

Supplementary Information for
“Circuit models for quasi-static bands in metallic photonic
crystals”

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SUPPLEMENTARY NOTE 1. BUNDLE CAPACITANCE MATRICES

We use two forms of the electrostatic matrix $\mathbf{M}$ in main-text Eq. (1). They give different mode counts at the Brillouin-zone centre.

Isolated bundle. With only inter-wire capacitances, $\mathbf{M}$ is the weighted graph Laplacian

$$\mathbf{M}_{ij} = \begin{cases} \sum_{l \neq i} C_{il}, & i = j, \\ -C_{ij}, & i \neq j, \end{cases} \quad (\text{S1})$$

with $C_{ij} = C_{ji} \geq 0$. For any real vector $\mathbf{v}$,

$$\mathbf{v}^\top \mathbf{M} \mathbf{v} = \sum_{i < j} C_{ij} (v_i - v_j)^2 \geq 0. \quad (\text{S2})$$

Thus $\mathbf{M}$ is positive semidefinite. Its null space contains the uniform vector and is one-dimensional when the capacitive graph is connected. The uniform vector is the global gauge freedom because a common shift of all node potentials stores no energy. At a permutation-selected zero with $\beta_n \neq 1$, the projected matrix $\bar{\mathbf{M}}$ is positive definite, whereas at $\beta_0 = 1$ the uniform gauge mode leaves $d_0 - 1 = t - 1$ dispersive pairs, with t the number of translation orbits. The two transverse photon-like modes of a triply periodic isolated bundle are additional and have no circuit counterpart. This statement concerns the triply periodic isolated-bundle geometry. The PEC shell supplies both a prescribed potential reference and the physical return-current conductor. The $\beta = 1$ common potential of the internal wires is therefore a nonzero wire-to-shell voltage rather than a global-potential gauge direction. It gives one conductor-supported multiconductor transmission-line branch. In a straight uniform bundle this branch is TEM; for the twisted periodic bundle we call it quasi-TEM.

Grounded (shielded) bundle. A PEC shell acts as the common reference conductor for the wire bundle. Grounding it adds the wire-to-shell capacitances,

$$\mathbf{M} \longrightarrow \mathbf{M} + \text{diag}(C_{10}, \dots, C_{N0}), \quad C_{i0} > 0. \quad (\text{S3})$$

The resulting Maxwell capacitance matrix is positive definite, so every permutation-selected zero, including $\beta_0 = 1$, gives all d_n branch pairs. The shell is a prescribed reference conductor, not an additional dynamical permutation orbit. In main-text Fig. 1 the six physically

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disconnected wires form one translation orbit with a six-cell return period, hence $d_n = 1$: one root is at Γ and five are at nonzero Bloch momentum. At Γ , Bloch periodicity leaves one internal potential component; with the grounded shell as reference, main-text Eq. (5) has $N = 2$ and gives $N_0 = 1$. For $\sigma = \text{id}$ this reduces to one TEM branch per wire. The isolated bundle has the same six permutation roots algebraically, but its $\beta = 1$ common mode is gauge; consequently only the five nonuniform quasi-static channels are physical, consistent with the isolated-bundle discussion in the main text.

SUPPLEMENTARY NOTE 2. PERMUTATION THEOREM AND VELOCITIES

Completeness of the zero-mode subspace. With the factorized inductive Bloch Laplacian $\mathbf{A}^\dagger \mathbf{L}^{-1} \mathbf{A}$, $\mathbf{A} = \mathbb{I} - \beta \mathbf{P}$, and positive-definite $\mathbf{L}^{-1}$, $\mathbf{A}^\dagger \mathbf{L}^{-1} \mathbf{A} \boldsymbol{\psi} = 0$ implies $\|\mathbf{A} \boldsymbol{\psi}\|_{\mathbf{L}^{-1}}^2 = 0$, hence $\mathbf{P} \boldsymbol{\psi} = \beta^{-1} \boldsymbol{\psi}$, for every admissible $\mathbf{L}$ including full mutual matrices. Because $\mathbf{P}$ is unitary, geometric and algebraic multiplicities coincide and no generalized (Jordan-type) inductive Bloch Laplacian null vectors exist; the same argument applies to the general bond incidence matrix $\mathbf{A}(\mathbf{k})$ of Methods, whose null space is fixed by loop consistency alone. At Γ , whether such a uniform null vector is a physical mode or a gauge direction is determined by the capacitance/reference structure as described in Supplementary Note 1 and main-text Eq. (5). For a permutation of order m not every m -th root of unity need appear: the zero-mode momenta are the union over translation orbits of the l_i -th roots of unity, with multiplicity at a root of order q equal to the number of translation orbits whose return periods l_i are divisible by q .

