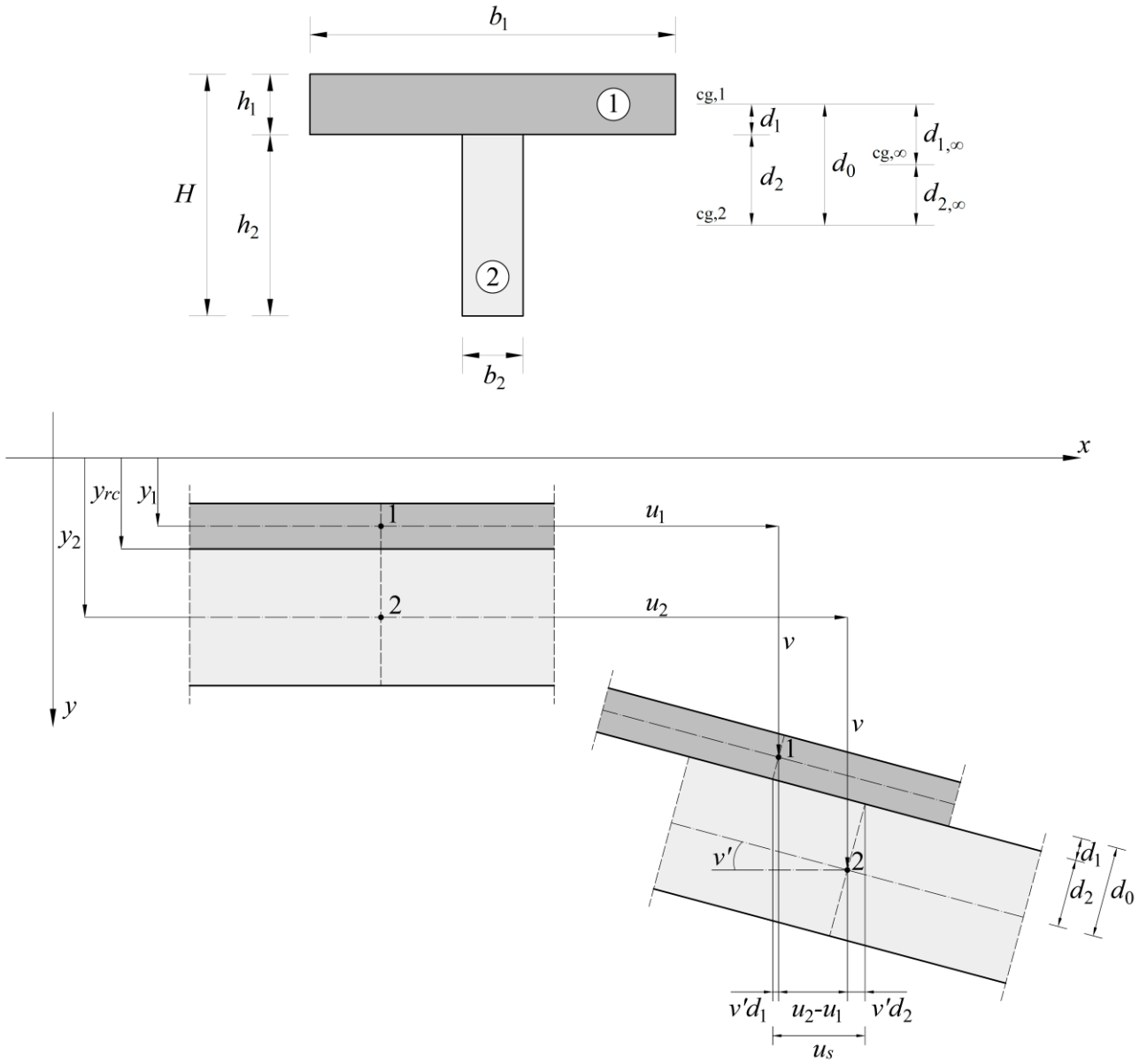
Supplementary material for

**Effects of end-slip restraints on two-layer beams with interlayer slip**

Beatrice TRAVASONI, Fabio MINGHINI, Nerio TULLINI

Department of Engineering; University of Ferrara, Italy

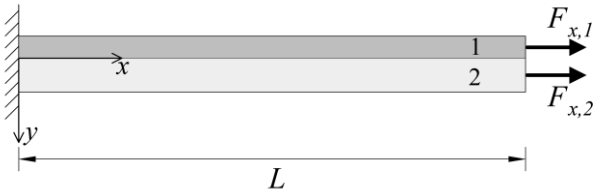
Reported in the following are 18 closed-form solutions for two-layer beams subjected to different boundary conditions and loading.




*Newmark kinematical model*

| *Parameters of the two-layer beam* | | |
|---|---|---|
| $E_iA_i$ | for $i$ = 1, 2 | Axial rigidity of the layer $i$. |
| $E_iJ_i$ | for $i$ = 1, 2 | Flexural rigidity of the layer $i$. |
| $EJ_0 = E_1J_1 + E_2J_2$ | | Flexural rigidity for non-composite beams. |
| $EJ_\infty = EJ_0 + d_0^2 \Big/ \left( \frac{1}{E_1A_1} + \frac{1}{E_2A_2} \right)$ | | Flexural rigidity for fully composite beams. |
| $\alpha^2 = k_h \left( \frac{1}{E_1A_1} + \frac{1}{E_2A_2} + \frac{d_0^2}{EJ_0} \right)$ | | Parameter proportional to the slip modulus $k_h$, characterizing the degree of composite action. |
| $\hat{\alpha} = \hat{k}_h \left( \frac{1}{E_1A_1} + \frac{1}{E_2A_2} + \frac{d_0^2}{EJ_0} \right)$ | | Parameter proportional to the end-slip stiffness $\hat{k}_h$, characterizing the degree of composite action for two-layer beam fastened at the beam ends only. |
| $\hat{\kappa} = \frac{\hat{k}_h}{k_hL}$ | | Ratio between the end-slip stiffness $\hat{k}_h$ and the resultant $k_hL$ of the distributed springs with stiffness $k_h$. |
| $d_{1,\infty} = d_0 \frac{E_2A_2}{E_1A_1 + E_2A_2}$, $d_{2,\infty} = d_0 \frac{E_1A_1}{E_1A_1 + E_2A_2}$ | | Distance of the centroid of the fully composite beam ($\alpha L = \infty$) to the centroid of the layer 1 and 2. |

## *S.1. Cantilever two-layer beam subjected to end axial forces $F_{x,1}$ and $F_{x,2}$*




*Boundary conditions*

$v(0) = u_1(0) = u_2(0) = v'(0) = 0$

$$E_1A_1u_1'(L) - \hat{k}_h u_s(L) = F_{x,1}, \qquad E_2A_2u_2'(L) + \hat{k}_h\, u_s(L) = F_{x,2}$$

$$EJ_0v''(L) + \hat{k}_h d_0 u_s(L) = 0, \qquad EJ_0v'''(L) - k_h d_0 u_s(L) = 0$$

*End displacements and rotation for fully composite beams* ($\alpha L = \infty$)

$$u_\infty(L) = \frac{(F_{x,1} + F_{x,2})L}{E_1A_1 + E_2A_2}, \qquad v_\infty(L) = \frac{C_\infty L^2}{2EJ_\infty}, \qquad \varphi_\infty(L) = \frac{C_\infty L}{EJ_\infty}$$

where $C_\infty = F_{x,1}d_{1,\infty} - F_{x,2}d_{2,\infty}$

*Constants and functions*

$$\hat{c}_1 = 1 + \alpha L\ \hat{\kappa}\ \tanh(\alpha L), \qquad f(x) = \frac{\sinh(\alpha x)}{\alpha L\ \hat{c}_1 \cosh(\alpha L)}, \qquad g(x) = f'(x)\,L = \frac{\cosh(\alpha x)}{\hat{c}_1 \cosh(\alpha L)}$$

*Displacement field*

$$u_1(x) = u_\infty(L)\frac{x}{L} + \varphi_\infty(L)\, d_{1,\infty}\left[\frac{x}{L} + \frac{f(x)}{EJ_\infty/EJ_0 - 1}\right], \quad u_2(x) = u_\infty(L)\frac{x}{L} - \varphi_\infty(L)\, d_{2,\infty}\left[\frac{x}{L} + \frac{f(x)}{EJ_\infty/EJ_0 - 1}\right]$$

$$v(x) = v_\infty(L)\left[\left(\frac{x}{L}\right)^2 - 2\,\frac{\cosh(\alpha x) - 1}{(\alpha L)^2 \hat{c}_1 \cosh(\alpha L)}\right]$$

$$u_s(x) = -\left(\frac{F_{x,1}}{E_1A_1} - \frac{F_{x,2}}{E_2A_2}\right) L\ f(x) = -\frac{C_\infty L\, d_0}{EJ_\infty - EJ_0}\ f(x) = -\frac{\alpha C_\infty}{k_h d_0}\,\frac{EJ_0}{EJ_\infty}\,\frac{\sinh(\alpha x)}{\hat{c}_1 \cosh(\alpha L)}$$

*Stress resultants*

$$N_1(x) = E_1A_1\ u_1'(x) = \frac{E_1A_1}{L}u_\infty(L) + \frac{C_\infty}{d_0}\left\{1 - \frac{EJ_0}{EJ_\infty}[1 - g(x)]\right\}$$

$$N_2(x) = E_2A_2\ u_2'(x) = \frac{E_2A_2}{L}u_\infty(L) - \frac{C_\infty}{d_0}\left\{1 - \frac{EJ_0}{EJ_\infty}[1 - g(x)]\right\}$$

$$M_i(x) = -E_iJ_i\ v''(x) = -C_\infty\,\frac{E_iJ_i}{EJ_\infty}[1 - g(x)] \qquad \text{for } i = 1, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\, d_{2,\infty} - N_1(x)\, d_{1,\infty}] = -C_\infty$

$$T_1(x) = -T_2(x) = M_1'(x) + k_h\, d_1\, u_s(x) = -k_h u_s(x)\,\frac{d_1E_2J_2 - d_2E_1J_1}{EJ_0}$$

$T(x) = T_1 + T_2 = 0$

***S.1. Cantilever two-layer beam subjected to end axial forces $F_{x,1}$ and $F_{x,2}$ (continued)***

*Cantilever two-layer beam fastened at the beam end only* ($\alpha L = 0$)

$$\hat{u}_1(x) = u_\infty(L)\frac{x}{L} + \varphi_\infty(L)\, d_{1,\infty}\, \frac{x}{L}\left[1 + \frac{1}{(1+\hat{\alpha}L)(EJ_\infty / EJ_0 - 1)}\right]$$

$$\hat{u}_2(x) = u_\infty(L)\frac{x}{L} - \varphi_\infty(L)\, d_{2,\infty}\, \frac{x}{L}\left[1 + \frac{1}{(1+\hat{\alpha}L)(EJ_\infty / EJ_0 - 1)}\right]$$

$$\hat{v}(x) = v_\infty(L)\left(\frac{x}{L}\right)^2 \frac{\hat{\alpha}L}{1+\hat{\alpha}L}$$

$$\hat{u}_s(x) = -\left(\frac{F_{x,1}}{E_1 A_1} - \frac{F_{x,2}}{E_2 A_2}\right)\frac{x}{1+\hat{\alpha}L}$$

$$\hat{N}_1(x) = \frac{E_1 A_1}{L} u_\infty(L) + \frac{C_\infty}{d_0}\left\{1 - \frac{EJ_0}{EJ_\infty}\,\frac{\hat{\alpha}L}{1+\hat{\alpha}L}\right\}$$

$$\hat{N}_2(x) = \frac{E_2 A_2}{L} u_\infty(L) - \frac{C_\infty}{d_0}\left\{1 - \frac{EJ_0}{EJ_\infty}\,\frac{\hat{\alpha}L}{1+\hat{\alpha}L}\right\}$$

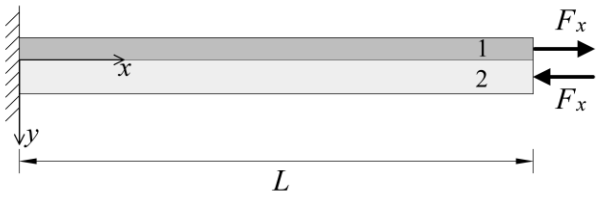
$$\hat{M}_i(x) = -C_\infty \frac{E_i J_i}{EJ_\infty}\,\frac{\hat{\alpha}L}{1+\hat{\alpha}L} \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1(x) + \hat{M}_2(x) + [\hat{N}_2(x)\, d_{2,\infty} - \hat{N}_1(x)\, d_{1,\infty}] = -C_\infty$$

$$\hat{T}_i(x) = 0 \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x) + \hat{T}_2(x) = 0$$

## *S.2. Cantilever two-layer beam subjected to a clockwise end couple $F_x\, d_0$*




*Boundary conditions*

$v(0) = u_1(0) = u_2(0) = v'(0) = 0$

$$E_1A_1u_1'(L) - \hat{k}_h u_s(L) = F_x\,, \qquad E_2A_2u_2'(L) + \hat{k}_h\, u_s(L) = -F_x$$

$$EJ_0 v''(L) + \hat{k}_h d_0 u_s(L) = 0\,, \qquad EJ_0 v'''(L) - k_h d_0 u_s(L) = 0$$

*End vertical displacement and rotation for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(L) = \frac{F_x d_0 L^2}{2EJ_\infty}\,, \qquad \varphi_\infty(L) = \frac{F_x d_0 L}{EJ_\infty}$$

*and end axial displacements for non-composite beams* ($\alpha L = 0$)

$$u_{1,0}(L) = \frac{F_x L}{E_1A_1}\,, \qquad u_{2,0}(L) = -\frac{F_x L}{E_2A_2}$$

*Constants and functions*

$$\hat{c}_1 = 1 + \alpha L\ \hat{\kappa}\ \tanh(\alpha L)\,, \qquad f(x) = \frac{\sinh(\alpha x)}{\alpha L\ \hat{c}_1 \cosh(\alpha L)}\,, \qquad g(x) = f'(x)\,L = \frac{\cosh(\alpha x)}{\hat{c}_1 \cosh(\alpha L)}$$

*Displacement field*

$$u_1(x) = \varphi_\infty(L)\, d_{1,\infty}\left[\frac{x}{L} + \left(\frac{EJ_\infty}{EJ_0} - 1\right)^{-1} f(x)\right] = u_{1,0}(L)\left[\frac{x}{L} - \frac{EJ_0}{EJ_\infty}\left(\frac{x}{L} - f(x)\right)\right], \qquad u_2(x) = -\frac{E_1A_1}{E_2A_2}u_1(x)$$

$$v(x) = v_\infty(L)\left[\left(\frac{x}{L}\right)^2 - 2\,\frac{\cosh(\alpha x) - 1}{(\alpha L)^2\,\hat{c}_1 \cosh(\alpha L)}\right]$$

$$u_s(x) = [u_{2,0}(L) - u_{1,0}(L)]\ f(x) = -F_x L\left(\frac{1}{E_1A_1} + \frac{1}{E_2A_2}\right) f(x) = -\frac{F_x d_0^2\, L}{EJ_\infty - EJ_0}\ f(x) = -\frac{\alpha F_x EJ_0}{k_h EJ_\infty}\,\frac{\sinh(\alpha x)}{\hat{c}_1 \cosh(\alpha L)}$$

*Stress resultants*

$$N_1(x) = -N_2(x) = E_1A_1\, u_1'(x) = F_x\left(1 - \frac{EJ_0}{EJ_\infty}\,[1 - g(x)]\right)$$

$$M_i(x) = -E_iJ_i\ v''(x) = -F_x d_0\ \frac{E_iJ_i}{EJ_\infty}\,[1 - g(x)] \qquad \text{for } i = 1,\, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\, d_{2,\infty} - N_1(x)\, d_{1,\infty}] = -F_x\, d_0$

$$T_1(x) = -T_2(x) = M_1'(x) + k_h\, d_1\, u_s(x) = -k_h u_s(x)\,\frac{d_1E_2J_2 - d_2E_1J_1}{EJ_0}$$

