Supplementary Information:

Mathematical Proof of the Hilbert-to-Liouville Space Transformation

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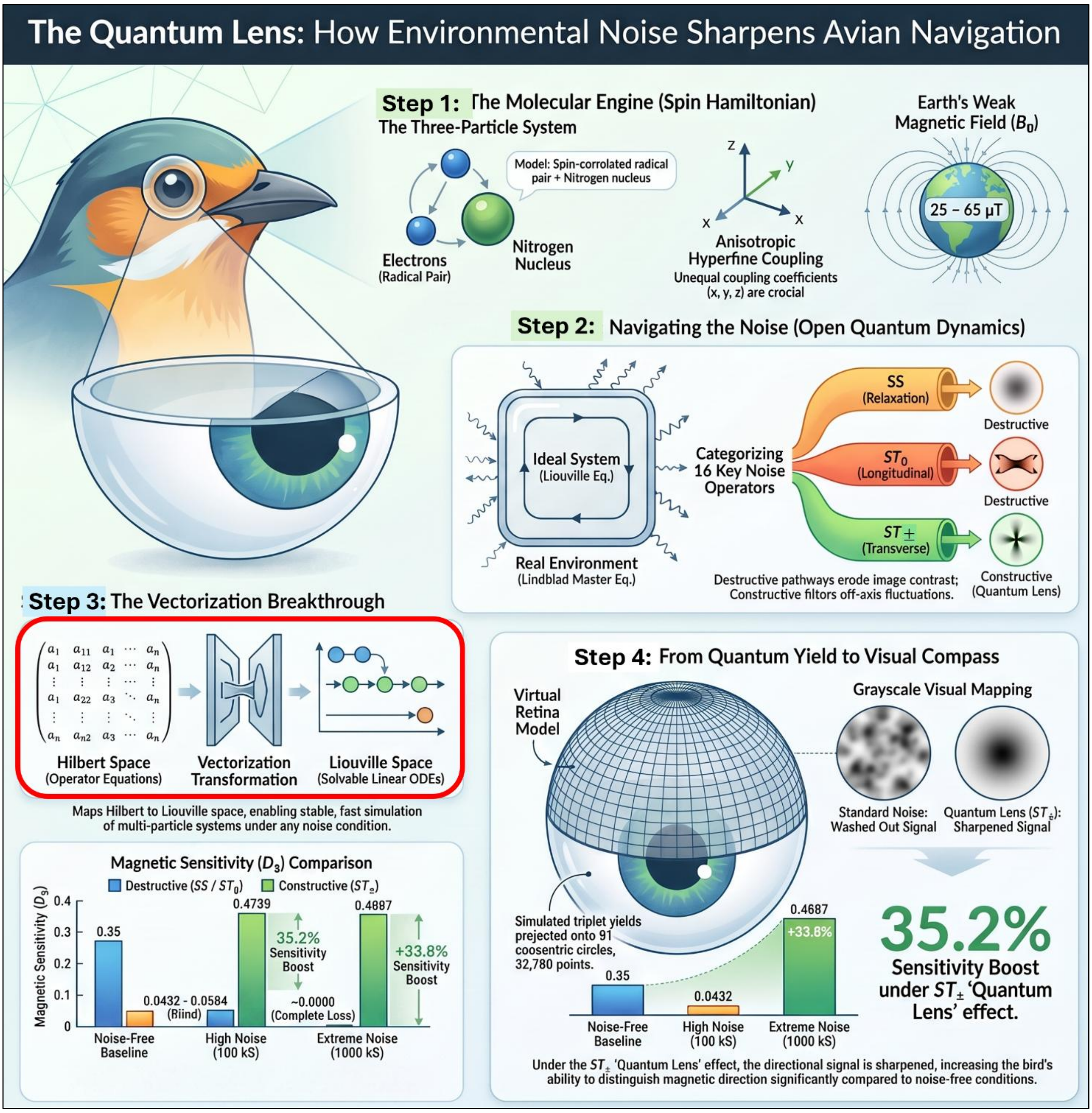

