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Stress Analysis of Anisotropic Inclusion Problems Using Complex-Variable Methods

Liming Chen · Seiichi Nomura*

Abstract This work presents an analytical method for deriving the stress fields in an elastic medium containing a circular anisotropic inclusion embedded in an infinitely extended isotropic matrix and subjected to uniform far-field stresses.

Stress functions are employed to represent the stress distributions. While the Airy stress function has been widely used to derive stress fields in two-dimensional isotropic materials, Lekhnitskii's formalism provides an alternative approach for two-dimensional anisotropic materials. In this formulation, the stress function is expressed as the real part of the sum of two analytic functions of independent complex variables, which are determined from the roots of the characteristic equation associated with the anisotropic elastic constants.

In the present study, the conventional Airy stress function is used for the isotropic matrix, whereas Lekhnitskii's formalism is employed for the anisotropic inclusion. The stress function for the matrix is represented by a Laurent series, while that for the inclusion is expressed as a Taylor series. The unknown complex coefficients in these series are determined by enforcing the continuity of displacements and tractions across the matrix–inclusion interface and also satisfying the far-field boundary conditions.

The resulting stress fields are obtained in closed form, providing a novel analytical solution to the circular anisotropic inclusion problem. The proposed approach can also be extended to related problems, including thermal stress and heat conduction, demonstrating its potential applicability to a broad range of engineering and materials-science problems.

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1 Introduction

The usefulness of composite materials lies in their capability to be tailored for specific applications by virtue of anisotropy. Despite this advantage, the stress analysis for anisotropic composite materials presents a challenge with limited resources available in the literature to address the complexity. This challenge is particularly pronounced when considering composites comprising matrix and inclusion (fiber) phases.

Significant research exists on stress analysis for composites with isotropic constituents, pioneered by the work of Eshelby [1]. However, when one of the constituents exhibits anisotropy, the complexity aggravates.

For stress analysis in 2-D, the Airy stress function has proven effective for isotropic materials [2,3]. For 2-D anisotropic materials, Lekhnitskii [4,5] derived formulas using complex variables related to the anisotropic material constants.

This paper presents an analytical method aimed at obtaining the stress fields in an elastic medium containing an anisotropic circular inclusion placed within an infinitely extended isotropic matrix and subjected to far-field stresses.

Historically, the Airy stress function has been mainly used as a tool for 2-D isotropic materials. In Lekhnitskii's formulation for anisotropic materials, the stress function is expressed as the real part of the sum of two analytic functions of two independent complex variables related to the roots of the characteristic equation associated with the anisotropic elastic constants. In this paper, the Airy stress function for isotropic materials is used for the matrix phase, while Lekhnitskii's complex-variable representation of the Airy stress function is used for the inclusion phase. The stress function for the matrix phase is expressed by the Laurent series while the Taylor series is employed for the inclusion phase. The unknown complex coefficients of the series can be determined by satisfying the boundary conditions and the continuity conditions for the displacement and traction across the matrix-inclusion interface.

Although complex-variable methods have been used in elasticity for many decades, their implementation and algebraic manipulation remain challenging. Without a computer algebra system, obtaining the results reported in this paper would have been impossible. The present method was implemented for an inclusion problem where both the inclusion and matrix are isotropic [6], making this paper a sequel to that work.

2 Modeling

2.1 Stress Functions

In a state of 2-D stress (plane stress or plane strain), the strain-stress relationship for an anisotropic material is expressed as

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{26} \\ \beta_{16} & \beta_{26} & \beta_{66} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}, \tag{1}$$

where $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$ are the in-plane strain components, $(\sigma_x, \sigma_y, \tau_{xy})$ are the in-plane stress components and β_{ij} is defined as

$$\beta_{ij} = \begin{cases} s_{ij}, & \text{plane stress,} \\ s_{ij} - \dfrac{s_{i3}s_{j3}}{s_{33}}, & \text{plane strain,} \end{cases} \tag{2}$$

where s_{ij} is the contracted elastic compliance.