$T(x) = T_1 + T_2 = 0$

***S.2. Cantilever two-layer beam subjected to a clockwise end couple $F_x$ $d_0$ (continued)***

*Cantilever two-layer beam fastened at the beam end only* ($\alpha L = 0$)

$$\hat{u}_1(x) = \varphi_\infty(L)\, d_{1,\infty} \frac{x}{L}\left[1 + \frac{1}{(1+\hat{\alpha}L)(EJ_\infty / EJ_0 - 1)}\right] = u_{1,0}(L)\frac{x}{L}\left[1 + \left(1 - \frac{EJ_0}{EJ_\infty}\right)\hat{\alpha}L\right]\frac{1}{1+\hat{\alpha}L}$$

$$\hat{u}_2(x) = -\varphi_\infty(L)\, d_{2,\infty} \frac{x}{L}\left[1 + \frac{1}{(1+\hat{\alpha}L)(EJ_\infty / EJ_0 - 1)}\right]$$

$$\hat{v}(x) = v_\infty(L)\left(\frac{x}{L}\right)^2 \frac{\hat{\alpha}L}{1+\hat{\alpha}L}$$

$$\hat{u}_s(x) = -F_x\left(\frac{1}{E_1A_1} + \frac{1}{E_2A_2}\right)\frac{x}{1+\hat{\alpha}L}$$

$$\hat{N}_1(x) = -\hat{N}_2(x) = F_x\left\{1 - \frac{EJ_0}{EJ_\infty}\frac{\hat{\alpha}L}{1+\hat{\alpha}L}\right\}$$

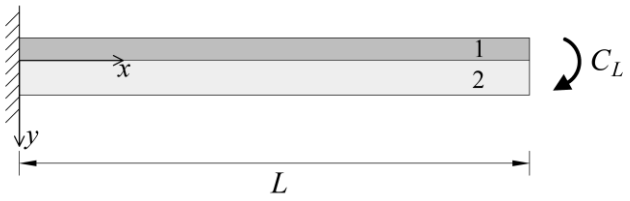
$$\hat{M}_i(x) = -F_x d_0 \frac{E_i J_i}{EJ_\infty}\frac{\hat{\alpha}L}{1+\hat{\alpha}L} \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1(x) + \hat{M}_2(x) + [\hat{N}_2(x)\, d_{2,\infty} - \hat{N}_1(x)\, d_{1,\infty}] = -F_x d_0$$

$$\hat{T}_i(x) = 0 \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x) + \hat{T}_2(x) = 0$$

## *S.3. Cantilever two-layer beam subjected to a clockwise end couple $C_L$*




*Boundary conditions*

$v(0) = u_1(0) = u_2(0) = v'(0) = 0$

$$E_1A_1u_1'(L) - \hat{k}_h u_s(L) = 0\,, \qquad E_2A_2u_2'(L) + \hat{k}_h\, u_s(L) = 0$$

$$EJ_0v''(L) + \hat{k}_h d_0 u_s(L) = C_L\,, \qquad EJ_0v'''(L) - k_h d_0 u_s(L) = 0$$

*End vertical displacement and rotation for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(L) = \frac{C_L L^2}{2EJ_\infty}, \qquad \varphi_\infty(L) = \frac{C_L L}{EJ_\infty}$$

*Constants and functions*

$$\hat{c}_1 = 1 + \alpha L\ \hat{\kappa}\ \tanh(\alpha L)\,, \qquad f(x) = \frac{\sinh(\alpha x)}{\alpha L\ \hat{c}_1 \cosh(\alpha L)}\,, \qquad g(x) = f'(x)\,L = \frac{\cosh(\alpha x)}{\hat{c}_1 \cosh(\alpha L)}$$

*Displacement field*

$$u_1(x) = \varphi_\infty(L)\, d_{1,\infty}\left[\frac{x}{L} - f(x)\right] \qquad u_2(x) = -\varphi_\infty(L)\, d_{2,\infty}\left[\frac{x}{L} - f(x)\right]$$

$$v(x) = v_\infty(L)\left[\left(\frac{x}{L}\right)^2 + 2\left(\frac{EJ_\infty}{EJ_0} - 1\right)\frac{\cosh(\alpha x) - 1}{(\alpha L)^2 \hat{c}_1 \cosh(\alpha L)}\right]$$

$$u_s(x) = \frac{C_L L\, d_0}{EJ_0}\ f(x) = \frac{\alpha C_L}{k_h d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\frac{\sinh(\alpha x)}{\hat{c}_1 \cosh(\alpha L)}$$

*Stress resultants*

$$N_1(x) = -N_2(x) = E_1A_1u_1'(x) = \frac{C_L}{d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left[1 - g(x)\right]$$

$$M_i(x) = -E_iJ_i\ v''(x) = -C_L\ \frac{E_iJ_i}{EJ_\infty}\left[1 + \left(\frac{EJ_\infty}{EJ_0} - 1\right) g(x)\right] \qquad \text{for } i = 1, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\, d_{2,\infty} - N_1(x)\, d_{1,\infty}] = -C_L$

$$T_1(x) = -T_2(x) = M_1'(x) + k_h\, d_1\, u_s(x) = k_h u_s(x)\frac{d_1E_2J_2 - d_2E_1J_1}{EJ_0}$$

$T(x) = T_1 + T_2 = 0$

***S.3. Cantilever two-layer beam subjected to a clockwise end couple $C_L$ (continued)***

*Cantilever two-layer beam fastened at the beam end only* ($\alpha L = 0$)

$$\hat{u}_1(x) = \varphi_\infty(L)\, d_{1,\infty} \frac{x}{L} \frac{\hat{\alpha}L}{1+\hat{\alpha}L}$$

$$\hat{u}_2(x) = -\varphi_\infty(L)\, d_{2,\infty} \frac{x}{L} \frac{\hat{\alpha}L}{1+\hat{\alpha}L}$$

$$\hat{v}(x) = v_\infty(L)\left(\frac{x}{L}\right)^2\left[1+\left(\frac{EJ_\infty}{EJ_0}-1\right)\frac{1}{1+\hat{\alpha}L}\right]$$

$$\hat{u}_s(x) = \frac{C_L\, d_0}{EJ_0} \frac{x}{1+\hat{\alpha}L}$$

$$\hat{N}_1(x) = -\hat{N}_2(x) = \frac{C_L}{d_0}\left(1-\frac{EJ_0}{EJ_\infty}\right)\frac{\hat{\alpha}L}{1+\hat{\alpha}L}$$

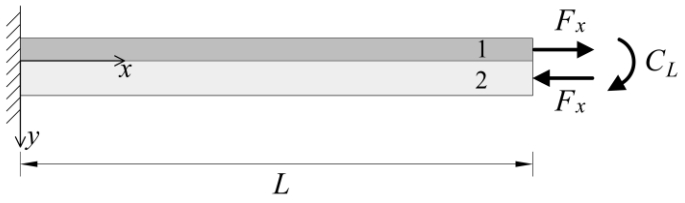
$$\hat{M}_i(x) = -C_L\, \frac{E_i J_i}{EJ_\infty}\left[1+\left(\frac{EJ_\infty}{EJ_0}-1\right)\frac{1}{1+\hat{\alpha}L}\right] \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1(x) + \hat{M}_2(x) + [\hat{N}_2(x)\, d_{2,\infty} - \hat{N}_1(x)\, d_{1,\infty}] = -C_L$$

$$\hat{T}_i(x) = 0 \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x) + \hat{T}_2(x) = 0$$

## *S.4. Cantilever two-layer beam subjected to a clockwise end couple $C_L$ + $F_x$ $d_0$*




*Boundary conditions*

$v(0) = u_1(0) = u_2(0) = v'(0) = 0$

$$E_1A_1u_1'(L) - \hat{k}_h u_s(L) = -F_x\,, \qquad E_2A_2u_2'(L) + \hat{k}_h\, u_s(L) = F_x$$

$$EJ_0v''(L) + \hat{k}_h d_0 u_s(L) = C_L\,, \qquad EJ_0v'''(L) - k_h d_0 u_s(L) = 0$$

*End vertical displacement and rotation for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(L) = \frac{(C_L + F_x d_0)\,L^2}{2EJ_\infty}\,, \qquad \varphi_\infty(L) = \frac{(C_L + F_x d_0)\,L}{EJ_\infty}$$

*End vertical displacement and end axial displacement for non-composite beams* ($\alpha L = 0$)

$$v_0(L) = \frac{C_L L^2}{2EJ_0}$$

$$u_{1,0}(L) = \frac{F_x L}{E_1A_1}\,, \qquad u_{2,0}(L) = -\frac{F_x L}{E_2A_2}$$

*Constants and functions*

$$\hat{c}_1 = 1 + \alpha L\ \hat{\kappa}\ \tanh(\alpha L)\,, \qquad f(x) = \frac{\sinh(\alpha x)}{\alpha L\ \hat{c}_1 \cosh(\alpha L)}\,, \qquad g(x) = f'(x)\,L = \frac{\cosh(\alpha x)}{\hat{c}_1 \cosh(\alpha L)}$$

*Displacement field*

$$u_1(x) = \varphi_\infty(L)\,d_{1,\infty}\frac{x}{L} + (u_{1,0}(L) - \varphi_\infty(L)\,d_{1,\infty})\,f(x)\,, \qquad u_2(x) = -\frac{E_1A_1}{E_2A_2}u_1(x)$$

$$v(x) = v_\infty(L)\left(\frac{x}{L}\right)^2 - 2\left[v_\infty(L) - v_0(L)\right]\frac{\cosh(\alpha x) - 1}{(\alpha L)^2\,\hat{c}_1\cosh(\alpha L)}$$

$$u_s(x) = -\left(u_{1,0} - u_{2,0} - 2v_0(L)\frac{d_0}{L}\right)f(x) = -\frac{d_0\ L}{(1 - EJ_0/EJ_\infty)}\left(\frac{C_L + F_x\,d_0}{EJ_\infty} - \frac{C_L}{EJ_0}\right)f(x)$$

*Stress resultants*

$$N_1(x) = -N_2(x) = E_1A_1\ u_1'(x) = \frac{E_1A_1}{L}\left[\varphi_\infty(L)\,d_{1,\infty} + (u_{1,0}(L) - \varphi_\infty(L)\,d_{1,\infty})\,g(x)\right]$$

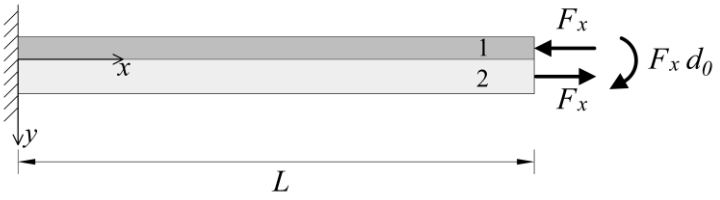
$$M_i(x) = -E_iJ_i\ v''(x) = -E_iJ_i\left(\frac{C_L + F_x\,d_0}{EJ_\infty} - \left(\frac{C_L + F_x\,d_0}{EJ_\infty} - \frac{C_L}{EJ_0}\right)g(x)\right) \qquad \text{for } i = 1, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\,d_{2,\infty} - N_1(x)\,d_{1,\infty}] = -(F_x\,d_0 + C_L)$

$$T_1(x) = -T_2(x) = M_1'(x) + k_h\,d_1\,u_s(x) = k_h u_s(x)\,\frac{d_1E_2J_2 - d_2E_1J_1}{EJ_0}$$

$T(x) = T_1 + T_2 = 0$

***S.5. Cantilever two-layer beam subjected to a clockwise end couple $C_L - F_x\, d_0$, with $C_L = F_x\, d_0$***




*Boundary conditions*

$v(0) = u_1(0) = u_2(0) = v'(0) = 0$

$$E_1A_1u_1'(L) - \hat{k}_h u_s(L) = -F_x\,, \qquad E_2A_2u_2'(L) + \hat{k}_h\, u_s(L) = F_x$$

$$EJ_0v''(L) + \hat{k}_h d_0 u_s(L) = F_x d_0\,, \qquad EJ_0v'''(L) - k_h d_0 u_s(L) = 0$$

*End axial displacement for non-composite beams* ($\alpha L = 0$)

$$u_{1,0}(L) = -\frac{F_x L}{E_1 A_1}\,, \qquad u_{2,0}(L) = \frac{F_x L}{E_2 A_2}$$

*Constants and functions*

$$\hat{c}_1 = 1 + \alpha L\ \hat{\kappa}\ \tanh(\alpha L)\,, \qquad f(x) = \frac{\sinh(\alpha x)}{\alpha L\ \hat{c}_1 \cosh(\alpha L)}\,, \qquad g(x) = f'(x)\,L = \frac{\cosh(\alpha x)}{\hat{c}_1 \cosh(\alpha L)}$$

*Displacement field*

$$u_1(x) = u_{1,0}(L)\ f(x), \qquad u_2(x) = u_{2,0}(L)\ f(x)$$

$$v(x) = \frac{2F_x d_o L^2}{EJ_0}\,\frac{\sinh^2(\alpha x/2)}{(\alpha L)^2\,\hat{c}_1 \cosh(\alpha L)}$$

$$u_s(x) = \left(\frac{d_0^2}{EJ_0} + \frac{1}{E_1A_1} + \frac{1}{E_2A_2}\right) F_x L\ f(x) = \frac{F_x}{k_h L}\,(\alpha L)^2\ f(x)$$

*Stress resultants*

$$N_1(x) = -N_2(x) = E_1A_1\ u_1'(x) = -F_x\ g(x)$$

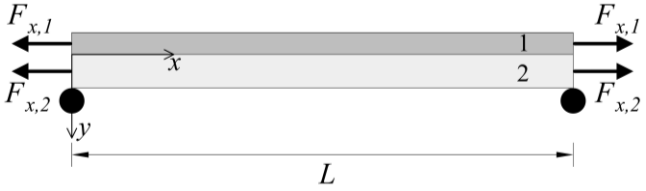
$$M_i(x) = -E_iJ_i\ v''(x) = -F_x d_0\ \frac{E_iJ_i}{EJ_0}\ g(x) \qquad \text{for } i = 1, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\ d_{2,\infty} - N_1(x)\ d_{1,\infty}] = 0$

$$T_1(x) = -T_2(x) = M_1'(x) + k_h\, d_1\, u_s(x) = -k_h u_s(x)\,\frac{d_1E_2J_2 - d_2E_1J_1}{EJ_0}$$

$T(x) = T_1 + T_2 = 0$

## *S.6. Simply supported two-layer beam subjected to symmetric end axial forces $F_{x,1}$ and $F_{x,2}$*




*Boundary conditions*

$E_1A_1u_1'(0)+\hat{k}_h u_s(0)=F_{x,1}$, $\qquad E_2A_2u_2'(0)-\hat{k}_h\,u_s(0)=F_{x,2}$

$v(0)=0$, $\qquad EJ_0v''(0)-\hat{k}_h d_0 u_s(0)=0$

$u_1(L/2)=u_2(L/2)=v'(L/2)=0$, $\qquad EJ_0v'''(L/2)-k_h d_0 u_s(L/2)=0$

*Displacements and rotation for fully composite beams* ($\alpha L=\infty$)

$$u_\infty(L)=-u_\infty(0)=\frac{(F_{x,1}+F_{x,2})L}{2\,(E_1A_1+E_2A_2)},\qquad v_\infty(L/2)=\frac{C_\infty L^2}{8EJ_\infty},\qquad \varphi_\infty(0)=-\varphi_\infty(L)=\frac{C_\infty L}{2EJ_\infty}$$

where $C_\infty=F_{x,2}d_{2,\infty}-F_{x,1}d_{1,\infty}$

*Constants and functions*

$\hat{c}_1=1+\alpha L\;\hat{\kappa}\tanh(\alpha L/2)$,

$$f(x)=\frac{\sinh[\alpha L/2\,(1-2x/L)]}{\alpha L\;\hat{c}_1\cosh(\alpha L/2)},\qquad g(x)=-f'(x)L=\frac{\cosh[\alpha L/2\,(1-2x/L)]}{\hat{c}_1\cosh(\alpha L/2)}$$

*Displacement field for $x\le L/2$*

$$u_1(x)=-u_\infty(L)\left(1-2\frac{x}{L}\right)+\varphi_\infty(0)\,d_{1,\infty}\left[1-2\frac{x}{L}+\frac{2f(x)}{EJ_\infty/EJ_0-1}\right],$$

$$u_2(x)=-u_\infty(L)\left(1-2\frac{x}{L}\right)-\varphi_\infty(0)\,d_{2,\infty}\left[1-2\frac{x}{L}+\frac{2f(x)}{EJ_\infty/EJ_0-1}\right]$$

$$v(x)=v_\infty(L/2)\left[\frac{4x}{L}\left(1-\frac{x}{L}\right)-16\,\frac{\sinh(\alpha x/2)\sinh[\alpha L/2\,(1-x/L)]}{(\alpha L)^2\hat{c}_1\cosh(\alpha L/2)}\right]$$

$$u_s(x)=-\left(\frac{F_{x,1}L}{E_1A_1}-\frac{F_{x,2}L}{E_2A_2}\right)f(x)=-\frac{C_\infty L\,d_0}{EJ_\infty-EJ_0}\;f(x)=-\frac{\alpha C_\infty}{k_h d_0}\,\frac{EJ_0}{EJ_\infty}\,\frac{\sinh[\alpha L/2\,(1-2x/L)]}{\hat{c}_1\cosh(\alpha L/2)}$$

