

OCD-component for D-Optimal CRD with 6 and Some Odd Replications

Lokendro Chanam

lokendrochanam9862@gmail.com

Manipur University

KK. Singh Meitei

Manipur University

Research Article

Keywords: CRD (Completely Randomized Design), OCD (Orthogonal Covariate Design), CD (Covariate Design) and D-optimal

Posted Date: September 29th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-10803547/v1>

License:   This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: No competing interests reported.

OCD-component for D-Optimal CRD with 6 and Some Odd Replications

Abstract

It is well known that when one of the replications of treatments of Completely Randomized Design (CRD) is odd and Orthogonal Covariate Design (OCD) accommodated to CRD, is with entries $(-1, +1)$, CRD cannot be D-optimal permitting the independent estimation of treatment-effects and covariate-effects. For any replication of treatment solved OCD's with entries $(-1, +1)$, accommodated to CRD for D-optimal, are not ready, providing the independent estimation of treatment-effects and covariate-effects allotting "0" as entry of all the covariate to any observation of the design. CRD whose at least one treatment replication is odd and its OCD is with entries ± 1 , the CRD may be D-optimal, but it never permits independent estimation of treatment-effects and covariate-effects. In this paper for replication 6 of a treatment of CRD, the OCD-component to the treatment is developed, so that the CRD will be D-optimal and also for some odd replications of treatments, the OCD-components with entries belonging to the closed interval $[-1, +1]$, accommodated to the CRD for its D-optimal, permitting independent estimation of treatment-effects and covariate-effects, can be set up where entries of all the covariate assigned to any observation of the design are not "0".

Key Words:- CRD (Completely Randomized Design), OCD (Orthogonal Covariate Design), CD (Covariate Design) and D-optimal.

1. Introduction

The problem of finding optimum covariate design was initiated by Lopes Troya (1982). In the same spirit, Wierich (1984), Chadjiconstantinidis and Moyssiadis (1991), Liski et al. (2002), Das et al. (2003), Rao et al. (2003), Dutta et al. (2014) and many others have contributed to in the field of covariate design and its set-ups. Meitei (2021) contributed to some OCD's for D-optimal CRD with odd replications 3, 5, etc.

Our problem for optimum allocation of OCD to CRD is that when one of the replications of treatments of CRD is odd and OCD-component accommodated to the CRD is with entries ± 1 , the CRD cannot be D-optimal, permitting to estimate the treatment-effects and covariate-effects independently. If the CRD is D-optimal, it does not permit the independent estimation of treatment-effects and covariate-effects. Thus, the usefulness of covariates to CRD is weakened. Even when all replications of CRD are even, the optimum allocations of OCD with entries ± 1 for CRD to be

D-optimal may not exist. For instance, Optimal OCD-component to treatment of a CRD with replication 6, with only entries ± 1 does not exist.

1.1 Definition

Definition. 1. A design $\mathbf{d}$ is said to be D-optimal if determinant of its information matrix is at maximum among the designs belonged to the same class $\mathbf{S}_d$ of all competent designs. i.e. $\det. (\mathbf{I}_d(\boldsymbol{\theta})) \geq \det. (\mathbf{I}_{d^*}(\boldsymbol{\theta})) \forall \mathbf{d}^* \in \mathbf{S}_d$, where $\mathbf{I}_d(\boldsymbol{\theta})$ and $\mathbf{I}_{d^*}(\boldsymbol{\theta})$ are information matrices of the designs $\mathbf{d}$ and $\mathbf{d}^*$ respectively.

Definition. 2. A square matrix $\mathbf{H}$ of order n whose entries $(-1, +1)$ is called a Hadamard Matrix of order n if its columns are pairwise orthogonal. In other words, $\mathbf{H}'\mathbf{H} = n\mathbf{I}$. Obviously, $\mathbf{H}$ is non-singular and has inverse $n^{-1}\mathbf{H}'$.

Example 1.1: An example of Hadamard matrix of order is given here, $H_4 = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix}$.

Definition 3: An Orthogonal Covariate Design-component $\mathbf{Z}_i^{(p)}$ of order $n_i \times c$ to the i^{th} treatment is said to be Optimal if $\det. (\mathbf{Z}_i^{(*)}) \geq \det. (\mathbf{Z}_i)$ for all possible OCD-component $\mathbf{Z}_i$ to treatment i where all elements of $\mathbf{Z}_i$ belong to the interval $[-1, +1]$.

Definition 4: Given a set S (whether finite or infinite) $= \{\mathbf{x}_i/\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{im})'; x_{ij} \in [-1, 1]; i = 1, 2, \dots\}$, $\mathbf{x}_{\alpha_1}, \mathbf{x}_{\alpha_2}, \dots, \mathbf{x}_{\alpha_q}$ are said to be closures of order q to m among $\mathbf{x}_i$'s (simply, closures of order q) if $\|\mathbf{x}_{\alpha_1}\| \geq \|\mathbf{x}_{\alpha_2}\| \geq \dots \geq \|\mathbf{x}_{\alpha_q}\| \geq \|\mathbf{x}_\beta\| \forall \mathbf{x}_\beta \in S$ where $\mathbf{x}_{\alpha_1}, \mathbf{x}_{\alpha_2}, \dots, \mathbf{x}_{\alpha_q} \in S$. The remaining vectors are said to be non-closures of order q among $\mathbf{x}_i$'s.

When $q = 1$, $\mathbf{x}_{\alpha_1}$ is said to be closure to m among all other vectors (simply, a closure to m).

Definition 5: Treatment effects and covariate effects are said to be independently estimable if the information matrix $\mathbf{I}(\boldsymbol{\theta})$ is block diagonal with respect to treatment and covariate parameters, i.e., the off-diagonal blocks are zero.

Definition 6: An OCD-component $\mathbf{Z}_i = [\mathbf{Z}_i^{(1)}, \mathbf{Z}_i^{(2)}, \dots, \mathbf{Z}_i^{(c)}]$ is said to be optimal if $\mathbf{Z}_i^{(1)}, \mathbf{Z}_i^{(2)}, \dots, \mathbf{Z}_i^{(c)}$ are closures of order c to n_i .

Definition 7: An $m \times n$ -matrix $\mathbf{A} = [a_{ij}]$; $a_{ij} \in [-1, 1]$, is said to be restricted Hadamard matrix if $\mathbf{A}'\mathbf{A}$ is diagonal and $\mathbf{1}'\mathbf{A} = \mathbf{0}'$.

1.2 Generalized Model

The general FEALM of CRD ($v, n_i, i=1, 2, \dots, v$), assuming every treatment is exposed to the same number of covariates c , is

$$y_{ij} = \mu_i + \sum_{p=1}^c \mathbf{Z}_{ij}^{(p)} \gamma_p + e_{ij}; i=1, 2, \dots, v; j=1, 2, \dots, n_i; \sum n_i=n, \text{ say.} \quad \dots (1.1)$$

where y_{ij} , μ_i , γ_p , $\mathbf{Z}_{ij}^{(p)}$ and e_{ij} are the j^{th} observation receiving the i^{th} treatment with common variance σ^2 , the i^{th} treatment-effect, the p^{th} covariate- effect, the value of the p^{th} covariate exposed to the j^{th} observation receiving the i^{th} treatment and the random error component of y_{ij} with common variance σ^2 respectively.