Fig. S1 Conceptual framework and multi-scale biophysical cascade of the noise-assisted avian quantum compass. The schematic illustrates the comprehensive simulation workflow in four operational steps: (Step 1) The three-particle molecular engine modeling a spin-correlated radical pair coupled to a nitrogen nucleus under Earth's weak geomagnetic field (25–65 μT); (Step 2) Systematic categorization of the 16 active Lindblad noise operators into destructive (SS and ST_0) and constructive ($ST_\pm$) dephasing pathways; (Step 3) The vectorization breakthrough mapping the density matrix from Hilbert to Liouville space to yield solvable linear ODEs; and (Step 4) Projection of triplet yields onto a virtual retina model, demonstrating how transverse dephasing ($ST_{\pm}$) acts as a "quantum lens" to enhance magnetic sensitivity by up to 35.2%.

Document Overview

This Supplementary Information document establishes the mathematical foundations and analytical proofs supporting the findings presented in the main manuscript. Specifically, we present the comprehensive mathematical derivations validating the Hilbert-to-Liouville space transformation and vectorized operator formulations highlighted in Step 3 of Fig. S1. While the main text outlines this vectorization breakthrough as a means to bypass traditional computational and scaling bottlenecks, this supplementary framework provides the explicit proofs for the vectorized Hamiltonian propagator L_H and the vectorized dissipative propagator L_D using Kronecker tensor products. By deriving these operators for arbitrary N-dimensional systems and validating their general applicability via mathematical induction for any arbitrary n-particle spin-coupled system, this document mathematically anchors the solvable, linear first-order ordinary differential equations (ODEs) that underpin our open-system avian quantum dynamics model.

Introduction and Theoretical Framework

As detailed in the Methods Section of the main text, incorporating quantum decoherence effects into the numerical model of the avian virtual compass requires reformulating the governing non-unitary Lindblad master equation[15], originally defined in Eq. (3) of the main text:

$$\frac{d}{dt}\hat{\boldsymbol{\rho}}(t) = -\frac{i}{\hbar}\left[\hat{\mathbf{H}}, \hat{\boldsymbol{\rho}}\right] + L_D(\hat{\boldsymbol{\rho}}) \tag{S1}$$

where $\hat{\boldsymbol{\rho}}(t)$ represents the system density operator, $\hat{\mathbf{H}}$ denotes the spin Hamiltonian, and $L_D(\hat{\boldsymbol{\rho}})$ signifies the dissipative propagator (or Lindblad dissipator) defined as

$$L_D(\hat{\boldsymbol{\rho}}) = \frac{1}{2}\sum_{j=1}^{2^{2n}} \lambda_j\left(\left[\hat{\mathbf{V}}_j\hat{\boldsymbol{\rho}}(t),\ \hat{\mathbf{V}}_j^{\dagger}\right] + \left[\hat{\mathbf{V}}_j,\ \hat{\boldsymbol{\rho}}(t)\hat{\mathbf{V}}_j^{\dagger}\right]\right) \tag{S2}$$

Expanding the commutator terms in Eq. (S2) yields the familiar form:

$$L_D(\hat{\boldsymbol{\rho}}) = \frac{1}{2}\sum_{j=1}^{2^{2n}} \lambda_j\left(2\hat{\mathbf{V}}_j\hat{\boldsymbol{\rho}}(t)\hat{\mathbf{V}}_j^{\dagger} - \hat{\mathbf{V}}_j^{\dagger}\hat{\mathbf{V}}_j\hat{\boldsymbol{\rho}}(t) - \hat{\boldsymbol{\rho}}(t)\hat{\mathbf{V}}_j^{\dagger}\hat{\mathbf{V}}_j\right) \tag{S3}$$

where $\hat{\mathbf{V}}_j$ denotes the transition operators representing specific decoherence channels, and λ_j represents their corresponding dephasing or relaxation rates.