*Stress resultants for $x\le L/2$*

$$N_1(x)=E_1A_1\,u_1'(x)=\frac{2E_1A_1}{L}u_\infty(L)-\frac{C_\infty}{d_0}\left\{1-\frac{EJ_0}{EJ_\infty}[1-g(x)]\right\}$$

$$N_2(x)=E_2A_2\,u_2'(x)=\frac{2E_2A_2}{L}u_\infty(L)+\frac{C_\infty}{d_0}\left\{1-\frac{EJ_0}{EJ_\infty}[1-g(x)]\right\}$$

$$M_i(x)=-E_iJ_i\,v''(x)=C_\infty\frac{E_iJ_i}{EJ_\infty}[1-g(x)]\qquad \text{for } i=1,\,2$$

$M(x)=M_1(x)+M_2(x)+[N_2(x)\,d_{2,\infty}-N_1(x)\,d_{1,\infty}]=C_\infty$

$$T_1(x)=-T_2(x)=M_1'(x)+k_h\,d_1\,u_s(x)=k_hu_s(x)\,\frac{d_1E_2J_2-d_2E_1J_1}{EJ_0}$$

$T(x)=T_1+T_2=0$

***S.6. Simply supported two-layer beam subjected to $F_{x,1}$ and $F_{x,2}$ (continued)***

*Simply supported two-layer beam fastened at both beam ends only* ($\alpha L = 0$) *for* $x \le L/2$

$$\hat{u}_1(x) = \left\{ -u_\infty(L) + \varphi_\infty(0)\, d_{1,\infty} \left[ 1 + \frac{2}{(2+\hat{\alpha}L)(EJ_\infty / EJ_0 - 1)} \right] \right\} \left( 1 - \frac{2x}{L} \right)$$

$$\hat{u}_2(x) = \left\{ -u_\infty(L) - \varphi_\infty(0)\, d_{2,\infty} \left[ 1 + \frac{2}{(2+\hat{\alpha}L)(EJ_\infty / EJ_0 - 1)} \right] \right\} \left( 1 - \frac{2x}{L} \right)$$

$$\hat{v}(x) = v_\infty(L/2) \frac{4x}{L} \left( 1 - \frac{x}{L} \right) \frac{\hat{\alpha}L}{2+\hat{\alpha}L}$$

$$\hat{u}_s(x) = \left( \frac{F_{x,1}L}{E_1 A_1} - \frac{F_{x,2}L}{E_2 A_2} \right) \left( 1 - \frac{2x}{L} \right) \frac{1}{2+\hat{\alpha}L}$$

$$\hat{N}_1(x) = \frac{2E_1 A_1}{L} u_\infty(L) - \frac{C_\infty}{d_0} \left\{ 1 - \frac{EJ_0}{EJ_\infty} \frac{\hat{\alpha}L}{2+\hat{\alpha}L} \right\}$$

$$\hat{N}_2(x) = \frac{2E_2 A_2}{L} u_\infty(L) + \frac{C_\infty}{d_0} \left\{ 1 - \frac{EJ_0}{EJ_\infty} \frac{\hat{\alpha}L}{2+\hat{\alpha}L} \right\}$$

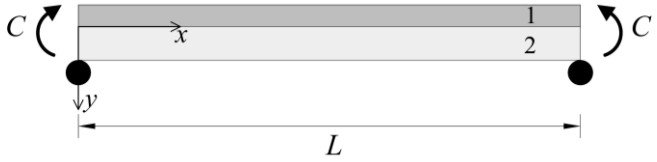
$$\hat{M}_i(x) = C_\infty \frac{E_i J_i}{EJ_\infty} \frac{\hat{\alpha}L}{2+\hat{\alpha}L} \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1(x) + \hat{M}_2(x) + [\hat{N}_2(x)\, d_{2,\infty} - \hat{N}_1(x)\, d_{1,\infty}] = C_\infty$$

$$\hat{T}_i(x) = 0 \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x) + \hat{T}_2(x) = 0$$

## *S.7. Simply supported two-layer beam subjected to symmetric end couples C*




*Boundary conditions*

$E_1 A_1 u_1'(0) + \hat{k}_h u_s(0) = 0$, $\quad E_2 A_2 u_2'(0) - \hat{k}_h\, u_s(0) = 0$

$v(0) = 0$, $\quad EJ_0 v''(0) - \hat{k}_h d_0 u_s(0) = -C$

$u_1(L/2) = u_2(L/2) = v'(L/2) = 0$, $\quad EJ_0 v'''(L/2) - k_h d_0 u_s(L/2) = 0$

*Displacements and rotation for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(L/2) = \frac{CL^2}{8EJ_\infty}, \qquad \varphi_\infty(0) = -\varphi_\infty(L) = \frac{CL}{2EJ_\infty}$$

*Constants and functions*

$$\hat{c}_1 = 1 + \alpha L\ \hat{\kappa}\ \tanh(\alpha L/2),$$

$$f(x) = \frac{\sinh[\alpha L/2\,(1-2x/L)]}{\alpha L\ \hat{c}_1 \cosh(\alpha L/2)}, \qquad g(x) = -f'(x)L = \frac{\cosh[\alpha L/2\,(1-2x/L)]}{\hat{c}_1 \cosh(\alpha L/2)}$$

*Displacement field for $x \le L/2$*

$$u_1(x) = \varphi_\infty(0)\ d_{1,\infty}\left(1 - 2\frac{x}{L} - 2f(x)\right),$$

$$u_2(x) = -\varphi_\infty(0)\ d_{2,\infty}\left(1 - 2\frac{x}{L} - 2f(x)\right)$$

$$v(x) = v_\infty(L/2)\left[\frac{4x}{L}\left(1-\frac{x}{L}\right) + 16\left(\frac{EJ_\infty}{EJ_0} - 1\right)\frac{\sinh(\alpha x/2)\sinh[\alpha L/2\,(1-x/L)]}{(\alpha L)^2\,\hat{c}_1\cosh(\alpha L/2)}\right]$$

$$u_s(x) = \frac{CLd_0}{EJ_0}\ f(x) = \frac{\alpha C}{k_h d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\frac{\sinh[\alpha L/2\,(1-2x/L)]}{\hat{c}_1\cosh(\alpha L/2)}$$

*Stress resultants for $x \le L/2$*

$$N_1(x) = -N_2(x) = E_1 A_1\ u_1'(x) = -\frac{C}{d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)[1 - g(x)]$$

$$M_i(x) = -E_i J_i\ v''(x) = C\,\frac{E_i J_i}{EJ_\infty}\left[1 + \left(\frac{EJ_\infty}{EJ_0} - 1\right)g(x)\right] \qquad \text{for } i = 1, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\ d_{2,\infty} - N_1(x)\ d_{1,\infty}] = C$

$$T_1(x) = -T_2(x) = M_1'(x) + k_h\, d_1\, u_s(x) = k_h u_s(x)\ \frac{d_1 E_2 J_2 - d_2 E_1 J_1}{EJ_0}$$

$T(x) = T_1 + T_2 = 0$

## *S.7. Simply supported two-layer beam subjected to symmetric end couples C (continued)*

*Simply supported two-layer beam fastened at both beam ends only* ($\alpha L = 0$) *for* $x \le L/2$

$$\hat{u}_1(x) = \varphi_\infty(0)\, d_{1,\infty} \left(1 - \frac{2x}{L}\right) \frac{\hat{\alpha} L}{2 + \hat{\alpha} L}$$

$$\hat{u}_2(x) = -\varphi_\infty(0)\, d_{2,\infty} \left(1 - \frac{2x}{L}\right) \frac{\hat{\alpha} L}{2 + \hat{\alpha} L}$$

$$\hat{v}(x) = v_\infty(L/2) \frac{4x}{L} \left(1 - \frac{x}{L}\right) \left(2 \frac{EJ_\infty}{EJ_0} + \hat{\alpha} L\right) \frac{1}{2 + \hat{\alpha} L}$$

$$\hat{u}_s(x) = \frac{C L d_0}{EJ_0} \left(1 - \frac{2x}{L}\right) \frac{1}{2 + \hat{\alpha} L}$$

$$\hat{N}_1(x) = -\hat{N}_2(x) = -\frac{C}{d_0} \left(1 - \frac{EJ_0}{EJ_\infty}\right) \frac{\hat{\alpha} L}{2 + \hat{\alpha} L}$$

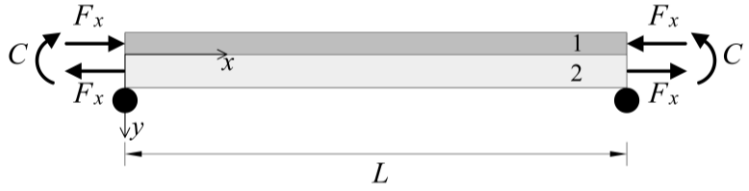
$$\hat{M}_i(x) = C \frac{E_i J_i}{EJ_\infty} \left[1 + \left(\frac{EJ_\infty}{EJ_0} - 1\right) \frac{2}{2 + \hat{\alpha} L}\right] \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1(x) + \hat{M}_2(x) + [\hat{N}_2(x)\, d_{2,\infty} - \hat{N}_1(x)\, d_{1,\infty}] = C$$

$$\hat{T}_i(x) = 0 \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x) + \hat{T}_2(x) = 0$$

## *S.8. Simply supported two-layer beam subjected to symmetric end couples C + $F_x$ $d_0$*




*Boundary conditions*

$E_1A_1u_1'(0)+\hat{k}_h u_s(0)=-F_x$, $\qquad E_2A_2u_2'(0)-\hat{k}_h\, u_s(0)=F_x$

$v(0) = 0$, $\qquad EJ_0v''(0)-\hat{k}_h d_0 u_s(0)=-C$

$u_1(L/2) = u_2(L/2) = v'(L/2) = 0$, $\qquad EJ_0v'''(L/2)-k_h d_0 u_s(L/2)=0$

*Displacements and rotation for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(L/2)=\frac{(C+F_x d_0)\,L^2}{8EJ_\infty}, \qquad \varphi_\infty(0)=-\varphi_\infty(L)=\frac{(C+F_x d_0)\,L}{2EJ_\infty}$$

*Constants and functions*

$$\hat{c}_1=1+\alpha L\;\hat{\kappa}\tanh(\alpha L/2),$$

$$f(x)=\frac{\sinh[\alpha L/2\,(1-2x/L)]}{\alpha L\,\hat{c}_1\cosh(\alpha L/2)}, \qquad g(x)=-f'(x)L=\frac{\cosh[\alpha L/2\,(1-2x/L)]}{\hat{c}_1\cosh(\alpha L/2)}$$

*Displacement field for $x \le L/2$*

$$u_1(x)=\varphi_\infty(0)\,d_{1,\infty}\left(1-2\frac{x}{L}\right)+\left(\frac{C+F_x d_0}{EJ_\infty}-\frac{C}{EJ_0}\right)\frac{L\,d_{1,\infty}\,f(x)}{EJ_\infty/EJ_0-1},$$

$$u_2(x)=-\varphi_\infty(0)\,d_{2,\infty}\left(1-2\frac{x}{L}\right)-\left(\frac{C+F_x d_0}{EJ_\infty}-\frac{C}{EJ_0}\right)\frac{L\,d_{2,\infty}\,f(x)}{EJ_\infty/EJ_0-1}$$

$$v(x)=v_\infty(L/2)\,\frac{4x}{L}\left(1-\frac{x}{L}\right)-\left(\frac{C+F_x d_0}{EJ_\infty}-\frac{C}{EJ_0}\right)\frac{2L^2\sinh(\alpha x/2)\sinh[\alpha L/2\,(1-x/L)]}{(\alpha L)^2\hat{c}_1\cosh(\alpha L/2)}$$

$$u_s(x)=-\left(\frac{C+F_x d_0}{EJ_\infty}-\frac{C}{EJ_0}\right)\frac{L\,d_0\;f(x)}{1-EJ_0/EJ_\infty}$$

*Stress resultants for $x \le L/2$*

$$N_1(x)=-N_2(x)=E_1A_1\,u_1'(x)=-\frac{E_1A_1}{L}\,d_{1,\infty}\left[2\varphi_\infty(0)+\left(\frac{C+F_x d_0}{EJ_\infty}-\frac{C}{EJ_0}\right)\frac{L\,g(x)}{EJ_\infty/EJ_0-1}\right]$$

$$M_i(x)=-E_iJ_i\,v''(x)=(C+F_x d_0)\,\frac{E_iJ_i}{EJ_\infty}[1-g(x)]+C\,\frac{E_iJ_i}{EJ_0}\,g(x) \qquad \text{for } i = 1, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\,d_{2,\infty} - N_1(x)\,d_{1,\infty}] = C + F_x\,d_0$

$$T_1(x)=-T_2(x)=M_1'(x)+k_h\,d_1\,u_s(x)=k_h u_s(x)\,\frac{d_1E_2J_2-d_2E_1J_1}{EJ_0}$$

$T(x) = T_1 + T_2 = 0$

## *S.9. Cantilever two-layer beam subjected to a vertical end point load $F_y$*




*Boundary conditions*

$v(0) = u_1(0) = u_2(0) = v'(0) = 0$

$$E_1 A_1 u_1'(L) - \hat{k}_h u_s(L) = 0\,, \qquad E_2 A_2 u_2'(L) + \hat{k}_h\, u_s(L) = 0$$

$$EJ_0 v''(L) + \hat{k}_h d_0 u_s(L) = 0\,, \qquad EJ_0 v'''(L) - k_h d_0 u_s(L) = -F_y$$

*End vertical displacement and rotation for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(L) = \frac{F_y L^3}{3EJ_\infty}\,, \qquad \varphi_\infty(L) = \frac{F_y L^2}{2EJ_\infty}$$

*and end rotation for non-composite beams ($\alpha L = 0$)*

$$\varphi_0(L) = \frac{F_y L^2}{2EJ_0}$$

*Constants and functions*

$$\hat{c}_1 = \tanh(\alpha L)\,[1 + \alpha L\ \hat{\kappa}\ \tanh(\alpha L/2)]/[1 + \alpha L\ \hat{\kappa}\ \tanh(\alpha L)]$$

$$f(x) = \cosh(\alpha x) - \hat{c}_1 \sinh(\alpha x)\,, \qquad g(x) = f'(x)/\alpha = \sinh(\alpha x) - \hat{c}_1 \cosh(\alpha x)$$

*Displacement field*

$$u_1(x) = \varphi_\infty(L)\, d_{1,\infty} \left[ \frac{2x}{L} - \left(\frac{x}{L}\right)^2 + \frac{2}{(\alpha L)^2}\,(f(x) - 1) \right], \qquad u_2(x) = -\frac{E_1 A_1}{E_2 A_2} u_1(x)$$

$$v(x) = \frac{v_\infty(L)}{2} \left(\frac{x}{L}\right)^2 \left(3 - \frac{x}{L}\right) + \frac{F_y}{\alpha^3 EJ_\infty} \left(\frac{EJ_\infty}{EJ_0} - 1\right) \left[\alpha x - \hat{c}_1 - g(x)\right]$$

$$u_s(x) = 2\varphi_0(L)\, d_0 \frac{1 - f(x)}{(\alpha L)^2} = \frac{F_y}{k_h d_0} \left(1 - \frac{EJ_0}{EJ_\infty}\right) [1 - f(x)]$$

*Stress resultants*

$$N_1(x) = -N_2(x) = E_1 A_1\, u_1'(x) = F_y \frac{L}{d_0} \left(1 - \frac{EJ_0}{EJ_\infty}\right) \left(1 - \frac{x}{L} + \frac{g(x)}{\alpha L}\right)$$

$$M_i(x) = -E_i J_i\, v''(x) = -F_y L \frac{E_i J_i}{EJ_\infty} \left[1 - \frac{x}{L} + \frac{1}{\alpha L} \left(1 - \frac{EJ_\infty}{EJ_0}\right) g(x)\right] \qquad \text{for } i = 1, 2$$

$$M(x) = M_1(x) + M_2(x) + [N_2(x)\, d_{2,\infty} - N_1(x)\, d_{1,\infty}] = -F_y L\,(1 - x/L)$$

$$T_i(x) = M_i'(x) + k_h\, d_i\, u_s(x) = F_y \frac{E_i J_i}{EJ_0} + k_h u_s(x) \frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$T(x) = T_1 + T_2 = F_y$$