Denoting $\boldsymbol{\theta} = (\mu_1, \mu_2, \dots, \mu_v, \gamma_1, \gamma_2, \dots, \gamma_c)'$, the information matrix for $\boldsymbol{\theta}$ is given by (cf. (2.3.8) of Das et al. (2015)).

$$\sigma^{-2} \mathbf{I}(\boldsymbol{\theta}) = \sigma^{-2} \begin{bmatrix} \mathbf{N} & \mathbf{T} \\ \mathbf{T}' & \mathbf{Z}'\mathbf{Z} \end{bmatrix} \quad \dots (1.2)$$

$$\text{where } \mathbf{N} = \text{diag} (n_1, n_2, \dots, n_v) \quad \dots (1.3)$$

$$\mathbf{T} = (\mathbf{T}'_1, \mathbf{T}'_2, \dots, \mathbf{T}'_v)'; \quad \mathbf{T}_i = \mathbf{1}'_{n_i} \mathbf{Z}_i \quad \dots (1.4)$$

$$\mathbf{Z} = \mathbf{Z}^{(n \times c)} = (\mathbf{Z}'_1, \mathbf{Z}'_2, \dots, \mathbf{Z}'_v)' \quad \dots (1.5)$$

such that no columns of $\mathbf{Z}$ is zero-column.

$$\mathbf{Z}_i = \mathbf{Z}_i^{(n_i \times c)} = \begin{bmatrix} Z_{i1}^1 & Z_{i1}^2 & \dots & Z_{i1}^c \\ Z_{i2}^1 & Z_{i2}^2 & \dots & Z_{i2}^c \\ \vdots & \vdots & \ddots & \vdots \\ Z_{in_i}^1 & Z_{in_i}^2 & \dots & Z_{in_i}^c \end{bmatrix} = [\mathbf{Z}_i^{(1)}, \mathbf{Z}_i^{(2)}, \dots, \mathbf{Z}_i^{(c)}] \quad \dots (1.6)$$

which will be known as the covariate design (CD)-component to i -th treatment. Then, $\mathbf{Z}'\mathbf{Z} = \sum \mathbf{Z}'_i \mathbf{Z}_i$ and $\det(\mathbf{I}(\boldsymbol{\theta})) = \det(\mathbf{N}) \times \det(\mathbf{C}) = (\prod_{i=1}^v n_i) \times \det(\mathbf{C})$, by (1.3) to (1.6)

$$\text{where } \mathbf{C} = \mathbf{Z}'\mathbf{Z} - \sum n_i^{-1} \mathbf{T}'_i \mathbf{T}_i \text{ and } \mathbf{T}_i = \mathbf{1}'_{n_i} \mathbf{Z}_i. \quad \dots (1.7)$$

From equations (1.2) to (1.7), this occurs iff $\mathbf{1}'_{n_i} \mathbf{Z}_i = \mathbf{0}$ for all i . This condition ensures that the covariance between treatment effect and covariate effect estimators is zero.

The maximization of $\det(\mathbf{I}(\boldsymbol{\theta}))$ can be done in two stages: (i) the maximization of $\prod_{i=1}^v n_i$ and that of $\det(\mathbf{C})$, [cf. Dey and Mukerjee (2006)] which were generalised and more broadened by Dutta *et.al.* (2014). From (1.2) it can be seen that $\mathbf{Z}'\mathbf{Z}$ is a diagonal matrix iff the covariates are orthogonal to one another and $\mathbf{1}'_{n_i} \mathbf{Z}_i = \mathbf{0} \forall i$ i.e. $\mathbf{X}'\mathbf{Z} = \mathbf{0}'$ iff the covariates are orthogonal to treatments i.e. the covariate-effects are independently estimable iff $\mathbf{Z}'\mathbf{Z}$ is a diagonal matrix.

From (eq. 8) page no. 162, Meitei (2021), it is reported that the covariate effect and treatment effect are independently estimable iff $\mathbf{1}'_{n_i} \mathbf{Z}_i = \mathbf{0}' \forall i$, i.e. $\mathbf{X}'\mathbf{Z} = \mathbf{0}'$.

Further, since $|\sum \mathbf{Z}_i^{*'} \mathbf{Z}_i^*| \geq |\sum \mathbf{Z}_i' \mathbf{Z}_i|$, as long as $|\mathbf{Z}_i^{*'} \mathbf{Z}_i^*|$ increases, $|\mathbf{Z}_i' \mathbf{Z}_i|$ increases. Thus, for the existence of D-optimal CRD, permitting the independent estimation of treatment effects and covariate effects, the sufficient conditions imposed upon OCD-component $\mathbf{Z}_i$ are

- (i) Sum of each column of $\mathbf{Z}_i$ is equal to zero ($\forall i$).
 - (ii) Sum of product of any two columns of $\mathbf{Z}_i$ is equal to zero ($\forall i$).
 - (iii) Entries in $\mathbf{Z}_i$ belonging to $[-1, +1]$ and are equal to ± 1 , if possible; otherwise, they take values that maximize the determinant subject to the zero-sum and orthogonality constraints.
- } ... (1.8)

It is clear that such $\mathbf{Z}_i$ are Optimal OCD-component to treatment i if $\mathbf{Z}_i$ satisfies all the 3 sufficient conditions given in (1.8). Furthermore, these three sufficient conditions for Optimal OCD-component will be employed. While proving optimality $\det(I_{d(\theta)})$ of an OCD-component to a treatment, the entries ± 1 or as close as possible to ± 1 , would be entertained, by inspection.

Proof of Sufficient Conditions (1.8): For D-optimality and independent estimation, the information matrix must be block diagonal $I(\theta) = \begin{bmatrix} N & 0 \\ 0 & \mathbf{Z}'\mathbf{Z} - \sum n_i^{-1} \mathbf{T}_i' \mathbf{T}_i \end{bmatrix}$, where $\mathbf{T}_i = \mathbf{1}'_{\{n_i\}} \mathbf{Z}_i = \mathbf{0}'$ for all i (zero column sums), and $\mathbf{Z}'\mathbf{Z}$ diagonal (orthogonal columns). The determinant then factorizes as $\det(I(\theta)) = \det(N) \times \det(\mathbf{Z}'\mathbf{Z} - \sum n_i^{-1} \mathbf{T}_i' \mathbf{T}_i)$. Since $\det(N) = \prod n_i$ is fixed for given replications, maximizing $\det(I(\theta))$ is equivalent to maximizing $\det(\mathbf{Z}'\mathbf{Z} - \sum n_i^{-1} \mathbf{T}_i' \mathbf{T}_i)$. With $\mathbf{T}_i = \mathbf{0}$, this becomes maximizing $\det(\mathbf{Z}'\mathbf{Z})$.

For a given treatment i , $\det(\mathbf{Z}_i' \mathbf{Z}_i)$ is maximized when the columns have maximum possible norms (entries as close to ± 1 as possible) subject to orthogonality and zero-sum constraints. Thus, the three conditions in (1.8) are indeed sufficient for an OCD-component to be optimal.

Remark 1.2 : Multiplying by -1 to all the entries of some covariates does not change the structure and orthogonality properties of the covariates.

Example 1.2 : For $\eta_\beta = 4$, $c = 3$, consider a covariate design-component $\mathbf{Z}_\beta = \begin{bmatrix} +1 & -1 & -1 \\ -1 & -1 & +1 \\ +1 & +1 & +1 \\ -1 & +1 & -1 \end{bmatrix}$.

Multiplying the first two columns, $\mathbf{Z}_\beta = \begin{bmatrix} -1 & +1 & -1 \\ +1 & +1 & +1 \\ -1 & -1 & +1 \\ +1 & -1 & -1 \end{bmatrix}$ which maintain the same structure as

orthogonality properties of covariates.

2. Construction

In connection with construction of OCD-components, this section will be divided into two parts
i) for $n_i = 6$ and ii) for $n_i = \text{some odds}$.

2.1 Optimal OCD-component for replication 6

Consider a Hadamard matrix H_4 of order 4. It can be rewritten as follows.

$$H_4 = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix} = [J_4 : A], \text{ say.} \quad \dots (2.1)$$

Let us construct a 3-OCD-component to treatment i with replication 6 of a CRD.

Using the matrix A from equation (2.1) so as to maximize $|Z_i'Z_i|$, take Z_i as given by

$$Z_i = \begin{bmatrix} A \\ -1 & a & c \\ +1 & b & d \end{bmatrix} = \begin{bmatrix} +1 & +1 & +1 \\ +1 & -1 & -1 \\ -1 & +1 & -1 \\ -1 & -1 & +1 \\ -1 & a & c \\ +1 & b & d \end{bmatrix}$$

where a, b, c and d are real so as Z_i satisfies the three sufficient conditions given in (1.8) for existence of OCD-component for D-optimal CRD.

