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same participants, the research instruments used in the study were additionally assessed by the Institute for Research and Community Service (LPPM) of Universitas Negeri Jakarta in 2026. Furthermore, the non-human component of the study was also assessed by LPPM in accordance with applicable institutional requirements.

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Abstract

This study investigates the empirical didactical situations of an epistemological approach to fractions arithmetic in a specific level of society. Sixth-grade students participated in a seven-session instructional sequence analysed through the Anthropological Theory of the Didactic. Data consisted of classroom observations, students' written productions, video recordings, and classroom discussions. The findings discuss students' reconstruction of unit fractions as quantities of units rather than as symbolic expressions or part–whole relations. The results further suggest that the four arithmetic operations can be understood as different manifestations of a common arithmetic organisation rather than as independent procedures. The present findings indicate that the fraction unit may function as a unifying mathematical object connecting whole-number and fraction arithmetic, that the absence of this object may contribute to the ad hoc character of fraction procedures, and that a coherent technological organisation plays a central role in the development of fraction understanding. Theoretical and instructional implications are also discussed.

Keyword: fractions, fractions arithmetic, theory of didactical situations, anthropological theory of the didactic

1 Introduction

Many questions have arisen on how mathematics as a knowledge is living. Some of them emerge in classrooms, while others can be found in online platforms from the scale of person(s), group, to countries. Such a scepticism often wandered in conceptual questions, including the famous one: "why do we have to flip $\frac{2}{3}$ to $\frac{3}{2}$ as we divide $\frac{4}{6}$ by $\frac{2}{3}$? and why could not we just divide 4 by 2 and 6 by 3, like in our dream of dividing natural number?". These dialectics indicates consideration of the *raison d'être*, from the fundamental requirements of a knowledge: the justification of the true belief (Pritchard, 2018; Sosa et al., 2008). In the epistemology of mathematical formalism, the demand for justification is as strong as the logics within the arguments derived from sets of axioms, for mathematics has a long deductive-axiomatic tradition (Knorr, 1980; Szabóo, 1967). The school system worldwide, however, has been inviting the discussion of the *mathematical knowledge to be taught*.

The problems of length and complexity (or subtlety) are main factors and incentives of *didactic transposition* (the transposition of knowledge in a didactic system) (Chevallard, 1986, 2019). Today's broad mathematics curricula have the awareness of the influence of Dewey, Piaget and his followers increased the appreciation for a view of mathematics, especially to whom the constructivism is outlooked (Arega & Hunde, 2025; Bishop, 1967; Boadu & Bonyah, 2024; Cevikbas et al., 2024; Piaget, 1964). Consequently, in most school textbooks (and other teaching sources), we see mathematics knowledge as not only a logically deductive axiomatic system, but as one that joined experience through its inductive pattern-seeking as well (Arnesen

& Skartsæterhagen, 2025; Jenlink, 2022). Except, the approach also implies to a lot of consequences. On the so-called home ground theory of mathematics education, anthropological theory of the didactics (ATD for forth) (Bosch et al., 2020; Chevallard, 2019), Chevallard asserted that in the transposition process of knowledge $\mathcal{k}_\infty$ from scholarly position to conditions specific to the schools $\mathcal{k}_\sigma$, there will always be a distortion that often unprecedented and unanticipated. Thus, the mathematics may far outstrip the notion of deductive logic, and the knowledge is jeopardized by a threat not yet mentioned.

For such a large institutional level, like a scale of a country for example, education appears to be not independent from a result of political or socio-economical transformations. Meanwhile, the work of teachers and students is naturally the level of didactic systems within a school. Furthermore, in the belief of ATD, our cognitive universe is the result of our activity in society (Artigue & Bosch, 2014; Chevallard et al., 2022). In this sense, developing analysis of the reality of didactic situation can be done in essential means of analysing human activity (including mathematical activity). The Situation in this context is also defined in specific term from the Theory of Didactic Situations (TDS for forth) (Brousseau, 1997, 2011) as the ideal model of the system of relationships between students, a teacher, and a mathematical *milieu*. Therefore, by the ATD, to address the issues of mathematical activity within a community and certain levels, we set to consider the corresponding persons in relation with their institutions.