***S.9. Cantilever two-layer beam subjected to a vertical end point load $F_y$ (continued)***

*Cantilever two-layer beam fastened at the beam end only* ($\alpha L = 0$)

$$\hat{u}_1(x) = \varphi_\infty(L)\, d_{1,\infty} \frac{\hat{\alpha}L}{(1+\hat{\alpha}L)} \frac{x}{L}$$

$$\hat{u}_2(x) = -\varphi_\infty(L)\, d_{2,\infty} \frac{\hat{\alpha}L}{(1+\hat{\alpha}L)} \frac{x}{L}$$

$$\hat{v}(x) = \frac{v_\infty(L)}{2}\left(\frac{x}{L}\right)^2 \left[3 - \frac{x}{L} + \left(\frac{EJ_\infty}{EJ_0} - 1\right)\left(\frac{6+3\hat{\alpha}L}{2(1+\hat{\alpha}L)} - \frac{x}{L}\right)\right]$$

$$\hat{u}_s(x) = \varphi_0(L)\, d_0 \frac{x}{L}\left[\frac{2+\hat{\alpha}L}{1+\hat{\alpha}L} - \frac{x}{L}\right]$$

$$\hat{N}_1(x) = -\hat{N}_2(x) = F_y \frac{L}{2d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\frac{\hat{\alpha}L}{(1+\hat{\alpha}L)}$$

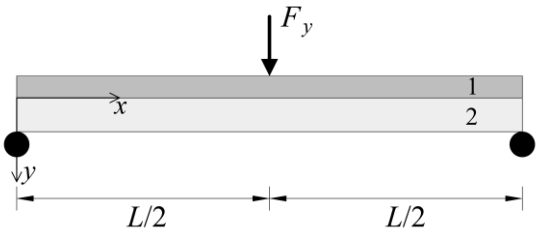
$$\hat{M}_i(x) = -F_y L \frac{E_i J_i}{EJ_\infty}\left[1 - \frac{x}{L} + \left(\frac{EJ_\infty}{EJ_0} - 1\right)\left(\frac{2+\hat{\alpha}L}{2(1+\hat{\alpha}L)} - \frac{x}{L}\right)\right] \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1(x) + \hat{M}_2(x) + [\hat{N}_2(x)\, d_{2,\infty} - \hat{N}_1(x)\, d_{1,\infty}] = -F_y L\left(1 - \frac{x}{L}\right)$$

$$\hat{T}_i(x) = F_y \frac{E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x) + \hat{T}_2(x) = F_y$$

## *S.10. Simply supported two-layer beam subjected to a point load $F_y$ at midspan*




*Boundary conditions*

$E_1A_1u_1'(0) + \hat{k}_h u_s(0) = 0$, $\qquad E_2A_2u_2'(0) - \hat{k}_h\, u_s(0) = 0$

$v(0) = 0$, $\qquad EJ_0v''(0) - \hat{k}_h d_0 u_s(0) = 0$

$u_1(L/2) = u_2(L/2) = v'(L/2) = 0$, $\qquad EJ_0v'''(L/2) = -F_y/2$

*Vertical displacement at midspan and end rotation for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(L/2) = \frac{F_y L^3}{48EJ_\infty}, \qquad \varphi_\infty(0) = \frac{F_y L^2}{16EJ_\infty}$$

*Constants and functions*

$$\hat{c}_1 = 1 + \alpha L\ \hat{\kappa}\ \tanh(\alpha L/2), \qquad \hat{c}_2 = 1 + \alpha L\ \hat{\kappa}\ \sinh(\alpha L/2)$$

$$f(x) = \frac{\hat{c}_2 \sinh(\alpha x) - \alpha L\ \hat{\kappa}\,[\cosh(\alpha L/2) - 1]\cosh(\alpha x)}{\alpha L\ \hat{c}_1 \cosh(\alpha L/2)}$$

$$g(x) = f'(x)L = \frac{\hat{c}_2 \cosh(\alpha x) - \alpha L\ \hat{\kappa}\,[\cosh(\alpha L/2) - 1]\sinh(\alpha x)}{\hat{c}_1 \cosh(\alpha L/2)}$$

*Displacement field for $x \le L/2$*

$$u_1(x) = \varphi_\infty(0)\ d_{1,\infty} \left\{1 - 4\left(\frac{x}{L}\right)^2 - \frac{8}{(\alpha L)^2}\,[1 - g(x)]\right\}, \qquad u_2(x) = -\frac{E_1A_1}{E_2A_2}u_1(x)$$

$$v(x) = v_\infty(L/2)\,\frac{x}{L}\left[3 - 4\left(\frac{x}{L}\right)^2\right] + \frac{F_y L}{2\alpha^2 EJ_\infty}\left(\frac{EJ_\infty}{EJ_0} - 1\right)\left\{\frac{x}{L} - \frac{\hat{\kappa}\,[\cosh(\alpha L/2) - 1]}{\hat{c}_1 \cosh(\alpha L/2)} - f(x)\right\}$$

$$u_s(x) = \frac{F_y d_0}{2\alpha^2 EJ_0}\,[1 - g(x)] = \frac{F_y}{2k_h d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)[1 - g(x)]$$

*Stress resultants for $x \le L/2$*

$$N_1(x) = -N_2(x) = E_1A_1\ u_1'(x) = -\frac{F_y L}{2d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left(\frac{x}{L} - f(x)\right)$$

$$M_i(x) = -E_i J_i\ v''(x) = \frac{F_y L}{2}\,\frac{E_i J_i}{EJ_\infty}\left[\frac{x}{L} + \left(\frac{EJ_\infty}{EJ_0} - 1\right) f(x)\right] \qquad \text{for } i = 1, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\ d_{2,\infty} - N_1(x)\ d_{1,\infty}] = F_y\ x/2$

$$T_i(x) = M_i'(x) + k_h\, d_i\, u_s(x) = \frac{F_y}{2}\,\frac{E_i J_i}{EJ_0} + k_h u_s(x)\,\frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$T(x) = T_1 + T_2 = F_y/2$

***S.10. Simply supported two-layer beam subjected to a point load $F_y$ at midspan (continued)***

*Simply supported two-layer beam fastened at both beam ends only* ($\alpha L = 0$) *for* $x \le L/2$

$$\hat{u}_1(x) = \varphi_\infty(0)\, d_{1,\infty} \frac{\hat{\alpha}L}{(2+\hat{\alpha}L)}\left(1-\frac{2x}{L}\right)$$

$$\hat{u}_2(x) = -\varphi_\infty(0)\, d_{2,\infty} \frac{\hat{\alpha}L}{(2+\hat{\alpha}L)}\left(1-\frac{2x}{L}\right)$$

$$\hat{v}(x) = v_\infty(L/2)\frac{x}{L}\left\{\frac{EJ_\infty}{EJ_0}\left[3-4\left(\frac{x}{L}\right)^2\right]-\left(\frac{EJ_\infty}{EJ_0}-1\right)\frac{3\hat{\alpha}L}{2+\hat{\alpha}L}\left(1-\frac{x}{L}\right)\right\}$$

$$\hat{u}_s(x) = \frac{F_y L^2 d_0}{8EJ_0}\left(1-\frac{2x}{L}\right)\left[\frac{1}{2+\hat{\alpha}L}+\frac{x}{L}\right]$$

$$\hat{N}_1(x) = -\hat{N}_2(x) = -F_y \frac{L}{8d_0}\left(1-\frac{EJ_0}{EJ_\infty}\right)\frac{\hat{\alpha}L}{(2+\hat{\alpha}L)}$$

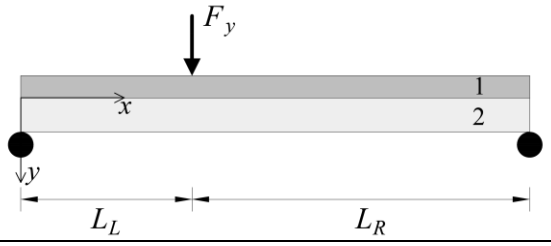
$$\hat{M}_i(x) = \frac{F_y L}{4}\frac{E_i J_i}{EJ_0}\left[\frac{2x}{L}-\frac{\hat{\alpha}L}{2(2+\hat{\alpha}L)}\left(1-\frac{EJ_0}{EJ_\infty}\right)\right] \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1(x)+\hat{M}_2(x)+[\hat{N}_2(x)\, d_{2,\infty}-\hat{N}_1(x)\, d_{1,\infty}] = \frac{F_y x}{2}$$

$$\hat{T}_i(x) = \frac{F_y}{2}\frac{E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x)+\hat{T}_2(x) = \frac{F_y}{2}$$

## *S.11. Simply supported two-layer beam subjected to a point load $F_y$*




*Boundary conditions*

$E_1A_1u'_{1,L}(0)+\hat{k}_h u_{s,L}(0)=0$, $\quad E_1A_1u'_{1,R}(L)-\hat{k}_h u_{s,R}(L)=0$

$E_2A_2u'_{2,L}(0)-\hat{k}_h u_{s,L}(0)=0$, $\quad E_2A_2u'_{2,R}(L)+\hat{k}_h u_{s,R}(L)=0$

$v_L(0)=0$, $\quad v_R(L)=0$

$EJ_0v''_L(0)-\hat{k}_h d_0 u_{s,L}(0)=0$, $\quad EJ_0v''_R(L)+\hat{k}_h d_0 u_{s,R}(L)=0$

$u_{1,L}(L_L)=u_{1,R}(L_L)$, $\quad u_{2,R}(L_L)=u_{2,R}(L_L)$

$u'_{1,L}(L_L)=u'_{1,R}(L_L)$, $\quad u'_{2,R}(L_L)=u'_{2,R}(L_L)$

$v_L(L_L)=v_R(L_L)$, $\quad v'_L(L_L)=v'_R(L_L)$

$EJ_0v'''_L(L_L)-k_h d_0 u_{s,L}(L_L)=-F_yL_R/L$, $\quad EJ_0v'''_R(L_L)-k_h d_0 u_{s,R}(L_L)=F_yL_L/L$

*Vertical displacement for fully composite beams* ($\alpha L=\infty$)

$$v_{L,\infty}(x)=\frac{F_y L_R}{6EJ_\infty}\frac{x}{L}(2L_LL_R+L_L^2-x^2) \qquad \text{for } 0\le x\le L_L$$

$$v_{R,\infty}(x)=\frac{F_y L_L}{6EJ_\infty}\left(1-\frac{x}{L}\right)[2L_LL_R+L_R^2-(L-x)^2] \qquad \text{for } L_L\le x\le L$$

*Constants and functions*

$f(x)=\sinh(\alpha x)+\alpha L\,\hat{\kappa}\cosh(\alpha x)$, $\quad g(x)=f'(x)/\alpha=\cosh(\alpha x)+\alpha L\,\hat{\kappa}\sinh(\alpha x)$

$$\hat{c}_1=\frac{L_L}{L}\cosh(\alpha L_L/2)\sinh(\alpha L_R/2)-\frac{L_R}{L}\cosh(\alpha L_R/2)\sinh(\alpha L_L/2)$$

$$\hat{c}_2=\frac{\hat{\kappa}}{f(L/2)g(L/2)}\left[\sinh(\alpha L_R/2)g(L_R/2)-\frac{L_R}{L}\sinh(\alpha L/2)g(L/2)\right]$$

$$f_L(x)=\frac{1}{\alpha L_R\,f(L/2)g(L/2)}\{g(L_R/2)\sinh(\alpha L_R/2)f(x)-\alpha L_R\,\hat{\kappa}\,g(L/2)\sinh[\alpha(L/2-x)]\}$$

$$g_L(x)=L_R f'_L=\frac{1}{f(L/2)g(L/2)}\{g(L_R/2)\sinh(\alpha L_R/2)g(x)+\alpha L_R\,\hat{\kappa}\,g(L/2)\cosh[\alpha(L/2-x)]\}$$

$$f_R(x)=\frac{1}{\alpha L_L\,f(L/2)g(L/2)}\{g(L_L/2)\sinh(\alpha L_L/2)f(L-x)-\alpha L_L\,\hat{\kappa}\,g(L/2)\sinh[\alpha(L/2-x)]\}$$

$$g_R(x)=-L_L f'_R=\frac{1}{f(L/2)g(L/2)}\{g(L_L/2)\sinh(\alpha L_L/2)g(L-x)+\alpha L_L\hat{\kappa}g(L/2)\cosh[\alpha(L/2-x)]\}$$

***S.11. Simply supported two-layer beam subjected to a point load $F_y$ (continued)***

*Displacement field for* $0 \le x \le L_L$

$$u_{1,L}(x) = v'_{L,\infty}(x)\, d_{1,\infty} - \frac{F_y L^2\, d_{1,\infty}}{EJ_\infty (\alpha L)^2}\left[\frac{L_R}{L} + \left(\frac{EJ_\infty}{EJ_0} - 1\right)\frac{2\hat{\kappa}\hat{c}_1}{f(L/2)} - g_L(x)\right], \qquad u_{2,L}(x) = -\frac{E_1 A_1}{E_2 A_2} u_{1,L}(x)$$

$$v_L(x) = v_{L,\infty}(x) + \frac{F_y L^3}{EJ_\infty (\alpha L)^2}\left(\frac{EJ_\infty}{EJ_0} - 1\right)\left[\left(\frac{L_R}{L} - \frac{2\hat{\kappa}\hat{c}_1}{f(L/2)}\right)\frac{x}{L} + \hat{c}_2 - \frac{L_R}{L}\, f_L(x)\right]$$

$$u_{s,L}(x) = \frac{F_y L^2 d_0}{EJ_0 (\alpha L)^2}\left[\frac{L_R}{L} - g_L(x)\right]$$

*Displacement field for* $L_L \le x \le L$

$$u_{1,R}(x) = v'_{R,\infty}(x)\, d_{1,\infty} + \frac{F_y L^2\, d_{1,\infty}}{EJ_\infty (\alpha L)^2}\left[\frac{L_L}{L} - \left(\frac{EJ_\infty}{EJ_0} - 1\right)\frac{2\hat{\kappa}\hat{c}_1}{f(L/2)} - g_R(x)\right], \qquad u_{2,R}(x) = -\frac{E_1 A_1}{E_2 A_2} u_{1,R}(x)$$

$$v_R(x) = v_{R,\infty}(x) + \frac{F_y L^3}{EJ_\infty (\alpha L)^2}\left(\frac{EJ_\infty}{EJ_0} - 1\right)\left[\frac{L_L}{L}\left(1 - \frac{x}{L}\right) - \frac{2\hat{\kappa}\hat{c}_1}{f(L/2)}\frac{x}{L} + \hat{c}_2 - \frac{L_L}{L}\, f_R(x)\right]$$

$$u_{s,L}(x) = -\frac{F_y L^2 d_0}{EJ_0 (\alpha L)^2}\left[\frac{L_L}{L} - g_R(x)\right]$$

*Stress resultants for* $0 \le x \le L_L$

$$N_{1,L}(x) = -N_{2,L}(x) = E_1 A_1\, u'_{1,L}(x) = -\frac{F_y L_R}{d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left[\frac{x}{L} - f_L(x)\right]$$

$$M_{i,L}(x) = -E_i J_i\, v''_L(x) = F_y L_R\, \frac{E_i J_i}{EJ_\infty}\left[\frac{x}{L} + \left(\frac{EJ_\infty}{EJ_0} - 1\right) f_L(x)\right] \qquad \text{for } i = 1, 2$$

$M_L(x) = M_{1,L}(x) + M_{2,L}(x) + [N_{2,L}(x)\, d_{2,\infty} - N_{1,L}(x)\, d_{1,\infty}] = F_y\, L_R\, x/L$

$$T_{i,L}(x) = F_y\, \frac{L_R}{L}\, \frac{E_i J_i}{EJ_0} + k_h u_{s,L}(x)\, \frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$T_L(x) = T_{1,L} + T_{2,L} = F_y\, L_R/L$

*Stress resultants for* $L_L \le x \le L$

$$N_{1,R}(x) = -N_{2,R}(x) = E_1 A_1\, u'_{1,R}(x) = -\frac{F_y L_L}{d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left[1 - \frac{x}{L} - f_R(x)\right]$$

$$M_{i,R}(x) = -E_i J_i\, v''_R(x) = F_y L_L\, \frac{E_i J_i}{EJ_\infty}\left[1 - \frac{x}{L} + \left(\frac{EJ_\infty}{EJ_0} - 1\right) f_R(x)\right] \qquad \text{for } i = 1, 2$$

$M_R(x) = M_{1,R}(x) + M_{2,R}(x) + [N_{2,R}(x)\, d_{2,\infty} - N_{1,R}(x)\, d_{1,\infty}] = F_y\, L_L\, (1 - x/L)$

$$T_{i,R}(x) = -F_y\, \frac{L_L}{L}\, \frac{E_i J_i}{EJ_0} + k_h u_{s,R}(x)\, \frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$T_R(x) = T_{1,R} + T_{2,R} = F_y\, L_L/L$

