N^* \simeq 11$ –14 at the enhanced illustrative $T_{c0}^{(m)}/\mu = 0.133$ (larger, and hence better controlled, for weaker coupling; the flagged $J = 2$, $\Lambda/\mu = 20$ coupled entry of Table IV (App. I) is excluded, having $\mu/\Gamma_N^* \simeq 5.9$). We do not claim a controlled error bar: $\Gamma_N/\mu \simeq 7$ –9% is power counting, and the clean coupling hierarchy that generates the competition is itself an $\simeq 11\%$ effect, so vertex, crossed-diagram and fixed-density corrections of comparable size could shift the numerical crossing rate appreciably. The *existence* of the crossing is governed by Eq. (28) rather than by these specific numbers.

Sharpening a materials estimate for a named compound, via *ab initio* band structures that fix μ, v_z, α and the node multiplicity, is the natural next step toward direct experimental

contact.

Status of the results. The logical standing of each ingredient was set out in Sec. II and is not repeated here. We recall only the two boundaries that matter for reading the numbers above. First, every disorder result is exact *given* the valley-diagonal, intra-pocket momentum-independent kernel of the small-pocket window Eq. (5), and is not established for generic forward-peaked or intervalley disorder. Second, the residual-DOS *classification* is robust within that projected Born kernel, whereas its explicit coefficients are conditional on fixed $\tilde{\Delta}$ and the reflection-symmetric branch. The principal open problem remains the full node-resolved anomalous impurity vertex.

Beyond the bulk thermodynamics of this work, the disorder robustness of the Majorana/Fermi-arc surface states—which the present disorder-averaged bulk self-energy does not address—and the non-equilibrium (Keldysh) transport signatures are natural follow-up studies.

XIII. RESEARCH FUNDING

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XIV. AUTHOR CONTRIBUTIONS STATEMENT

E.M. and R.S. contributed equally to the work conceptualization, model development, and funding acquisition. Both authors participated in text writing. Both authors reviewed the manuscript.

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Appendix A: Density of states

In this section, we present the details on the calculation of the density of states for a multi-Weyl semimetal, whose energy spectrum is $E(\mathbf{q}) = \sqrt{v_z^2 q_z^2 + \alpha^2 q_\perp^{2J}}$. Therefore, the density of states is defined

$$\begin{aligned}
N(\varepsilon) &= \int \frac{d^3 q}{(2\pi)^3} \delta \left(\varepsilon - \sqrt{v_z^2 q_z^2 + \alpha^2 q_\perp^{2J}} \right) \\
&= 2 \int \frac{d^2 q_\perp}{(2\pi)^3} \int_0^\infty dq_z \frac{\delta \left(q_z - \sqrt{\frac{\varepsilon^2 - \alpha^2 q_\perp^{2J}}{v_z^2}} \right)}{\left| \frac{\partial \varepsilon}{\partial q_z} \right|} \\
&= \frac{2|\varepsilon|}{v_z} \int_0^\infty \frac{dq_\perp q_\perp}{(2\pi)^2} \frac{1}{\sqrt{\varepsilon^2 - \alpha^2 q_\perp^{2J}}} \Theta(\varepsilon^2 - \alpha^2 q_\perp^{2J}), \tag{A1}
\end{aligned}$$

with $\Theta(x)$ the Heaviside theta function, and in the last line we integrated over the azimuthal angle in the plane.

Finally, we introduce the change of variables $x = \alpha^2 q_\perp^{2J} / \varepsilon^2$, for which $dx = 2J \alpha^2 q_\perp^{2J-1} \varepsilon^{-2} dq_\perp$ and hence $q_\perp dq_\perp = \frac{1}{2J} (\varepsilon^2 / \alpha^2)^{1/J} x^{1/J-1} dx$, such that we end up with the integral expression

$$\begin{aligned}
N(\varepsilon) &= \frac{|\varepsilon|^{2/J}}{4\pi^2 J v_z \alpha^{2/J}} \int_0^1 dx x^{1/J-1} (1-x)^{-1/2} \\
&= \frac{|\varepsilon|^{2/J}}{4\pi^2 J v_z \alpha^{2/J}} B\left(\frac{1}{J}, \frac{1}{2}\right), \tag{A2}
\end{aligned}$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function, with $\Gamma(x)$ the Gamma function. The factor $1/J$ from the Jacobian ensures the correct normalization: for $J = 1$ with $\alpha = v_z = v$ this reduces to the standard Weyl result $N(\varepsilon) = \varepsilon^2 / (2\pi^2 v^3)$.

Appendix B: Angular averages on the Fermi Surface

In this appendix, we present the mathematical details to compute angular averages over the Fermi surface, defined as

$$\langle F(\cos \psi) \rangle_{FS} \equiv \frac{\int (\cos \psi)^{2/J-1} F(\cos \psi) d\psi}{\int (\cos \psi)^{2/J-1} d\psi}, \quad (\text{B1})$$

where $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $F(\cos \psi)$ is an arbitrary function of $\cos \psi$.

In order to evaluate these integrals, it is convenient to make the change of variables $t = \cos^2 \psi$, such that $t \in [0, 1]$, and hence the expression is written in the equivalent form

$$\langle F(\cos \psi) \rangle_{FS} \equiv \frac{\int_0^1 dt t^{1/J-1} (1-t)^{-1/2} F(\sqrt{t})}{\int_0^1 dt t^{1/J-1} (1-t)^{-1/2}}. \quad (\text{B2})$$

In particular, let us consider the following explicit cases which are used in the main text

$$\begin{aligned} \langle \cos^2 \psi \rangle_{FS} &= \frac{\int (\cos \psi)^{2/J+1} d\psi}{\int (\cos \psi)^{2/J-1} d\psi} \\ &= \frac{\int_0^1 dt t^{1/J} (1-t)^{-1/2}}{\int_0^1 dt t^{1/J-1} (1-t)^{-1/2}} = \frac{B(\frac{1}{J} + 1, \frac{1}{2})}{B(\frac{1}{J}, \frac{1}{2})} = \frac{2}{J+2}, \end{aligned} \quad (\text{B3})$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function, and $\Gamma(x)$ the Gamma function. Similarly, we can obtain the following results

$$\langle \sin^2 \psi \rangle_{FS} = \langle 1 - \cos^2 \psi \rangle_{FS} = 1 - \frac{2}{J+2} = \frac{J}{J+2}, \quad (\text{B4})$$

and

$$\langle \cos^4 \psi \rangle_{FS} = \frac{\int_0^1 dt t^{1/J+1} (1-t)^{-1/2}}{\int_0^1 dt t^{1/J-1} (1-t)^{-1/2}} = \frac{B(\frac{1}{J} + 2, \frac{1}{2})}{B(\frac{1}{J}, \frac{1}{2})} = \frac{(2/J)(2/J+2)}{(2/J+1)(2/J+3)}. \quad (\text{B5})$$

The inverse-gap moment governing the residual-DOS threshold (App. F) is, in the same variables,

$$\left\langle \frac{1}{\cos \psi} \right\rangle_{FS} = \frac{\int_0^1 dt t^{1/J-3/2} (1-t)^{-1/2}}{\int_0^1 dt t^{1/J-1} (1-t)^{-1/2}} = \frac{B(\frac{1}{J} - \frac{1}{2}, \frac{1}{2})}{B(\frac{1}{J}, \frac{1}{2})} \quad (\text{B6})$$

which is finite *only* when $\frac{1}{J} - \frac{1}{2} > 0$, i.e. $J = 1$ (giving $\frac{\pi}{2}$); for $J \geq 2$ the Beta integral diverges at the node, so no threshold exists—the analytic origin of the charge-dependent onset in Table I.

Appendix C: Impurity self-energy, the two-rate structure, and the Abrikosov–Gor’kov equation

We collect here the derivation of the generalized Abrikosov–Gor’kov gap equation used in Secs. III and the main text, making the rate conventions explicit. Work in the projected conduction band with the Matsubara Nambu propagator

$$\mathcal{G}_0^{-1}(\hat{\mathbf{k}}, \xi, i\omega_n) = i\omega_n\tau_0 - \xi\tau_3 - \Delta_a f_a(\hat{\mathbf{k}})\tau_1, \quad \omega_n = (2n+1)\pi k_B T, \quad (\text{C1})$$

with τ_i Nambu matrices and the monopole phase gauged into τ_1 *in a local gauge patch* (globally τ_1, τ_2 ; see below). We work throughout in the valley-diagonal, intra-pocket momentum-independent limit defined by the two-scale window $q_{\text{intra}}^{\text{max}}\xi_{\text{dis}} \ll 1 \ll |2\mathbf{Q}|\xi_{\text{dis}}$ of Eq. (5), in which $W(\mathbf{k}_F - \mathbf{k}'_F) \simeq W(0) \equiv n_{\text{imp}}|V|^2$ across one Fermi surface while inter-node processes are suppressed. This is a projected disorder ensemble, not generic scalar point disorder; for a finite-range correlator the factor $W(\mathbf{k}_F - \mathbf{k}'_F)/W(0)$ would have to be retained inside the angular integral below and the kernel spectrum recomputed. The Born self-energy in the band basis then carries the intra-node coherence factor $O(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \langle u(\hat{\mathbf{k}})|u(\hat{\mathbf{k}}')\rangle$,

$$\Sigma(\hat{\mathbf{k}}, i\omega_n) = W(0) \int \frac{d^3 k'}{(2\pi)^3} |O(\hat{\mathbf{k}}, \hat{\mathbf{k}}')|^2 \tau_3 \mathcal{G}(\hat{\mathbf{k}}', i\omega_n) \tau_3. \quad (\text{C2})$$