To circumvent the computational bottlenecks of solving the Lindblad equation in its operator form, we establish an isomorphic vectorization mapping[17] that projects the density matrix from Hilbert space to Liouville space. This mathematical transformation restructures the $N \times N$ density matrix $\boldsymbol{\rho}$ (where the dimension $N = 2^n$) into an N^2-dimensional column vector, denoted as $|\boldsymbol{\rho}(t)\rangle\rangle$:

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & \rho_{NN} \end{bmatrix} \xleftrightarrow[\text{Hilbert} - \text{Liouville}]{} |\boldsymbol{\rho}\rangle\rangle = \begin{bmatrix} \rho_{11} \\ \rho_{21} \\ \vdots \\ \rho_{N1} \\ \vdots \\ \rho_{1N} \\ \rho_{2N} \\ \vdots \\ \rho_{NN} \end{bmatrix} \tag{S4}$$

In terms of the column vectors $\boldsymbol{\rho}_i$, the above expression can be rewritten as

$$\boldsymbol{\rho} = [\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2 \quad \cdots \quad \boldsymbol{\rho}_N] \xleftrightarrow[\text{Hilbert} - \text{Liouville}]{} |\boldsymbol{\rho}\rangle\rangle = \begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \\ \vdots \\ \boldsymbol{\rho}_N \end{bmatrix} \tag{S5}$$

Under this vectorized representation, the time derivative of the density matrix operator is mapped as:

$$\frac{d}{dt}\widehat{\boldsymbol{\rho}}(t) = L_H(\widehat{\boldsymbol{\rho}}) + L_D(\widehat{\boldsymbol{\rho}}) \xleftrightarrow{\text{Hilbert} - \text{Liouville}} \frac{d}{dt}|\boldsymbol{\rho}\rangle\rangle = \mathcal{L}_H|\boldsymbol{\rho}\rangle\rangle + \mathcal{L}_D|\boldsymbol{\rho}\rangle\rangle \tag{S6}$$

wherein the vectorized Hamiltonian propagator $\mathcal{L}_H$ and the vectorized dissipative propagator $\mathcal{L}_D$ are formulated in Liouville space using Kronecker tensor products:

$$\mathcal{L}_H = -\frac{i}{\hbar}(\mathbf{I} \otimes \mathbf{H} - \mathbf{H}^T \otimes \mathbf{I}) \tag{S7}$$

$$\mathcal{L}_D = \frac{1}{2}\sum_{j=1}^{2^{2n}} \lambda_j \left(2\overline{\mathbf{V}}_j \otimes \mathbf{V}_j - \mathbf{I} \otimes \mathbf{V}_j^\dagger \mathbf{V}_j - \left(\mathbf{V}_j^\dagger \mathbf{V}_j\right)^T \otimes \mathbf{I}\right) \tag{S8}$$

Here, $\mathbf{V}_j$ is the matrix representation of the decoherence operator $\widehat{\mathbf{V}}_j$, while $\overline{\mathbf{V}}_j$ and $\mathbf{V}_j^\dagger$ denote its complex conjugate and conjugate transpose, respectively. These formulations establish the mathematical proofs for the vectorized operators $\mathcal{L}_H$ and $\mathcal{L}_D$ presented in Eqs. (12), (13), and (14) in the main text

This linear system representation is highly efficient for standard ordinary differential equation (ODE) solvers. In the following two sections, we will prove Eq. (S7) and Eq. (S8), respectively, for arbitrary operator dimensions $N = 2^n$, where n is the number of coupled spin particles.

Vectorization of the Hamiltonian Propagator

We first prove the vectorization of the Hamiltonian propagator $L_H(\widehat{\boldsymbol{\rho}})$ given by the base case for a 2-dimensional system ($N = 2$). Let the matrix representations of the Hamiltonian operator $\widehat{\mathbf{H}}$ and the density operator $\boldsymbol{\rho}$ be expressed as

$$\mathbf{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \qquad \boldsymbol{\rho} = \begin{bmatrix} \rho_1 & \rho_3 \\ \rho_2 & \rho_4 \end{bmatrix} \tag{S9}$$