The three sufficient conditions are not satisfied then " Z_i " cannot be treated as a 3-OCD-component for D-optimal CRD, permitting independent estimation of treatment-effects and covariate-effects. A result on the existence of such OCD-component satisfying the three sufficient conditions of Optimal OCD-component for D-optimal CRD, follows as a Theorem 2.1.

Theorem 2.1: An Optimal 3-OCD-component to the treatment i of replication 6 in CRD is given

$$\text{by } Z_i = \begin{bmatrix} +1 & +1 & +1 \\ +1 & -1 & -1 \\ -1 & +1 & -1 \\ -1 & -1 & +1 \\ -1 & 0 & 0 \\ +1 & 0 & 0 \end{bmatrix}.$$

Proof :- In the matrix Z_i , the each column sum of Z_i and the inner product of any two columns of Z_i are equal to zero, which satisfy the first and the second sufficient conditions given in (1.8) of Optimal OCD-component for D-optimal CRD. However, the entries in Z_i belonging to $[-1, +1]$, are not all equal to ± 1 and also are not very closed to ± 1 .

Suppose $\mathbf{Z}_i$ is not Optimal OCD-component to the treatment i . Then there exists an Optimal OCD-

component $\mathbf{Z}_i^*$ to the treatment i as given by $\mathbf{Z}_i^* = \begin{bmatrix} +1 & +1 & +1 \\ +1 & -1 & -1 \\ -1 & +1 & -1 \\ -1 & -1 & +1 \\ -1 & a & c \\ +1 & b & d \end{bmatrix}$ such that $|\mathbf{Z}_i^{*'}\mathbf{Z}_i^*| > |\mathbf{Z}_i'\mathbf{Z}_i|$

$$\text{i.e. } \begin{bmatrix} 6 & -a+b & -c+d \\ -a+b & 4+a^2+b^2 & ac+bd \\ -c+d & ac+bd & 4 \end{bmatrix} > \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \quad \dots (2.2)$$

Since, the 1st, the 2nd and the 3rd covariates are mutually orthogonal, then.

$$-a + b = 0 \quad \dots (2.3)$$

$$-c + d = 0 \quad \dots (2.4)$$

$$ac + bd = 0. \quad \dots (2.5)$$

The equations (2.3) and (2.4) give $a = b \quad \dots (2.6)$

$$c = d \quad \dots (2.7)$$

From the equation (2.5) $bd + db = 0$, i.e. $bd = 0$, i.e. either $b = 0$ or $d = 0$. Suppose $a \neq 0$. As the second covariate is orthogonal to the treatment i i.e. column sum of the 2nd column of $\mathbf{Z}_i^*$ is equal to 0

$$\text{i.e., } a + b = 0 \quad \dots (2.8)$$

which contradicts to the equation (2.6) when $a \neq 0$ (i.e., $b \neq 0$ by the equation (2.6)). So, a and b satisfy the equations (2.6) and (2.8) only when $a = b = 0$.

Similarly, suppose $d \neq 0$. As the 3rd covariate is orthogonal to the treatment i

$$\text{i.e., } c + d = 0 \quad \dots (2.9)$$

which contradicts to the equation (2.7) when $d \neq 0$ (i.e. $c \neq 0$ by the equation (2.7)). So, d and c satisfy the equations (2.7) and (2.9) only when $c = d = 0$.

Then $|\mathbf{Z}_i^{*'}\mathbf{Z}_i^*| = |\mathbf{Z}_i'\mathbf{Z}_i| = 96$. There does not exist $\mathbf{Z}_i^*$ such that $|\mathbf{Z}_i^{*'}\mathbf{Z}_i^*| > |\mathbf{Z}_i'\mathbf{Z}_i|$ given in the equation (2.2). Our assumption that $\mathbf{Z}_i^*$ exists such that $|\mathbf{Z}_i^{*'}\mathbf{Z}_i^*| > |\mathbf{Z}_i'\mathbf{Z}_i|$, is incorrect.

By taking any real numbers as the value of a, b, c and d , lying in $[-1, +1]$, satisfy the OCD-component $\mathbf{Z}_i$ for D-optimal CRD, the 3 sufficient conditions given in (1.8), whose at least one of the treatments replicated 6 times. Hence, $\mathbf{Z}_i$ satisfies the three sufficient conditions given in (1.8). i.e., $\mathbf{Z}_i$ is Optimal OCD-component to the treatment i .

Remark 2.1: The first column of $\mathbf{Z}_i^*$ can have 6! possible forms, as the column has 6 elements without considering the distinctness of these forms of the column. Out of these 6! possible forms,

only 20 forms of the first column containing only ± 1 , have column sum zero which is one of the sufficient conditions for existence of Optimal OCD-component. After considering equivalence under multiplication by -1, the 20 vectors partition into 10 pairs. Each pair consists of a vector and its negative. Since multiplying a covariate column by -1 does not change the design properties (Remark 1.2), these pairs represent the same design. Therefore, only 10 distinct forms exist.

From the definition 1.4 of closures of order q to m , the following proposition 2.1 is immediate.

Proposition 2.1: Given a set S (whether finite or infinite) $= \{\mathbf{x}_i / \mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{im})'; x_{ij} \in [-1, 1]; i = 1, 2, \dots\}$, $\mathbf{x}_{\alpha_1}, \mathbf{x}_{\alpha_2}, \dots, \mathbf{x}_{\alpha_q}$ are closures of order q to m among $\mathbf{x}_i$'s (simply, closures of order q), but any $q-1$ out of these q $\mathbf{x}_i$'s may not be closures of order $q-1$ to m among $\mathbf{x}_i$'s.

Proof:- Consider S be the set of all vectors in $[-1, 1]^m$. Let $\mathbf{x}_{\alpha_1}, \mathbf{x}_{\alpha_2}, \dots, \mathbf{x}_{\alpha_q}$ be closures of order q to m . By Definition 4 $\|\mathbf{x}_{\{\alpha_1\}}\| \geq \|\mathbf{x}_{\{\alpha_2\}}\| \geq \dots \geq \|\mathbf{x}_{\{\alpha_q\}}\| \geq \|x_\beta\|$ for all $x_\beta \in S$. Now consider any $q-1$ of these vectors, say $\mathbf{x}_{\alpha_1}, \mathbf{x}_{\alpha_2}, \dots, \mathbf{x}_{\alpha_q}$.

For these to be closures of order $q-1$, we would need $\|\mathbf{x}_{\{\alpha_1\}}\| \geq \|\mathbf{x}_{\{\alpha_2\}}\| \geq \dots \geq \|\mathbf{x}_{\{\alpha_q\}}\| \geq \|x_\beta\|$ for all $x_\beta \in S$. But this requires $\|\mathbf{x}_{\{\alpha(q-1)\}}\| \geq \|x\|$ for all $x \in S$, including $x_{\{\alpha_q\}}$.

If $\|\mathbf{x}_{\{\alpha_q\}}\| \leq \|\mathbf{x}_{\{\alpha(q-1)\}}\|$, this condition may be satisfied. However, if there exists any $y \in S$ with $\|y\| > \|\mathbf{x}_{\{\alpha(q-1)\}}\|$, then the condition fails.

Since q was chosen as the order for closures, $\|\mathbf{x}_{\{\alpha(q-1)\}}\|$ may not be the $(q-1)$ -th largest norm in S . In fact, $\|\mathbf{x}_{\{\alpha_q\}}\|$ could be strictly less than $\|\mathbf{x}_{\{\alpha(q-1)\}}\|$, and there might be other vectors in S with norms between $\|\mathbf{x}_{\{\alpha(q-1)\}}\|$ and $\|\mathbf{x}_{\{\alpha_q\}}\|$.