In the ξ' integration the $\xi'\tau_3$ term is dropped as odd about the Fermi surface; this is the standard quasiclassical (particle–hole-symmetric) approximation and is not exact for the cutoff-sensitive power-law DOS, a caveat that matters only for the $J = 1$ UV-dominated feedback. Likewise, writing the chiral gap as $-\Delta_a f_a \tau_1$ fixes a gauge patchwise; the globally correct form is $-(\text{Re } \Delta_a f_a)\tau_1 + (\text{Im } \Delta_a f_a)\tau_2$ with a momentum-dependent connection, which does not affect the angle-averaged rates below. Writing $\int d^3 k' \rightarrow N_0 \int d\xi' \langle \cdots \rangle_{\text{FS}}$ and using $\int d\xi' \mathcal{G} = -\pi \text{sgn}(\omega_n) (i\tilde{\omega}_n\tau_0 - \tilde{\Delta}\tau_1)/\sqrt{\tilde{\omega}_n^2 + \tilde{\Delta}^2|f'|^2}$ together with $\tau_3\tau_0\tau_3 = \tau_0$, $\tau_3\tau_1\tau_3 = -\tau_1$, the dressed frequency and gap obey

$$\tilde{\omega}_n = \omega_n + \Gamma_b \left\langle |O(\hat{\mathbf{k}}, \hat{\mathbf{k}}')|^2 \frac{\tilde{\omega}_n}{\sqrt{\tilde{\omega}_n^2 + \tilde{\Delta}^2|f_a(\hat{\mathbf{k}}')|^2}} \right\rangle_{\hat{\mathbf{k}}'}, \quad (\text{C3})$$

$$\tilde{\Delta}(\hat{\mathbf{k}}) = \Delta f_a(\hat{\mathbf{k}}) + \Gamma_b \left\langle |O(\hat{\mathbf{k}}, \hat{\mathbf{k}}')|^2 \frac{\tilde{\Delta}(\hat{\mathbf{k}}')}{\sqrt{\tilde{\omega}_n^2 + \tilde{\Delta}^2|f_a(\hat{\mathbf{k}}')|^2}} \right\rangle_{\hat{\mathbf{k}}'}, \quad (\text{C4})$$

with the bare rate $\Gamma_b \equiv \pi N_0 n_{\text{imp}}|V|^2 = 2\pi\gamma N_0$ and $\gamma \equiv \frac{1}{2}n_{\text{imp}}|V|^2$. The coherence kernel $|O|^2 = \frac{1}{2}(1 + \mathbf{d} \cdot \mathbf{d}')$ acts diagonally on each pairing channel, $\langle |O(\hat{\mathbf{k}}, \hat{\mathbf{k}}')|^2 f_a(\hat{\mathbf{k}}') \rangle_{\hat{\mathbf{k}}'} = \kappa_a f_a(\hat{\mathbf{k}})$

(Sec. III), with the isotropic eigenvalue $\kappa_0 = \langle |O|^2 \rangle = \frac{1}{2}$ and the monopole eigenvalue $\kappa_m = \frac{1}{4} \langle \cos^2 \psi \rangle = \frac{1}{2(J+2)}$. Setting $\tilde{\Delta}(\hat{\mathbf{k}}) = \tilde{\Delta}_a f_a(\hat{\mathbf{k}})$ —so $\tilde{\Delta}_a$ is the coefficient of the *unnormalized* basis function f_a (the gap amplitude, as in Sec. II), not a normalized-channel value—and linearizing as $T \rightarrow T_c$ (where $\tilde{\Delta} \rightarrow 0$, so each denominator reduces to $|\tilde{\omega}_n|$ with *no* replacement of $|f_a|^2$), Eqs. (C3)–(C4) collapse to

$$\tilde{\omega}_n = \omega_n + \Gamma_N \text{sgn}(\omega_n), \quad \tilde{\Delta}_a = \Delta + \Gamma_b \kappa_a \frac{\tilde{\Delta}_a}{|\tilde{\omega}_n|}, \quad (\text{C5})$$

where the normalization $\langle |f_a|^2 \rangle$ (1 for s , $2/(J+2)$ for the monopole) enters only the clean coupling $g_a N_0 \langle |f_a|^2 \rangle$ and hence $T_{c0}^{(a)}$, not the AG denominators. These exhibit *two distinct rates*: the normal (frequency) rate $\Gamma_N \equiv \Gamma_b \kappa_0 = \Gamma_b/2 = \pi\gamma N_0 = \hbar/2\tau$ and the anomalous (gap) rate $\Gamma_b \kappa_a$. In magnetic AG [46] these coincide with opposite sign; here scalar disorder gives $\kappa_s = \kappa_0$ (conventional) and $\kappa_m < \kappa_0$ (monopole). The Cooper-channel pair susceptibility depends on $\tilde{\Delta}_a/\Delta$ and $\tilde{\omega}_n/\omega_n$ only through their *difference*, so the effective pair-breaking rate is

$$\Gamma_a^{\text{pb}} = \Gamma_N - \Gamma_b \kappa_a = \Gamma_N (1 - \eta_a), \quad \eta_a \equiv \frac{\kappa_a}{\kappa_0}. \quad (\text{C6})$$

For the conventional channel $\eta_s = 1$, so $\Gamma_s^{\text{pb}} = 0$: the frequency and gap renormalizations cancel identically and T_c is unshifted (Anderson theorem). For the monopole channel $\eta_m = 1/(J+2)$, so $\Gamma_m^{\text{pb}} = \Gamma_N (J+1)/(J+2)$. Linearizing the gap equation at T_c and summing over Matsubara frequencies gives the digamma law Eq. (26), with $T_c \rightarrow 0$ at the critical rate $\Gamma_c = \frac{\pi}{2} e^{-\gamma_E} k_B T_{c0} \simeq 0.882 k_B T_{c0} = \Delta_{00}/2$, where $\Delta_{00} = \pi e^{-\gamma_E} k_B T_{c0} = 1.764 k_B T_{c0}$ is the clean gap and γ_E Euler's constant. The rate is closed self-consistently by $\Gamma_N = \pi\gamma N(\mu; \Gamma_N)$ (Lorentzian broadening scheme), whose DOS feedback is analyzed in Sec. IX.

Appendix D: Status of the criterion and its derivation

Sufficiency follows by continuity. Necessity and uniqueness follow from a monotonicity property of the AG scaling function, which we now reduce analytically. Define the scaling function explicitly through the AG equation with all conventions displayed,

$$\ln \frac{1}{u} = \psi\left(\frac{1}{2} + \frac{R(u)}{2\pi u}\right) - \psi\left(\frac{1}{2}\right), \quad u \equiv \frac{T_c}{T_{c0}}, \quad R \equiv \frac{\Gamma^{\text{pb}}}{T_{c0}}, \quad (\text{D1})$$

so that R is strictly decreasing with $R(1) = 0$ and $R(0^+) = \frac{\pi}{2} e^{-\gamma_E} = 0.8819$. Introducing $z \equiv R(u)/(2\pi u)$ and $\Phi(z) \equiv \psi(\frac{1}{2} + z) - \psi(\frac{1}{2})$, Eq. (D1) reads $\Phi(z) = -\ln u$, so $z(u)$ is

strictly decreasing, and

$$Q(u) \equiv \frac{R(u)}{u} = 2\pi z(u), \quad L(u) \equiv \frac{d \ln Q}{d \ln u} = -\frac{1}{z \Phi'(z)} = -\frac{1}{z \psi'(\frac{1}{2} + z)}. \quad (\text{D2})$$

Because z decreases with u , L is strictly decreasing in u if and only if $h(x) \equiv (x - \frac{1}{2}) \psi'(x)$ is strictly increasing for $x = \frac{1}{2} + z > \frac{1}{2}$, i.e. iff (with ψ' and ψ'' the trigamma and tetragamma functions)

$$h'(x) = \psi'(x) + (x - \frac{1}{2})\psi''(x) = \sum_{n=0}^{\infty} \frac{n+1-x}{(n+x)^3} > 0. \quad (\text{D3})$$

For $\frac{1}{2} < x \leq 1$ every term of the series is non-negative, so Eq. (D3) holds *a fortiori*. For large x the Stirling expansion gives $h'(x) = 1/(6x^3) + O(x^{-4}) > 0$. In the intermediate range we verify Eq. (D3) numerically ($h' = 0.443, 3.88 \times 10^{-2}, 1.77 \times 10^{-3}, 1.37 \times 10^{-6}$ at $x = 1, 2, 5, 50$; positive on a 2×10^5 -point scan of $x \in [\frac{1}{2} + 10^{-4}, 500]$, with $h' \sim 1/(6x^3) > 0$ controlling $x > 500$), but we have *not* found a closed-form proof of positivity for all $x > 1$. Accordingly we state Eq. (28) as *necessary and sufficient given the monotonicity of the AG scaling function* established analytically at both ends of the interval and verified numerically in between—not as a fully proved “if and only if.” Granting that monotonicity, $\Psi(T) \equiv \Gamma_N^{(m)}(T)/\Gamma_N^{(s)}(T)$ obeys

$$\Gamma_N^{(a)}(T) = \frac{T}{1 - \eta_a} Q\left(\frac{T}{T_{c0}^{(a)}}\right), \quad \Psi = \frac{1 - \eta_s}{1 - \eta_m} \frac{Q(r u_s)}{Q(u_s)}, \quad \frac{d \ln \Psi}{d \ln u_s} = L(r u_s) - L(u_s) > 0, \quad (\text{D4})$$

with $u_s = T/T_{c0}^{(s)} \in (0, 1)$ and $r u_s = T/T_{c0}^{(m)} < u_s$ (so both arguments of Q lie in its domain $(0, 1)$). Hence Ψ increases strictly from $[(1 - \eta_s)/(1 - \eta_m)]/r$ at $T \rightarrow 0$ to ∞ at $T \rightarrow T_{c0}^{(s)}$, and $\Psi = 1$ has exactly one root—no crossing and re-crossing—precisely when Eq. (28) holds. See Fig. 1.