The matrix form of the Hamiltonian propagator $L_H(\widehat{\boldsymbol{\rho}})$ is defined as

$$\begin{aligned} L_H(\hat{\rho}) &= -\frac{i}{\hbar}[\mathbf{H}, \boldsymbol{\rho}] = -\frac{i}{\hbar}\left(\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} \rho_1 & \rho_3 \\ \rho_2 & \rho_4 \end{bmatrix} - \begin{bmatrix} \rho_1 & \rho_3 \\ \rho_2 & \rho_4 \end{bmatrix}\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\right) \\ &= -\frac{i}{\hbar}\begin{bmatrix} H_{12}\rho_2 - \rho_3 H_{21} & (H_{11} - H_{22})\rho_3 + H_{12}\rho_4 - \rho_1 H_{12} \\ H_{21}\rho_1 + (H_{22} - H_{11})\rho_2 - H_{21}\rho_4 & H_{21}\rho_3 - \rho_2 H_{12} \end{bmatrix} \end{aligned} \tag{S10}$$

Through the Hilbert − Liouville transformation given by Eq. (S4), the vectorization of $L_H(\hat{\rho})$ becomes

$$|L_H(\hat{\rho})\rangle\rangle = -\frac{i}{\hbar}\begin{bmatrix} H_{12}\rho_2 - \rho_3 H_{21} \\ H_{21}\rho_1 + (H_{22} - H_{11})\rho_2 - H_{21}\rho_4 \\ (H_{11} - H_{22})\rho_3 + H_{12}\rho_4 - \rho_1 H_{12} \\ H_{21}\rho_3 - \rho_2 H_{12} \end{bmatrix}$$

$$= -\frac{i}{\hbar}\begin{bmatrix} 0 & H_{12} & -H_{21} & 0 \\ H_{21} & (H_{22} - H_{11}) & 0 & -H_{21} \\ -H_{12} & 0 & (H_{11} - H_{22}) & H_{12} \\ 0 & -H_{12} & H_{21} & 0 \end{bmatrix}|\rho\rangle\rangle$$

$$= -\frac{i}{\hbar}\left(\begin{bmatrix} H_{11} & H_{12} & 0 & 0 \\ H_{21} & H_{22} & 0 & 0 \\ 0 & 0 & H_{11} & H_{12} \\ 0 & 0 & H_{21} & H_{22} \end{bmatrix} - \begin{bmatrix} H_{11} & 0 & H_{21} & 0 \\ 0 & H_{11} & 0 & H_{21} \\ H_{12} & 0 & H_{22} & 0 \\ 0 & H_{12} & 0 & H_{22} \end{bmatrix}\right)|\rho\rangle\rangle$$

$$= -\frac{i}{\hbar}(\mathbf{I}_2 \otimes \mathbf{H} - \mathbf{H}^T \otimes \mathbf{I}_2)|\rho\rangle\rangle = \mathcal{L}_H|\rho\rangle\rangle$$

This completes the proof of the vectorization of the Hamiltonian propagator $L_H(\hat{\rho})$ for the case of $N = 2$.

$$-\frac{i}{\hbar}[\hat{\mathbf{H}}, \hat{\boldsymbol{\rho}}] \underset{\text{Hilbert} - \text{Liouville}}{\longleftrightarrow} -\frac{i}{\hbar}(\mathbf{I}_2 \otimes \mathbf{H} - \mathbf{H}^T \otimes \mathbf{I}_2)|\rho\rangle\rangle \tag{S11}$$

Subsequently, we generalize this proof to an arbitrary dimension $N = 2^n > 2$. Beginning with the right-hand side of the vectorized relation in Eq. (S11) and utilizing the column-partitioning notation in Eq. (S5), we obtain:

$$-\frac{i}{\hbar}(\mathbf{I}_N \otimes \mathbf{H} - \mathbf{H}^T \otimes \mathbf{I}_N)|\rho\rangle\rangle = -\frac{i}{\hbar}(\mathbf{I}_N \otimes \mathbf{H} - \mathbf{H}^T \otimes \mathbf{I}_N)\begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \\ \vdots \\ \boldsymbol{\rho}_N \end{bmatrix} \tag{S12}$$