Velocity formula with Schur correction. Let β_n be a permutation-selected zero with null space V_n , $\mathbf{\Pi}$ the orthogonal projector onto V_n , $\mathbf{Q} = \mathbb{I} - \mathbf{\Pi}$, and $\mathbf{A}_0 = \mathbb{I} - \beta_n \mathbf{P}$. The reduced problem of Methods uses

$$\begin{aligned} \bar{\mathbf{A}}_{\text{eff}} &= \mathbf{\Pi} \mathbf{L}^{-1} \mathbf{\Pi} - \bar{\mathbf{\Sigma}}, \\ \bar{\mathbf{\Sigma}} &= \mathbf{\Pi} \mathbf{L}^{-1} \mathbf{A}_0 [\mathbf{Q} \mathbf{A}_0^\dagger \mathbf{L}^{-1} \mathbf{A}_0 \mathbf{Q}]^+ \mathbf{A}_0^\dagger \mathbf{L}^{-1} \mathbf{\Pi}, \end{aligned} \tag{S4}$$

where $[\cdot]^+$ is the inverse on the range of $\mathbf{Q}$. The pseudoinverse expression is exact: $\mathbf{A}_0$ is normal, so the off-diagonal coupling lies in $V_n^\perp$, and the restricted block is bounded below near β_n because the theorem gives a fixed finite set of zeros. If $[\mathbf{L}^{-1}, \mathbf{P}] = 0$ (identical wires, or inductances uniform along each cycle) then $\bar{\mathbf{\Sigma}} = 0$. For inductance matrices with mutual coupling the correction is essential: in the validation below the projected formula

TABLE S1. **Representative quasi-static bands determined by the wiring.** The second column lists $N; \{\mathbf{W}_\alpha\}; \text{rank } \mathbf{W}$, where $\mathbf{W}_\alpha$ are independent return-translation vectors. Figure references are to the main text. The $\omega = 0$ set combines the finite- $\mathbf{k}$ condition of main-text Eq. (3) with the physical Γ count of main-text Eq. (5). The fourth column lists the physical low-frequency mode content at Γ , and the last column gives the corresponding leading Γ -point dispersion. Photon-like branches are linear, whereas screened branches can be quadratic. A screened branch that is quadratic at Γ is generally linear normal to a generic nonzero $\omega = 0$ sheet. Main-text Eqs. (9) and (10) give these dispersions for the triply periodic structures, and main-text Eq. (12) gives the two three-rod branches; the shielded-bundle photon-like branch follows directly from main-text Eq. (2).

Structure	$N; \{\mathbf{W}_\alpha\}; \text{rank } \mathbf{W}$	$\omega = 0$ set	Γ modes	Γ dispersion
Straight rod $\parallel y$ (Fig. 3a)	1; $\{a\hat{y}\}; 1$	$k_y = 0$	2 photon-like	2 linear
Connected crossed rods (Fig. 3b)	1; $\{a\hat{x}, a\hat{y}\}; 2$	$k_x = k_y = 0$	uniform-field-derived (photon-like)	1 linear
Connected 3D mesh	1; $\{a\hat{x}, a\hat{y}, a\hat{z}\}; 3$	none	none below ω_p	—
Disconnected crossed rods (Fig. 3c)	2; $\{a\hat{x}, a\hat{y}\}; 2$	$k_x k_y = 0$	1 photon-like + 1 screened	1 linear + 1 quadratic
Three disconnected rods (Fig. S2)	3; $\{a\hat{x}, a\hat{y}, a\hat{z}\}; 3$	$k_x k_y k_z = 0$	2 screened	2 quadratic
Twisted-pair crystal (Fig. 2b)	2; $\{2a\hat{x}, 2a\hat{y}\}; 2$	$k_x a \in \{0, \pi\}$ or $k_y a \in \{0, \pi\}$	1 photon-like + 1 screened	1 linear + 1 quadratic
Shielded six-wire bundle (Fig. 1a)	2; $\{6h\hat{z}\}; 1$	$k_z h / \pi = 0, \pm 1/3, \pm 2/3, 1$	1 quasi-TEM transmission-line	1 linear

without $\bar{\Sigma}$ misestimates velocities by 11%, 15%, 25% and 29% for (12)(34)(56), (123)(456), (123456) and (1234)(56) respectively, while Eq. (S4) reproduces the numerical slopes to within 1.1×10^{-5} . Both reductions are evaluated by the supplied script.