Appendix E: Specific-heat jump of the nodal monopole channel

Near T_c the clean weak-coupling free-energy density of a single channel with uniform amplitude Δ_a and form factor f_a is $\delta F = a_a(T)|\Delta_a|^2 + \frac{1}{2}b_a|\Delta_a|^4$, with

$$a_a(T) = N_0 \langle |f_a|^2 \rangle \ln \frac{T}{T_c^{(a)}} \simeq N_0 \langle |f_a|^2 \rangle \frac{T - T_c}{T_c}, \quad b_a = N_0 \langle |f_a|^4 \rangle \frac{7\zeta(3)}{8\pi^2 (k_B T_c)^2}. \quad (\text{E1})$$

Minimizing gives $|\Delta_a|^2 = -a_a/b_a$ and $\delta F_{\min} = -a_a^2/2b_a$ below T_c , so the specific-heat jump $\Delta C = -T d^2\delta F_{\min}/dT^2|_{T_c} = T_c [a'_a(T_c)]^2/b_a$ is

$$\Delta C = \frac{8\pi^2 k_B^2 N_0}{7\zeta(3)} T_c \frac{\langle |f_a|^2 \rangle^2}{\langle |f_a|^4 \rangle}. \quad (\text{E2})$$

Dividing by the Sommerfeld normal value $C_n = \frac{2\pi^2}{3} N_0 k_B^2 T_c$,

$$\left. \frac{\Delta C}{C_n} \right|_{T_c} = \frac{12}{7\zeta(3)} \frac{\langle |f_a|^2 \rangle^2}{\langle |f_a|^4 \rangle}. \quad (\text{E3})$$

For the gapped conventional channel $f_s = 1$ this is the BCS value $12/[7\zeta(3)] = 1.43$. For the point-nodal monopole gap $f_m = \cos \psi e^{iJ\phi}$ the Fermi-surface averages are $\langle \cos^2 \psi \rangle = 2/(J+2)$ [Eq. (24)] and $\langle \cos^4 \psi \rangle = (2/J)(2/J+2)/[(2/J+1)(2/J+3)]$, so $\langle \cos^2 \psi \rangle^2 / \langle \cos^4 \psi \rangle = \frac{5}{6}, \frac{2}{3}, \frac{11}{20}$ and

$$\frac{\Delta C}{C_n} = 1.19, 0.95, 0.79 \quad (J = 1, 2, 3), \quad (\text{E4})$$

the more anisotropic gap at higher charge giving a smaller jump. These are the clean ($\gamma \rightarrow 0$) values. On general grounds the *absolute* discontinuity (not the normalized ratio $\Delta C/C_n$) vanishes as $T_c^{(m)} \rightarrow 0$ at $\Gamma_N^{(m0)}$; the *finite-disorder* jump requires the disorder-dressed anisotropic Ginzburg–Landau quartic, which we do not compute; we therefore give no finite- γ jump curve.

Appendix F: Angle-resolved zero-energy equations and the scalar reduction

Here we derive the residual-DOS equation Eq. (32) of the main text from the *full angle-resolved* self-energy. We show that the zero-energy frequency self-energy is angle-independent on the reflection-symmetric branch (with no odd solution bifurcating in the range used), and we display the closed nonlinear system that fixes the renormalized amplitude. Real-frequency continuation ($i\omega_n \rightarrow E + i0^+$) of the Born equations Eqs. (C3)–(C4) gives, at general angle,

$$\tilde{\omega}(\hat{\mathbf{k}}, E) = E + \Gamma_b \left\langle |O(\hat{\mathbf{k}}, \hat{\mathbf{k}}')|^2 \frac{\tilde{\omega}(\hat{\mathbf{k}}', E)}{D(\hat{\mathbf{k}}', E)} \right\rangle_{\hat{\mathbf{k}}'}, \quad (\text{F1})$$

$$\tilde{\Delta}(\hat{\mathbf{k}}, E) = \Delta f_m(\hat{\mathbf{k}}) + \Gamma_b \left\langle C(\hat{\mathbf{k}}, \hat{\mathbf{k}}') \frac{\tilde{\Delta}(\hat{\mathbf{k}}', E)}{D(\hat{\mathbf{k}}', E)} \right\rangle_{\hat{\mathbf{k}}'}, \quad (\text{F2})$$

with $D = \sqrt{\tilde{\Delta}^2 - \tilde{\omega}^2}$ (or its analytic continuation) and $N(E)/N_0 = \text{Re}\langle \tilde{\omega}/D \rangle$. The azimuthal average of the coherence factor is $\langle |O|^2 \rangle_{\phi'} = \frac{1}{2}(1 + \sin \psi \sin \psi')$: the transverse,

J -fold-winding part $\propto \cos \psi \cos \psi' \cos[J(\phi - \phi')]$ integrates to zero over ϕ' . Consider the zero-energy limit $E \rightarrow 0$, where $\tilde{\omega}(\hat{\mathbf{k}}, 0) = i\Gamma_0(\hat{\mathbf{k}})$ and, in the rank-one sector, $\tilde{\Delta}(\hat{\mathbf{k}}, 0) = \tilde{\Delta} \cos \psi$: we restrict to the rank-one angular structure and parameterize its spectral amplitude by a fixed $\tilde{\Delta}$; no self-consistent anomalous equation is solved. The frequency equation becomes

$$\Gamma_0(\psi) = \Gamma_b \left\langle \frac{1}{2} (1 + \sin \psi \sin \psi') \frac{\Gamma_0(\psi')}{\sqrt{\Gamma_0(\psi')^2 + \tilde{\Delta}^2 \cos^2 \psi'}} \right\rangle_{\psi'}. \quad (\text{F3})$$

Angle independence of Γ_0 without assuming it. Rather than assume Γ_0 is even, decompose it in the rank-two $m = 0$ sector spanned by the azimuthally averaged kernel, $\Gamma_0(\psi) = A + B \sin \psi$. Writing $\mathcal{I}_0 = \langle \Gamma_0/D \rangle$ and $\mathcal{I}_1 = \langle \sin \psi' \Gamma_0/D \rangle$ with $D(\psi') = \sqrt{\Gamma_0(\psi')^2 + \tilde{\Delta}^2 \cos^2 \psi'}$, the frequency equation gives the closed pair

$$A = \Gamma_N \mathcal{I}_0, \quad B = \Gamma_N \mathcal{I}_1. \quad (\text{F4})$$

Now $\mathcal{I}_1$ is an average of $\sin \psi'$ against $\Gamma_0(\psi')/D(\psi')$ over the measure $(\cos \psi')^{2/J-1}$, which is even in ψ' . For the reflection-symmetric branch $B = 0$ the integrand is even, so $\mathcal{I}_1 = 0$ and $B = 0$ is *self-consistent*: $\Gamma_0 = A$ is exactly angle-independent. Linearizing Eq. (F4) about this branch gives the odd-channel response explicitly. With $x(\psi') = A + B \sin \psi'$ and $D = \sqrt{x^2 + \tilde{\Delta}^2 \cos^2 \psi'}$, one has $\partial_B(x/D) = \sin \psi' \partial_x(x/D) = \sin \psi' \tilde{\Delta}^2 \cos^2 \psi' / D^3$, so

$$\delta B = \Gamma_N \chi_1 \delta B, \quad \chi_1 = \left\langle \frac{\sin^2 \psi' \tilde{\Delta}^2 \cos^2 \psi'}{D_0^3} \right\rangle, \quad D_0 = \sqrt{A^2 + \tilde{\Delta}^2 \cos^2 \psi'}, \quad (\text{F5})$$

which is dimensionally 1/energy, so $\Gamma_N \chi_1$ is dimensionless. (The A equation receives no linear-order shift from δB , since its integrand ∂_B is odd and the measure even, so the two channels decouple at this order.) Evaluating Eq. (F5) on the symmetric solution $A(\Gamma_N)$, we find $\Gamma_N \chi_1 < 1$ throughout, with the maximum $\Gamma_N \chi_1 \simeq 0.75$ at $J = 2$, $\Gamma_N/\tilde{\Delta} = 0.2$ (values 0.16–0.39 for $J = 1$, 0.09–0.75 for $J = 2$, 0.07–0.49 for $J = 3$ over $\Gamma_N/\tilde{\Delta} \in [0.2, 1.2]$, corresponding to $A/\tilde{\Delta} \in [0, 1.05]$). The maximum is 0.75, below the bifurcation threshold 1, so *no continuous odd bifurcation occurs in the parameter range studied*. This establishes local stability of the reflection-symmetric branch; it does not exclude a disconnected odd solution or a discontinuous change of the self-energy outside the iteration basin.