$$= -\frac{i}{\hbar}\left(\begin{bmatrix} \mathbf{H} & 0 & \cdots & 0 \\ 0 & \mathbf{H} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H} \end{bmatrix} - \begin{bmatrix} H_{11}\mathbf{I}_N & H_{21}\mathbf{I}_N & \cdots & H_{N1}\mathbf{I}_N \\ H_{12}\mathbf{I}_N & H_{22}\mathbf{I}_N & \cdots & H_{N2}\mathbf{I}_N \\ \vdots & \vdots & \ddots & \vdots \\ H_{1N}\mathbf{I}_N & H_{2N}\mathbf{I}_N & \cdots & H_{NN}\mathbf{I}_N \end{bmatrix}\right)\begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \\ \vdots \\ \boldsymbol{\rho}_N \end{bmatrix}$$

$$= -\frac{i}{\hbar}\left(\begin{bmatrix} \mathbf{H}\boldsymbol{\rho}_1 \\ \mathbf{H}\boldsymbol{\rho}_2 \\ \vdots \\ \mathbf{H}\boldsymbol{\rho}_N \end{bmatrix} - \begin{bmatrix} \sum H_{i1}\boldsymbol{\rho}_i \\ \sum H_{i2}\boldsymbol{\rho}_i \\ \vdots \\ \sum H_{iN}\boldsymbol{\rho}_i \end{bmatrix}\right)$$

By applying the inverse mapping to return from Liouville space to Hilbert space, this column vector is transformed back into its corresponding square-matrix representation:

$$-\frac{i}{\hbar}\left([\mathbf{H}\boldsymbol{\rho}_1 \quad \mathbf{H}\boldsymbol{\rho}_2 \quad \cdots \quad \mathbf{H}\boldsymbol{\rho}_N] - \left[\sum H_{i1}\boldsymbol{\rho}_i \quad \sum H_{i2}\boldsymbol{\rho}_i \quad \cdots \quad \sum H_{iN}\boldsymbol{\rho}_i\right]\right)$$

$$= -\frac{i}{\hbar}\left(\mathbf{H}[\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2 \quad \cdots \quad \boldsymbol{\rho}_N] - [\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2 \quad \cdots \quad \boldsymbol{\rho}_N]\begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N1} & H_{N2} & \cdots & H_{NN} \end{bmatrix}\right)$$

$$= -\frac{i}{\hbar}(\mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H}) = L_H(\hat{\boldsymbol{\rho}}) \tag{S13}$$

This demonstrates the exact equivalence between the matrix and vectorized forms, thereby proving that the Hamiltonian propagator $L_H(\hat{\boldsymbol{\rho}})$ is successfully recovered from its vector form $\mathcal{L}_H|\rho\rangle\rangle$ for any arbitrary dimension N.

Vectorization of the Lindblad Dissipator

Due to the linear property of the Lindblad dissipator by Eq. (S2) with respect to the density operator $\boldsymbol{\rho}$, each summated term in Eq. (S2) operates independently. Consequently, to establish the general validity of the transformation for an arbitrary dimension N, it is sufficient to prove the mapping for a single, arbitrary decoherence operator $\hat{\mathbf{V}}$ of dimension $N \times N$ acting on the density matrix $\boldsymbol{\rho}$:

$$L_D(\hat{\boldsymbol{\rho}}) = [\hat{\mathbf{V}}\hat{\boldsymbol{\rho}}, \hat{\mathbf{V}}^\dagger] + [\hat{\mathbf{V}}, \hat{\boldsymbol{\rho}}\hat{\mathbf{V}}^\dagger] = 2\hat{\mathbf{V}}\hat{\boldsymbol{\rho}}\hat{\mathbf{V}}^\dagger - \hat{\mathbf{V}}^\dagger\hat{\mathbf{V}}\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\rho}}\hat{\mathbf{V}}^\dagger\hat{\mathbf{V}}$$
$$\leftrightarrow \mathcal{L}_D|\boldsymbol{\rho}\rangle\rangle = (2\overline{\mathbf{V}} \otimes \mathbf{V} - \mathbf{I}_N \otimes \mathbf{V}^\dagger\mathbf{V} - (\mathbf{V}^\dagger\mathbf{V})^T \otimes \mathbf{I}_N)|\boldsymbol{\rho}\rangle\rangle \tag{S14}$$

where $\mathbf{I}_N$ denotes the $N \times N$ identity matrix, and $\otimes$ denotes the Kronecker tensor product. While Eq. (S14) holds for any arbitrary $\hat{\mathbf{V}}$, in our physical model we restrict the summation to the 16 active symmetry-driven decoherence operators derived in the main text to reduce computational complexity.