Representative structures. Supplementary Table S1 collects the static counts, $\omega = 0$ touching sets, and leading Γ -point dispersions for the representative geometries used in the main text.

Validation across cycle structures. We validate the theorem for five representative permutations of six wire labels using the exact circuit model (Supplementary Table S2; Supplementary Fig. S1 plots four of them, the identity case being listed in the table only). The calculations use a grounded positive-definite $\mathbf{M}$ with random capacitances and either uniform inductances or a random symmetric positive-definite $\mathbf{L}$ with mutual coupling. In every case the computed $\omega = 0$ solutions occur exactly at the predicted root-of-unity phases with the predicted multiplicities—including the multiplicities at the shared roots of (1234)(56) and the two-fold degeneracies of (123)(456)—with $|\omega^2| < 10^{-15}$ at the predicted momenta, and no touchings anywhere else. A dense scan of the lowest band over 4801 Bloch phases,

excluding neighborhoods of width 5×10^{-3} around the roots, gives smallest off-root inductive Bloch Laplacian eigenvalues of 2.5×10^{-4} for uniform $\mathbf{L}$ and 9.3×10^{-6} for random mutual $\mathbf{L}$, both well above the zero tolerance. Numerically extracted slopes agree with the Schur-corrected formula, Eq. (S4), to within 1.1×10^{-5} relative accuracy at every root of (123456), (123)(456), (1234)(56) and (12)(34)(56) with random $\mathbf{M}$ and mutual $\mathbf{L}$. These calculations use the stated circuit model with deterministic seeds and are not full-wave simulations. In the deterministic validation cases, omitting the Schur term produces maximum relative errors of 10.90%, 15.04%, 24.63%, and 29.36% for the four listed permutations. These values are example-specific and are not universal bounds. Including the Schur correction reduces the discrepancy to the 10^{-5} level; the largest value produced by the supplied finite-difference extraction is 1.21×10^{-5} . In full-wave COMSOL simulations the pinning is demonstrated for the regular and irregular 6-cycle bundles (main-text Fig. 1d,f) and for the 2-cycle twisted crystal (main-text Fig. 2f); the reference structure of disconnected crossed rods is also simulated full-wave in main-text Fig. 2e. For every q , the validated multiplicities summed over the q -th roots of unity equal $\sum_i \gcd(q, l_i)$, the number of dynamical bundle conductors of the q -cell supercell (excluding the prescribed reference shell), consistent with the supercell counting identity of Methods Eq. (15).

Validation of the three-dimensional rule. Supplementary Table S3 tests main-text Eq. (3) in multiple lattice directions. All cases use a random symmetric positive-definite $\mathbf{L}$ with mutual coupling. (i) *Twisted-pair crystal* (main-text Fig. 2b): four strands, x -bonds swapping strands 1,2 and y -bonds swapping strands 3,4, each displaced by one cell. The two quotient conductors have return-translation lattices $2a\mathbb{Z}\hat{x}$ and $2a\mathbb{Z}\hat{y}$; on a 5×5 grid of $(k_x a, k_y a) \in [0, \pi]^2$ at two values of $k_z a$ the stiffness nullity equals the predicted count—1 on each sheet $k_x a$ or $k_y a \in \{0, \pi\}$, 2 at intersections, 0 elsewhere—at every point, with smallest generic-momentum eigenvalue 7.8×10^{-2} . (ii) *Two-axis permutations*: four conductors with $\sigma_x = (12)(34)$, $\sigma_y = (13)(24)$, a single orbit with stabilizer $2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}$ in primitive-cell integer coordinates; the nullity is 1 exactly on the four lines $(k_x a, k_y a) \in \{0, \pi\}^2$ and 0 at all other sampled momenta, confirming that the multiplicity is one per orbit rather than one per momentum condition. (iii) *Annihilator structure*: two conductors with $\sigma_x = \sigma_y = (12)$; in primitive-cell integer coordinates the return-translation lattice is $\{(m, n, 0) : m+n \text{ even}\}$, and the nullity is 1 at $(k_x a, k_y a) = (0, 0)$ and (π, π) but 0 at $(\pi, 0)$ and $(0, \pi)$ —the zero set is determined by the full return-translation lattice rather than by separate conditions for each

TABLE S2. Circuit-model validation of the permutation theorem for six-wire bundles (grounded $\mathbf{M}$ with random C_{ij} , C_{i0}). Zero phases are $k_z h/\pi$; the multiplicities are listed in the same order as the phases. “max $|\omega^2|$ ” is the largest residual eigenvalue at the predicted zeros over both inductance classes (uniform and random mutual); “min gap” is the smallest off-root stiffness eigenvalue in the 4801-phase scan; “vel. err.” is the largest relative deviation of numerically extracted slopes from Eq. (S4) (random mutual $\mathbf{L}$).