The strain compatibility condition for 2-D is expressed as

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{3}$$

Substituting Eq. (1) into Eq. (3) yields

$$\begin{aligned} &\beta_{11}\frac{\partial^2 \sigma_x}{\partial y^2} + \beta_{12}\frac{\partial^2 \sigma_y}{\partial y^2} + \beta_{16}\frac{\partial^2 \tau_{xy}}{\partial y^2} + \beta_{12}\frac{\partial^2 \sigma_x}{\partial x^2} + \beta_{22}\frac{\partial^2 \sigma_y}{\partial x^2} + \beta_{26}\frac{\partial^2 \tau_{xy}}{\partial x^2} \\ &\quad = \beta_{16}\frac{\partial^2 \sigma_x}{\partial x \partial y} + \beta_{26}\frac{\partial^2 \sigma_y}{\partial x \partial y} + \beta_{66}\frac{\partial^2 \tau_{xy}}{\partial x \partial y}. \end{aligned} \tag{4}$$

The stress equilibrium equations in a state of plane stress without a body force are expressed as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \tag{5}$$

Equations (5) are automatically satisfied if the stress components are derived from a single function, F, as

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \tag{6}$$

The function F is called the Airy stress function [7].

Substituting Eq. (6) into Eq. (4) yields a fourth-order differential equation for F as

$$\beta_{11}\frac{\partial^4 F}{\partial y^4} - 2\beta_{16}\frac{\partial^4 F}{\partial x \partial y^3} + (2\beta_{12} + \beta_{66})\frac{\partial^4 F}{\partial x^2 \partial y^2} - 2\beta_{26}\frac{\partial^4 F}{\partial x^3 \partial y} + \beta_{22}\frac{\partial^4 F}{\partial x^4} = 0. \tag{7}$$

Equation (7) can be written as

$$L_4 F = 0, \tag{8}$$

where the 4th-order differential operator, L_4, is defined as

$$L_4 \equiv \beta_{11}\frac{\partial^4}{\partial y^4} - 2\beta_{16}\frac{\partial^4}{\partial x \partial y^3} + (2\beta_{12}+\beta_{66})\frac{\partial^4}{\partial x^2 \partial y^2} - 2\beta_{26}\frac{\partial^4}{\partial x^3 \partial y} + \beta_{22}\frac{\partial^4}{\partial x^4}. \tag{9}$$

2.2 Complex Variable Approach

The characteristic equation for Eq. (8) is in the form of

$$\beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12}+\beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22} = 0. \tag{10}$$

Equation (10) has four complex roots, forming two pairs of complex conjugates, as follows

$$\begin{aligned} \mu_k \equiv \alpha_k + i\beta_k, \qquad & \bar{\mu}_k \equiv \alpha_k - i\beta_k, \\ \beta_k \geq 0, \qquad & k = 1, 2 \end{aligned} \tag{11}$$

The general solution to Eq. (8) is expressed as

$$F = F_1(x+\mu_1 y) + F_2(x+\mu_2 y) + F_3(x+\bar{\mu}_1 y) + F_4(x+\bar{\mu}_2 y), \tag{12}$$

where $F_1 \sim F_4$ are four independent complex functions of the argument.

Introducing a general complex variable, z_k, and its conjugate, $\bar{z}_k$, defined as

$$z_k = x + \mu_k y, \qquad \bar{z}_k = x + \bar{\mu}_k y. \tag{13}$$

Additionally, since F is real, the right-hand side of Eq. (12) must consist of two complex-conjugate pairs and can therefore be reduced to

$$F = 2\,\mathrm{Re}\left(F_1(z_1) + F_2(z_2)\right), \tag{14}$$

where F_1 and F_2 are independent functions of z_1 and z_2, respectively, and $\mathrm{Re}(f)$ is the real part of a complex function f.