## *S.11. Simply supported two-layer beam subjected to a point load $F_y$ (continued)*

*Displacement field for* $\hat{k}_h = 0$ *and* $0 \le x \le L_L$

$$u_{1,L}(x) = v'_{L,\infty}(x)\, d_{1,\infty} + \frac{F_y L^2\, d_{1,\infty}}{EJ_\infty\,(\alpha L)^2}\left( -\frac{L_R}{L} + \frac{\sinh(\alpha L_R)\cosh(\alpha x)}{\sinh(\alpha L)} \right), \qquad u_{2,L}(x) = -\frac{E_1 A_1}{E_2 A_2}\, u_{1,L}(x)$$

$$v_L(x) = v_{L,\infty}(x) + \frac{F_y L^3}{EJ_\infty (\alpha L)^2}\left( \frac{EJ_\infty}{EJ_0} - 1 \right)\left( \frac{L_R}{L}\left( \frac{x}{L} \right) - \frac{\sinh(\alpha L_R)\sinh(\alpha x)}{\alpha L \sinh(\alpha L)} \right)$$

$$u_{s,L}(x) = \frac{F_y L^2 d_0}{EJ_0 (\alpha L)^2}\left( \frac{L_R}{L} - \frac{\sinh(\alpha L_R)\cosh(\alpha x)}{\sinh(\alpha L)} \right)$$

*Displacement field for* $\hat{k}_h = 0$ *and* $L_L \le x \le L$

$$u_{1,R}(x) = v'_{R,\infty}(x)\, d_{1,\infty} + \frac{F_y L^2\, d_{1,\infty}}{EJ_\infty\,(\alpha L)^2}\left( \frac{L_L}{L} - \frac{\sinh(\alpha L_L)\cosh[\alpha(L-x)]}{\sinh(\alpha L)} \right), \qquad u_{2,R}(x) = -\frac{E_1 A_1}{E_2 A_2}\, u_{1,R}(x)$$

$$v_R(x) = v_{R,\infty}(x) + \frac{F_y L^3}{EJ_\infty (\alpha L)^2}\left( \frac{EJ_\infty}{EJ_0} - 1 \right)\left( \frac{L_L}{L}\left( 1 - \frac{x}{L} \right) - \frac{\sinh(\alpha L_L)\sinh[\alpha(L-x)]}{\alpha L \sinh(\alpha L)} \right)$$

$$u_{s,R}(x) = -\frac{F_y L^2 d_0}{EJ_0 (\alpha L)^2}\left( \frac{L_L}{L} - \frac{\sinh(\alpha L_L)\cosh[\alpha(L-x)]}{\sinh(\alpha L)} \right)$$

*Stress resultants for* $\hat{k}_h = 0$ *and* $0 \le x \le L_L$

$$N_{1,L}(x) = -N_{2,L}(x) = E_1 A_1\, u'_{1,L}(x) = -\frac{F_y L_R}{d_0}\left( 1 - \frac{EJ_0}{EJ_\infty} \right)\left[ \frac{x}{L} - \frac{\sinh(\alpha L_R)\sinh(\alpha x)}{\alpha L_R \sinh(\alpha L)} \right]$$

$$M_{i,L}(x) = F_y L_R\, \frac{E_i J_i}{EJ_\infty}\left[ \frac{x}{L} + \left( \frac{EJ_\infty}{EJ_0} - 1 \right) \frac{\sinh(\alpha L_R)\sinh(\alpha x)}{\alpha L_R \sinh(\alpha L)} \right] \qquad \text{for } i = 1, 2$$

$$M_L(x) = M_{1,L}(x) + M_{2,L}(x) + [N_{2,L}(x)\, d_{2,\infty} - N_{1,L}(x)\, d_{1,\infty}] = F_y\, L_R\, x/L$$

$$T_{i,L}(x) = F_y\, \frac{L_R}{L}\, \frac{E_i J_i}{EJ_0} + k_h u_{s,L}(x)\, \frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$T_L(x) = T_{1,L} + T_{2,L} = F_y\, L_R / L$$

*Stress resultants for* $\hat{k}_h = 0$ *and* $L_L \le x \le L$

$$N_{1,R}(x) = -N_{2,R}(x) = -\frac{F_y L_L}{d_0}\left( 1 - \frac{EJ_0}{EJ_\infty} \right)\left[ 1 - \frac{x}{L} - \frac{\sinh(\alpha L_L)\sinh[\alpha(L-x)]}{\alpha L_L \sinh(\alpha L)} \right]$$

$$M_{i,R}(x) = F_y L_L\, \frac{E_i J_i}{EJ_\infty}\left[ 1 - \frac{x}{L} + \left( \frac{EJ_\infty}{EJ_0} - 1 \right) \frac{\sinh(\alpha L_L)\sinh[\alpha(L-x)]}{\alpha L_L \sinh(\alpha L)} \right] \qquad \text{for } i = 1, 2$$

$$M_R(x) = M_{1,R}(x) + M_{2,R}(x) + [N_{2,R}(x)\, d_{2,\infty} - N_{1,R}(x)\, d_{1,\infty}] = F_y\, L_L\, (1 - x/L)$$

$$T_{i,R}(x) = -F_y\, \frac{L_L}{L}\, \frac{E_i J_i}{EJ_0} + k_h u_{s,R}(x)\, \frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$T_R(x) = T_{1,R} + T_{2,R} = F_y\, L_L / L$$

## *S.11. Simply supported two-layer beam subjected to a point load $F_y$ (continued)*

*Simply supported two-layer beam fastened at both beam ends only* ($\alpha L = 0$)

$$\hat{u}_1(x) = \frac{F_y L_L L_R\, d_{1,\infty}}{12EJ_\infty}\left[\frac{EJ_\infty}{EJ_0}\frac{L_R - L_L}{L} + \frac{3\hat{\alpha}L}{2+\hat{\alpha}L}\left(1-2\frac{x}{L}\right)\right] \qquad \text{for } 0 \le x \le L$$

$$\hat{u}_2(x) = -\frac{F_y L_L L_R\, d_{2,\infty}}{12EJ_\infty}\left[\frac{EJ_\infty}{EJ_0}\frac{L_R - L_L}{L} + \frac{3\hat{\alpha}L}{2+\hat{\alpha}L}\left(1-2\frac{x}{L}\right)\right] \qquad \text{for } 0 \le x \le L$$

$$\hat{v}_L(x) = \frac{F_y L_L L_R L}{12EJ_\infty}\frac{x}{L}\left\{\frac{EJ_\infty}{EJ_0}\left[\frac{L_R - L_L}{L} + 3 - 2\frac{L}{L_L}\left(\frac{x}{L}\right)^2\right] - \left(\frac{EJ_\infty}{EJ_0} - 1\right)\frac{3\hat{\alpha}L}{2+\hat{\alpha}L}\left(1-\frac{x}{L}\right)\right\} \qquad \text{for } 0 \le x \le L_L$$

$$\hat{u}_{s,L}(x) = \frac{F_y L_L L_R\, d_0}{2EJ_0}\left[\frac{1}{2+\hat{\alpha}L}\left(1-\frac{2x}{L}\right) + \frac{x}{L}\left(1-\frac{L}{L_L}\frac{x}{L}\right)\right] \qquad \text{for } 0 \le x \le L_L$$

$$\hat{v}_R(x) = -\frac{F_y L_L L_R L}{12EJ_\infty}\left(1-\frac{x}{L}\right)\left\{\frac{EJ_\infty}{EJ_0}\left[\frac{2L}{L_R}\left(\frac{L_L}{L} - \frac{x}{L}\right)^2 - \frac{4x}{L}\right] + \left(\frac{EJ_\infty}{EJ_0} - 1\right)\frac{3\hat{\alpha}L}{2+\hat{\alpha}L}\frac{x}{L}\right\} \qquad \text{for } L_L \le x \le L$$

$$\hat{u}_{s,R}(x) = \frac{F_y L_L L_R\, d_0}{2EJ_0}\left\{\frac{1}{2+\hat{\alpha}L}\left(1-\frac{2x}{L}\right) + \frac{L_L}{L}\left(1-\frac{L}{L_L}\frac{x}{L}\right)\left[1 + \frac{L_L}{L_R}\left(1-\frac{L}{L_L}\frac{x}{L}\right)\right]\right\} \qquad \text{for } L_L \le x \le L$$

$$\hat{N}_1(x) = -\hat{N}_2(x) = -F_y\frac{L_L L_R}{2Ld_0}\left(1-\frac{EJ_0}{EJ_\infty}\right)\frac{\hat{\alpha}L}{(2+\hat{\alpha}L)} \qquad \text{for } 0 \le x \le L$$

$$\hat{M}_{i,L}(x) = \frac{F_y L_L L_R}{L}\frac{E_i J_i}{EJ_0}\left[\frac{L}{L_L}\frac{x}{L} - \frac{\hat{\alpha}L}{2(2+\hat{\alpha}L)}\left(1-\frac{EJ_0}{EJ_\infty}\right)\right] \qquad \text{for } i = 1, 2 \text{ and } 0 \le x \le L_L$$

$$\hat{M}_L(x) = \hat{M}_{1,L}(x) + \hat{M}_{2,L}(x) + [\hat{N}_2(x)\, d_{2,\infty} - \hat{N}_1(x)\, d_{1,\infty}] = \frac{F_y L_R x}{L} \qquad \text{for } 0 \le x \le L$$

$$\hat{T}_{i,L}(x) = \frac{F_y L_R}{L}\frac{E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2 \text{ and } 0 \le x \le L_L$$

$$\hat{T}_L(x) = \hat{T}_{1,L}(x) + \hat{T}_{2,L}(x) = \frac{F_y L_R}{L} \qquad \text{for } 0 \le x \le L$$

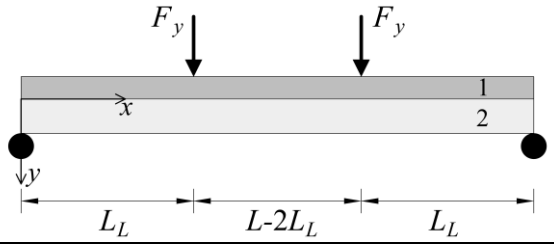
$$\hat{M}_{i,R}(x) = \frac{F_y L_L L_R}{L}\frac{E_i J_i}{EJ_0}\left[\frac{L}{L_R}\left(1-\frac{x}{L}\right) - \frac{\hat{\alpha}L}{2(2+\hat{\alpha}L)}\left(1-\frac{EJ_0}{EJ_\infty}\right)\right] \qquad \text{for } i = 1, 2 \text{ and } L_L \le x \le L$$

$$\hat{M}_R(x) = \hat{M}_{1,R}(x) + \hat{M}_{2,R}(x) + [\hat{N}_2(x)\, d_{2,\infty} - \hat{N}_1(x)\, d_{1,\infty}] = F_y L_L\left(1-\frac{x}{L}\right) \qquad \text{for } L_L \le x \le L$$

$$\hat{T}_{i,R}(x) = -\frac{F_y L_L}{L}\frac{E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2 \text{ and } L_L \le x \le L$$

$$\hat{T}_R(x) = \hat{T}_{1,R}(x) + \hat{T}_{2,R}(x) = -\frac{F_y L_L}{L} \qquad \text{for } 0 \le x \le L$$

## *S.12. Simply supported two-layer beam subjected to two point load $F_y$*




*Boundary conditions*

$E_1A_1u'_{1,L}(0)+\hat{k}_h u_{s,L}(0)=0$, $\quad u_{1,C}(L/2)=0$

$E_2A_2u'_{2,L}(0)-\hat{k}_h u_{s,L}(0)=0$, $\quad u_{2,C}(L/2)=0$

$v_L(0)=0$, $\quad v'_C(L/2)=0$

$EJ_0v''_L(0)-\hat{k}_h d_0 u_{s,L}(0)=0$, $\quad v'''_C(L/2)=0$

$u_{1,L}(L_L)=u_{1,C}(L_L)$, $\quad u_{2,L}(L_L)=u_{2,C}(L_L)$

$u'_{1,L}(L_L)=u'_{1,C}(L_L)$, $\quad u'_{2,L}(L_L)=u'_{2,C}(L_L)$

$v_L(L_L)=v_C(L_L)$, $\quad v'_L(L_L)=v'_C(L_L)$

$EJ_0v'''_L(L_L)-k_h d_0 u_{s,L}(L_L)=-F_y$, $\quad EJ_0v'''_C(L_L)-k_h d_0 u_{s,C}(L_L)=0$

*Vertical displacement for fully composite beams* ($\alpha L=\infty$)

$$v_{L,\infty}(x)=\frac{F_y L^3}{6EJ_\infty}\frac{x}{L}\left[3\frac{L_L}{L}\left(1-\frac{L_L}{L}\right)-\left(\frac{x}{L}\right)^2\right] \qquad \text{for } x\le L_L$$

$$v_{C,\infty}(x)=\frac{F_y L^3}{6EJ_\infty}\frac{L_L}{L}\left[-\left(\frac{L_L}{L}\right)^2+3\frac{x}{L}\left(1-\frac{x}{L}\right)\right] \qquad \text{for } L_L\le x\le L/2$$

*Constants and functions*

$\hat{c}_1=1+\alpha L\,\hat{\kappa}\tanh(\alpha L/2)$, $\quad \hat{c}_2=1+\alpha L\,\hat{\kappa}\tanh(\alpha L_L/2)$

$$\hat{c}_3=\frac{\hat{\kappa}}{\cosh(\alpha L/2)}\left\{2\sinh(\alpha L_L/2)\sinh[\alpha/2(L-L_L)]-\alpha L_L\sinh(\alpha L/2)\right\}$$

$$f_L(x)=\frac{\cosh[\alpha(L/2-L_L)]\sinh(\alpha x)}{\alpha L\,\hat{c}_1\cosh(\alpha L/2)}$$

$$\hat{f}_L(x)=\frac{2\hat{\kappa}}{\hat{c}_1}\frac{\sinh(\alpha x/2)}{\cosh(\alpha L/2)}\left\{\sinh[\alpha/2(L-x)]+\cosh[\alpha(L/2-L_L)]\sinh(\alpha x/2)\right\}$$

$$g_L(x)=L\,f'_L(x)=\frac{\cosh[\alpha(L/2-L_L)]\cosh(\alpha x)}{\hat{c}_1\cosh(\alpha L/2)}$$

$$\hat{g}_L(x)=L\,\hat{f}'_L(x)=\frac{\alpha L\,\hat{\kappa}}{\hat{c}_1\cosh(\alpha L/2)}\left\{\sinh[\alpha(L/2-x)]+\cosh[\alpha(L/2-L_L)]\sinh(\alpha x)\right\}$$

$$f_C(x)=\frac{\hat{c}_2\sinh(\alpha L_L)\cosh[\alpha(L/2-x)]}{\alpha L\,\hat{c}_1\cosh(\alpha L/2)}$$

$$g_C(x)=-L\,f'_C(x)=\frac{\hat{c}_2\sinh(\alpha L_L)\sinh[\alpha(L/2-x)]}{\hat{c}_1\cosh(\alpha L/2)}$$

***S.12. Simply supported two-layer beam subjected to two point load $F_y$ (continued)***

*Displacement field for $0 \le x \le L_L$*

$$u_{1,L}(x) = v'_{L,\infty}(x)\, d_{1,\infty} - \frac{F_y L^2\, d_{1,\infty}}{EJ_\infty} \frac{1 - g_L(x) - \hat{g}_L(x)}{(\alpha L)^2}, \qquad u_{2,L}(x) = -\frac{E_1 A_1}{E_2 A_2} u_{1,L}(x)$$

$$v_L(x) = v_{L,\infty}(x) + \frac{F_y L^3}{EJ_\infty (\alpha L)^2} \left( \frac{EJ_\infty}{EJ_0} - 1 \right) \left[ \frac{x}{L} - f_L(x) - \hat{f}_L(x) \right], \qquad u_{s,L}(x) = \frac{F_y L^2\, d_0}{EJ_0} \frac{1 - g_L(x) - \hat{g}_L(x)}{(\alpha L)^2}$$

*Displacement field for $L_L \le x \le L/2$*

$$u_{1,C}(x) = v'_{C,\infty}(x)\, d_{1,\infty} - \frac{F_y L^2\, d_{1,\infty}}{EJ_\infty} \frac{g_C(x)}{(\alpha L)^2}, \qquad u_{2,C}(x) = -\frac{E_1 A_1}{E_2 A_2} u_{1,C}(x)$$

$$v_C(x) = v_{C,\infty}(x) + \frac{F_y L^3}{EJ_\infty (\alpha L)^2} \left( \frac{EJ_\infty}{EJ_0} - 1 \right) \left[ \frac{L_L}{\hat{c}_1 L} - \frac{\hat{c}_3}{\hat{c}_1} - f_C(x) \right], \qquad u_{s,C}(x) = \frac{F_y L^2\, d_0}{EJ_0} \frac{g_C(x)}{(\alpha L)^2}$$