We first verify the mapping from $\mathcal{L}_D|\boldsymbol{\rho}\rangle\rangle$ to $L_D(\hat{\boldsymbol{\rho}})$ in Eq. (S14) for a two-dimensional system ($N = 2$) as the base case, and subsequently extend the proof to an arbitrary N-dimensional system. Let the matrix representations of the decoherence operator $\hat{\mathbf{V}}$ and the density operator $\hat{\boldsymbol{\rho}}$ be formulated as:

$$\mathbf{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \quad \boldsymbol{\rho} = \begin{bmatrix} \rho_1 & \rho_3 \\ \rho_2 & \rho_4 \end{bmatrix} = [\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2] \tag{S15}$$

where $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are the first and second columns of the density matrix $\boldsymbol{\rho}$, respectively. We define an auxiliary matrix $\mathbf{W}$:

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \mathbf{V}^\dagger\mathbf{V} = \begin{bmatrix} \bar{v}_{11} & \bar{v}_{21} \\ \bar{v}_{12} & \bar{v}_{22} \end{bmatrix}\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

The vectorized representation of the Lindblad dissipator $\mathcal{L}_D|\boldsymbol{\rho}\rangle\rangle$ can then be expressed:

$$\mathcal{L}_D|\boldsymbol{\rho}\rangle\rangle = (2\overline{\mathbf{V}} \otimes \mathbf{V} - \mathbf{I}_2 \otimes \mathbf{V}^\dagger\mathbf{V} - (\mathbf{V}^\dagger\mathbf{V})^T \otimes \mathbf{I}_2)\begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \end{bmatrix}$$

$$= \left(2\begin{bmatrix} \bar{v}_{11}\mathbf{V} & \bar{v}_{12}\mathbf{V} \\ \bar{v}_{21}\mathbf{V} & \bar{v}_{22}\mathbf{V} \end{bmatrix} - \begin{bmatrix} \mathbf{W} & 0 \\ 0 & \mathbf{W} \end{bmatrix} - \begin{bmatrix} w_{11}\mathbf{I}_2 & w_{21}\mathbf{I}_2 \\ w_{12}\mathbf{I}_2 & w_{22}\mathbf{I}_2 \end{bmatrix}\right)\begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \end{bmatrix}$$

$$= 2\begin{bmatrix} \bar{v}_{11}\mathbf{V}\boldsymbol{\rho}_1 + \bar{v}_{12}\mathbf{V}\boldsymbol{\rho}_2 \\ \bar{v}_{21}\mathbf{V}\boldsymbol{\rho}_1 + \bar{v}_{22}\mathbf{V}\boldsymbol{\rho}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{W}\boldsymbol{\rho}_1 \\ \mathbf{W}\boldsymbol{\rho}_2 \end{bmatrix} - \begin{bmatrix} w_{11}\boldsymbol{\rho}_1 + w_{21}\boldsymbol{\rho}_2 \\ w_{12}\boldsymbol{\rho}_1 + w_{22}\boldsymbol{\rho}_2 \end{bmatrix}$$

By applying the inverse transformation of Eq. (S5), the above vector form can be transformed to the following matrix form

$$2[\bar{v}_{11}\boldsymbol{\rho}_1 + \bar{v}_{12}\boldsymbol{\rho}_2 \quad \bar{v}_{21}\boldsymbol{\rho}_1 + \bar{v}_{22}\boldsymbol{\rho}_2] - \mathbf{W}[\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2] - [w_{11}\boldsymbol{\rho}_1 + w_{21}\boldsymbol{\rho}_2 \quad w_{12}\boldsymbol{\rho}_1 + w_{22}\boldsymbol{\rho}_2]$$

$$= 2\mathbf{V}[\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2]\begin{bmatrix} \bar{v}_{11} & \bar{v}_{21} \\ \bar{v}_{12} & \bar{v}_{22} \end{bmatrix} - \mathbf{W}[\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2] - [\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2]\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

$$= 2\mathbf{V}\boldsymbol{\rho}\mathbf{V}^\dagger - \mathbf{V}^\dagger\mathbf{V}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{V}^\dagger\mathbf{V} = L_D(\widehat{\boldsymbol{\rho}}) \tag{S16}$$