With Eqs. (14) and (6), the stress components are expressed as

$$\sigma_x = 2\,\mathrm{Re}\left(\mu_1^2 \frac{\mathrm{d}^2 F_1(z_1)}{\mathrm{d}z_1^2} + \mu_2^2 \frac{\mathrm{d}^2 F_2(z_2)}{\mathrm{d}z_2^2}\right), \tag{15}$$

$$\sigma_y = 2\,\mathrm{Re}\left(\frac{\mathrm{d}^2 F_1(z_1)}{\mathrm{d}z_1^2} + \frac{\mathrm{d}^2 F_2(z_2)}{\mathrm{d}z_2^2}\right), \tag{16}$$

$$\tau_{xy} = -2\,\mathrm{Re}\left(\mu_1 \frac{\mathrm{d}^2 F_1(z_1)}{\mathrm{d}z_1^2} + \mu_2 \frac{\mathrm{d}^2 F_2(z_2)}{\mathrm{d}z_2^2}\right). \tag{17}$$

Substituting Eqs. (15)–(17) into Eq. (1) and integrating the result yield the expression of the displacements as

$$u = 2\,\mathrm{Re}\left(a_1\frac{\mathrm{d}F_1(z_1)}{\mathrm{d}z_1} + a_2\frac{\mathrm{d}F_2(z_2)}{\mathrm{d}z_2}\right) - \omega_3 y + u_0, \tag{18}$$

$$v = 2\,\mathrm{Re}\left(b_1\frac{\mathrm{d}F_1(z_1)}{\mathrm{d}z_1} + b_2\frac{\mathrm{d}F_2(z_2)}{\mathrm{d}z_2}\right) + \omega_3 x + v_0, \tag{19}$$

where ω_3 is the rigid body rotation around the z-axis, u_0 and v_0 are the rigid body translations in the x- and the y-direction, respectively. The symbols a_k and b_k are defined as

$$\begin{aligned} a_k &= \beta_{11}\mu_k^2 + \beta_{12} - \beta_{16}\mu_k, \\ b_k &= \beta_{12}\mu_k + \frac{\beta_{22}}{\mu_k} - \beta_{26}. \end{aligned} \qquad k = 1, 2. \tag{20}$$

If the characteristic equation of Eq. (10) has repeating roots, i.e.,

$$\mu_1 = \mu_2 = \alpha + i\beta, \qquad \bar{\mu}_1 = \bar{\mu}_2 = \alpha - i\beta, \tag{21}$$

the solution to Eq. (7) is expressed as

$$F = F_1(z_1) + \bar{z}_1 F_2(z_1) + F_3(\bar{z}_1) + z_1 F_4(\bar{z}_1), \tag{22}$$

where $F_1 \sim F_4$ are four independent complex functions of the argument. Again, using the condition that F must be real, it can be shown that Eq. (22) is reduced to

$$F = 2\,\mathrm{Re}\left(F_1(z_1) + \bar{z}_1 F_2(z_1)\right), \tag{23}$$

using two independent functions.

For isotropic materials, it follows

$$\begin{aligned} \mu_1 = \mu_2 = i, \qquad & \bar{\mu}_1 = \bar{\mu}_2 = -i, \\ \alpha = 0, \qquad & \beta = 1, \\ z_1 = x + iy = z, \qquad & \bar{z}_1 = x - iy = \bar{z}. \end{aligned} \tag{24}$$

Therefore, F can be expressed as

$$F = 2\,\mathrm{Re}\left(F_1(z) + \bar{z} F_2(z)\right), \tag{25}$$

or commonly written as

$$F = \mathrm{Re}\left(\chi(z) + \bar{z}\gamma(z)\right), \tag{26}$$

where $\chi(z)$ and $\gamma(z)$ are independent analytic functions of z. Equation (26) is the well-known elasticity solution for an isotropic material. The complex representation of the x- and y-displacements for isotropic material is expressed as [3,6,7]

$$u + iv = \frac{1}{2\mu}\left[\kappa\gamma(z) - z\overline{\gamma'(z)} - \overline{\chi'(z)}\right], \tag{27}$$

where μ is the shear modulus and κ is the Kolosov parameter expressed as

$$\kappa = \begin{cases} \dfrac{3-\nu}{1+\nu}, & \text{plane stress,} \\ 3-4\nu, & \text{plane strain,} \end{cases} \tag{28}$$

where ν is Poisson's ratio.