Scope: disorder model. Equation (32) is derived with the same valley-diagonal, intra-pocket momentum-independent kernel used throughout, i.e. within the two-scale window Eq. (5). It is therefore part of the same physical disorder model as the pair-breaking

analysis, and not an independent result. Should the intra-pocket kernel acquire a genuine forward-scattering factor $W(\mathbf{k}_F - \mathbf{k}'_F)/W(0)$, the angular average $\langle 1/|f_m| \rangle$ entering Eq. (32) is reweighted and the onset classification must be rechecked; we have not done so.

Scope: fixed $\tilde{\Delta}$. Everything in this appendix, and Fig. 3, concerns the *first* equation only—the zero-energy frequency self-consistency at a *fixed* renormalized amplitude $\tilde{\Delta}$. We plot $N(0)/N_0$ against $\Gamma_N/\tilde{\Delta}$, so no gap equation is needed and none is solved. Obtaining $\tilde{\Delta}$ itself would require the full finite-Matsubara system for $\tilde{\omega}_n$ and $\tilde{\Delta}_n$ together with the order-parameter equation; that system is *not* determined by the zero-energy equations alone and we do not implement it here. Including it would reparametrize the abscissa of Fig. 3 but not alter the classification below, which depends only on the convergence of $\langle 1/|f_m| \rangle$.

Asymptotics with complete constants. For $J = 2$ the normalized measure is uniform, $d\psi/\pi$, and the average is a complete elliptic integral,

$$\left\langle \frac{1}{\sqrt{\tilde{\Delta}^2 \cos^2 \psi + \Gamma_0^2}} \right\rangle = \frac{2}{\pi \sqrt{\tilde{\Delta}^2 + \Gamma_0^2}} K(k), \quad k^2 = \frac{\tilde{\Delta}^2}{\tilde{\Delta}^2 + \Gamma_0^2}, \quad (\text{F6})$$

and $K(k) \rightarrow \ln(4/k')$ with $k' = \Gamma_0/\sqrt{\tilde{\Delta}^2 + \Gamma_0^2}$ gives $\langle \cdots \rangle \simeq (2/\pi\tilde{\Delta}) \ln(4\tilde{\Delta}/\Gamma_0)$; the self-consistency $1 = \Gamma_N \langle \cdots \rangle$ then yields the *complete* asymptotic

$$\Gamma_0 = 4\tilde{\Delta} \exp\left[-\frac{\pi\tilde{\Delta}}{2\Gamma_N}\right] \quad (J = 2). \quad (\text{F7})$$

For $J = 3$ the near-node region $t = \cos^2 \psi \sim (\Gamma_0/\tilde{\Delta})^2$ dominates; rescaling $t = (\Gamma_0/\tilde{\Delta})^2 u$ gives $\langle \cdots \rangle \rightarrow (2/\tilde{\Delta})(\tilde{\Delta}/\Gamma_0)^{1/3}$, the coefficient being a Beta-function ratio that we evaluate in closed form. Explicitly, with $t = \cos^2 \psi$ the normalized measure is $t^{1/J-1}(1-t)^{-1/2}dt/B(\frac{1}{J}, \frac{1}{2})$. The map $t = \cos^2 \psi$ is two-to-one from $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ onto $t \in [0, 1]$, so *both* nodal poles ($\psi = \pm\frac{\pi}{2}$, i.e. $t \rightarrow 0$) are covered; the resulting factor of two appears identically in the numerator and in the normalization $B(\frac{1}{J}, \frac{1}{2})$ and therefore cancels, which is why no extra factor of two survives, and rescaling $t = (\Gamma_0/\tilde{\Delta})^2 u$ in the node-dominated region gives $\int_0^\infty u^{1/3-1}(1+u)^{-1/2}du = B(\frac{1}{3}, \frac{1}{6})$. Hence the coefficient is

$$\frac{B(\frac{1}{3}, \frac{1}{6})}{B(\frac{1}{3}, \frac{1}{2})} = \frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{1}{2})} \cdot \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{2})} = \frac{\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{2})^2} = \frac{\pi/\sin(\pi/6)}{\pi} = \frac{1}{\sin(\pi/6)} = 2, \quad (\text{F8})$$

using the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ and $\Gamma(\frac{1}{2})^2 = \pi$; whence

$$\Gamma_0 = 8 \frac{\Gamma_N^3}{\tilde{\Delta}^2} \quad (J = 3). \quad (\text{F9})$$

The $J = 1$ threshold is $\Gamma_N^c = 2\tilde{\Delta}/\pi$ from the finite $\langle 1/\cos\psi \rangle = \pi/2$. Note the frequency equation carries the *normal* rate Γ_N on its right-hand side by construction; the anomalous rate would enter only through the effective spectral amplitude $\tilde{\Delta}$, which we hold *fixed*: a self-consistent $\tilde{\Delta}$ would reparameterize the abscissa $\Gamma_N/\tilde{\Delta}$ but is not computed here.

Appendix G: A band-sewing construction of the conventional channel and a sufficient condition for Anderson protection

In the main text, the conventional channel is introduced at the projected conduction-band level through the constant form factor

$$f_s(\mathbf{k}) = 1. \quad (\text{G1})$$

In a multivalley system, however, the constancy of the projected form factor is not by itself sufficient to establish Anderson protection. One must also verify that the impurity vertices appearing in the normal and anomalous self-energies are related in such a way that their coherence kernels coincide. In this Appendix, we give an explicit band-sewing realization of Eq. (G1) and formulate a sufficient condition under which the corresponding conventional channel has

$$\eta_s = 1, \quad \Gamma_s^{\text{pb}} = 0. \quad (\text{G2})$$

The result applies to an explicitly specified class of scalar disorder that preserves the antiunitary relation between the two states forming a Cooper pair. It should not be interpreted as a general theorem for arbitrary valley-dependent or intervalley disorder.

1. Explicit conduction spinors and the parity-dependent projection

Since the parity dependence of $M_m^{(J)}$ is one of the technical claims of this work, we record the projection explicitly. In the convention of Eq. (1) the chirality enters only the σ_z term, so

$$h_\chi(\mathbf{q}) = \mathbf{h}_\chi \cdot \boldsymbol{\sigma}, \quad \mathbf{h}_\chi = (\alpha q_\perp^J \cos J\phi, \alpha q_\perp^J \sin J\phi, \chi v_z q_z), \quad (\text{G3})$$

and the conduction eigenvector, with $\cos \Theta = \chi v_z q_z / |\mathbf{h}|$ and $\sin \Theta = \cos \psi$ on the Fermi surface, is

$$u_\chi(\mathbf{q}) = \begin{pmatrix} \cos(\Theta/2) \\ \sin(\Theta/2) e^{iJ\phi} \end{pmatrix}. \quad (\text{G4})$$

The partner state follows from $\mathbf{h}_-(-\mathbf{q})$. Under $\mathbf{q} \rightarrow -\mathbf{q}$ one has $\phi \rightarrow \phi + \pi$, hence $\cos J\phi \rightarrow (-1)^J \cos J\phi$ and $\sin J\phi \rightarrow (-1)^J \sin J\phi$, while the σ_z component becomes $-v_z(-q_z) = +v_z q_z$. Therefore

$$\mathbf{h}_-(-\mathbf{q}) = ((-1)^J \alpha q_\perp^J \cos J\phi, (-1)^J \alpha q_\perp^J \sin J\phi, v_z q_z), \quad (\text{G5})$$

so that $\mathbf{h}_-(-\mathbf{q}) = \mathbf{h}_+(\mathbf{q})$ for even J , whereas for odd J the transverse part is reversed, i.e. the azimuth is shifted by π . Writing $c = \cos(\Theta/2)$, $s = \sin(\Theta/2)$, this gives $u_-(-\mathbf{q}) = (c, s e^{iJ\phi})^\top$ for even J and $u_-(-\mathbf{q}) = (c, -s e^{iJ\phi})^\top$ for odd J . Evaluating $f_m = u_+^\dagger(\mathbf{q}) M u_-^*(-\mathbf{q})$ for the two constant candidates yields

$$\begin{aligned} J \text{ even : } & u_+^\dagger \sigma_x u_-^* = 2cs e^{-iJ\phi} = \cos \psi e^{-iJ\phi}, & u_+^\dagger (i\sigma_y) u_-^* &= 0, \\ J \text{ odd : } & u_+^\dagger (i\sigma_y) u_-^* = -2cs e^{-iJ\phi} = -\cos \psi e^{-iJ\phi}, & u_+^\dagger \sigma_x u_-^* &= 0, \end{aligned} \quad (\text{G6})$$

using $2cs = \sin \Theta = \cos \psi$. The vanishing entries are the content of the parity dependence: the complementary matrix has *no* overlap with the inter-node conduction pair, so a single momentum-independent choice cannot serve both parities. The assignment $M_m^{(J)} = i\sigma_y$ (odd J), σ_x (even J) of Eq. (9) is exactly the one dictated by the antiunitary bookkeeping of Eq. (11), since the antisymmetric $i\sigma_y$ pairs Θ_J -partners only when $\Theta_J^2 = -1$.