*Stress resultants for $0 \le x \le L_L$*

$$N_{1,L}(x) = -N_{2,L}(x) = E_1 A_1\, u'_{1,L}(x) = -\frac{F_y L}{d_0} \left( 1 - \frac{EJ_0}{EJ_\infty} \right) \left( \frac{x}{L} - f_L(x) - \frac{\hat{f}''_L(x)}{\alpha^2} \right)$$

$$M_{i,L}(x) = -E_i J_i\, v''_L(x) = F_y L \frac{E_i J_i}{EJ_\infty} \left[ \frac{x}{L} + \left( \frac{EJ_\infty}{EJ_0} - 1 \right) \left( f_L(x) + \frac{\hat{f}''_L(x)}{\alpha^2} \right) \right] \qquad \text{for } i = 1, 2$$

$$M_L(x) = M_{1,L}(x) + M_{2,L}(x) + [N_{2,L}(x)\, d_{2,\infty} - N_{1,L}(x)\, d_{1,\infty}] = F_y\, x$$

$$T_{i,L}(x) = M'_{i,L}(x) + k_h\, d_i\, u_{s,L}(x) = F_y \frac{E_i J_i}{EJ_0} + k_h u_{s,L}(x) \frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$T_L(x) = T_{1,L} + T_{2,L} = F_y$$

*Stress resultants for $L_L \le x \le L/2$*

$$N_{1,C}(x) = -N_{2,C}(x) = E_1 A_1\, u'_{1,C}(x) = -\frac{F_y L}{d_0} \left( 1 - \frac{EJ_0}{EJ_\infty} \right) \left[ \frac{L_L}{L} - f_C(x) \right]$$

$$M_{i,C}(x) = -E_i J_i\, v''_C(x) = F_y L \frac{E_i J_i}{EJ_\infty} \left[ \frac{L_L}{L} + \left( \frac{EJ_\infty}{EJ_0} - 1 \right) f_C(x) \right] \qquad \text{for } i = 1, 2$$

$$M_C(x) = M_{1,C}(x) + M_{2,C}(x) + [N_{2,C}(x)\, d_{2,\infty} - N_{1,C}(x)\, d_{1,\infty}] = F_y\, L_L$$

$$T_{i,C}(x) = M'_{i,C}(x) + k_h\, d_i\, u_{s,C}(x) = k_h u_{s,C}(x) \frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$T_C(x) = T_{1,C} + T_{2,C} = 0$$

## *S.12. Simply supported two-layer beam subjected to two point load $F_y$ (continued)*

*Constants and functions for* $\hat{k}_h = 0$

$$\hat{c}_1 = \hat{c}_2 = 1, \qquad \hat{c}_3 = 0, \qquad \hat{f}_L(x) = \hat{g}_L(x) = 0$$

$$f_L(x) = \frac{\cosh[\alpha(L/2 - L_L)]\sinh(\alpha x)}{\alpha L \cosh(\alpha L/2)}, \qquad g_L(x) = L\, f'_L(x) = \frac{\cosh[\alpha(L/2 - L_L)]\cosh(\alpha x)}{\cosh(\alpha L/2)}$$

$$f_C(x) = \frac{\sinh(\alpha L_L)\cosh[\alpha(L/2 - x)]}{\alpha L \cosh(\alpha L/2)}, \qquad g_C(x) = -L\, f'_C(x) = \frac{\sinh(\alpha L_L)\sinh[\alpha(L/2 - x)]}{\cosh(\alpha L/2)}$$

*Displacement field for* $\hat{k}_h = 0$ *and* $x \le L_L$

$$u_{1,L}(x) = v'_{L,\infty}(x)\, d_{1,\infty} - \frac{F_y L^2\, d_{1,\infty}}{EJ_\infty} \frac{1 - g_L(x)}{(\alpha L)^2}, \qquad u_{2,L}(x) = -\frac{E_1 A_1}{E_2 A_2} u_{1,L}(x)$$

$$v_L(x) = v_{L,\infty}(x) + \frac{F_y L^3}{EJ_\infty (\alpha L)^2}\left(\frac{EJ_\infty}{EJ_0} - 1\right)\left[\frac{x}{L} - f_L(x)\right], \qquad u_{s,L}(x) = \frac{F_y L^2\, d_0}{EJ_0} \frac{1 - g_L(x)}{(\alpha L)^2}$$

*Displacement field for* $\hat{k}_h = 0$ *and* $L_L \le x \le L/2$

$$u_{1,C}(x) = v'_{C,\infty}(x)\, d_{1,\infty} - \frac{F_y L^2\, d_{1,\infty}}{EJ_\infty} \frac{g_C(x)}{(\alpha L)^2}, \qquad u_{2,C}(x) = -\frac{E_1 A_1}{E_2 A_2} u_{1,C}(x)$$

$$v_C(x) = v_{C,\infty}(x) + \frac{F_y L^3}{EJ_\infty (\alpha L)^2}\left(\frac{EJ_\infty}{EJ_0} - 1\right)\left[\frac{L_L}{L} - f_C(x)\right], \qquad u_{s,C}(x) = \frac{F_y L^2\, d_0}{EJ_0} \frac{g_C(x)}{(\alpha L)^2}$$

*Stress resultants for* $\hat{k}_h = 0$ *and* $x \le L_L$

$$N_{1,L}(x) = -N_{2,L}(x) = E_1 A_1\, u'_{1,L}(x) = -\frac{F_y L}{d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left(\frac{x}{L} - f_L(x)\right)$$

$$M_{i,L}(x) = -E_i J_i\, v''_L(x) = F_y L\, \frac{E_i J_i}{EJ_\infty}\left[\frac{x}{L} + \left(\frac{EJ_\infty}{EJ_0} - 1\right) f_L(x)\right] \qquad \text{for } i = 1, 2$$

$$M(x) = M_{1,L}(x) + M_{2,L}(x) + [N_{2,L}(x)\, d_{2,\infty} - N_{1,L}(x)\, d_{1,\infty}] = F_y\, x$$

$$T_{i,L}(x) = M'_{i,L}(x) + k_h\, d_i\, u_{s,L}(x) = F_y\, \frac{E_i J_i}{EJ_0} + k_h u_{s,L}(x)\, \frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$T_L(x) = T_{1,L} + T_{2,L} = F_y$$

*Stress resultants for* $\hat{k}_h = 0$ *and* $L_L \le x \le L/2$

$$N_{1,C}(x) = -N_{2,C}(x) = E_1 A_1\, u'_{1,C}(x) = -\frac{F_y L}{d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left(\frac{L_L}{L} - f_C(x)\right)$$

$$M_{i,C}(x) = -E_i J_i\, v''_C(x) = F_y L\, \frac{E_i J_i}{EJ_\infty}\left[\frac{L_L}{L} + \left(\frac{EJ_\infty}{EJ_0} - 1\right) f_C(x)\right] \qquad \text{for } i = 1, 2$$

$$M(x) = M_{1,L}(x) + M_{2,L}(x) + [N_{2,L}(x)\, d_{2,\infty} - N_{1,L}(x)\, d_{1,\infty}] = F_y\, L_L$$

$$T_{i,C}(x) = M'_{i,C}(x) + k_h\, d_i\, u_{s,C}(x) = k_h u_{s,C}(x)\, \frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$T_C(x) = T_{1,C} + T_{2,C} = 0$$

## *S.12. Simply supported two-layer beam subjected to two point load $F_y$ (continued)*

*Simply supported two-layer beam fastened at both beam ends only* ($\alpha L = 0$)

$$\hat{u}_1(x) = \frac{F_y L^2\, d_{1,\infty}}{EJ_\infty} \frac{\hat{\alpha}L}{2(2+\hat{\alpha}L)} \frac{L_L}{L}\left(1-\frac{L_L}{L}\right)\left(1-2\frac{x}{L}\right) \qquad \text{for } x \le L/2$$

$$\hat{u}_2(x) = -\frac{F_y L^2\, d_{2,\infty}}{EJ_\infty} \frac{\hat{\alpha}L}{2(2+\hat{\alpha}L)} \frac{L_L}{L}\left(1-\frac{L_L}{L}\right)\left(1-2\frac{x}{L}\right) \qquad \text{for } x \le L/2$$

$$\hat{v}_L(x) = \frac{F_y L^3}{6EJ_0}\frac{x}{L}\left\{\frac{3}{2+\hat{\alpha}L}\frac{L_L}{L}\left(1-\frac{L_L}{L}\right)\left[2+\hat{\alpha}x+\hat{\alpha}L\frac{EJ_0}{EJ_\infty}\left(1-\frac{x}{L}\right)\right]-\left(\frac{x}{L}\right)^2\right\} \qquad \text{for } x \le L_L$$

$$\hat{u}_{s,L}(x) = \frac{F_y L^2 d_0}{2EJ_0}\left[2\frac{L_L}{L}\left(1-\frac{L_L}{L}\right)\frac{1+\hat{\alpha}x}{2+\hat{\alpha}L}-\left(\frac{x}{L}\right)^2\right] \qquad \text{for } x \le L_L$$

$$\hat{v}_C(x) = \frac{F_y L^3}{6EJ_0}\frac{L_L}{L}\left\{-\left(\frac{L_L}{L}\right)^2+\frac{1}{2+\hat{\alpha}L}\left\{6+3\hat{\alpha}L\left[\frac{L_L}{L}+\frac{EJ_0}{EJ_\infty}\left(1-\frac{L_L}{L}\right)\right]\right\}\frac{x}{L}\left(1-\frac{x}{L}\right)\right\} \qquad \text{for } L_L \le x \le L/2$$

$$\hat{u}_{s,C}(x) = \frac{F_y L^2 d_0}{2EJ_0}\frac{L_L}{L}\left[2\left(1-\frac{L_L}{L}\right)\frac{1+\hat{\alpha}x}{2+\hat{\alpha}L}+\frac{L_L}{L}-2\frac{x}{L}\right] \qquad \text{for } L_L \le x \le L/2$$

$$\hat{N}_1(x) = -\hat{N}_2(x) = E_1 A_1\, \hat{u}_1'(x) = -\frac{F_y L_L}{d_0}\left(1-\frac{L_L}{L}\right)\left(1-\frac{EJ_0}{EJ_\infty}\right)\frac{\hat{\alpha}L}{2+\hat{\alpha}L} \qquad \text{for } x \le L/2$$

$$\hat{M}_{i,L}(x) = F_y L\,\frac{E_i J_i}{EJ_\infty}\left[-\frac{L_L}{L}\left(1-\frac{L_L}{L}\right)\left(\frac{EJ_\infty}{EJ_0}-1\right)\frac{\hat{\alpha}L}{2+\hat{\alpha}L}+\frac{EJ_\infty}{EJ_0}\frac{x}{L}\right] \qquad \text{for } i = 1, 2 \text{ and } x \le L/2$$

$$\hat{M}_L(x) = \hat{M}_{1,L}(x)+\hat{M}_{2,L}(x)+[\hat{N}_2(x)\,d_{2,\infty}-\hat{N}_1\,d_{1,\infty}] = F_y x \qquad \text{for } x \le L_L$$

$$\hat{T}_{i,L}(x) = F_y\,\frac{E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2 \text{ and } x \le L_L$$

$$\hat{T}_L(x) = \hat{T}_{1,L}(x)+\hat{T}_{2,L}(x) = F_y \qquad \text{for } x \le L_L$$

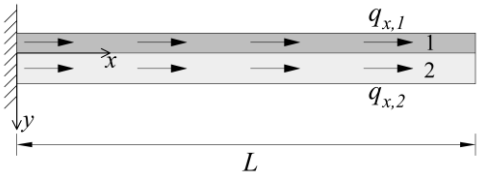
$$\hat{M}_{i,C}(x) = F_y L_L\,\frac{E_i J_i}{EJ_\infty}\left[\frac{\hat{\alpha}L}{2+\hat{\alpha}L}\left(1-\frac{L_L}{L}\right)+\frac{EJ_\infty}{EJ_0}\frac{2+\hat{\alpha}L_L}{2+\hat{\alpha}L}\right] \qquad \text{for } i = 1, 2 \text{ and } x \ge L_L$$

$$\hat{M}_C(x) = \hat{M}_{1,C}(x)+\hat{M}_{2,C}(x)+[\hat{N}_2(x)\,d_{2,\infty}-\hat{N}_1\,d_{1,\infty}] = F_y L_L \qquad \text{for } L_L \le x \le L/2$$

$$\hat{T}_{i,C}(x) = 0 \qquad \text{for } i = 1, 2 \text{ and } x \ge L_L$$

$$\hat{T}_C(x) = \hat{T}_{1,C}(x)+\hat{T}_{2,C}(x) = 0 \qquad \text{for } L_L \le x \le L/2$$

## *S.13. Cantilever two-layer beam subjected to distributed axial loads $q_{x1}$ and $q_{x2}$*




*Boundary conditions*

$v(0) = u_1(0) = u_2(0) = v'(0) = 0$

$$E_1A_1u_1'(L) - \hat{k}_h u_s(L) = 0\,, \qquad E_2A_2u_2'(L) + \hat{k}_h\, u_s(L) = 0$$

$$EJ_0v''(L) + \hat{k}_h d_0 u_s(L) = 0\,, \qquad EJ_0v'''(L) - k_h d_0 u_s(L) = 0$$

*End displacements and rotation for fully composite beams* ($\alpha L = \infty$)

$$u_\infty(L) = \frac{(q_{x,1}+q_{x,2})L^2}{2(E_1A_1+E_2A_2)}, \qquad v_\infty(L) = \frac{C_\infty L^3}{3EJ_\infty}, \qquad \varphi_\infty(L) = \frac{C_\infty L^2}{2EJ_\infty}$$

where $C_\infty = q_{x,1}d_{1,\infty} - q_{x,2}d_{2,\infty}$

*Constants and functions*

$$\hat{c}_1 = 1 + \alpha L\,\hat{\kappa}\tanh(\alpha L)\,, \qquad \hat{c}_2 = 1 + \alpha L\,\hat{\kappa}\tanh(\alpha L/2)$$

$$f(x) = \sinh(\alpha x) - \frac{\hat{c}_2}{\hat{c}_1}\tanh(\alpha L)\cosh(\alpha x)\,, \qquad g(x) = f'(x)/\alpha = \cosh(\alpha x) - \frac{\hat{c}_2}{\hat{c}_1}\tanh(\alpha L)\sinh(\alpha x)$$

*Displacement field*

$$u_1(x) = (u_\infty(L) + \varphi_\infty d_{1,\infty})\frac{x}{L}\left(2 - \frac{x}{L}\right) + \frac{2\varphi_\infty d_{1,\infty}}{(\alpha L)^2\left(EJ_\infty/EJ_0 - 1\right)}\left[1 - g(x)\right]$$

$$u_2(x) = (u_\infty(L) - \varphi_\infty d_{2,\infty})\frac{x}{L}\left(2 - \frac{x}{L}\right) - \frac{2\varphi_\infty d_{2,\infty}}{(\alpha L)^2\left(EJ_\infty/EJ_0 - 1\right)}\left[1 - g(x)\right]$$

$$v(x) = \frac{v_\infty(L)}{2}\left(\left(\frac{x}{L}\right)^2\left(3 - \frac{x}{L}\right) - \frac{6}{(\alpha L)^3}\left[\alpha x + \frac{\hat{c}_2}{\hat{c}_1}\tanh(\alpha L)(\cosh(\alpha x) - 1) - \sinh(\alpha x)\right]\right)$$

$$u_s(x) = \left(\frac{q_{x,2}}{E_2A_2} - \frac{q_{x,1}}{E_1A_1}\right)\frac{L^2}{(\alpha L)^2}\left[1 - g(x)\right] = -\frac{C_\infty L^2 d_0}{(\alpha L)^2(EJ_\infty - EJ_0)}\left[1 - g(x)\right] = -\frac{C_\infty}{k_h d_0}\frac{EJ_0}{EJ_\infty}\left[1 - g(x)\right]$$