This confirms the exact correspondence to the operator-form Lindblad dissipator in Hilbert space.

This proof can be straightforwardly generalized to an arbitrary N-dimensional system. Specifically, the $N \times N$ density matrix $\boldsymbol{\rho}$ is partitioned into N distinct column vectors:

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & \rho_{NN} \end{bmatrix} = [\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2 \quad \cdots \quad \boldsymbol{\rho}_N]$$

and the related N-dimensional matrices are defined as

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1N} \\ w_{21} & w_{22} & \cdots & w_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N1} & w_{N2} & \cdots & w_{NN} \end{bmatrix} = \mathbf{V}^\dagger\mathbf{V} = \begin{bmatrix} \bar{v}_{11} & \bar{v}_{21} & \cdots & \bar{v}_{N1} \\ \bar{v}_{12} & \bar{v}_{22} & \cdots & \bar{v}_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{v}_{1N} & \bar{v}_{2N} & \cdots & \bar{v}_{NN} \end{bmatrix}\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1N} \\ v_{21} & v_{22} & \cdots & v_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ v_{N1} & v_{N2} & \cdots & v_{NN} \end{bmatrix}$$

To prove the validity of the transformation for general N-dimensional systems, we evaluate the vectorized expression of the Lindblad dissipator in Eq. (S14):

$$\mathcal{L}_D|\boldsymbol{\rho}\rangle\rangle = (2\overline{\mathbf{V}} \otimes \mathbf{V} - \mathbf{I}_N \otimes \mathbf{V}^\dagger\mathbf{V} - (\mathbf{V}^\dagger\mathbf{V})^T \otimes \mathbf{I}_N)\begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \\ \vdots \\ \boldsymbol{\rho}_N \end{bmatrix} \tag{S17}$$

$$= \left(2\begin{bmatrix} \bar{v}_{11}\mathbf{V} & \bar{v}_{12}\mathbf{V} & \cdots & \bar{v}_{1N}\mathbf{V} \\ \bar{v}_{21}\mathbf{V} & \bar{v}_{22}\mathbf{V} & \cdots & \bar{v}_{2N}\mathbf{V} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{v}_{N1}\mathbf{V} & \bar{v}_{N2}\mathbf{V} & \cdots & \bar{v}_{NN}\mathbf{V} \end{bmatrix} - \begin{bmatrix} \mathbf{W} & 0 & \cdots & 0 \\ 0 & \mathbf{W} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{W} \end{bmatrix} - \begin{bmatrix} w_{11}\mathbf{I}_N & w_{21}\mathbf{I}_N & \cdots & w_{N1}\mathbf{I}_N \\ w_{12}\mathbf{I}_N & w_{22}\mathbf{I}_N & \cdots & w_{N2}\mathbf{I}_N \\ \vdots & \vdots & \ddots & \vdots \\ w_{1N}\mathbf{I}_N & w_{2N}\mathbf{I}_N & \cdots & w_{NN}\mathbf{I}_N \end{bmatrix}\right)\begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \\ \vdots \\ \boldsymbol{\rho}_N \end{bmatrix}$$

$$= 2\begin{bmatrix} \sum \bar{v}_{1i}\mathbf{V}\boldsymbol{\rho}_i \\ \sum \bar{v}_{2i}\mathbf{V}\boldsymbol{\rho}_i \\ \vdots \\ \sum \bar{v}_{Ni}\mathbf{V}\boldsymbol{\rho}_i \end{bmatrix} - \begin{bmatrix} \mathbf{W}\boldsymbol{\rho}_1 \\ \mathbf{W}\boldsymbol{\rho}_2 \\ \vdots \\ \mathbf{W}\boldsymbol{\rho}_N \end{bmatrix} - \begin{bmatrix} \sum w_{i1}\boldsymbol{\rho}_i \\ \sum w_{i2}\boldsymbol{\rho}_i \\ \vdots \\ \sum w_{iN}\boldsymbol{\rho}_i \end{bmatrix}$$