It is worth recording the gauge behaviour explicitly, since the projection involves a complex conjugation and is therefore easily confused with an ordinary band overlap. Under independent gauge transformations of the two conduction spinors, $u_\pm \rightarrow e^{i\theta_\pm} u_\pm$, the definition $f_m = u_+^\dagger(\mathbf{q}) M u_-^*(-\mathbf{q})$ gives

$$f_m \longrightarrow e^{-i[\theta_+(\mathbf{q}) + \theta_-(-\mathbf{q})]} f_m, \quad (\text{G7})$$

i.e. the two phases enter with the *same* sign and *add*. This is the signature of the conjugation on the partner state: an intra-node overlap $u^\dagger u'$ would instead produce the difference $\theta - \theta'$. Consequently $|f_m|$ is gauge invariant while $\arg f_m$ is not, and only gauge-invariant combinations may be quoted as physical.

Two remarks on conventions follow. First, the sense of the azimuthal winding, $e^{-iJ\phi}$ in Eq. (G6), and the overall sign are fixed by the gauge choice Eq. (G4) for the two spinors, consistently with Eq. (G7); the gauge-invariant content is $|f_m| = \cos\psi$ together with $|\text{winding}| = J$, which is what enters every result of this paper. In the main text we quote $f_m = \cos\psi e^{iJ\phi}$. The two differ by $\phi \rightarrow -\phi$, which is a reflection of the azimuthal coordinate: it maps $\mathbf{h}_+$ to its mirror image and therefore corresponds to the opposite handedness of the transverse frame, or equivalently to the opposite labelling of which node carries monopole charge $+J$. Since the sign of the winding is fixed by that labelling and not by any physical quantity, no result of this paper depends on it. Second, $|f_m| = \cos\psi$ holds for both parities, so the kernel eigenvalue $\eta_m = 1/(J+2)$ and all thermodynamic results are independent of the parity assignment; only the identification of which constant matrix realizes the channel depends on it.

2. Band-sewing realization of a constant projected gap

Let

$$|u_+(\mathbf{k})\rangle, \quad |u_-(-\mathbf{k})\rangle \quad (\text{G8})$$

denote normalized conduction-band spinors associated with the two Weyl nodes participating in the zero-center-of-mass internode Cooper pair. The spinors act in the orbital-pseudospin space of the normal-state Hamiltonian and obey

$$\langle u_\chi(\mathbf{k}) | u_\chi(\mathbf{k}) \rangle = 1, \quad \chi = \pm. \quad (\text{G9})$$

An explicit pseudospin matrix producing a constant projected gap is

$$M_s(\mathbf{k}) \equiv |u_+(\mathbf{k})\rangle \langle u_-^*(-\mathbf{k})| = u_+(\mathbf{k}) u_-^T(-\mathbf{k}). \quad (\text{G10})$$

Its conduction-band projection is identically unity:

$$\begin{aligned} f_s(\mathbf{k}) &= u_+^\dagger(\mathbf{k}) M_s(\mathbf{k}) u_-^*(-\mathbf{k}) \\ &= [u_+^\dagger(\mathbf{k}) u_+(\mathbf{k})] [u_-^T(-\mathbf{k}) u_-^*(-\mathbf{k})] \\ &= 1. \end{aligned} \quad (\text{G11})$$

Thus, Eq. (G10) gives an explicit realization of the constant form factor assumed for the conventional channel. The corresponding gap matrix in node $\otimes$ pseudospin space may be

written as

$$\widehat{\Delta}_s(\mathbf{k}) = \Delta_s \begin{pmatrix} 0 & M_s(\mathbf{k}) \\ -M_s^T(-\mathbf{k}) & 0 \end{pmatrix}_{\text{node}}, \quad (\text{G12})$$

or, equivalently,

$$\widehat{\Delta}_s(\mathbf{k}) = \Delta_s [\tau_+ \otimes M_s(\mathbf{k}) - \tau_- \otimes M_s^T(-\mathbf{k})], \quad (\text{G13})$$

where

$$\tau_{\pm} = \frac{\tau_x \pm i\tau_y}{2} \quad (\text{G14})$$

act in node space. By construction,

$$\widehat{\Delta}_s^T(-\mathbf{k}) = -\widehat{\Delta}_s(\mathbf{k}), \quad (\text{G15})$$

so the full gap obeys the fermionic antisymmetry condition. Equation (G10) is generally momentum dependent and gauge covariant. It should therefore be interpreted as an explicit band-sewing realization of the effective conventional channel, and not necessarily as the projection of a local, momentum-independent orbital interaction. A momentum-independent microscopic realization exists only when the two conduction-band spinors are connected by a momentum-independent sewing matrix, as discussed below.

3. Antiunitary sewing relation

Suppose that the two members of the Cooper pair are related by an antiunitary operator

$$\mathcal{A} = \mathcal{U}_A \mathcal{K}, \quad (\text{G16})$$

where $\mathcal{K}$ denotes complex conjugation and $\mathcal{U}_A$ acts in the complete internal Hilbert space. In general, $\mathcal{U}_A$ may contain node, orbital-pseudospin, and physical-spin matrices. We assume

$$|u_-(-\mathbf{k})\rangle = e^{i\chi(\mathbf{k})} \mathcal{A} |u_+(\mathbf{k})\rangle = e^{i\chi(\mathbf{k})} \mathcal{U}_A |u_+^*(\mathbf{k})\rangle. \quad (\text{G17})$$

The phase $\chi(\mathbf{k})$ is a gauge-dependent sewing phase. *Gauge convention.* Two equivalent bookkeepings are possible and must not be mixed. (i) In the *covariant* realization Eq. (G10), $M_s(\mathbf{k})$ absorbs the sewing phase, so $f_s \equiv 1$ identically and the residual phase resides in the gap matrix. (ii) In the *constant-matrix* realization $M_s = U_A$ with the phase kept explicit in Eq. (G17), one finds $f_s(\mathbf{k}) = e^{-i\chi(\mathbf{k})}$ up to a sign, so the form factor carries the phase. Only the gauge-invariant product of gap and vertex factors is physical, and it is the same in both.

In what follows we adopt convention (ii), in which the phase is displayed explicitly, so that its cancellation against the vertex phase is visible; in convention (i) the same cancellation is automatic because $M_s(\mathbf{k})$ is covariant. Equations (G11) and (G21) are statements in convention (i), i.e. after the sewing phase has been gauged away. Under independent Bloch-state gauge transformations,

$$|u_\chi(\mathbf{k})\rangle \longrightarrow e^{i\alpha_\chi(\mathbf{k})} |u_\chi(\mathbf{k})\rangle, \quad (\text{G18})$$

the matrix $M_s(\mathbf{k})$ in Eq. (G10) transforms covariantly, while its projected form factor Eq. (G11) remains invariant. If the sewing relation can be chosen in the form

$$u_-^*(-\mathbf{k}) = U_A^\dagger u_+(\mathbf{k}), \quad (\text{G19})$$

with a momentum-independent unitary matrix U_A , then a simpler realization of the conventional channel is

$$M_s = U_A. \quad (\text{G20})$$

Indeed,

$$u_+^\dagger(\mathbf{k}) M_s u_-^*(-\mathbf{k}) = u_+^\dagger(\mathbf{k}) u_+(\mathbf{k}) = 1. \quad (\text{G21})$$

Whether such a momentum-independent matrix exists depends on the detailed node, orbital, and spin structure of the microscopic model. For the multi-Weyl Hamiltonian of the main text, the pseudospin-only antiunitary Θ_J satisfies

$$\Theta_J^2 = \begin{cases} -1, & J \text{ odd}, \\ +1, & J \text{ even}. \end{cases} \quad (\text{G22})$$

Consequently, a single momentum-independent pseudospin matrix need not provide a fermionically antisymmetric conventional pair for both charge parities. The required antisymmetry may instead reside in node space, physical-spin space, or a combination of internal sectors. This is one reason why the full node $\otimes$ pseudospin construction is essential.

4. No-go for a momentum-independent orbital sewing matrix

For the specific two-band model of the main text we can be sharper: *no* momentum-independent pseudospin-only U_A exists, for any J . Requiring Eq. (G19) for all $\mathbf{k}$ is equivalent to $U_A h_+(\mathbf{k})^* U_A^\dagger = h_-(-\mathbf{k})$. Using $h_+(\mathbf{k})^* = \text{Re } z \sigma_x - \text{Im } z \sigma_y + k_z \sigma_z$ and $h_-(-\mathbf{k}) =$

$-(-1)^J [\text{Re } z \sigma_x + \text{Im } z \sigma_y] + k_z \sigma_z$ with $z = (k_x + ik_y)^J$, this demands, for even J , $U_A \sigma_x U_A^\dagger = -\sigma_x$ together with σ_y, σ_z left invariant; but invariance of σ_y and σ_z forces $U_A \propto \mathbb{K}$, which contradicts the sign reversal of σ_x . For odd J the same argument applies with $\sigma_x \leftrightarrow \sigma_y$. The covariant construction Eq. (G10) therefore remains available and exact, but a *local, momentum-independent* realization requires internal structure beyond the orbital pseudospin. Whether enlarging the internal space by physical spin supplies it is examined in App. G7; we find that it does not.