*Stress resultants*

$$N_1(x) = E_1A_1\,u_1'(x) = \frac{E_1A_1\,L}{E_1A_1+E_2A_2}(q_{x,1}+q_{x,2})\left(1 - \frac{x}{L}\right) + \frac{C_\infty L}{d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left[1 - \frac{x}{L} - \frac{f(x)}{\alpha L\,(EJ_\infty/EJ_0 - 1)}\right]$$

$$N_2(x) = E_2A_2\,u_2'(x) = \frac{E_2A_2\,L}{E_1A_1+E_2A_2}(q_{x,1}+q_{x,2})\left(1 - \frac{x}{L}\right) - \frac{C_\infty L}{d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left[1 - \frac{x}{L} - \frac{f(x)}{\alpha L\,(EJ_\infty/EJ_0 - 1)}\right]$$

$$M_i(x) = -E_iJ_i\,v''(x) = -\frac{EJ_i}{EJ_\infty}C_\infty L\left(1 - \frac{x}{L} + \frac{f(x)}{\alpha L}\right) \qquad \text{for } i = 1, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\,d_{2,\infty} - N_1(x)\,d_{1,\infty}] = -C_\infty\,L\,(1 - x/L)$

$$T_i(x) = \frac{C_\infty}{d_0EJ_\infty}\left[1 - g(x)\right]\left(d_0E_1J_1 - d_1EJ_0\right) \qquad \text{for } i = 1, 2$$

$T(x) = T_1 + T_2 = 0$

***S.13. Cantilever two-layer beam subjected to distributed axial loads $q_{x1}$ and $q_{x2}$ (continued)***

*Cantilever two-layer beam fastened at the beam end only* ($\alpha L = 0$)

$$\hat{u}_1(x) = (u_\infty(L) + \varphi_\infty d_{1,\infty})\frac{x}{L}\left(2 - \frac{x}{L}\right) + \frac{\varphi_\infty d_{1,\infty}}{(EJ_\infty / EJ_0 - 1)}\frac{x}{L}\left(\frac{2+\hat{\alpha}L}{1+\hat{\alpha}L} - \frac{x}{L}\right)$$

$$\hat{u}_2(x) = (u_\infty(L) - \varphi_\infty d_{2,\infty})\frac{x}{L}\left(2 - \frac{x}{L}\right) - \frac{\varphi_\infty d_{2,\infty}}{(EJ_\infty / EJ_0 - 1)}\frac{x}{L}\left(\frac{2+\hat{\alpha}L}{1+\hat{\alpha}L} - \frac{x}{L}\right)$$

$$\hat{v}(x) = v_\infty(L/2)\frac{3\hat{\alpha}L}{4(1+\hat{\alpha}L)}\left(\frac{x}{L}\right)^2$$

$$\hat{u}_s(x) = -\frac{C_\infty L^2 d_0}{2(EJ_\infty - EJ_0)}\frac{x}{L}\left(\frac{2+\hat{\alpha}L}{1+\hat{\alpha}L} - \frac{x}{L}\right) = \frac{1}{2}\left(\frac{q_{x,2}L^2}{E_2A_2} - \frac{q_{x,1}L^2}{E_1A_1}\right)\frac{x}{L}\left(\frac{2+\hat{\alpha}L}{1+\hat{\alpha}L} - \frac{x}{L}\right)$$

$$\hat{N}_1(x) = \frac{E_1A_1\,L}{E_1A_1 + E_2A_2}(q_{x,1} + q_{x,2})\left(1 - \frac{x}{L}\right) + \frac{C_\infty L}{d_0}\frac{EJ_0}{EJ_\infty}\left[\left(\frac{EJ_\infty}{EJ_0} - 1\right)\left(1 - \frac{x}{L}\right) + \frac{2+\hat{\alpha}L}{2(1+\hat{\alpha}L)} - \frac{x}{L}\right]$$

$$\hat{N}_2(x) = \frac{E_2A_2\,L}{E_1A_1 + E_2A_2}(q_{x,1} + q_{x,2})\left(1 - \frac{x}{L}\right) - \frac{C_\infty L}{d_0}\frac{EJ_0}{EJ_\infty}\left[\left(\frac{EJ_\infty}{EJ_0} - 1\right)\left(1 - \frac{x}{L}\right) + \frac{2+\hat{\alpha}L}{2(1+\hat{\alpha}L)} - \frac{x}{L}\right]$$

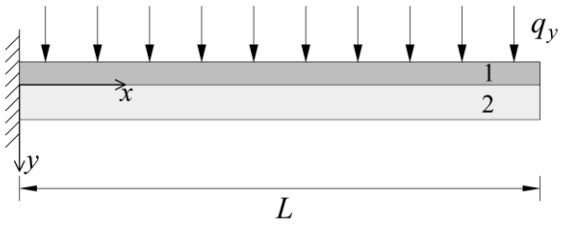
$$\hat{M}_i(x) = -C_\infty L\frac{E_iJ_i}{EJ_\infty}\frac{\hat{\alpha}L}{2(1+\hat{\alpha}L)} \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1(x) + \hat{M}_2(x) + [\hat{N}_2(x)\,d_{2,\infty} - \hat{N}_1(x)\,d_{1,\infty}] = -C_\infty L\left(1 - \frac{x}{L}\right)$$

$$\hat{T}_i(x) = 0 \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x) + \hat{T}_2(x) = 0$$

## *S.14. Cantilever two-layer beam subjected to a distributed load $q_y$*




*Boundary conditions*

$v(0) = u_1(0) = u_2(0) = v'(0) = 0$

$$E_1A_1u_1'(L) - \hat{k}_h u_s(L) = 0\,, \qquad E_2A_2u_2'(L) + \hat{k}_h\, u_s(L) = 0$$

$$EJ_0v''(L) + \hat{k}_h d_0 u_s(L) = 0\,, \qquad EJ_0v'''(L) - k_h d_0 u_s(L) = 0$$

*End vertical displacement and rotation for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(L) = \frac{q_y L^4}{8EJ_\infty}\,, \qquad \varphi_\infty(L) = \frac{q_y L^3}{6EJ_\infty}$$

*Constants and functions*

$$\hat{c}_1 = 1 + \alpha L\ \hat{\kappa}\ \tanh(\alpha L)\,, \qquad \hat{c}_2 = [1 + \alpha L \sinh(\alpha L)]/(\hat{c}_1 \cosh(\alpha L)) + (\alpha L)^2\ \hat{\kappa}/\hat{c}_1$$

$$f(x) = \frac{1}{(\alpha L)^2}\left[\alpha L \sinh(\alpha x) - \hat{c}_2 \cosh(\alpha x)\right], \qquad g(x) = f'(x)L = \frac{1}{\alpha L}\left[\alpha L \cosh(\alpha x) - \hat{c}_2 \sinh(\alpha x)\right]$$

*Displacement field*

$$u_1(x) = \varphi_\infty(L)\, d_{1,\infty}\left[3\frac{x}{L} - 3\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3 - \frac{6}{(\alpha L)^2}\left(1 - \frac{x}{L} - g(x)\right)\right], \qquad u_2(x) = -\frac{E_1A_1}{E_2A_2}u_1(x)$$

$$v(x) = \frac{v_\infty(L)}{3}\left(\frac{x}{L}\right)^2\left[6 - 4\frac{x}{L} + \left(\frac{x}{L}\right)^2\right] + \frac{q_y L^4}{(\alpha L)^2 EJ_\infty}\left(\frac{EJ_\infty}{EJ_0} - 1\right)\left[\frac{x}{L}\left(1 - \frac{x}{2L}\right) - \frac{\hat{c}_2}{(\alpha L)^2} - f(x)\right]$$

$$u_s(x) = \frac{q_y L^3 d_0}{EJ_0}\,\frac{1}{(\alpha L)^2}\left[1 - \frac{x}{L} - g(x)\right] = \frac{q_y L}{k_h d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left[1 - \frac{x}{L} - g(x)\right]$$

*Stress resultants*

$$N_1(x) = -N_2(x) = E_1A_1\, u_1'(x) = \frac{q_y L^2}{2d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left[\left(1 - \frac{x}{L}\right)^2 + \frac{2}{(\alpha L)^2} + 2f(x)\right]$$

$$M_i(x) = -E_iJ_i\, v''(x) = -\frac{q_y L^2}{2}\,\frac{E_iJ_i}{EJ_\infty}\left[\left(1 - \frac{x}{L}\right)^2 - 2\left(\frac{EJ_\infty}{EJ_0} - 1\right)\left(\frac{1}{(\alpha L)^2} + f(x)\right)\right] \qquad \text{for } i = 1, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\, d_{2,\infty} - N_1(x)\, d_{1,\infty}] = -\,q_y\, L^2/2\ (1 - x/L)^2$

$$T_i(x) = M_i'(x) + k_h\, d_i\, u_s(x) = q_y L\,\frac{E_iJ_i}{EJ_0}\left(1 - \frac{x}{L}\right) + k_h u_s(x)\,\frac{d_iEJ_0 - d_0E_iJ_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$T(x) = T_1 + T_2 = q_y\, L\ (1 - x/L)$

## *S.14. Cantilever two-layer beam subjected to a distributed load $q_y$ (continued)*

*Cantilever two-layer beam fastened at the beam end only* ($\alpha L = 0$)

$$\hat{u}_1(x) = \varphi_\infty(L)\, d_{1,\infty}\, \frac{\hat{\alpha}L}{(1+\hat{\alpha}L)}\, \frac{x}{L}$$

$$\hat{u}_2(x) = -\frac{E_1 A_1}{E_2 A_2}\, \hat{u}_1(x)$$

$$\hat{v}(x) = \frac{v_\infty(L)}{3} \left(\frac{x}{L}\right)^2 \left\{ 6 - 4\frac{x}{L} + \left(\frac{x}{L}\right)^2 + \left(\frac{EJ_\infty}{EJ_0} - 1\right) \left[ 6 - 4\frac{x}{L} + \left(\frac{x}{L}\right)^2 - \frac{2\hat{\alpha}L}{1+\hat{\alpha}L} \right] \right\}$$

$$\hat{u}_s(x) = \frac{q_y L^3 d_0}{6EJ_0}\, \frac{x}{L} \left[ 3 - 3\frac{x}{L} + \left(\frac{x}{L}\right)^2 - \frac{\hat{\alpha}L}{1+\hat{\alpha}L} \right]$$

$$\hat{N}_1(x) = -\hat{N}_2(x) = \frac{q_y L^2}{6d_0} \left(1 - \frac{EJ_0}{EJ_\infty}\right) \frac{\hat{\alpha}L}{1+\hat{\alpha}L}$$

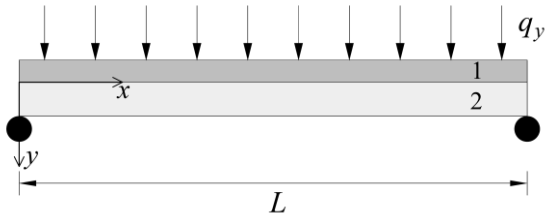
$$\hat{M}_i(x) = -\frac{q_y L^2}{2}\, \frac{E_i J_i}{EJ_\infty} \left\{ \left(1 - \frac{x}{L}\right)^2 + \left(\frac{EJ_\infty}{EJ_0} - 1\right) \left[ \left(1 - \frac{x}{L}\right)^2 - \frac{\hat{\alpha}L}{3(1+\hat{\alpha}L)} \right] \right\} \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1(x) + \hat{M}_2(x) + [\hat{N}_2(x)\, d_{2,\infty} - \hat{N}_1(x)\, d_{1,\infty}] = -\frac{q_y L^2}{2} \left(1 - \frac{x}{L}\right)^2$$

$$\hat{T}_i(x) = q_y L \left(1 - \frac{x}{L}\right) \frac{E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x) + \hat{T}_2(x) = q_y L \left(1 - \frac{x}{L}\right)$$

## *S.15. Simply supported two-layer beam subjected to a distributed load $q_y$*




*Boundary conditions*

$E_1A_1u_1'(0) + \hat{k}_h u_s(0) = 0$, $\qquad E_2A_2u_2'(0) - \hat{k}_h\, u_s(0) = 0$

$EJ_0 v''(0) - \hat{k}_h d_0 u_s(0) = 0$, $\qquad v(0) = 0$

$u_1(L/2) = u_2(L/2) = v'(L/2) = v'''(L/2) = 0$

*Vertical displacement at midspan and end rotation for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(L/2) = \frac{5}{384}\frac{q_y L^4}{EJ_\infty}, \qquad \varphi_\infty(0) = \frac{q_y L^3}{24EJ_\infty}$$

*Constants and functions*

$$\hat{c}_1 = [1 + (\alpha L)^2\,\hat{\kappa}/2]/[1 + \alpha L\,\hat{\kappa}\tanh(\alpha L/2)]$$

$$f(x) = \hat{c}_1 \sinh[\alpha(x - L/2)]/[\alpha L \cosh(\alpha L/2)], \quad g(x) = f'(x)L = \hat{c}_1 \cosh[\alpha(x - L/2)]/\cosh(\alpha L/2)$$

*Displacement field for $x \le L/2$*

$$u_1(x) = \varphi_\infty(0)\, d_{1,\infty} \left\{1 - 6\left(\frac{x}{L}\right)^2 + 4\left(\frac{x}{L}\right)^3 - \frac{24}{(\alpha L)^2}\left[\frac{1}{2} - \frac{x}{L} + f(x)\right]\right\}, \qquad u_2(x) = -\frac{E_1A_1}{E_2A_2}u_1(x)$$

$$v(x) = \frac{q_y L^4}{24EJ_\infty}\left\{\frac{x}{L}\left[1 - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right] + \frac{12}{(\alpha L)^2}\left(\frac{EJ_\infty}{EJ_0} - 1\right)\left[\frac{x}{L}\left(1 - \frac{x}{L}\right) - \frac{2}{(\alpha L)^2}[\hat{c}_1 - g(x)]\right]\right\}$$

$$u_s(x) = \frac{q_y L^3 d_0}{EJ_0}\frac{1}{(\alpha L)^2}\left[\frac{1}{2} - \frac{x}{L} + f(x)\right] = \frac{q_y L}{k_h d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left[\frac{1}{2} - \frac{x}{L} + f(x)\right]$$

*Stress resultants for $x \le L/2$*

$$N_1(x) = -N_2(x) = E_1A_1\, u_1'(x) = -\frac{q_y L^2}{2d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\left\{\frac{x}{L}\left(1 - \frac{x}{L}\right) - \frac{2\,[1 - g(x)]}{(\alpha L)^2}\right\}$$

$$M_i(x) = -E_iJ_i\, v''(x) = \frac{q_y L^2}{8}\frac{E_iJ_i}{EJ_\infty}\left\{4\frac{x}{L}\left(1 - \frac{x}{L}\right) + \left(\frac{EJ_\infty}{EJ_0} - 1\right)\frac{8\,[1 - g(x)]}{(\alpha L)^2}\right\} \qquad \text{for } i = 1, 2$$

$M(x) = M_1(x) + M_2(x) + [N_2(x)\, d_{2,\infty} - N_1(x)\, d_{1,\infty}] = q_y\, x\, (L - x)/2$

$$T_i(x) = M_i'(x) + k_h\, d_i\, u_s(x) = \frac{q_y L}{2}\frac{E_iJ_i}{EJ_0}\left(1 - \frac{2x}{L}\right) + k_h u_s(x)\,\frac{d_iEJ_0 - d_0E_iJ_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$T(x) = T_1 + T_2 = q_y\,(L - 2x)/2$

## *S.15. Simply supported two-layer beam subjected to a distributed load $q_y$ (continued)*

*Simply supported two-layer beam fastened at both beam ends only* ($\alpha L = 0$) *for* $x \le L/2$

$$\hat{u}_1(x) = \varphi_\infty(0)\, d_{1,\infty} \frac{\hat{\alpha}L}{(2+\hat{\alpha}L)}\left(1-\frac{2x}{L}\right)$$

$$\hat{u}_2(x) = -\frac{E_1 A_1}{E_2 A_2}\hat{u}_1(x)$$

$$\hat{v}(x) = \frac{q_y L^4}{24 E J_\infty}\frac{x}{L}\left(1-\frac{x}{L}\right)\left\{1+\frac{x}{L}-\left(\frac{x}{L}\right)^2+\left(\frac{EJ_\infty}{EJ_0}-1\right)\left[1+\frac{x}{L}-\left(\frac{x}{L}\right)^2-\frac{\hat{\alpha}L}{2+\hat{\alpha}L}\right]\right\}$$

$$\hat{u}_s(x) = \frac{q_y L^3 d_0}{12 E J_0}\left(1-\frac{2x}{L}\right)\left[\frac{x}{L}-\left(\frac{x}{L}\right)^2+\frac{1}{2+\hat{\alpha}L}\right]$$

$$\hat{N}_1(x) = -\hat{N}_2(x) = -\frac{q_y L^2}{12 d_0}\left(1-\frac{EJ_0}{EJ_\infty}\right)\frac{\hat{\alpha}L}{2+\hat{\alpha}L}$$