σ	zero phases	mult.	max $ \omega^2 $	min gap
(123456)	$0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}$	1, 1, 1, 1, 1, 1	$< 10^{-15}$	2×10^{-5}
(123)(456)	$0, \frac{2}{3}, \frac{4}{3}$	2, 2, 2	$< 10^{-15}$	2×10^{-5}
(12)(34)(56)	0, 1	3, 3	$< 10^{-15}$	2×10^{-5}
(1234)(56)	$0, \frac{1}{2}, 1, \frac{3}{2}$	2, 1, 2, 1	$< 10^{-15}$	2×10^{-5}
identity	0	6	$< 10^{-15}$	9×10^{-6}
vel. err. (Schur-corrected): $\leq 1.1 \times 10^{-5}$ at all roots tested				

TABLE S3. Validation of the three-dimensional permutation rule (random mutual $\mathbf{L}$ throughout). “nullity” is the number of stiffness eigenvalues below 10^{-10} ; the prediction is main-text Eq. (3) evaluated from the return-translation lattices.

wiring (σ_x, σ_y)	momenta tested	nullity = prediction
twisted pairs	$5 \times 5 \times 2$ grid + generic	yes, all points
(12)(34), (13)(24)	$\{0, \pi, 1.1, 2.2\}^2$	yes; 1 only at $\{0, \pi\}^2$
(12), (12)	$\{0, \pi\}^2$	yes; 0 at $(\pi, 0), (0, \pi)$

direction.

SUPPLEMENTARY NOTE 3. SCREENED-MODE DISPERSIONS

All calculations in this note solve the exact Maxwell–Telegraph eigenproblem, main-text Eq. (7), in units $\varepsilon_0 = \mu_0 = a = 1$. Physical branches are distinguished from the static and gauge kernel of the first-order operator using the circuit-energy fraction criterion stated in Supplementary Note 5. The circuit-energy fraction is then used only to classify the resulting eigenpairs as circuit dominated or field dominated and, where appropriate, to select the

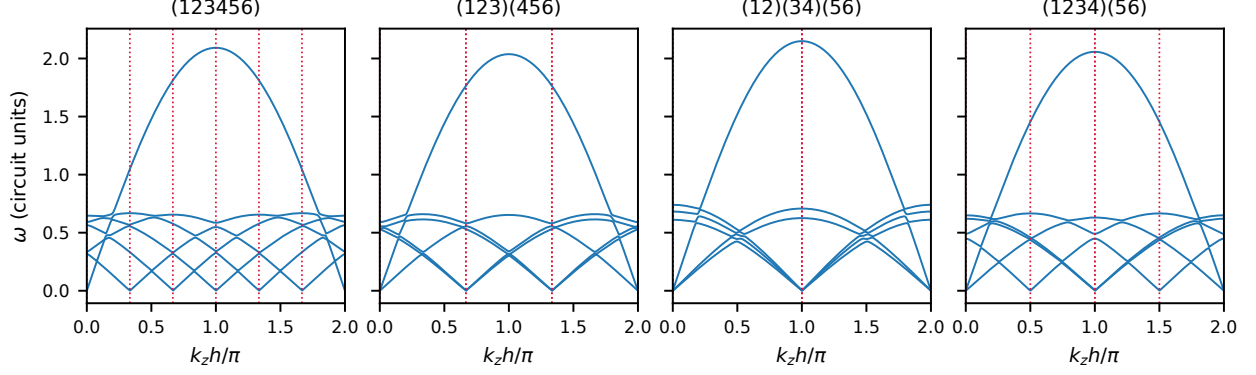


FIG. S1. **Exact circuit-model bands for four permutation cycle structures** (grounded six-wire bundle; parameters of Supplementary Note 5). Dotted vertical lines mark the zeros predicted by the permutation theorem, main-text Eqs. (3) and (4); in every case the computed bands touch $\omega = 0$ exactly there, with the predicted multiplicities. The panels are calculated from the circuit model with deterministic seeds and are not full-wave simulations.