5. Normal and anomalous impurity vertices

Let the disorder potential in the full internal Hilbert space be denoted by

$$\hat{V}_{\mathbf{k}\mathbf{k}'}.$$
(G23)

The projected particle-line matrix element at the positive-chirality node is

$$U_+(\mathbf{k}, \mathbf{k}') = \langle u_+(\mathbf{k}) | \hat{V}_{\mathbf{k}\mathbf{k}'} | u_+(\mathbf{k}') \rangle.$$
(G24)

The corresponding matrix element for the partner state is

$$U_-(-\mathbf{k}, -\mathbf{k}') = \langle u_-(-\mathbf{k}) | \hat{V}_{-\mathbf{k}, -\mathbf{k}'} | u_-(-\mathbf{k}') \rangle.$$
(G25)

The normal coherence kernel is proportional to

$$K_N(\mathbf{k}, \mathbf{k}') = \overline{|U_+(\mathbf{k}, \mathbf{k}')|^2},$$
(G26)

where the overline denotes the disorder average. The anomalous Born self-energy contains the particle vertex at the positive node and the hole vertex at the negative node. In the Nambu convention used here, the projected anomalous kernel is proportional to

$$K_s^{\text{an}}(\mathbf{k}, \mathbf{k}') = U_+(\mathbf{k}, \mathbf{k}') U_-(-\mathbf{k}, -\mathbf{k}') \frac{f_s(\mathbf{k}')}{f_s(\mathbf{k})}.$$
(G27)

Equivalently, since the band-isotropic sewing form factor has unit magnitude, one may write

$$K_s^{\text{an}}(\mathbf{k}, \mathbf{k}') = U_+(\mathbf{k}, \mathbf{k}') U_-(-\mathbf{k}, -\mathbf{k}') f_s(\mathbf{k}') f_s^*(\mathbf{k}).$$
(G28)

The absence of a complex conjugate on U_- follows after combining the hole vertex with the reversed momentum indices and using Hermiticity:

$$U_-^*(-\mathbf{k}', -\mathbf{k}) = U_-(-\mathbf{k}, -\mathbf{k}').$$
(G29)

If the disorder preserves the pairing antiunitary,

$$\mathcal{A}\hat{V}_{\mathbf{k}\mathbf{k}'}\mathcal{A}^{-1} = \hat{V}_{-\mathbf{k},-\mathbf{k}'}, \quad (\text{G30})$$

then

$$U_-(-\mathbf{k}, -\mathbf{k}') = e^{i[\chi(\mathbf{k}')-\chi(\mathbf{k})]}U_+^*(\mathbf{k}, \mathbf{k}'). \quad (\text{G31})$$

In the constant-matrix sewing convention, $f_s(\mathbf{k}) = e^{-i\chi(\mathbf{k})}$, up to a momentum-independent phase. Consequently,

$$\begin{aligned} K_s^{\text{an}}(\mathbf{k}, \mathbf{k}') &= e^{i[\chi(\mathbf{k}')-\chi(\mathbf{k})]}|U_+(\mathbf{k}, \mathbf{k}')|^2 e^{-i[\chi(\mathbf{k}')-\chi(\mathbf{k})]} \\ &= |U_+(\mathbf{k}, \mathbf{k}')|^2 = K_N(\mathbf{k}, \mathbf{k}'). \end{aligned} \quad (\text{G32})$$

Thus the normal and anomalous impurity kernels coincide pointwise.

Equation (G32) is the pointwise coherence-kernel identity required for Anderson protection. An equivalent full-matrix sufficient condition may be written schematically as

$$\boxed{\hat{V}_{\mathbf{k}\mathbf{k}'}\hat{\Delta}_s(\mathbf{k}')\hat{V}_{-\mathbf{k},-\mathbf{k}'}^T = W_N(\mathbf{k}, \mathbf{k}')\hat{\Delta}_s(\mathbf{k})}, \quad (\text{G33})$$

where $W_N(\mathbf{k}, \mathbf{k}')$ is the same scalar weight entering the normal self-energy. Equation (G33) should be understood after projection onto the paired low-energy subspace. A related superconducting-fitness condition is

$$F_V(\mathbf{k}, \mathbf{k}') \equiv \hat{V}_{\mathbf{k}\mathbf{k}'}\hat{\Delta}_s(\mathbf{k}') - \hat{\Delta}_s(\mathbf{k})\hat{V}_{-\mathbf{k},-\mathbf{k}'}^* = 0. \quad (\text{G34})$$

The vanishing of F_V expresses compatibility between the internal structure of the impurity potential and that of the Cooper pair. It is a sufficient condition for the equality of the normal and anomalous impurity renormalizations.

6. Cancellation in the linearized gap equation

Let κ_0 denote the eigenvalue of the normal impurity kernel acting on the constant form factor, and let κ_s denote the corresponding anomalous eigenvalue. Equation (G32) implies

$$\kappa_s = \kappa_0. \quad (\text{G35})$$

Therefore,

$$\eta_s \equiv \frac{\kappa_s}{\kappa_0} = 1. \quad (\text{G36})$$

The linearized Born equations then have the same impurity rate in the frequency and gap channels:

$$\tilde{\omega}_n = \omega_n + \Gamma_N \text{sgn}(\omega_n), \quad (\text{G37})$$

$$\tilde{\Delta}_{s,n} = \Delta_s + \Gamma_N \frac{\tilde{\Delta}_{s,n}}{|\tilde{\omega}_n|}. \quad (\text{G38})$$

Solving Eq. (G38) gives

$$\frac{\tilde{\Delta}_{s,n}}{|\tilde{\omega}_n|} = \frac{\Delta_s}{|\omega_n|}. \quad (\text{G39})$$

The impurity rate consequently drops out of the linearized pairing susceptibility. At fixed normal-state density of states,

$$T_c^{(s)} = T_{c0}^{(s)}, \quad \Gamma_s^{\text{pb}} = \Gamma_N(1 - \eta_s) = 0. \quad (\text{G40})$$

This is the generalized Anderson cancellation for the conventional band-sewn channel.

7. Example: valley-even scalar disorder

A simple disorder class satisfying the required condition is a scalar potential that acts identically on the two nodes,

$$\hat{V}_0(\mathbf{r}) = V_0(\mathbf{r}) \tau_0 \otimes \sigma_0. \quad (\text{G41})$$

If the full pairing antiunitary exchanges the two paired nodes and leaves $\hat{V}_0$ invariant, then Eq. (G30) holds. The particle and partner-state matrix elements are antiunitary conjugates, and the normal and anomalous coherence kernels coincide. Under these conditions,

$$\eta_s = 1 \quad (\text{G42})$$

follows exactly within the Born and projected-band framework. This result requires the same random scalar field to act on the two paired nodes. It need not hold if the disorder contains valley-dependent components. For example, consider valley-odd disorder,

$$\hat{V}_z(\mathbf{r}) = V_z(\mathbf{r}) \tau_z \otimes \sigma_0. \quad (\text{G43})$$

A node-exchanging antiunitary generally changes the sign of $\hat{V}_z$:

$$\mathcal{A} \hat{V}_z \mathcal{A}^{-1} = -\hat{V}_z. \quad (\text{G44})$$

The two members of the Cooper pair then experience opposite random potentials. The identity in Eq. (G32) need not hold, and the conventional channel may acquire a finite pair-breaking rate,

$$\eta_s < 1, \quad \Gamma_s^{\text{pb}} > 0. \quad (\text{G45})$$

The same caution applies to intervalley disorder proportional to τ_x or τ_y , for which the complete valley-space impurity ladder must be calculated.

8. On a possible realization with explicit physical spin, and why we do not claim one

The previous subsection showed that no momentum-independent pseudospin-only sewing matrix exists for this model. It is natural to ask whether enlarging the internal space by physical spin repairs this. If physical spin is retained as an additional two-component degree of freedom, a momentum-independent candidate gap is

$$\hat{\Delta}_s = \Delta_s \tau_x \otimes \sigma_0 \otimes i s_y, \quad (\text{G46})$$

where s_i acts in physical-spin space. Since

$$\tau_x^T = \tau_x, \quad \sigma_0^T = \sigma_0, \quad (i s_y)^T = -i s_y, \quad (\text{G47})$$

the full matrix is antisymmetric, $\hat{\Delta}_s^T = -\hat{\Delta}_s$, for *both* charge parities, since the antisymmetry now resides in the spin sector rather than in the pseudospin sector. This removes the even/odd- J *antisymmetry* obstruction encountered above.

We emphasize, however, that this does *not* by itself constitute a microscopic realization, and we withdraw any such claim. The projection of Eq. (G46) onto the conduction bands is

$$f_s(\mathbf{k}) = u_+^\dagger(\mathbf{k}) \sigma_0 u_-^*(-\mathbf{k}), \quad (\text{G48})$$

which equals unity only if the *orbital* sewing relation $u_-(-\mathbf{k}) = u_+^*(\mathbf{k})$ holds. The spin-singlet factor $i s_y$ supplies fermionic antisymmetry, but it is a spectator in the orbital sector and therefore cannot establish that relation; indeed, the no-go argument of App. G 4 shows that for the two-band Hamiltonian Eq. (1) the required momentum-independent orbital sewing does *not* exist. Asserting $u_-(-\mathbf{k}) = u_+^*(\mathbf{k})$ would therefore contradict that no-go rather than evade it. A genuine local realization would require an enlarged normal-state Hamiltonian in

which a momentum-independent antiunitary $\mathcal{A}$ satisfies $\mathcal{A} \mathcal{H}(\mathbf{k}) \mathcal{A}^{-1} = \mathcal{H}(-\mathbf{k})$ and maps the positive-node conduction state onto its negative-node partner globally, including all gauge patches. We have not constructed such a model, and we do not claim one.