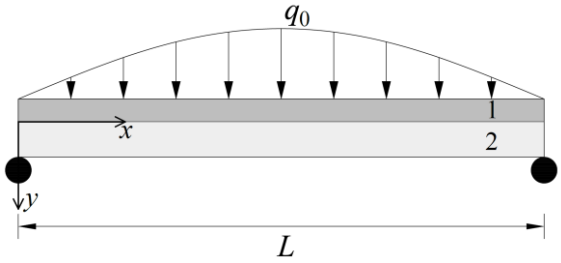
$$\hat{M}_i(x) = \frac{q_y L^2}{8}\frac{E_i J_i}{EJ_\infty}\left\{4\frac{x}{L}\left(1-\frac{x}{L}\right)+\left(\frac{EJ_\infty}{EJ_0}-1\right)\left[4\frac{x}{L}\left(1-\frac{x}{L}\right)-\frac{2\hat{\alpha}L}{3(2+\hat{\alpha}L)}\right]\right\} \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1(x)+\hat{M}_2(x)+[\hat{N}_2(x)\, d_{2,\infty}-\hat{N}_1(x)\, d_{1,\infty}] = \frac{q_y L^2}{2}\frac{x}{L}\left(1-\frac{x}{L}\right)$$

$$\hat{T}_i(x) = q_y L\left(\frac{1}{2}-\frac{x}{L}\right)\frac{E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x)+\hat{T}_2(x) = q_y L\left(\frac{1}{2}-\frac{x}{L}\right)$$

## *S.16. Simply supported two-layer beam subjected to a sinusoidal load $q_0 \sin(\pi x/L)$*




*Boundary conditions*

$E_1 A_1 u_1'(0) + \hat{k}_h u_s(0) = 0$,  $E_2 A_2 u_2'(0) - \hat{k}_h\, u_s(0) = 0$

$EJ_0 v''(0) - \hat{k}_h d_0 u_s(0) = 0$,  $v(0) = 0$

$u_1(L/2) = u_2(L/2) = v'(L/2) = v'''(L/2) = 0$

*Vertical displacement at midspan and end rotation for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(L/2) = \frac{1}{\pi^4}\frac{q_0 L^4}{EJ_\infty}, \qquad \varphi_\infty(0) = \frac{1}{\pi^3}\frac{q_0 L^3}{EJ_\infty}$$

*Constants and functions*

$$\hat{c}_1 = 1 + \alpha L\, \hat{\kappa} \tanh(\alpha L/2), \qquad \hat{c}_2 = \frac{\hat{\kappa}}{\hat{c}_1[(\alpha L)^2 + \pi^2]\cosh(\alpha L/2)}$$

$$EJ_{\text{ef}} = EJ_\infty \frac{(\alpha L)^2 + \pi^2}{\beta}, \qquad \beta = (\alpha L)^2 + \pi^2 \frac{EJ_\infty}{EJ_0}$$

$$q_y(x) = q_0 \sin(\pi x/L)$$

*Displacement field for particular solution (also referred to gamma method)*

$$u_{1,p}(x) = v_p'(x)\, d_{1,\infty} \frac{(\alpha L)^2}{\beta}, \qquad u_{2,p}(x) = -\frac{E_1 A_1}{E_2 A_2} u_{1,p}(x)$$

$$v_p(x) = \frac{q_y(x) L^4}{\pi^4 EJ_{\text{ef}}}, \qquad v_p'(x) = \frac{q_0 L^3}{\pi^3 EJ_{\text{ef}}} \cos(\pi x/L)$$

$$u_{s,p}(x) = \frac{q_0 L^3}{\pi EJ_0} \frac{d_0 \cos(\pi x/L)}{(\alpha L)^2 + \pi^2}$$

*Displacement field*

$$u_1(x) = u_{1,p}(x) + \varphi_\infty(0)\, d_{1,\infty}\, \pi^2\, \alpha L\, \hat{c}_2 \sinh[\alpha(L/2 - x)], \qquad u_2(x) = -\frac{E_1 A_1}{E_2 A_2} u_1(x)$$

$$v(x) = v_p(x) - 2 v_\infty(L/2)\left(\frac{EJ_\infty}{EJ_0} - 1\right)\pi^3\, \hat{c}_2 \sinh(\alpha x/2)\sinh[\alpha/2(L - x)]$$

$$u_s(x) = u_{s,p}(x) - u_{s,p}(0)\frac{\alpha L\, \hat{\kappa}}{\hat{c}_1 \cosh(\alpha L/2)} \sinh[\alpha(L/2 - x)]$$

## *S.16. Simply supported two-layer beam subjected to a sinusoidal load $q_0 \sin(\pi x/L)$ (continued)*

*Stress resultants*

$$N_1(x) = -N_2(x) = E_1 A_1\, u'_{1,p}(x) - \frac{q_0 L^2}{\pi d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)(\alpha L)^2\, \hat{c}_2 \cosh[\alpha(L/2 - x)]$$

$$M_i(x) = \frac{q_0 L^2}{\pi^2}\frac{E_i J_i}{EJ_\infty}\left[\frac{\beta \sin(\pi x/L)}{(\alpha L)^2 + \pi^2} - \left(\frac{EJ_\infty}{EJ_0} - 1\right)\pi(\alpha L)^2\, \hat{c}_2 \cosh[\alpha(L/2 - x)]\right] \qquad \text{for } i = 1, 2$$

$$M(x) = M_1 + M_2 + N_2\, d_{2,\infty} - N_1\, d_{1,\infty} = \frac{q_y(x) L^2}{\pi^2}$$

$$T_i(x) = \frac{q_0 L}{\pi}\frac{E_i J_i}{EJ_0}\cos(\pi x/L) + k_h u_s(x)\frac{d_i EJ_0 - d_0 E_i J_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$T(x) = T_1(x) + T_2(x) = \frac{q_0 L}{\pi}\cos(\pi x/L)$$

*Simply supported two-layer beam fastened at both beam ends only* ($\alpha L = 0$)

$$\hat{u}_1(x) = \varphi_\infty(0)\, d_{1,\infty}\frac{\hat{\alpha} L}{(2 + \hat{\alpha} L)}\left(1 - \frac{2x}{L}\right)$$

$$\hat{u}_2(x) = -\frac{E_1 A_1}{E_2 A_2}\hat{u}_1(x)$$

$$\hat{v}(x) = v_\infty(L/2)\left[\frac{EJ_\infty}{EJ_0}\sin(\pi x/L) - \left(\frac{EJ_\infty}{EJ_0} - 1\right)\frac{\pi \hat{\alpha} L}{2 + \hat{\alpha} L}\frac{x}{L}\left(1 - \frac{x}{L}\right)\right]$$

$$\hat{u}_s(x) = \frac{q_0 L^3 d_0}{\pi^3 EJ_0}\left[\cos(\pi x/L) - \frac{\hat{\alpha} L}{2 + \hat{\alpha} L}\left(1 - \frac{2x}{L}\right)\right]$$

$$\hat{N}_1(x) = -\hat{N}_2(x) = -\frac{q_y L^2}{12 d_0}\left(1 - \frac{EJ_0}{EJ_\infty}\right)\frac{\hat{\alpha} L}{2 + \hat{\alpha} L}$$

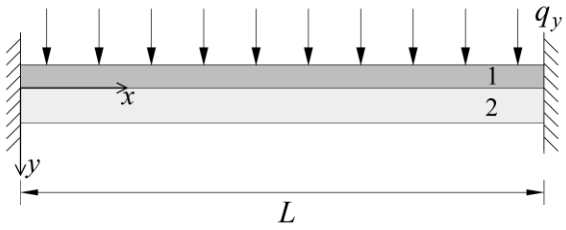
$$\hat{M}_i(x) = \frac{q_0 L^2}{\pi^2}\frac{E_i J_i}{EJ_\infty}\left[\frac{EJ_\infty}{EJ_0}\sin(\pi x/L) - \left(\frac{EJ_\infty}{EJ_0} - 1\right)\frac{2\hat{\alpha} L}{\pi(2 + \hat{\alpha} L)}\right] \qquad \text{for } i = 1, 2$$

$$\hat{M}(x) = \hat{M}_1 + \hat{M}_2 + \hat{N}_2\, d_{2,\infty} - \hat{N}_1\, d_{1,\infty} = \frac{q_y(x) L^2}{\pi^2}$$

$$\hat{T}_i(x) = \frac{q_0 L}{\pi}\frac{E_i J_i}{EJ_0}\cos(\pi x/L) \qquad \text{for } i = 1, 2$$

$$\hat{T}(x) = \hat{T}_1(x) + \hat{T}_2(x) = \frac{q_0 L}{\pi}\cos(\pi x/L)$$

## *S.17. Built-in two-layer beam subjected to a distributed load $q_y$*




*Boundary conditions*

$u_1(0) = u_2(0) = v(0) = v'(0) = 0,$ $\qquad u_1(L/2) = u_2(L/2) = v'(L/2) = v'''(L/2) = 0$

*Vertical displacement for fully composite beams* ($\alpha L = \infty$)

$$v_\infty(x) = \frac{q_y L^4}{24 E J_\infty}\left(\frac{x}{L}\right)^2\left(1-\frac{x}{L}\right)^2$$

*Functions*

$$f(x) = \sinh(\alpha x/2)\cosh[\alpha/2\,(L-x)]/\sinh(\alpha L/2)$$

$$g(x) = 2\,f'(x)/\alpha = \cosh[\alpha\,(L/2-x)]/\sinh(\alpha L/2)$$

*Displacement field*

$$u_1(x) = \frac{q_y L^3}{12 E J_\infty} d_{1,\infty}\left\{\frac{x}{L}\left(1-2\frac{x}{L}\right)\left(1-\frac{x}{L}\right)+\frac{12}{(\alpha L)^2}\left[\frac{x}{L}-f(x)\right]\right\}, \qquad u_2(x) = -\frac{E_1 A_1}{E_2 A_2}u_1(x)$$

$$v(x) = v_\infty(x) + \frac{q_y L^4}{2 E J_\infty}\frac{1}{(\alpha L)^3}\left(\frac{E J_\infty}{E J_0}-1\right)\left[\alpha x\left(1-\frac{x}{L}\right)-\coth(\alpha L/2)+g(x)\right]$$

$$u_s(x) = \frac{q_y L^3 d_0}{E J_0}\frac{1}{(\alpha L)^2}\left[-\frac{x}{L}+f(x)\right] = \frac{q_y L}{k_h d_0}\left(1-\frac{E J_0}{E J_\infty}\right)\left[-\frac{x}{L}+f(x)\right]$$

*Stress resultants*

$$N_1(x) = -N_2(x) = E_1 A_1\,u_1'(x) = \frac{q_y L^2}{12 d_0}\left(1-\frac{E J_0}{E J_\infty}\right)\left\{1-\frac{6x}{L}+6\left(\frac{x}{L}\right)^2+\frac{6[2-\alpha L\,g(x)]}{(\alpha L)^2}\right\}$$

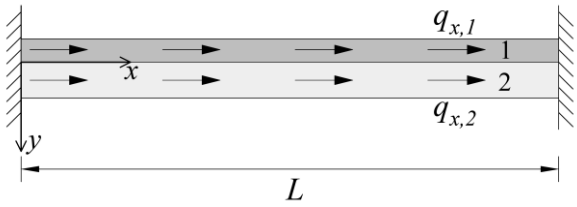
$$M_i(x) = -E_i J_i\,v''(x) = -\frac{q_y L^2}{12}\frac{E_i J_i}{E J_\infty}\left\{1-\frac{6x}{L}+6\left(\frac{x}{L}\right)^2-\left(\frac{E J_\infty}{E J_0}-1\right)\frac{6[2-\alpha L\,g(x)]}{(\alpha L)^2}\right\} \qquad \text{for } i = 1, 2$$

$$M(x) = M_1 + M_2 + N_2\,d_{2,\infty} - N_1\,d_{1,\infty} = -\frac{q_y L^2}{12}\left[1-\frac{6x}{L}+6\left(\frac{x}{L}\right)^2\right]$$

$$T_i(x) = M_i'(x) + k_h\,d_i\,u_s(x) = \frac{q_y L}{2}\frac{E_i J_i}{E J_0}\left(1-\frac{2x}{L}\right)+k_h u_s(x)\frac{d_i E J_0 - d_0 E_i J_i}{E J_0} \qquad \text{for } i = 1, 2$$

$T(x) = T_1 + T_2 = q_y\,L/2\,(1 - 2\,x/L)$

## *S.18. Built-in two-layer beam subjected to distributed axial loads $q_{x1}$ and $q_{x2}$*




*Boundary conditions*

$u_1(0) = u_2(0) = v(0) = v'(0) = 0,$ $\qquad u_1(L) = u_2(L) = v(L) = v'(L/2) = 0$

*Axial displacement for fully composite beams* ($\alpha L = \infty$) *and distributed couple*

$$u_\infty(x) = \frac{(q_{x,1}+q_{x,2})L^2}{2(E_1A_1+E_2A_2)}\frac{x}{L}\left(1-\frac{x}{L}\right), \qquad C_\infty = q_{x,1}d_{1,\infty} - q_{x,2}d_{2,\infty}$$

*Constant and functions*

$$c_h = 1+\frac{24}{(\alpha L)^3}\left(\frac{EJ_\infty}{EJ_0}-1\right)[\alpha L/2-\tanh(\alpha L/2)]$$

$$f(x) = \sinh(\alpha x/2)\sinh[\alpha/2(L-x)]/\cosh(\alpha L/2)$$

$$g(x) = 2f'(x)/\alpha = \sinh[\alpha(L/2-x)]/\cosh(\alpha L/2)$$

*Displacement field*

$$u_1(x) = \frac{u_\infty(x)}{c_h}+\frac{c_h-1}{c_h}\frac{q_{x,1}L^2}{2E_1A_1}\frac{x}{L}\left(1-\frac{x}{L}\right)+\frac{2C_\infty L^2}{d_0E_1A_1}\frac{f(x)}{(\alpha L)^2c_h}$$

$$u_2(x) = \frac{u_\infty(x)}{c_h}+\frac{c_h-1}{c_h}\frac{q_{x,2}L^2}{2E_2A_2}\frac{x}{L}\left(1-\frac{x}{L}\right)-\frac{2C_\infty L^2}{d_0E_2A_2}\frac{f(x)}{(\alpha L)^2c_h}$$

$$v(x) = -\frac{C_\infty L^3}{EJ_0}\frac{1}{(\alpha L)^3c_h}\left\{\alpha x\left[1-\frac{3x}{L}+2\left(\frac{x}{L}\right)^2\right]-\left[1-6\left(\frac{x}{L}\right)^2+4\left(\frac{x}{L}\right)^3\right]\tanh(\alpha L/2)+g(x)\right\}$$

$$u_s(x) = -\frac{2C_\infty}{k_hd_0c_h}f(x)$$

*Stress resultants*

$$N_1(x) = E_1A_1u_1'(x) = \left[\frac{E_1A_1}{E_1A_1+E_2A_2}\frac{(q_{x,1}+q_{x,2})L}{2c_h}+\frac{c_h-1}{c_h}\frac{q_{x,1}L}{2}\right]\left(1-\frac{2x}{L}\right)+\frac{C_\infty L}{d_0}\frac{g(x)}{\alpha L\,c_h}$$

$$N_2(x) = E_2A_2u_2'(x) = \left[\frac{E_2A_2}{E_1A_1+E_2A_2}\frac{(q_{x,1}+q_{x,2})L}{2c_h}+\frac{c_h-1}{c_h}\frac{q_{x,2}L}{2}\right]\left(1-\frac{2x}{L}\right)-\frac{C_\infty L}{d_0}\frac{g(x)}{\alpha L\,c_h}$$

$$M_i(x) = -E_iJ_i\,v''(x) = -\frac{C_\infty L}{(\alpha L)^3c_h}\frac{E_iJ_i}{EJ_0}\left\{12\left(1-\frac{2x}{L}\right)[\alpha L/2-\tanh(\alpha L/2)]-(\alpha L)^2\,g(x)\right\} \qquad \text{for } i = 1, 2$$

$$M(x) = M_1+M_2+N_2\,d_{2,\infty}-N_1\,d_{1,\infty} = -\frac{C_\infty L}{(\alpha L)^3c_h}\frac{EJ_\infty}{EJ_0}12\left(1-\frac{2x}{L}\right)[\alpha L/2-\tanh(\alpha L/2)]$$

$$T_i(x) = -\frac{C_\infty}{c_h}\frac{E_iJ_i}{EJ_0}\left\{1-\frac{24}{(\alpha L)^3}[\alpha L/2-\tanh(\alpha L/2)]\right\}+k_hu_s(x)\frac{d_iEJ_0-d_0E_iJ_i}{EJ_0} \qquad \text{for } i = 1, 2$$

$$T(x) = T_1+T_2 = -\frac{C_\infty}{c_h}\left\{1-\frac{24}{(\alpha L)^3}[\alpha L/2-\tanh(\alpha L/2)]\right\}$$