screened and plasmonic branches for plotting. Main-text Eq. (10) is a projected generalized eigenproblem for non-orthogonal conductor directions. The calculations for disconnected crossed rods and three rods below use orthogonal directions, for which $\mathbf{S}^\top \mathbf{S} = \mathbb{I}$. If the projected metric $\mathbf{\Pi}_\kappa \mathbf{S}^\top \mathbf{S} \mathbf{\Pi}_\kappa$ is singular, its null directions correspond to zero-net-current channels and must first be treated by the linear problem. Equation (10) applies to the complementary quadratic sector on which the projected magnetic metric is positive definite.

Quadratic band of disconnected crossed rods. Equation (11) is interpreted as a fixed-direction limit with $k_x^2 + k_y^2 > 0$. When $k_x = 0$ or $k_y = 0$, the branch lies on the corresponding zero-frequency plane at this order. On the k_z axis, both numerator and denominator vanish and the leading expression does not determine the dispersion; the full projected problem or higher-order terms must be used. For $\mathbf{C}' = C' \mathbb{I}$, $k_z = 0$, the closed form $\omega^2 = 2a^2 k_x^2 k_y^2 / (\mu_0 C')$ [main-text Eq. (11)] is confirmed with relative errors 5.9×10^{-4} , 1.5×10^{-4} , 3.7×10^{-5} at $|k_x|a = 0.02, 0.01, 0.005$ ($k_y = 0.6 k_x$): the error scales as $k^2 a^2$, as expected for a leading-order result. Out of plane, at $\mathbf{k}a = (0.01, 0.006, 0.02)$, the general form including the factor $k^2 / (k_x^2 + k_y^2)$ matches to 4.6×10^{-4} , whereas omitting the factor gives a result that differs by a factor of 3.9; the out-of-plane factor is therefore required. With a generic symmetric positive-definite matrix $\mathbf{C}'$ the closed form $\omega^2 = a^2 k_x^2 k_y^2 \sum_{ij} C'_{ij} / (\mu_0 \det \mathbf{C}')$ holds to 6.7×10^{-5} at $|\mathbf{k}|a \approx 7 \times 10^{-3}$; in the Maxwell-capacitance sign convention $\mathbf{C}' = \begin{pmatrix} C_s & -C_m \\ -C_m & C_s \end{pmatrix}$ this reads

$\omega^2 = 2a^2 k_x^2 k_y^2 / [(C_s + C_m)\mu_0]$: mutual capacitance softens the band. No term linear in $\mathbf{k}$ exists. At $\mathbf{k}a = (0.01, 10^{-4}, 0)$, the computed ω^2 agrees with the pure product form to 10^{-4} , so any possible linear contribution is at least four orders smaller than the product term.

Three disconnected orthogonal rods. We next apply the direction-resolved selection rule to the structure formed by three mutually orthogonal and electrically disconnected rods, shown in Fig. S2b. At the Γ point, the three conductor potentials contain one common-potential gauge direction and two independent relative-potential directions. Because the rods are disconnected, there is no uniform mean-field direction compatible with all three rod returns. The static count therefore gives

$$r_\phi = 2, \quad d_{\text{per}} - \text{rank}(W) = 3 - 3 = 0, \quad (\text{S5})$$

and hence

$$N_0 = 2, \quad (\text{S6})$$

The two physical static modes are consequently screened relative-potential modes rather than uniform-field-derived modes. For a generic propagation direction

$$\mathbf{k} = \epsilon \hat{\mathbf{k}}, \quad (\text{S7})$$

together with the continuity condition

$$\hat{K} J_0 = v C' V_0, \quad (\text{S8})$$

For three mutually orthogonal one-channel rods, the channel-direction matrix has full rank,

$$\mathbf{S} J_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{S9})$$

so that

$$\ker \mathbf{S} = \{0\}, \quad (\text{S10})$$

No nonzero zero-net-cell-current channel is therefore available to transport either relative-potential pattern linearly. For generic directions away from the coordinate planes and axes, the two screened branches are consequently predicted to have the leading behavior

$$\omega_\alpha(\mathbf{k}) = D_\alpha(\mathbf{k}) |\mathbf{k}|^2 + o(|\mathbf{k}|^2), \quad \alpha = 1, 2. \quad (\text{S11})$$