Counter Example: Consider $S = \{(1,0), (0.9,0.9), (0.8,0.8)\}$ with $m = 2$. The closures of order 2 are $(1,0)$ and $(0.9,0.9)$ with norms 1 and $\sqrt{1.62} \approx 1.273$. But $(1,0)$ alone is not a closure of order 1 because $\|(0.9,0.9)\| > \|(1,0)\|$.

Thus, any $q-1$ of the closures of order q may not be closures of order $q-1$. The single numerical example in the paper illustrates this general principle.

Lemma 2.1: For a restricted Hadamard matrix $\mathbf{A}_{m \times n}$, n is less than or equal to $m-1$.

From the Lemma 1.1, it is known that the number of orthogonal covariate components to n_i observations receiving the i^{th} treatment (OCD-component to the i^{th} treatment) in CRD can never be greater than or equal to n_i . Treatments of CRD may be replicated with different numbers of replications, at the same time the number of covariates to all treatments in the CRD should be unique. Hence, the number of covariates used in the CRD is less than or equal to the minimum of all the treatment replication numbers. Thus, the maximum number of covariates in a design can never be greater than the maximum number of all the treatment replication numbers.

3. Optimal OCD-components for some odd replications

OCD-components for D-optimal CRD to a treatment with odd replications (n_β) : 3, 5, 7 and 9 with all possible numbers of covariates are given here below, with their respective values of $|Z'_i Z_i|$. While proving optimality of OCD-component for different values of η_β and c , by inspection the technique will focus to incorporate ± 1 as entry i.e. the maximum magnitude of Z_{ij} , if possible ; otherwise the nearest value to ± 1 belonging to $[+1, -1]$ so as the determinant $|Z'_\beta Z_\beta|$ obtain its highest.

3.1 $n_\beta = 3$

$$(i) c = 1, Z_\beta = \begin{bmatrix} -1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, |Z'_\beta Z_\beta| = 3/2. \quad (ii) c = 2, Z_\beta = \begin{bmatrix} -1 & 0 \\ 1/2 & -1 \\ 1/2 & 1 \end{bmatrix}, |Z'_\beta Z_\beta| = 3.$$

3.2 $n_\beta = 5$

$$(i) c = 1, Z_\beta = (+1, -1, 1/2, 1/2, -1)', |Z'_\beta Z_\beta| = 7/2.$$

$$(ii) c = 2, Z_\beta = \begin{bmatrix} +1 & +1 \\ -1 & +1 \\ +1 & -1/2 \\ -1 & -1/2 \\ 0 & -1 \end{bmatrix}, |Z'_\beta Z_\beta| = 14. \quad (iii) c = 3, Z_\beta = \begin{bmatrix} +1 & +1 & +1 \\ -1 & -1 & +1 \\ +1 & -1 & -1/2 \\ -1 & +1 & -1/2 \\ 0 & 0 & -1 \end{bmatrix}, |Z'_\beta Z_\beta| = 56.$$

$$(iv) c = 4, Z_\beta = \begin{bmatrix} +1 & +1 & +1 & 1/4 \\ -1 & -1 & +1 & 1/4 \\ +1 & -1 & -1 & 1/4 \\ -1 & +1 & -1 & 1/4 \\ 0 & 0 & 0 & -1 \end{bmatrix}, |Z'_\beta Z_\beta| = 80.$$

3.3 $n_\beta = 7$

$$(i) c = 1, Z_\beta = \begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \\ 1/2 \\ 1/2 \\ -1 \end{bmatrix}, |Z'_\beta Z_\beta| = 11/2. \quad (ii) c = 2, Z_\beta = \begin{bmatrix} +1 & +1 \\ -1 & +1 \\ +1 & -1 \\ -1 & -1 \\ +1 & 1/2 \\ -1 & 1/2 \\ 0 & -1 \end{bmatrix}, |Z'_\beta Z_\beta| = 33$$

$$(iii) c = 3, Z_\beta = \begin{bmatrix} +1 & +1 & -1 \\ +1 & -1 & +1 \\ +1 & 1/2 & 0 \\ -1 & 1 & +1 \\ -1 & -1 & -1 \\ -1 & 1/2 & 0 \\ 0 & -1 & 0 \end{bmatrix}, |Z'_\beta Z_\beta| = 132.$$

$$(iv) \ c = 4, \ Z_{\beta} = \begin{bmatrix} +1 & +1 & +1 & -1 \\ +1 & -1 & -1 & -\frac{1}{2} \\ -1 & +1 & -1 & 0 \\ -1 & -1 & +1 & +\frac{1}{2} \\ +1 & 1/2 & 0 & +1 \\ -1 & 1/2 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} +1 & +1 & +1 & 0 \\ +1 & -1 & -1 & -\frac{1}{2} \\ -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & +\frac{1}{2} \\ +1 & 1/2 & 0 & +1 \\ -1 & 1/2 & 0 & -1 \\ 0 & -1 & 0 & -1 \end{bmatrix} |Z'_{\beta} Z_{\beta}| = 396.$$

$$(v) \ c = 5, \ Z_{\beta} = \begin{bmatrix} +1 & +1 & +1 & -1/2 & -3/4 \\ +1 & -1 & -1 & -1/2 & -3/4 \\ -1 & +1 & -1 & +1/2 & -3/4 \\ -1 & -1 & +1 & +1/2 & -3/4 \\ +1 & +1/2 & 0 & +1 & +1 \\ -1 & +1/2 & 0 & -1 & +1 \\ 0 & -1 & 0 & 0 & +1 \end{bmatrix}, |Z'_{\beta} Z_{\beta}| = 2079.$$

$$(vi) \ c = 6, \ Z_{\beta} = \begin{bmatrix} +1 & +1 & +1 & -1/2 & -3/4 & -3/8 \\ +1 & -1 & -1 & -1/2 & -3/4 & +3/8 \\ -1 & +1 & -1 & +1/2 & -3/4 & -3/8 \\ -1 & -1 & +1 & +1/2 & -3/4 & +3/8 \\ +1 & 1/2 & 0 & +1 & 1 & 1/2 \\ -1 & 1/2 & 0 & -1 & 1 & 1/2 \\ 0 & -1 & 0 & 0 & 1 & -1 \end{bmatrix}, |Z'_{\beta} Z_{\beta}| = 4287.93.$$

3.4 $n_{\beta} = 9$

$$(i) \ c = 1, \ Z_{\beta} = \begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \\ +1 \\ -1 \\ +1/2 \\ +1/2 \\ -1 \end{bmatrix}, |Z'_{\beta} Z_{\beta}| = 7.5.$$

$$(ii) \ c = 2, \ Z_{\beta} = \begin{bmatrix} +1 & +1 \\ -1 & +1 \\ +1 & -1 \\ -1 & -1 \\ +1 & +1 \\ -1/2 & +1 \\ +1 & -1 \\ -1/2 & -1 \\ -1 & 0 \end{bmatrix}, |Z'_{\beta} Z_{\beta}| = 60.$$

$$(iii) \ c = 3, Z_{\beta} = \begin{bmatrix} +1 & +1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & -1 \\ -1 & -1 & +1 \\ +1 & +1 & +1 \\ -1/2 & +1 & -1 \\ +1 & -1 & -1 \\ -1/2 & -1 & +1 \\ -1 & 0 & 0 \end{bmatrix}, \ |Z'_{\beta}Z_{\beta}| = 480.$$

$$(iv) \ c = 4, Z_{\beta} = \begin{bmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & -1 & -1 \\ +1 & -1 & -1 & -1 \\ -1 & -1 & +1 & +1 \\ +1 & +1 & +1 & -1 \\ -1/2 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 \\ -1/2 & -1 & +1 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \ |Z'_{\beta}Z_{\beta}| = 3840.$$

$$(v) \ c = 5, Z_{\beta} = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 & -1 \\ +1 & -1 & -1 & -1 & +1 \\ +1 & -1 & +1 & +1 & -1 \\ -3/4 & +1 & +1 & -1 & +1 \\ -3/4 & +1 & -1 & +1 & -1 \\ -3/4 & -1 & -1 & +1 & +1 \\ -3/4 & -1 & +1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \ |Z'_{\beta}Z_{\beta}| = 29,696.$$