2.3 Determination of Stress Functions for Inclusion and Matrix

An infinitely extended isotropic medium with an anisotropic circular inclusion at its center, having a radius of a, is considered (Fig. 1). The entire medium is subjected to a uniform far-field stress.

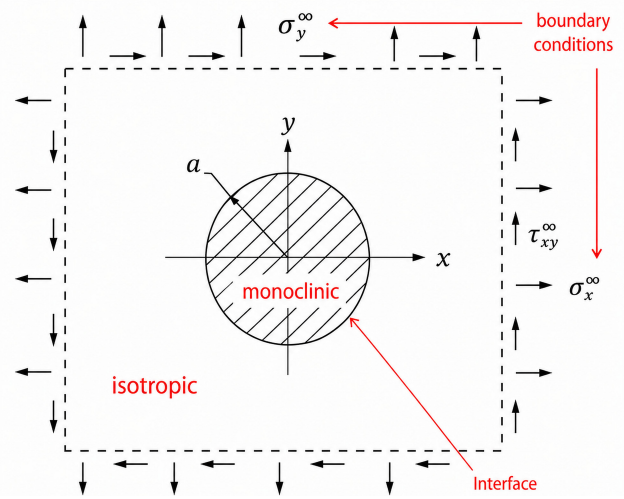


Fig. 1 An anisotropic inclusion in an isotropic matrix phase

As shown in the previous section, the stress functions for both the inclusion and matrix phases are expressed by analytic functions of the complex variables. These functions are to be determined from the topology of the domain and the boundary conditions [3,7]. In a simply-connected domain, they can be represented by the Taylor series. Hence, the analytic components of the Airy stress-function representation, $F_1(z_1)$ and $F_2(z_2)$, for the inclusion phase are expanded by the Taylor series as

$$F_1(z_1) = \sum_{i=1}^{\infty} A_i z_1^i, \qquad F_2(z_2) = \sum_{i=1}^{\infty} B_i z_2^i. \tag{29}$$

where A_i and B_i are unknown complex coefficients to be determined from the boundary and continuity conditions. As these analytic functions enter the stress expressions through differentiation, the index begins at $i = 1$.

In the doubly-connected domain of the matrix phase, the analytic functions $\chi(z)$ and $\gamma(z)$ in the isotropic formulation can be expressed by Laurent series as

$$\chi(z) = \widetilde{C}_1 \log z + \widetilde{C}_2 z \log z + \sum_{i=-\infty}^{\infty} C_i z^i, \tag{30}$$

$$\gamma(z) = \widetilde{D}_1 \log z + \widetilde{D}_2 z \log z + \sum_{i=-\infty}^{\infty} D_i z^i, \tag{31}$$

where C_i, D_i and $\widetilde{C}_1 \sim \widetilde{D}_2$ are unknown complex coefficients yet to be determined. The $\log z$ and $z \log z$ terms are added to account for the $1/z$ term after differentiations ($\log z$ for the first-order derivative and $z \log z$ for the second-order derivative).

The continuity conditions of the displacements and traction forces at $r = a$ must be satisfied as

$$u^{\text{matrix}} = u^{\text{inclusion}}, \qquad v^{\text{matrix}} = v^{\text{inclusion}}, \qquad r = a, \tag{32}$$

$$t_x^{\text{matrix}} = t_x^{\text{inclusion}}, \qquad t_y^{\text{matrix}} = t_y^{\text{inclusion}}, \qquad r = a, \tag{33}$$

where u and v are the displacements and t_x and t_y are the traction forces. In addition, the far-field stress components are assumed to be uniform as

$$(\sigma_x, \sigma_y, \tau_{xy}) \to (\sigma_x^\infty, \sigma_y^\infty, \tau_{xy}^\infty) \quad \text{as} \quad x \to \infty,\ y \to \infty. \tag{34}$$

The unknown coefficients in Eqs. (29)–(31) can be determined to satisfy Eqs. (32)–(34).