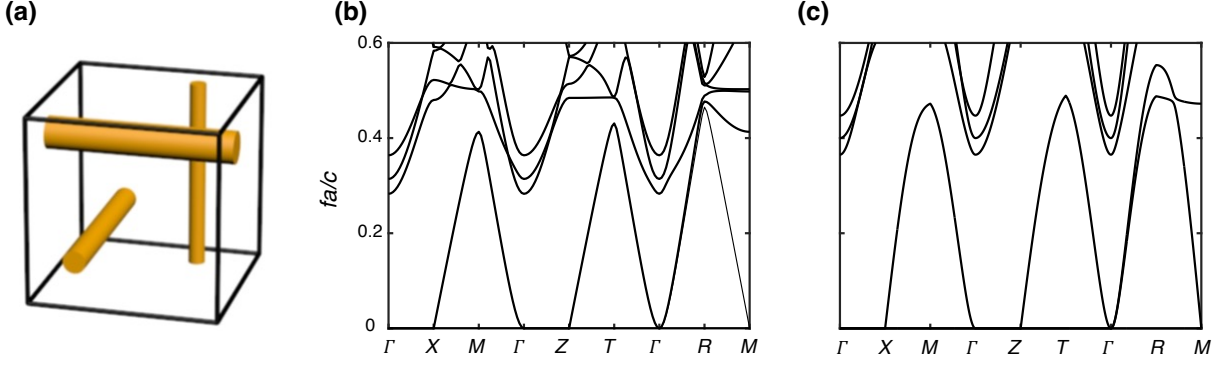


FIG. S2. **Three disconnected orthogonal rods: Maxwell–Telegraph prediction and full-wave comparison.** (a) Primitive cell containing three mutually orthogonal, electrically disconnected PEC rods directed along x , y , and z . The rods cross geometrically but remain electrically isolated. Low-frequency bands along the direction $\Gamma-X-M-\Gamma-Z-T-\Gamma-R-M$, calculated using (b) the Maxwell–Telegraph model and (c) the full-wave COMSOL eigenproblem. Two branches approach zero at Γ , consistent with the two relative-potential degrees of freedom obtained from the static count. Their vanishing slopes and fitted small- k behavior are consistent with the predicted $\omega \propto |\mathbf{k}|^2$ dispersion.

Special directions on which $\hat{K}$ loses rank require separate treatment and are not used for the scaling comparison below. Figure Sa,b compares this prediction with bands calculated using both the Maxwell–Telegraph model and the full-wave COMSOL eigenproblem. Both calculations show two low-frequency branches approaching zero at Γ , with no additional uniform-field-derived branch between these modes and the plasma-frequency sector over the frequency range shown.

In both calculations, the two lowest bands bend toward zero with vanishing slope at Γ , consistent with the predicted quadratic rather than linear dispersion. Because the smallest frequencies accessible to the full-wave eigensolver are limited by numerical resolution, the COMSOL data should be interpreted as supporting the quadratic prediction rather than independently establishing the exact asymptotic exponent.

Inductance independence. At $\mathbf{k}a = 0.01(1, 0.7, 0.45)$, the relative changes in the two screened eigenvalues of the cell with three disconnected rods are below 10^{-3} when $\mathbf{L}'$ is rescaled tenfold or replaced by a random matrix with mutual coupling, confirming that the leading screened coefficients contain $\mathbf{C}'$ and μ_0 only; the residual drift is the expected

$\mathcal{O}(k^2 a^2)$ correction.

Closed forms for three disconnected rods. For $\mathbf{C}' = C'\mathbb{I}$ and for a generic symmetric positive-definite $\mathbf{C}'$, the two invariants of main-text Eq. (12) and the projected spectrum of main-text Eq. (10) are confirmed with relative errors decreasing as $k^2 a^2$: at $|\mathbf{k}|a = 6.5 \times 10^{-3}$, the relative errors in the sum, product, and individual branches are, respectively, $(4.1, 7.7, 4.3) \times 10^{-5}$ for the scalar case and $(4.5, 8.3, 4.8) \times 10^{-5}$ for the generic case. Along the (111) diagonal with equal Cartesian components $k_x a = k_y a = k_z a = 0.01$ (that is, $|\mathbf{k}|a = 1.73 \times 10^{-2}$), the scalar-capacitance case has a relative splitting of only 6×10^{-12} , and the degenerate value agrees with $3a^2 k^4 / (\mu_0 C')$, in which k denotes the common Cartesian component, to 2.7×10^{-4} . This degeneracy is protected by coordinate-permutation symmetry; a generic capacitance matrix need not retain it. At $|\mathbf{k}|a = 0.39$ along $\hat{\mathbf{k}} \propto (1, 0.7, 0.45)$, with $\mathbf{C}' = 1.1\mathbb{I}$ and $\mathbf{L}' = 0.6\mathbb{I}$, the model spectrum contains exactly two screened branches below the cutoff ($\omega a/c = 0.054$ and 0.085), followed by the plasmonic cluster at 1.33 – 1.38 ($\omega_p a/c = 1.29$), with no additional uniform-field-derived branch between the screened branches and the plasmonic cluster..