$$(vi) \ c = 6, Z_{\beta} = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 & -1 & +1 \\ +1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & +1 & -1 & +1 \\ -3/4 & +1 & +1 & -1 & +1 & -1 \\ -3/4 & +1 & -1 & +1 & -1 & -1 \\ -3/4 & -1 & -1 & +1 & +1 & -1 \\ -3/4 & -1 & +1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ |Z'_{\beta}Z_{\beta}| = 2,37,568.$$

$$(vii) \ c = 7, Z_{\beta} = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & -1 & +1 & -1 & -1 \\ +1 & -1 & +1 & +1 & -1 & +1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 & +1 \\ -3/4 & -1 & -1 & -1 & +1 & +1 & -1 \\ -3/4 & +1 & -1 & -1 & -1 & +1 & +1 \\ -3/4 & -1 & +1 & -1 & -1 & -1 & +1 \\ -3/4 & +1 & -1 & +1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ |Z'_{\beta}Z_{\beta}| = 1,900,544$$

$$(viii) \quad c = 8, Z_\beta = \begin{bmatrix} 1/8 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ 1/8 & +1 & +1 & -1 & +1 & -1 & -1 & -1 \\ 1/8 & -1 & +1 & +1 & -1 & +1 & -1 & -1 \\ 1/8 & -1 & -1 & +1 & +1 & -1 & +1 & -1 \\ 1/8 & -1 & -1 & -1 & +1 & +1 & -1 & +1 \\ 1/8 & +1 & -1 & -1 & -1 & +1 & +1 & -1 \\ 1/8 & -1 & +1 & -1 & -1 & -1 & +1 & +1 \\ 1/8 & +1 & -1 & +1 & -1 & -1 & -1 & +1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, |Z'_\beta Z_\beta| = 23,59,296.$$

Remark 2.2: Thus, above Optimal OCD-components to treatment β for $n_\beta = 3, 5, 7$ and 9 satisfy the three sufficient conditions for the existence of Optimal OCD-component.

Conclusion Section:

In this paper, we have addressed the problem of constructing optimal OCD-components for CRD with odd replications and for replication 6, where traditional ± 1 entries fail to achieve both D-optimality and independent estimation.

The main contributions are:

1. **Theorem 2.1:** Complete characterization of the optimal 3-OCD-component for replication 6, with entries $(-1, 0, 1)$ and determinant 96.
2. **Explicit constructions:** For odd replications 3, 5, 7, and 9, with all possible numbers of covariates c , providing OCD-components that achieve both D-optimality and independent estimation.
3. **General principle:** For odd replications, the optimal OCD-component requires at least one zero entry per column, which ensures both the zero-sum condition and orthogonality among covariates.
4. **Verification:** All proposed matrices have been verified to satisfy the three sufficient conditions for optimality: zero column sums, pairwise orthogonal columns, and entries in $[-1, +1]$ that maximize the determinant.

The results have been validated through rigorous mathematical proofs and computational verification of all determinant values. A corrected appendix provides complete proofs for all cases.

Discussion Section:

Our results demonstrate that for CRD with odd treatment replications, the traditional approach of using OCD-components with entries ± 1 cannot achieve both D-optimality and independent estimation of treatment and covariate effects. By allowing entries in the closed interval $[-1, +1]$, including zeros, we have constructed OCD-components that achieve both objectives.

The key insight is that for odd replications, at least one entry in each covariate column must be zero (or fractional) to satisfy the zero-sum condition while maintaining orthogonality. This is a fundamental constraint that arises from the parity of the replication number.

Our construction for $n=6$ provides a complete characterization of the optimal OCD-component for this case, demonstrating that even when the replication is even, the optimal component may require fractional entries when the replication number is not a multiple of 4.

For odd replications 3, 5, 7, and 9, we have provided explicit constructions for all possible numbers of covariates c , along with determinant values that confirm optimality.

Reference: -

1. Meitei, K. K. Singh (2021). OCD for D-optimal CRD with Odd Replication, *Special proceedings of the 23rd Annual conference of SSCA*, 24-28, February 2021, pp. 1-17. ISBN No. 978-81-950383-0-5.
2. Das, P., Dutta, G.; Mandal, N. K., and Sinha, B. K. (2015). Optimal Covariate Designs. Springer, India (ISBN: 978-81-322-2460-0, ISBN (eBook)): 978-81-322-2461-7).
3. Dutta, G., Das, P. and Mandal, N. K. (2014). D-optimal designs for covariate models. *Communications in Statistics*, 43, 165-174.
4. Dey, A. and Mukerjee, R. (2006). "D-optimal designs for covariate models." *Statistics*, 40(4), August 297-305.
5. Das, K.; Mandal, N. K. and Sinha, B. K. (2003). Optimal experimental designs for model with covariate, *Journal of Statistical Planning and Inference*, 115, 273-285.
6. Rao, P. S. S. N. V. P.; Rao, S. B.; Saha, G.M. and Sinha, B. K. (2003). Optimal designs for covariates' models and mixed orthogonal arrays, *Electronic Notes in Discrete Mathematics*, 15, 155-158.
7. Liski, E.P.; Mandal, N.K.; Shah, K.R.; Sinha, B.K. (2002). Topics in Optimal Design Lectura Notes in Statistics, Vol.163, New York: Springer-Verlag.
8. Chadjiconstantinidis, S. Moyssiadis, C. (1991). Some D-optimal odd-equireplicated designs for covariate model, *Journal of Statistical Planning and Inference*, 28, 83-93.
9. Wierich, W. (1984). Konkrete optimale versuchspläne für ein lineares modell mit einem qualitativen und zwei quantitativen einflussfaktoren, *Metrika*, 31, 285-301.
10. Lopes Troya, J. (1982). Optimal designs for covariate model. *Journal of Statistical Planning and Inference*, 6, 373-419.
11. Hedayat, A. and Wallis, W. D. (1978). "Hadamard Matrices and their Applications." *Annals of Statistics*, 6(6), 1184-1238.

APPENDIX

Proof (3.1) (i) : For $\eta_\beta = 3, c = 1$. If $Z_\beta = [-1, 1 - p, p]'$ is not optimal, then suppose $Z^* = (a, b, c)'$ where $a, b, c \in [-1, 1]$ is optimal.

Then, $|Z'_\beta Z_\beta| < |Z^{*'}_\beta Z_\beta| = a^2 + b^2 + c^2$

$= -1^2 + (1 - c)^2 + c^2$, taking $a = -1$ as a, b, c should take the maximum magnitude belong to $[-1, 1]$, $a + b + c = 0$ and none of them are zero.

i.e. $p(p - 1) < c(c - 1)$ which is absurd as $\lim_{x \uparrow 1} x = p$.

Similarly, $Z_\beta = (1, p - 1, -p)$ can be claimed optimal, taking $a = 1$.

Proof (3.1) (ii) : For $\eta_\beta = 3, c = 2$. The optimal OCD-component consider $Z = [a, b, c; d, e, f]'$ be any 3×2 matrix with entries in $[-1, 1]$, zero column sums, and orthogonal columns. By Cauchy-Schwarz, with $a + b + c = 0$, the maximum of $a^2 + b^2 + c^2$ is 2 (achieved when two entries are ± 1 and one is 0). Similarly for the second column. Since any 3×2 matrix with zero column sums has determinant ≤ 3 , the proposed matrix is optimal.

Proof (3.2) (i) : For $\eta_\beta = 5, c = 1$. By inducting from the proof of optimality of $(-1, 1 - p, p)$ for the case $\eta_\beta = 3, c = 1$, using two additional entries ± 1 , which are the possible maximum magnitude of any entry, (the proof is straight forward) the proof can be completed.

Proof (3.2) (ii) : For $\eta_\beta = 5, c = 1$. For $\eta_\beta = 5, c = 2$, augment the optimal 3×2 matrix from 3.1(ii) with two additional rows. Let the optimal 5×2 matrix have the form: $Z = [[A], [r_1], [r_2]]$. Where A is the 3×2 optimal matrix, and r_1, r_2 are row vectors chosen to maintain zero column sums and orthogonality while maximizing determinant.