3 Results and Discussion

After solving for all the unknown coefficients subject to the boundary and continuity conditions, it was found that A_2, B_2, C_{-1}, C_2, D_{-2}, D_1 and D_2 are the only non-zero coefficients while all other coefficients are zero. The resulting analytic functions for the anisotropic inclusion phase take the form

$$F_1(z_1) = (a[1] + a[2]i) z_1^2, \qquad F_2(z_2) = (b[1] + b[2]i) z_2^2, \tag{35}$$

and the complex potentials for the isotropic matrix phase are given by

$$\chi(z) = \frac{c[1] + c[2]i}{z} + \frac{\sigma_x^\infty + \sigma_y^\infty}{4} z, \tag{36}$$

$$\gamma(z) = \frac{d[1] + d[2]i}{z^2} + \left(i \frac{\tau_{xy}^\infty}{2} + \frac{1}{4}(\sigma_y^\infty - \sigma_x^\infty) \right) z^2 + d[3] \log z. \tag{37}$$

where $a[1] \sim d[3]$ are real-valued unknown constants to be determined from the continuity and boundary conditions of Eqs. (32)–(34). These nine unknown constants can be solved exactly using symbolic-capable software [8]. The resulting expressions are too lengthy to be presented here, but they are completely manageable and available upon request. Figure 2 shows a screenshot of a portion of the symbolic output.

```
σxinclusion
(1 + κm) (2 β1^2 β11 β2^2 κm σx∞ + 2 β22 κm σx∞ + β1^2 β11 β2^2 κm^2 σx∞ + β22 κm^2 σx∞ -
  4 β1^2 β11 β12 β2^2 μm σx∞ + 4 β1^2 β11 β22 μm σx∞ - 4 β12 β22 μm σx∞ + 4 β11 β2^2 β22 μm σx∞ +
  2 β1^2 β11 β22 κm μm σx∞ + 2 β11 β2^2 β22 κm μm σx∞ + 6 β1^2 β11 β2^2 β22 κm μm σx∞ +
  6 β22^2 κm μm σx∞ - 4 β1^2 β11 β12^2 β2^2 μm^2 σx∞ + 4 β1^2 β11 β12 β22 μm^2 σx∞ - 4 β12^2 β22 μm^2 σx∞ +
  4 β11 β12 β2^2 β22 μm^2 σx∞ - 12 β1^2 β11 β12 β2^2 β22 μm^2 σx∞ + 12 β1^2 β11 β22^2 μm^2 σx∞ -
  12 β12 β22^2 μm^2 σx∞ + 12 β11 β2^2 β22^2 μm^2 σx∞ + 4 β1^2 β11 β16 β2^2 β26 μm^2 σx∞ +
  12 β1^2 β11 β2^2 β26^2 μm^2 σx∞ - 2 β1^2 β11 β2^2 κm σy∞ - 2 β22 κm σy∞ + β1^2 β11 β2^2 κm^2 σy∞ +
  β22 κm^2 σy∞ + 4 β1^2 β11 β12 β2^2 μm σy∞ - 4 β1^2 β11 β22 μm σy∞ + 4 β12 β22 μm σy∞ -
  4 β11 β2^2 β22 μm σy∞ - 8 β1^2 β11 β12 β2^2 κm μm σy∞ + 2 β1^2 β11 β22 κm μm σy∞ -
  8 β12 β22 κm μm σy∞ + 2 β11 β2^2 β22 κm μm σy∞ - 2 β1^2 β11 β2^2 β22 κm μm σy∞ - 2 β22^2 κm μm σy∞ +
  12 β1^2 β11 β12^2 β2^2 μm^2 σy∞ - 12 β1^2 β11 β12 β22 μm^2 σy∞ + 12 β12^2 β22 μm^2 σy∞ -
  12 β11 β12 β2^2 β22 μm^2 σy∞ + 4 β1^2 β11 β12 β2^2 β22 μm^2 σy∞ - 4 β1^2 β11 β22^2 μm^2 σy∞ +
  4 β12 β22^2 μm^2 σy∞ - 4 β11 β2^2 β22^2 μm^2 σy∞ - 12 β1^2 β11 β16 β2^2 β26 μm^2 σy∞ -
  4 β1^2 β11 β2^2 β26^2 μm^2 σy∞ + 8 α2 β1^2 β11 β22 μm^2 (- ((β16 + 3 β26) σx∞) + (3 β16 + β26) σy∞) +
  8 α1 β11 β22 μm (α2 (2 + κm + 2 β12 μm + 6 β22 μm) σx∞ +
```

Fig. 2 Partial output of the solution from *Mathematica*

3.1 Illustrative Example

To demonstrate the closed-form solution, an illustrative example is considered for a circular anisotropic inclusion embedded in an isotropic matrix under plane-stress loading. All the input values are normalized hence dimensionless.