Equal-normalization parallel current channels are linear. For two conductors with the same normalized current direction $\hat{\mathbf{x}}$ and mutual $\mathbf{C}'$ and $\mathbf{L}'$, the zero-net-current subspace is one-dimensional, $\mathcal{U} = \text{span}\{(1, -1)^\top\}$. Main-text Eq. (9) therefore gives the leading coefficient $\omega = v|k_x|[1 + \mathcal{O}(k^2 a^2)]$, where $v^2 = \mathbf{u}^\top \mathbf{C}'^{-1} \mathbf{u} / \mathbf{u}^\top \mathbf{L}' \mathbf{u}$ and $\mathbf{u} = (1, -1)^\top$. The relative errors are $\leq 8 \times 10^{-6}$ at $k_\perp = 0$ for $k_x a = 0.02, 0.01, 0.005$, and the branch remains flat across the transverse zone to within 0.5% up to $k_\perp a = 0.4$ in either transverse direction. The difference from the product form for disconnected crossed rods at the same momenta follows from the selection rule. This test uses equal current-channel normalization; it does not identify unequal return-translation multiplicities with unequal vectors in $\mathbf{S}$.

Channel extension for connected crossed rods. A connected conductor with several independent current directions requires separate conductor-potential and directed-current coordinates. Let $\mathbf{V} \in \mathbb{C}^{N_c}$ contain the conductor potentials and $\mathbf{J} \in \mathbb{C}^M$ the directed channel currents. With assignment matrix $\mathbf{B} \in \mathbb{R}^{N_c \times M}$, direction matrix $\mathbf{G} = (\hat{\mathbf{g}}_1 \cdots \hat{\mathbf{g}}_M)$, and $\mathbf{K}_g = \text{diag}(\mathbf{k} \cdot \hat{\mathbf{g}}_m)$, the plane-wave circuit equations are

$$\mathbf{K}_g \mathbf{B}^\top \mathbf{V} = \omega \mathbf{L}'_{\text{ch}} \mathbf{J} - i \mathbf{G}^\top \mathbf{E}, \quad \mathbf{B} \mathbf{K}_g \mathbf{J} = \omega \mathbf{C}'_c \mathbf{V}, \quad \mathbf{j} = \mathbf{G} \mathbf{J} / a^2. \quad (\text{S12})$$

The voltage–current blocks $\mathbf{B} \mathbf{K}_g$ and $\mathbf{K}_g \mathbf{B}^\top$ are mutually adjoint, so the coupled first-order

problem remains Hermitian for real $\mathbf{k}$. Main-text Eq. (7) is recovered for one channel per conductor, $\mathbf{B} = \mathbb{I}$ and $\mathbf{G} = \mathbf{S}$. Main-text Fig. 3e uses $N_c = 1$, $M = 2$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \end{pmatrix}$, and $\mathbf{G} = \begin{pmatrix} \hat{x} & \hat{y} \end{pmatrix}$, separating the single equipotential conductor of the connected crossed rods from its two directed current channels. The supplied `verify_mt.py` script includes a matched unit-parameter sanity check of Eq. (S12); for that test the lowest positive branch reproduces $\omega = \sqrt{k_x^2 + k_y^2}$ at two values of k_z to machine precision. This matched-case identity is not used as a general closed form for arbitrary channel parameters.