By transforming this column-vector representation back to its square-matrix format, we retrieve:

$$2\left[\sum \bar{v}_{1i}\mathbf{V}\boldsymbol{\rho}_i \quad \sum \bar{v}_{2i}\mathbf{V}\boldsymbol{\rho}_i \quad \cdots \quad \sum \bar{v}_{Ni}\mathbf{V}\boldsymbol{\rho}_i\right] - \mathbf{W}[\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2 \quad \cdots \quad \boldsymbol{\rho}_N]$$

$$-\left[\sum w_{i1}\boldsymbol{\rho}_i \quad \sum w_{i2}\boldsymbol{\rho}_i \quad \cdots \quad \sum w_{iN}\boldsymbol{\rho}_i\right]$$

$$= 2\mathbf{V}[\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2 \quad \cdots \quad \boldsymbol{\rho}_N]\begin{bmatrix} \bar{v}_{11} & \bar{v}_{21} & \cdots & \bar{v}_{N1} \\ \bar{v}_{12} & \bar{v}_{22} & \cdots & \bar{v}_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{v}_{1N} & \bar{v}_{2N} & \cdots & \bar{v}_{NN} \end{bmatrix} - \mathbf{W}[\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2 \quad \cdots \quad \boldsymbol{\rho}_N]$$

$$
-\begin{bmatrix} \boldsymbol{\rho}_1 & \boldsymbol{\rho}_2 & \cdots & \boldsymbol{\rho}_N \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1N} \\ w_{21} & w_{22} & \cdots & w_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N1} & w_{N2} & \cdots & w_{NN} \end{bmatrix}
$$

$$
= 2\mathbf{V}\boldsymbol{\rho} - \mathbf{V}^\dagger\mathbf{V}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{V}^\dagger\mathbf{V} \tag{S18}
$$

where $\boldsymbol{\rho} = [\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2 \quad \cdots \quad \boldsymbol{\rho}_N]$ and $\mathbf{W} = \mathbf{V}^\dagger\mathbf{V}$ by definition. The mathematical transition from the vectorized form $\mathcal{L}_D|\boldsymbol{\rho}\rangle\rangle$ to the matrix form $L_D(\widehat{\boldsymbol{\rho}})$ establishes that the vectorized Lindblad dissipator is mathematically equivalent to its operator representation in Hilbert space, completing the proof for an arbitrary dimension N.

Implications for Avian Magnetoreception Models

Rigorous validation of the Hilbert-to-Liouville space transformation for arbitrary n-particle systems has profound implications for modeling avian magnetoreception[7,8,11]. In realistic cellular microenvironments, the radical-pair system (comprising two electron spins) is coupled to a dense network of surrounding nuclear spins (such as Nitrogen-14 or Hydrogen-1 nuclei) embedded within retinal cryptochromes[19,20]. Direct numerical integration of the governing master equation in its operator form faces severe computational limitations, as the system's dimensionality scales exponentially as $2^{2n} \times 2^{2n}$ in Hilbert space. By leveraging this proven state-space transformation, researchers can:

- Formulate Linear State-Space Systems: Convert complex operator differential equations into a highly tractable, vectorized system of linear first-order ODEs.
- Accelerate Numerical Computation: Implement highly optimized numerical integration algorithms (e.g., adaptive Runge-Kutta schemes) that execute several orders of magnitude faster than conventional operator-based solvers.
- Enable Scalable Quantum Modeling: Seamlessly scale simulations from simplified three-particle toy models to realistic, high-dimensional multi-nuclear spin networks, thereby facilitating a more comprehensive and biologically authentic understanding of magnetoreception in nature.