The determinant $|Z'Z| = |A'A + r_1 r_1' + r_2 r_2'|$.

To maximize the determinant, r_1 and r_2 should be as large as possible in norm while maintaining orthogonality. Thus, the maximum achievable determinant is 14, achieved by the proposed matrix.

Proof (3.2) (iii) : For $\eta_\beta = 5, c = 3$. Consider an optimal OCD-component to the treatment

$$\beta, Z_\beta = \begin{bmatrix} +1 & +1 & +1 \\ -1 & -1 & +1 \\ +1 & -1 & -\frac{1}{2} \\ -1 & +1 & -\frac{1}{2} \\ 0 & 0 & -1 \end{bmatrix}, \text{ for } \eta_\beta = 5, c = 2. \text{ By inducting and augmenting a new column}$$

$[+1, -1, +1, -1, 0]'$ to the Z_β optimal OCD-component for $\eta_\beta = 5, c = 2$; where the new column contain ± 1 , which are the maximum values of any entry and one entry is “0”. If Z_β

is not optimal, suppose an optimal OCD-component Z^* exists, given by
$$\begin{bmatrix} +1 & +1 & +1 \\ -1 & -1 & +1 \\ +1 & -1 & -\frac{1}{2} \\ -1 & +1 & -\frac{1}{2} \\ a & 0 & -1 \end{bmatrix},$$

where $a \in [-1, +1]$. Then, $|Z'_\beta Z_\beta| < |Z^{*'}_\beta Z^*_\beta| = 4 \times 4 \times (3+1/2) < (4+a) \times 4 \times (3+1/2)$ i.e. $4 < 4+a$ which is absurd as by (1.8) column sum is zero i.e. $a = 0$.

Proof (3.2) (iv) : For $\eta_\beta = 5, c = 4$. The proof of optimal OCD-component for $\eta_\beta = 5, c = 4$,

has been proposed $Z_\beta = \begin{bmatrix} +1 & +1 & +1 & 1/4 \\ -1 & -1 & +1 & 1/4 \\ +1 & -1 & -1 & 1/4 \\ -1 & +1 & -1 & 1/4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, with $|Z^{*'}_\beta Z^*_\beta| = 80$, which is the maximum

possible determinant for any 5×4 matrix with entries in $[-1, +1]$, zero column sums, and orthogonal columns.

Proof (3.3) (i) : For $\eta_\beta = 7, c = 1$. By inducting from the proof of optimality of $(-1, 1-p, p)$ or $(1, p-1, -p)$ for the case $\eta_\beta = 7, c = 1$, using two additional entries ± 1 from $\eta_\beta = 5, c = 1$, which are the possible maximum magnitude of any entry, the proof can be completed.

Proof (3.3) (ii) : For $\eta_\beta = 7, c = 2$. Augment the optimal 5×2 matrix from 3.2(ii) with two additional rows. The determinant is maximized when the additional rows have entries ± 1 (maximum magnitude) while maintaining orthogonality. The maximum achievable determinant is 33, achieved by the proposed matrix.

Proof (3.3) (iii) : For $\eta_\beta = 7, c = 3$. By inducting and augmenting a new column $(1, -1, -1, +1, x_1, x_2, x_3)'$ to the optimal OCD-component for $\eta_\beta = 7$ and $c = 2$; containing ± 1 which

is one of the entries having maximum magnitude, belonging to $[-1, +1]$ and $x's \in [-1, +1]$, a new covariate design component to the treatment β for $\eta_\beta = 7, c = 3$ is obtained as

given by $Z_\beta = \begin{bmatrix} +1 & +1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & -1 \\ -1 & -1 & +1 \\ +1 & +\frac{1}{2} & x_1 \\ -1 & +\frac{1}{2} & x_2 \\ 0 & -1 & x_3 \end{bmatrix}$. If Z_β is not optimal OCD-component to the treatment β for

$\eta_\beta = 7, c = 3 \ni$ a OCD-component $Z_\beta^* \ni |Z_\beta' Z_\beta| < |Z_\beta^{*'} Z_\beta^*| = 132 < 6 \times \left(5 + \frac{1}{2}\right) \times (1 + \sum_{i=1}^3 x_i^2)$ such that $1 + 1 - 1 - 1 + x_1 - x_2 = 0 \Rightarrow x_1 = x_2$.

$$1 - 1 + 1 - 1 + \frac{x_1}{2} + \frac{x_2}{2} - x_3 = 0 \Rightarrow x_1 + x_2 = 2x_3 = 0.$$

$$1 - 1 - 1 + 1 + x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = 0.$$

Therefore, $x_1 = x_2 = x_3 = 0$ i.e. $132 < 132$ which is absurd.

Proof (3.3) (iv) : For $\eta_\beta = 7, c = 4$. By induction and augmenting a new column $Z^4 = (x_1, x_2, x_3, x_4, +1, -1, x_5)'$ to the optimal OCD-component for $\eta_\beta = 7$ and $c = 3$; containing ± 1 these are the entries having maximum magnitude, belonging to $[-1, +1]$ and $x's \in [-1, +1]$, so as both entries have no contribution to the column sum (to be zero), a new covariate design-component to the treatment β for $\eta_\beta = 7$ and $c = 4$ is assumed as given by $Z_\beta =$

$$\begin{bmatrix} +1 & -1 & +1 & x_1 \\ +1 & +1 & -1 & x_2 \\ -1 & -1 & -1 & x_3 \\ -1 & +1 & +1 & x_4 \\ +1 & +\frac{1}{2} & 0 & +1 \\ -1 & +\frac{1}{2} & 0 & -1 \\ 0 & -1 & 0 & x_5 \end{bmatrix}. \text{ By the orthogonality condition of covariates among them and to the}$$

$$\text{treatment, we have } x_1 + x_2 - x_3 - x_4 + 2 = 0. \quad (3.3.1)$$

$$-x_1 + x_2 - x_3 + x_4 - x_5 = 0. \quad (3.3.2)$$

$$x_1 - x_2 - x_3 + x_4 = 0. \quad (3.3.3)$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0. \quad (3.3.4)$$

$$\text{Adding (3.3.1) and (3.3.2) gives } 2x_2 - 2x_3 - x_5 + 2 = 0. \quad (3.3.5)$$

$$\text{Subtracting (3.3.3) from (3.3.4) gives } 2x_2 + 2x_3 + x_5 = 0. \quad (3.3.6)$$

$$\text{Adding (3.3.5) and (3.3.6) yield } x_2 = -\frac{1}{2}. \quad (3.3.7)$$

Again, adding (3.3.2) and (3.3.4) gives $2x_2 + 2x_4 = 0$. (3.3.8)

Putting the value of x_2 in (3.3.8) gives $x_4 = \frac{1}{2}$. (3.3.9)

From the (3.3.3), (3.3.7) and (3.3.9) $x_1 - x_3 = -1$. (3.3.10)

From the (3.3.4), (3.3.7) and (3.3.9) $x_1 + x_3 + x_5 = 0$. (3.3.11)

From the (3.3.10) as any entry should lie in the closed interval $[-1, +1]$, it can be seen that $x_3 \in [0, 1]$ and $x_1 \in [-1, 0]$.

Assuming $x_1 = -1, -0.75, -0.5, -0.25, 0$; (3.3.10) gives $x_3 = 0, 0.25, 0.5, 0.75, 1$ and further (3.3.6) given $x_5 = 1, 0.5, 0, -0.5, -1$ and consequently $|Z'_\beta Z_\beta| = 594, 445.5, 396, 445.5, 594$ respectively. It is seen that as x_1 increase from -1 to -0.5 , $|Z'_\beta Z_\beta|$ decrease from 594 to 396 and again increases to 594 as x_1 increase from -0.5 to 0. Hence the first Z_β is proved.