Plasma branches. The mutual-inductance cutoffs $\omega_{p,\pm}$ of Methods agree with the exact eigenproblem to 8.7×10^{-11} relative accuracy at $|\mathbf{k}|a \rightarrow 0$. The second-order expansion about the symmetric cutoff contains the radicand $(c^2 - v_{\text{TL}}^2)^2(k_x^2 - k_y^2)^2$, *not* $(c^2 + v_{\text{TL}}^2)^2$: at $k_y = 0$ the roots are the exact on-wire plasmon $v_{\text{TL}}^2 k_x^2$ and transverse branch $c^2 k_x^2$, which the wrong sign would replace by a spurious flat branch; at $k_x = k_y$ the two forms coincide, so the error would not be detected along the diagonal. Neither $\omega^2 - \omega_p^2$ nor $\omega - \omega_p$ contains a term linear in $|\mathbf{k}|$: the linear-order secular condition is a perfect square multiplied by $\det \mathbf{C}' > 0$.

SUPPLEMENTARY NOTE 4. EFFECTIVE CIRCUIT

Equivalence of the two field-eliminated forms. Eliminating the field from the Maxwell sector dresses the inductance with the transverse projector of the radiated field: with $\boldsymbol{\kappa} = \mathbf{S}^\top \mathbf{k}$ and $\Delta = k_0^2 - k^2$,

$$\mathbf{L}_{\text{eff}}^{\text{exact}} = \mathbf{L}' - \frac{\mu}{a^2 \Delta} \left(\mathbf{S}^\top \mathbf{S} - \frac{\boldsymbol{\kappa} \boldsymbol{\kappa}^\top}{k_0^2} \right), \quad (\text{S13})$$

while the capacitance remains unchanged. Using $k_0^2 = \omega^2 \varepsilon \mu$ and the identity $\boldsymbol{\kappa} \boldsymbol{\kappa}^\top = \mathbf{K} \mathbf{1} \mathbf{1}^\top \mathbf{K}$, valid for any $\mathbf{S}$ since $\kappa_n = \mathbf{k} \cdot \hat{\mathbf{e}}_n$, the longitudinal term can be moved to the capacitive side and yields Methods Eq. (14) identically; the equivalence holds for every $\mathbf{S}$ (verified symbolically for generic $\mathbf{S}$ and against the exact eigenproblem) on the common domain $\Delta \neq 0$. This elimination is the Schur complement restricted to the current sector. It removes the sourceless $\mathbf{J} = 0$ solutions, which form the air cone $\omega = c|\mathbf{k}|$ at $\Delta = 0$; these solutions must be added separately from the homogeneous field sector.

SUPPLEMENTARY NOTE 5. NUMERICAL DETAILS

All lengths are expressed in units of the lattice constant a , with $\mu_0 = \varepsilon_0 = 1$ in the model calculations. The geometric parameters are stated in the main-text figure captions and Methods. The supplied scripts specify the numerical matrices, random seeds, scan grids, branch-selection tolerances, and cases with mutual coupling; the Maxwell–Telegraph checks include both scalar and generic positive-definite matrices.

Positive-frequency eigenpairs of the first-order Maxwell–Telegraph operator were obtained from the generalized eigenproblem using the cut $\omega > 10^{-9}$ in model units and were checked using the normalized eigenproblem residual. Exact zero-frequency global-potential, gauge, and static-coordinate directions were excluded separately. Where applicable, the Maxwell divergence constraints were used to distinguish propagating branches from unconstrained static solutions. For plotting and mode classification, we calculated the circuit-energy fraction $\Psi_{VJ}^\dagger \mathcal{Q}_{VJ} \Psi_{VJ} / (\Psi^\dagger \mathcal{Q} \Psi)$. A threshold greater than 1/2 was used to label an already identified physical eigenpair as circuit dominated and to distinguish screened and plasmonic branches from field-dominated solutions, including the sourceless air-cone sector. This energy-fraction threshold was a plotting and classification convention and was not a criterion of physicality.

Positive-frequency eigenpairs were identified from the generalized eigenproblem and checked using the normalized residual. Exact zero-frequency global-potential and static-coordinate directions were excluded separately. The circuit-energy fraction was used only to classify physical eigenpairs as circuit-dominated or field-dominated; it was not used as a criterion of physicality. When figures display only screened or plasma branches, the stated circuit-energy threshold is a plotting/classification convention.

Full-wave modeling. COMSOL calculations with PEC metal and Floquet boundary conditions give main-text Figs. 1d and 1f, main-text Figs. 2c and 2d, and Supplementary Fig. S2c. Main-text Fig. 1e is an LC calculation. Main-text Figs. 2e and 2f, Figs. 3d and 3f, Fig. 4, and Supplementary Fig. S2b use the one-current-channel-per-conductor Maxwell–Telegraph model. Main-text Fig. 3e uses the channel-resolved extension, and Supplementary Fig. S1 is an exact circuit calculation.