Accordingly, throughout this work the band-isotropic conventional channel is introduced *phenomenologically at the projected-band level*, and $\eta_s = 1$ is retained as an explicit assumption whose consequences, when relaxed, are quantified by Eq. (28). The band-sewing construction of App. G 2 remains available as an exact but momentum-dependent, gauge-covariant realization of a constant projected form factor; it should not be read as a local orbital s -wave order parameter.

9. Scope of the result

The construction above establishes two logically distinct statements. First, the band-sewing matrix

$$M_s(\mathbf{k}) = u_+(\mathbf{k})u_-^T(-\mathbf{k}) \quad (\text{G49})$$

provides an explicit gap matrix whose conduction-band projection is the constant form factor $f_s = 1$. Second, exact Anderson protection additionally requires the impurity potential to preserve the antiunitary sewing relation between the paired states. A sufficient condition is the equality of the normal and anomalous coherence kernels in Eq. (G32), or equivalently the vanishing disorder-fitness condition Eq. (G34). When this condition holds,

$$\eta_s = 1. \quad (\text{G50})$$

When it does not hold, the constancy of f_s alone does not prevent pair breaking, and the conventional channel must instead be described by an effective protection factor $\eta_s < 1$. This is the situation parameterized by the general two-channel crossing criterion in the main text.

Appendix H: Continuum Ginzburg–Landau estimate

Writing $\Delta(\hat{\mathbf{k}}) = \Delta_s f_s + \Delta_m f_m$ and expanding the free energy to quartic order in the continuum (C_∞) approximation, the quartic term is the Fermi-surface average $\mathcal{F}_4 = \frac{b}{2} \langle |\Delta(\hat{\mathbf{k}})|^4 \rangle$ with the single positive weak-coupling coefficient $b = 7\zeta(3)N_0/(8\pi^2 T_c^2)$. Because the two

channels carry different azimuthal winding, every term odd in $e^{iJ\phi}$ averages to zero and

$$\langle |\Delta|^4 \rangle = \beta_s |\Delta_s|^4 + \beta_m |\Delta_m|^4 + 4\beta_{sm} |\Delta_s|^2 |\Delta_m|^2, \quad (\text{H1})$$

with

$$\beta_s = \langle |f_s|^4 \rangle = 1, \quad \beta_m = \langle |f_m|^4 \rangle, \quad \beta_{sm} = \langle |f_s|^2 |f_m|^2 \rangle = \langle |f_m|^2 \rangle. \quad (\text{H2})$$

We emphasize the cross coefficient 4, which is the origin of the factor of two below and is easily mis-stated: expanding $|\Delta|^4 = (|\Delta_s f_s|^2 + |\Delta_m f_m|^2 + 2 \text{Re} \Delta_s^* \Delta_m f_s^* f_m)^2$, the term $2|\Delta_s f_s|^2 |\Delta_m f_m|^2$ and the surviving half of $4(\text{Re} \Delta_s^* \Delta_m f_s^* f_m)^2$ each contribute $2\beta_{sm}$. Comparing Eq. (H1) with the canonical two-component form $\frac{\tilde{\beta}_s}{2} |\Delta_s|^4 + \frac{\tilde{\beta}_m}{2} |\Delta_m|^4 + \tilde{\beta}_{sm} |\Delta_s|^2 |\Delta_m|^2$ gives $\tilde{\beta}_{s,m} = b\beta_{s,m}$ and $\tilde{\beta}_{sm} = 2b\beta_{sm}$. Homogeneous coexistence of the two condensates is a stable minimum iff $\tilde{\beta}_{sm} < \sqrt{\tilde{\beta}_s \tilde{\beta}_m}$, i.e. iff

$$R_{\text{clean}} \equiv \frac{2 \langle |f_m|^2 \rangle}{\sqrt{\langle |f_m|^4 \rangle}} < 1, \quad (\text{H3})$$

so that $R_{\text{clean}} > 1$ disfavors simple homogeneous coexistence. The ratio is invariant under rescaling $f_m \rightarrow c f_m$ and therefore independent of the normalization convention. Using the raw moments $\langle |f_m|^2 \rangle = 2/(J+2)$ and $\langle |f_m|^4 \rangle = (2/J)(2/J+2)/[(2/J+1)(2/J+3)]$ (App. B) gives

$$R_{\text{clean}} = 1.83, 1.63, 1.48 \quad (J = 1, 2, 3), \quad (\text{H4})$$

all exceeding unity. The correct reading of this is narrow: the continuum quartic overlap disfavors simple coexistence, but *no* conclusion concerning the order of the transition or the presence of intermediate phases can be drawn without the full point-group-resolved, disorder-dressed quartic functional. The continuum expansion drops the phase-locking term $\propto e^{2iJ\phi}$ that a discrete C_n restores whenever $n \mid 2J$ (e.g. C_4 at $J = 2$), a complete treatment requires the representation content of the two order parameters under the actual point group, and disorder renormalizes all quartic coefficients, which we do not compute. The order of the ordered-state transition is left open.

Appendix I: Lorentzian spectral broadening of the density of states and its feedback on the crossing

This appendix collects the disorder-induced density-of-states feedback. It is kept out of the main text because it is a phenomenological spectral convolution rather than a self-

consistent Born solution, because it is ultraviolet sensitive for $J = 1, 2$, and because none of the primary results uses it.

The scattering rate itself is fixed self-consistently within a Lorentzian spectral-broadening scheme, $\Gamma_N \equiv \hbar/2\tau = \pi\gamma N(\mu; \Gamma_N)$ with $N(\mu; \Gamma_N) = \int d\varepsilon N(\varepsilon) (\Gamma_N/\pi)/[(\mu - \varepsilon)^2 + \Gamma_N^2]$. We should be precise about what this is. Retaining only the imaginary part of the self-energy—i.e. a pure Lorentzian broadening of the density of states—is exact only if the real part $\text{Re } \Sigma$ and the associated shift of the chemical potential vanish, which is guaranteed by particle–hole symmetry. That symmetry is precisely what the asymmetric, cutoff-sensitive power-law density of states Eq. (2) violates: with $N(\varepsilon) \propto |\varepsilon|^{2/J}$ regulated at Λ , the Kramers–Kronig partner of the broadened spectrum generically produces a nonzero $\text{Re } \Sigma$ and hence a disorder-induced shift $\delta\mu$, which we do not compute. We therefore do *not* call this a self-consistent Born calculation. It should be read as a *phenomenological spectral (Lorentzian) convolution* that captures the disorder-induced filling of the pseudogap while omitting the real self-energy and the attendant chemical-potential renormalization. A consistent treatment would compute $\text{Re } \Sigma$ and $\text{Im } \Sigma$ with the same ultraviolet regularization and re-fix μ at fixed density; we have not done so, and this is one reason the feedback is kept out of the primary phase diagrams rather than propagated into them. A further restriction should be noted: for finite-range disorder the normal self-energy is generally momentum dependent, especially in the forward-scattering regime, so the scalar convolution written here is consistent only within the intra-pocket momentum-independent limit of Eq. (5). Two cautions apply to this convolution and we treat it accordingly. First, the Lorentzian (Cauchy) kernel has no finite second moment and the power law $N(\varepsilon) = N_J |\varepsilon|^{2/J}$ [Eq. (2)] is unbounded, so $N(\mu; \Gamma_N)$ *cannot* be reduced to a local moment expansion $N(\mu) + \frac{1}{2}\Gamma^2 N''(\mu) + \dots$; such an expansion is non-convergent here. We therefore regulate every broadening integral with the ultraviolet cutoff Λ of the effective model of Eq. (1) and solve the self-consistency *numerically at fixed μ* (Sec. IX). The result corrects the naive picture: $N(\mu; \Gamma_N)$ *risks* monotonically with disorder for *all* $J = 1, 2, 3$ within this cutoff-regulated continuum convolution (broadening fills in the pseudogap), not in the rise/flat/fall pattern that a curvature expansion would suggest. The strength and cutoff-dependence, however, differ sharply by J , controlled by the large-energy tail exponent $2/J - 2$ of the integrand $N(\varepsilon)/[(\mu - \varepsilon)^2 + \Gamma^2]$:

- $J = 1$ ($2/J - 2 = 0$): the integral is *ultraviolet divergent*, $N(\mu; \Gamma_N) \sim (\Gamma_N/\pi) N_1 \cdot 2\Lambda \propto \Lambda$, i.e. cutoff-dominated. This is *not* a controlled low-energy quantity—it signals the

breakdown of the unbounded continuum-DOS model—so we quote *no* numerical $J = 1$ DOS-feedback value and disable this feedback everywhere below (as for the phase diagram). The divergence is an artifact of the unbounded linear dispersion, not of the physics: on a lattice the Weyl cone terminates in a band of finite width W , the density of states turns over and vanishes at the band edges, and the Lorentzian convolution is cut off by the band rather than by an arbitrary Λ . A tight-binding regularization therefore replaces the arbitrary continuum cutoff by material-specific spectral weight distributed over a finite band: the convolution becomes a finite integral controlled by the full lattice density of states and the bandwidth, rather than by Λ . We deliberately do not reduce this to a scaling formula, since the result depends on the entire band structure and its DOS normalization, not on the nodal theory. The point is simply that the $J = 1$ pathology is a continuum artifact with a known lattice cure, and that a material specific lattice-based evaluation is the natural next step for quantitative $J = 1$ predictions.