For this example, the isotropic matrix is assumed to be epoxy resin, and the circular anisotropic inclusion of radius $a = 1$ is formed with an E-glass/epoxy unidirectional composite, which has 2/3 of the glass aligned with the x-axis and 1/3 of the glass aligned in a direction of 55° from the x-axis to obtain the monoclinic material characteristics [9]. The contracted elastic compliance components for the inclusion are then calculated from the material property values as [10]

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{26} \\ \beta_{16} & \beta_{26} & \beta_{66} \end{pmatrix} = \begin{pmatrix} 0.2227 & -0.0690 & -0.0222 \\ -0.0690 & 0.5200 & -0.2431 \\ -0.0222 & -0.2431 & 1.1942 \end{pmatrix}. \tag{38}$$

For the matrix phase, the Young modulus and the Poisson ratio are chosen as

$$E_m = 0.62, \qquad \nu_m = 0.35. \tag{39}$$

For the far-field stresses at infinity, the following normalized loads are applied:

$$\sigma_x^\infty = 1, \qquad \sigma_y^\infty = 0, \qquad \tau_{xy}^\infty = 0. \tag{40}$$

Figure 3 shows the resulting von Mises stress distribution around the inclusion. It is seen that the von Mises stress inside the inclusion is uniform but discontinuous across the interface between the inclusion and the matrix. In addition, the stress disturbance caused by the inclusion decreases rapidly with increasing distance from the origin.

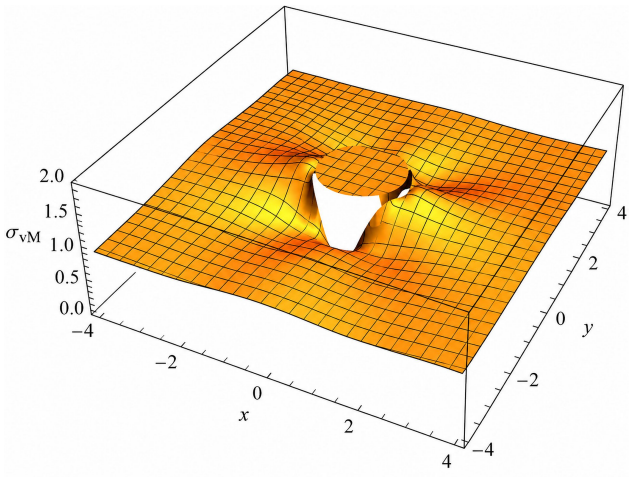


Fig. 3 The von Mises stress distribution

3.2 Finite-Element Verification

The closed-form solution was further verified by finite-element analysis using Abaqus software. A two-dimensional plane-stress model of the same circular anisotropic inclusion embedded in an isotropic matrix was analyzed under the normalized remote tensile stress of unity. Comparisons were made for the von Mises stress along the x- and y-axes, and along the matrix side of the circular interface.

For the loading and material properties considered here, the analytical von Mises stress inside the inclusion is uniform and given by

$$\sigma_{\text{in}} = 1.40551. \tag{41}$$

The corresponding matrix stresses with x and y measured in the normalized length are

$$\sigma_{\text{m-}x\text{axis}} = \sqrt{\frac{1.14605 - 2.24335x^2 - 0.586449x^4 + 2.22674x^6 + x^8}{x^8}}, \tag{42}$$

and

$$\sigma_{\text{m-}y\text{axis}} = \sqrt{\frac{1.14605 - 0.812774y^2 - 1.54016y^4 + 0.245321y^6 + y^8}{y^8}}. \tag{43}$$

The closed-form interface expression is considerably longer and is therefore not repeated here. Instead, it is evaluated directly from the analytical solution for the circumferential comparison below.