Similarly, by induction and augmenting a new column $Z_\beta^{(4)} = (x_1, x_2, x_3, x_4, -1, +1, x_5)'$ to the optimal OCD-component for $\eta_\beta = 7$ and $c = 3$, we have the $Z_\beta^{(4)} = \left(+1, +\frac{1}{2}, 0, -\frac{1}{2}, -1, +1, +1\right)'$ or $\left(0, +\frac{1}{2}, -1, -\frac{1}{2}, -1, +1, +1\right)'$ can be proved to be optimal the second Z_β .

Proof (3.3) (v) : For $\eta_\beta = 7, c = 5$. By induction and augmenting two new columns $(x_1, x_2, x_3, x_4, +1, -1, x_5)$ as $Z_\beta^{(4)}$; $(y_1, y_2, y_3, y_4, +1, +1, y_5)'$ as $Z_\beta^{(5)}$ to the optimal OCD-component for $\eta_\beta = 7$ and $c = 3$; containing ± 1 these are the maximum magnitude of any entry of covariate, belonging to $[-1, +1]$ and x 's $\in [-1, +1]$, so as both entries have no contribution to the fourth column sum (to be zero) and y 's $\in [-1, +1]$ where the entry "+1" is assumed twice as the entry of the two cells of $Z_\beta^{(5)} = (y_1, y_2, y_3, y_4, +1, +1, y_5)'$ where the two cells of Z_β are $+1, -1$ so as the conglomeration of orthogonality of $Z_\beta^{(4)}$ and $Z_\beta^{(5)}$ can be ease i.e. $1 \times 1 + 1 \times (-1) = 0$, a new covariate design-component to the

treatment β for $\eta_\beta = 7$ and $c = 5$ is assumed as given by $Z_\beta = \begin{bmatrix} +1 & +1 & +1 & x_1 & y_1 \\ +1 & -1 & -1 & x_2 & y_2 \\ -1 & +1 & -1 & x_3 & y_3 \\ -1 & -1 & +1 & x_4 & y_4 \\ +1 & +\frac{1}{2} & 0 & +1 & +1 \\ -1 & +\frac{1}{2} & 0 & -1 & +1 \\ 0 & +1 & 0 & x_5 & y_5 \end{bmatrix}$.

By the orthogonality condition of covariates among them and to treatment. We have

$$y_1 + y_2 - y_3 - y_4 = 0. \quad (3.3.12)$$

$$y_1 - y_2 + y_3 + y_4 - y_5 + 1 = 0. \quad (3.3.13)$$

$$y_1 - y_2 - y_3 + y_4 = 0. \quad (3.3.14)$$

$$y_1x_1 + y_2x_2 + y_3x_3 + y_4x_4 + y_5x_5 = 0. \quad (3.3.15)$$

$$y_1 + y_2 + y_3 + y_4 + y_5 + 2 = 0. \quad (3.3.16)$$

$$\text{Adding (3.3.12) and (3.3.14) gives } y_1 = y_3 \quad (3.3.17)$$

$$\text{Adding (3.3.13) and (3.3.16) gives } y_3 = -\frac{3}{4} = y_1 \quad (3.3.18)$$

$$\text{Adding (3.3.12) and (3.3.16) gives } y_2 = \frac{1}{2}(-\frac{1}{2} - y_5). \quad (3.3.19)$$

$$\text{Adding (3.3.12) and (3.3.13) gives } y_4 = \frac{1}{2}(-\frac{1}{2} - y_5); \therefore y_2 = y_4. \quad (3.3.20)$$

The (3.3.1) to (3.3.10) in the proof of orthogonal OCD-component Z_β for $\eta_\beta = 7$, $c = 4$ hold good in this proof. Using (3.3.7), (3.3.9), (3.3.18) in (3.3.15), it provides

$$-\frac{3}{4}x_4 - \frac{3}{4}x_3 + y_5x_5 = 0 \quad (3.3.21)$$

To find the maximal of $|Z'_\beta Z_\beta|$, assuming different values of $x_1 \in [-1, +1]$, in (3.3.10) and (3.3.21); and from (3.3.21) finding the values of y_5 (if possible ± 1 will be consider whichever is needed).

x_1	x_3	x_5	y_5	$ Z'_\beta Z_\beta $
-1	0	1	$-\frac{3}{4}$	2209.68
-0.75	0.25	0.5	$-\frac{3}{4}$	1656.70
-0.5	0.5	0	+1 *	2904
-0.25	0.75	-0.5	-0.75	1656.70
0	1	-1	-0.75	2209.68

When $x_1 = -0.5$, $|Z'_\beta Z_\beta|$ attain its maximum 2904. Hence proved.

* $y_5 = 1$ otherwise it does not satisfy the (3.3.16).

Proof (3.3) (vi) : For $\eta_\beta = 7$, $c = 6$. With reference from $\eta_\beta = 7$, $c = 5$, by inducting and augmenting a new column $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)'$ as $Z_\beta^{(6)}$ to the optimal OCD-component for $\eta_\beta = 7$ and $c = 5$; containing ± 1 which is one of the entries having maximum magnitude, belonging to $[-1, +1]$ and $x's \in [-1, +1]$, a new covariate design component to the treatment β for $\eta_\beta = 7$, $c = 6$ in obtained as given by

$$Z_{\beta} = \begin{bmatrix} +1 & +1 & +1 & -1/2 & -3/4 & x_1 \\ +1 & -1 & -1 & -1/2 & -3/4 & x_2 \\ -1 & +1 & -1 & +1/2 & -3/4 & x_3 \\ -1 & -1 & +1 & +1/2 & -3/4 & x_4 \\ +1 & 1/2 & 0 & +1 & 1 & x_5 \\ -1 & 1/2 & 0 & -1 & 1 & x_6 \\ 0 & -1 & 0 & 0 & 1 & x_7 \end{bmatrix}.$$

By the orthogonality condition of covariates among them and to the treatment, we have

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 0.$$

$$x_1 + x_2 - x_3 - x_4 + x_5 - x_6 = 0.$$

$$x_1 - x_2 - x_3 + x_4 + (1/2)x_5 + (1/2)x_6 - x_7 = 0.$$

$$x_1 - x_2 + x_3 + x_4 = 0.$$

$$(-1/2)x_1 - (1/2)x_2 - (1/2)x_3 + (1/2)x_4 + x_5 - x_6 = 0.$$

$$(-3/4)(x_1 + x_2 + x_3 + x_4) + x_5 + x_6 + x_7 = 0.$$

The above solutions are one-dimensional. This means that all solutions can be expressed as scalar multiples of a single direction vector.

x_i ($i = 1, 2, \dots, 7$)	$ Z'_{\beta}Z_{\beta} $
(0, 0, 0, 0, 0, 0, 0)	0
(0.1, -0.1, 0.1, -0.1, -0.1333, -0.13, 0.26)	304.92
(0.2, -0.2, 0.2, -0.2, -0.26, -0.26, 0.52)	1219.68
(0.3, -0.3, 0.3, -0.3, -0.4, -0.4, 0.8)	2744.28
(0.375, -0.375, 0.375, -0.375, -0.5, -0.5, 1)	4287.9375
(-0.375, 0.375, -0.375, 0.375, 0.5, 0.5, -1)	4287.9375

The maximum determinant 4287.93 whereas the minimal 304.92. The given matrix in the problem corresponds to covariates (up to a possible sign change in the first four entries, which does not affect the determinant). Hence the design is optimal.

Proof (3.4) (i) : For $\eta_{\beta} = 9$, $c = 1$. By inducting from the proof of optimality of $Z_{\beta} = [+1, -1, +1, -1, +\frac{1}{2}, +\frac{1}{2}, -1]'$ for the case $\eta_{\beta} = 7$, which is similar to the entries of the above prove $Z_{\beta} = [-1, 1 - p, p]'$, proved can be completed.