- $J = 2$ ($2/J - 2 = -1$): *logarithmically* cutoff-divergent, $N(\mu; \Gamma_N) - N(\mu) \propto \Gamma_N \ln(\Lambda/\Gamma_N)$ —mild but cutoff-sensitive ($N/N_0 \simeq 1.16$ – 1.29 over the same Λ range).
- $J = 3$ ($2/J - 2 = -4/3 < -1$): ultraviolet *convergent*, a small cutoff-insensitive rise ($N/N_0 \simeq 1.06$ – 1.10).

We emphasize that this feedback is *not* numerically negligible even for $J = 2$: a 16–29% change in N_0 enters $T_{c0} \propto e^{-1/\lambda}$ exponentially and can shift T_{c0} substantially. We therefore do not claim it is small; we *isolate* the pair-breaking mechanism by disabling the feedback in all phase diagrams, and report the DOS calculation separately. We also do *not* claim that the feedback cancels in the clean ratio $r = T_{c0}^{(s)}/T_{c0}^{(m)}$: in weak coupling $T_{c0}^{(a)} = A\Lambda \exp[-1/(g_a N \langle |f_a|^2 \rangle)]$, so

$$\ln r(N) = \frac{1}{\lambda_m(N)} - \frac{1}{\lambda_s(N)}, \quad (\text{I1})$$

which is *exponentially sensitive* to a common change of N whenever the two channel couplings differ—a 16–29% shift in N can move r appreciably. The decomposition used here is therefore a *deliberate simplification*, not a cancellation: we hold r fixed by construction in order to isolate the pair-breaking mechanism, and we report the DOS calculation separately. If the feedback were included, r would become γ -dependent and the crossing criterion

Eq. (28) would hold locally, as an implicit condition on $r(\Gamma_N)$, rather than at the displayed constant r . For $J = 1$ we make no quantitative DOS-feedback claim at all. (The near-node non-Anderson physics at $\varepsilon \rightarrow 0$ [37, 38] is regulated here by the finite Fermi energy, $\Gamma_N \ll \mu$; Sec. IX. All results below are at fixed μ ; fixing the carrier density instead would shift $\mu(\gamma)$, a refinement we do not pursue here.)

Coupled Lorentzian-DOS-plus-AG test: does the crossing survive the DOS feedback? Because the disorder-induced change of $N(\mu; \Gamma_N)$ is not physically separable from the pair-breaking it accompanies, the decoupling used above must be tested. We therefore solve a *coupled* problem for $J = 2, 3$: the self-consistent broadening rate $\Gamma_N = \pi\gamma N(\mu; \Gamma_N)$ self-consistently, with $N(\mu; \Gamma_N)$ fed back into *both* clean scales through $T_{c0}^{(a)}(\Gamma_N) = \frac{2e^{\gamma E}}{\pi} \Lambda e^{-1/[g_a N(\mu; \Gamma_N) \langle |f_a|^2 \rangle]}$, each channel then suppressed by its own AG rate. We are explicit about the scope of the word “coupled”: the normal-state DOS and the linearized T_c equations are solved together, whereas the ordered-state gap equation, the impurity T -matrix, and a fixed-density chemical potential are *not*. Couplings are fixed by matching the clean $T_{c0}^{(m)} = 0.133\mu$ and $r = 0.625$ at each cutoff, at fixed μ .

The crossing survives in every case tested (Fig. 6). Two features of the table matter. First, in the *local* coordinate the crossing barely moves, $\Gamma_N^*/T_{c0}^{(m)}(\Gamma_N^*) = 0.51\text{--}0.56$ throughout: most of the apparent displacement in the clean-normalized column is the rise of the pairing scale itself, not a change in the pair-breaking competition. Second, and consistently with Eq. (I1), the feedback *does* change the clean ratio, r drifting from 0.625 to 0.63–0.68.

We therefore state the conclusion narrowly. *These coupled solutions verify that the crossing survives for the chosen λ_m, λ_s , at fixed μ , for $J = 2, 3$, over the cutoff ranges listed.* We do *not* claim that a common DOS factor protects the crossing in general: because $\ln r(N) = 1/\lambda_m(N) - 1/\lambda_s(N)$ [Eq. (I1)] is exponentially sensitive whenever the channel couplings differ, a sufficiently strong feedback could in principle move r across the boundary Eq. (28). Note also that the rise of both T_{c0} ’s is a property of the regulated convolution used here, not a universal consequence of disorder. The *location* is moreover cutoff sensitive for the logarithmically divergent $J = 2$ case ($0.795 \rightarrow 1.269$ as Λ/μ doubles) and mild for the convergent $J = 3$ case; this is why the cutoff-independent decoupled presentation is retained for the quantitative phase diagrams, and why no such calculation is attempted for $J = 1$.

TABLE IV. Coupled Lorentzian-DOS-plus-AG diagnostics. Γ_N^* normalized both to the *clean* $T_{c0}^{(m)}$ and to the *local* disorder-dressed scale $T_{c0}^{(m)}(\Gamma_N^*)$, which is the coordinate actually entering the AG equation; $r(\Gamma_N^*)$ is the disorder-dressed clean ratio; T^* is the crossing temperature. The crossing rate is also given in units of μ , since that is the ratio controlling the validity of the projected Born treatment. We deliberately do *not* quote a coupled monopole-destruction rate: in every case tested the formal destruction point lies at $\Gamma_N \gtrsim 0.18\mu$ —and for the largest cutoffs at $\Gamma_N \sim 8\mu$ —i.e. far outside the regime $\Gamma_N \ll \mu$ in which the calculation is meaningful, so those numbers would be extrapolations of the scheme beyond its own domain rather than predictions. [†]The $J = 2$, $\Lambda/\mu = 20$ entry has $\Gamma_N^*/\mu = 0.169$, which is no longer comfortably within $\Gamma_N/\mu \ll 1$; that parameter set should not be used to support claims of controlled metallic behavior. All crossings occur at $T_c^{(m)} > 0$.

J	Λ/μ	$\frac{\Gamma_N^*}{T_{c0}^{(m)}}$	$\frac{\Gamma_N^*}{T_{c0}^{(m)}(\Gamma_N^*)}$	N/N_0	$r(\Gamma_N^*)$	$\frac{\Gamma_N^*}{\mu}$	monopole destruction
2	10	0.795	0.558	1.087	0.649	0.106	outside controlled range
2	20	1.269	0.514	1.214	0.679	0.169 [†]	outside controlled range
3	10	0.602	0.547	1.022	0.631	0.080	outside controlled range
3	20	0.659	0.540	1.040	0.636	0.088	outside controlled range
3	40	0.740	0.533	1.060	0.642	0.098	outside controlled range

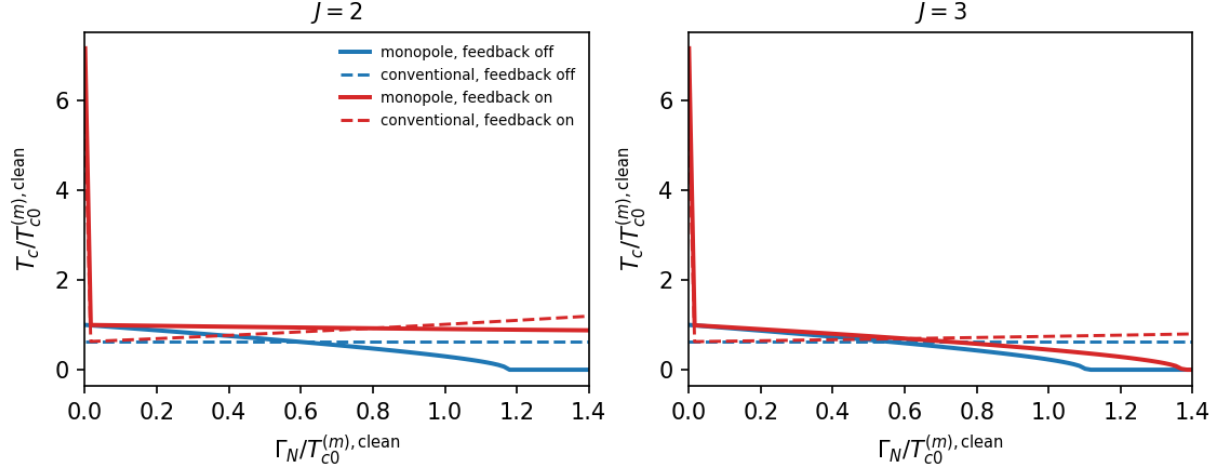


FIG. 6. Coupled Lorentzian-DOS-plus-AG instability crossing (red) compared with the decoupled treatment (blue), for $J = 2$ (left) and $J = 3$ (right) at $\Lambda/\mu = 10$. The calculation is self-consistent only within the restricted DOS-broadening scheme of Sec. IX, which retains the imaginary self-energy alone. Solid: monopole channel; dashed: conventional channel. Including the disorder-dressed $N(\mu; \Gamma_N)$ in both clean scales raises both curves and displaces the crossing to a larger rate, but the crossing persists. Temperatures and rates are normalized to the *clean* $T_{c0}^{(m)}$. Couplings are fixed by matching $T_{c0}^{(m)} = 0.133\mu$ and $r = 0.625$ in the clean limit at each cutoff, at fixed μ ; see Table IV for the diagnostics.