A 60×60 square domain with a centered inclusion of radius 1 was used to approximate the infinite medium. The matrix properties were given by Eq. (39), while the inclusion was defined using the monoclinic anisotropic elastic properties employed in the analytical model. Quadratic plane-stress

elements (CPS8R, with CPS6 where required) were refined near the interface. Uniform tractions were applied to produce $\sigma_x^{\infty} = 1$, and appropriate constraints were imposed to remove rigid-body motion. Mesh- and domain-refinement checks produced negligible changes in the quantities used for comparison.

Figure 4 compares the analytical and finite-element von Mises stresses along the x- and y-axes and along the matrix side of the circular inclusion–matrix interface. The FEA results reproduce the uniform inclusion stress and closely follow the analytical matrix distributions. Small oscillations occur immediately adjacent to the material interface, consistent with numerical stress recovery across the abrupt material-property discontinuity.

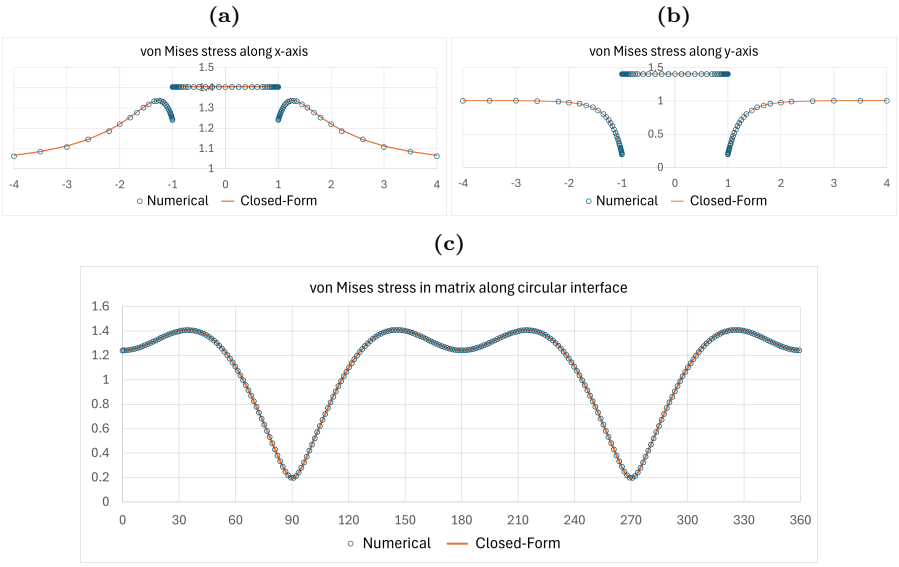


Fig. 4 Comparison of finite-element and analytical von Mises stresses along the x- and y-axes and along the matrix side of the circular inclusion–matrix interface

The circumferential results also agree closely in both magnitude and angular variation, including maxima near 35°, 145°, 215°, and 325° and minima near 90° and 270°. Overall, the finite-element results provide independent numerical verification of the closed-form solution.

4 Conclusions

In this study, an exact stress analysis was carried out for a 2-D isotropic medium containing a centrally located anisotropic inclusion of circular shape. The classical complex-variable theory, coupled with the Airy stress function and implemented using a computer algebra system, was shown to yield a novel closed-form solution. As an analytic function can be expanded by either the Taylor series or the Laurent series, the present method provides a unified approach to the general 2-D anisotropic stress analysis.

The derived expression for the stress field, though lengthy, is completely manageable. The method can be used for other fields, such as thermal stress analysis, where the medium is subjected to heat flux.

It is noted that if the matrix phase is anisotropic, the present approach is not applicable, and a more refined method is required. It is anticipated that the results will be reported in future publications.

Statements and Declarations

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Author Contributions *All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by L. Chen and S. Nomura. The first draft of the manuscript was written by L. Chen and S. Nomura and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.*
Data Availability *Data supporting the numerical verification are available upon request.*
Code Availability *The Abaqus model/input files and any symbolic-computation code are available upon request.*

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