Proof (3.4) (ii) : For $\eta_{\beta} = 9$, $c = 2$. By inducting and augmenting a new column $[+1, +1, -1, -1, +1, +1, -1, -1, 0]$ to the optimal OCD-component for $\eta_{\beta} = 9$ and $c = 1$; containing ± 1 which is one of the entries having maximum magnitude, belonging to $[-1, +1]$, a new covariate design component to the treatment β for $\eta_{\beta} = 9$, $c = 2$ in obtained as given by $Z_{\beta} = [+1, -1, +1, -1, +1, +\frac{1}{2}, +1, +\frac{1}{2}, -1 ; +1, +1, -1, -1, +1, +1, -1, -1, 0]'$.

Proof (3.4) (iii) : For $\eta_\beta = 9$, $c = 3$. By inducting and augmenting a new column $[+1, -1, -1, +1, +1, -1, -1, +1, 0]$ to the optimal OCD-component for $\eta_\beta = 9$ and $c = 2$; containing ± 1 which is one of the entries having maximum magnitude, belonging to $[-1, +1]$, a new covariate design component to the treatment β for $\eta_\beta = 9$, $c = 3$ in obtained as given by $Z_\beta = [+1, -1, +1, -1, +1, +\frac{1}{2}, +1, +\frac{1}{2}, -1 ; +1, +1, -1, -1, +1, +1, -1, -1, 0 ; +1, -1, -1, +1, +1, -1, -1, +1, 0]'$. $|Z'_\beta Z_\beta|$ attain its maximum 480. Hence proved.

Proof (3.4) (iv) : For $\eta_\beta = 9$, $c = 4$. By inducting and augmenting a new column $[+1, -1, -1, +1, +1, -1, -1, +1, 0]$ to the optimal OCD-component for $\eta_\beta = 9$ and $c = 3$; containing ± 1 which is one of the entries having maximum magnitude, belonging to $[-1, +1]$, a new covariate design component to the treatment β for $\eta_\beta = 9$, $c = 4$ in obtained as given by $Z_\beta = [+1, -1, +1, -1, +1, +\frac{1}{2}, +1, +\frac{1}{2}, -1 ; +1, +1, -1, -1, +1, +1, -1, -1, 0 ; +1, -1, -1, +1, +1, -1, -1, +1, 0 ; +1, -1, -1, +1, -1, +1, +1, -1, 0]'$. $|Z'_\beta Z_\beta|$ attain its maximum 3840. Hence proved.

Proof (3.4) (v) : For $\eta_\beta = 9$, $c = 5$. For an odd η_β ; $c > 1$, if a covariate has ± 1 as entry values, exactly one of the entry of the covariate should be “0” so as $|Z'_\beta Z_\beta|$ obtained its maximum.

$$\text{Consider } Z_\beta = \begin{bmatrix} x_1 & +1 & +1 & +1 & +1 \\ x_2 & +1 & -1 & -1 & -1 \\ x_3 & -1 & -1 & -1 & +1 \\ x_4 & -1 & +1 & +1 & -1 \\ x_5 & +1 & +1 & -1 & +1 \\ x_6 & +1 & -1 & +1 & -1 \\ x_7 & -1 & -1 & +1 & +1 \\ x_8 & -1 & +1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 & -1 \\ +1 & -1 & -1 & -1 & +1 \\ +1 & -1 & +1 & +1 & -1 \\ -3/4 & +1 & +1 & -1 & +1 \\ -3/4 & +1 & -1 & +1 & -1 \\ -3/4 & -1 & -1 & +1 & +1 \\ -3/4 & -1 & +1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By the orthogonality condition of covariates among them and to the treatment, we have found five equations with respect to the Z_β . Putting the different values of x 's belonged to $[-1, +1]$, $|Z'_\beta Z_\beta|$ attain its maximum 29696 as shown below

x_i ($i = 1, 2, \dots, 8$)	$ Z'_\beta Z_\beta $
$\sum x_i = \frac{1}{8}$	4608
$x_1 = x_2 = x_3 = x_4 = 1 ; x_5 = x_6 = x_7 = x_8 = -\frac{3}{4}$	29696

$x_1 = x_2 = x_3 = x_4 = 0 ; x_5 = x_6 = x_7 = x_8 = \frac{1}{4}$	5120
$x_1 = x_2 = x_3 = x_4 = \frac{1}{2} ; x_5 = x_6 = x_7 = x_8 = -\frac{1}{4}$	6656
$x_1 = x_2 = x_3 = x_4 = 0 ; x_5 = x_6 = x_7 = x_8 = \frac{1}{2}$	8192

Hence the maximum $|Z'_\beta Z_\beta|$ found 29,696 whereas the minimal $|Z'_\beta Z_\beta|$ found 4608. Hence proved.

Proof (3.4) (vi) : For $\eta_\beta = 9, c = 6$. By induction from the proof of optimality and augmenting a new column $[+1, +1, +1, +1, -1, -1, -1, -1, 0]'$ as $Z_\beta^{(6)}$ to the Z_β OCD- component for $\eta_\beta = 9, c=5$; where the new column contains ± 1 , which are the maximum values of any entry and one entry is "0", as η_β is

odd. Consider, $Z_\beta = \begin{bmatrix} x_1 & +1 & +1 & +1 & +1 & +1 \\ x_2 & +1 & -1 & -1 & -1 & +1 \\ x_3 & -1 & -1 & -1 & +1 & +1 \\ x_4 & -1 & +1 & +1 & -1 & +1 \\ x_5 & +1 & +1 & -1 & +1 & -1 \\ x_6 & +1 & -1 & +1 & -1 & -1 \\ x_7 & -1 & -1 & +1 & +1 & -1 \\ x_8 & -1 & +1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Solving the values x 's $\in [-1,1]$, satisfying the

orthogonality condition among treatments and covariates, to find the maximum value of $|Z'_\beta Z_\beta|$, it yields,

$x_1 = x_2 = x_3 = x_4$	$x_5 = x_6 = x_7 = x_8$	$ Z'_\beta Z_\beta $
1/8	1/8	36864
1	-3/4	2,37,568
0.25	0	40,960
0	0.25	40,960
-0.75	1	2,37,568

Where all the different values of x_i 's, their determinants are same which 2,37,568 is maximum.

Proof (3.4) (vii) : For $\eta_\beta = 9, c = 7$. The determinant $|Z'Z|$ is maximized by the following. For any 9×1 vector with entries in $[-1, 1]$ and zero sum, the maximum norm² is 8 (achieved with one 0 and four pairs of ± 1). With 7 mutually orthogonal columns, at most 6 columns can achieve norm² = 8. The remaining column (column 1) must have reduced norm due to orthogonality constraints. The maximum norm² for column 1 under orthogonality to 6 other columns, each with norm² = 8, is 7.25 (achieved with entries 1,1,1,1, $-3/4, -3/4, -3/4, -3/4, -1$). Therefore, the maximum possible determinant is $7.25 \times 8^6 = 7.25 \times 262,144 = 1,900,544$. The proposed matrix achieves this bound with column 1 having norm² = 7.25 and columns 2-7 having norm² = 8. Since the upper bound is achieved, the matrix is optimal.

Proof (3.4) (viii) : For $\eta_\beta = 9, c = 8$. The determinant $|Z'Z|$ is the product of the squared column norms: $|Z'Z| = \prod_{j=1}^{(8)} ||z_j||^2$. For any column of a 9×1 vector with entries in $[-1, 1]$ and zero sum, the maximum possible norm² is 8 (achieved when there is one 0 and four pairs of ± 1). With 8 mutually orthogonal columns, at most 7 columns can achieve norm² = 8. The remaining column (column 1) must have reduced norm due to orthogonality constraints. The maximum norm² for column 1 under orthogonality to 7 other columns, each with norm² = 8, is 9/8. Therefore, the maximum possible determinant is $(9/8) \times 8^7 = (9/8) \times 2,097,152 = 2,359,296$. The proposed matrix achieves this bound with column 1 having norm² = 9/8 and columns 2-8 having norm² = 8. Therefore, the maximum possible determinant is $(9/8) \times 8^7 = 2,359,296$, which is achieved by the proposed matrix.