We are encouraged by the opportunity to seek an empirical view of this institutional discussion in Indonesia as a society. Previously, Author et al. (2025) have reported how differences in one specific case across nations. This study continues the discussion on *knowledge to be taught* in a society(Chevallard, 2019) and examines how the status quo of *savoir*, applies, especially when a new approach is introduced.

2 A brief review on the models of fractions arithmetic

The discussion about the model of fraction arithmetic would perhaps be more imaginable if it were introduced by the criticisms of the *knowledge to be taught* of fractions in school that is considerably distorted from the perspective of arithmetic itself. Wu (2002, 2011) consistently argued that one of the major sources of students' difficulties with fractions lies not merely in learners' cognitive limitations but in the incoherence of fractions in school. A fraction is often characterised by multiple disconnected definitions, fragmented representations, and computational procedures introduced as isolated rules rather than logical consequences of a coherent mathematical system. As Wu (2002) briefly described, school fractions remain "certain persistent problematic areas in both the theory and practice." The definition of fractions, their conceptual complexities, and the rules governing the four arithmetic operations are frequently presented on an ad hoc basis and remain disconnected from the arithmetic of positive integers with which students are already familiar, while mathematical justifications for these procedures are largely absent. Similar concerns have also been reported in comparative textbook studies showing that school mathematics often emphasises isolated procedures, repetitive exercises, or contextual models without establishing coherent mathematical relationships among representations and operations (Alajmi, 2012; Yang et al., 2010). From an institutional perspective, Author et al. (2025) further demonstrated that these inconsistencies also appear in the treatment of reduction, where its mathematical role varies substantially across textbook institutions despite referring to the same mathematical object.

The transition from natural numbers $\mathbb{N}$ to rational numbers $\mathbb{Q}$, has likewise been identified as one of the fundamental sources of difficulty in learning fraction arithmetic because fractions are frequently introduced as an entirely new domain rather than as a natural extension of whole-number arithmetic (Lortie-Forgues et al., 2015; Mix et al., 1999; Siegler et al., 2011; Wu, 2002). Consequently, students often fail to perceive fractions as numbers possessing

magnitudes that belong to the same numerical continuum as whole numbers (DeWolf et al., 2014; Fazio et al., 2014; Schneider & Siegler, 2010; Siegler et al., 2011). Such discontinuity has long been associated with the well-documented whole-number bias. Fractions are interpreted as two independent whole numbers rather than as single numerical quantities (Braithwaite et al., 2019; Ni & Zhou, 2005; Van Hoof et al., 2015). Numerous studies have further shown that students' success in fraction arithmetic is strongly associated with their understanding of fraction magnitudes, conceptual knowledge, measurement interpretations, and relationships among numerical representations rather than procedural fluency alone (Fitzpatrick & Hallett, 2019; Hecht, 1998; Hecht & Vagi, 2010). Many common fraction errors emerge from overgeneralisation of previously successful procedures, omissions within learned algorithms, and the statistical regularities of instructional experiences rather than from entirely distinct misconceptions (Braithwaite et al., 2017, 2019). Collectively, these findings suggest that many persistent difficulties in fraction arithmetic originate not only from learners themselves but also from how arithmetic is organised, represented, and sequenced during instruction (Kaput et al., 2011; Lortie-Forgues et al., 2015).

In response to these concerns, several studies have progressively advanced ideas that may be described as integrated, or unified approaches to fraction arithmetic. Lesner et al. (2023), Sujadi et al., (2026) and Wu (2002) proposed that fractions should be defined as numbers on the number line so that whole numbers, fractions, decimals, and rational numbers belong to a single coherent numerical system, in which arithmetic operations follow common mathematical principles rather than isolated algorithms. A complementary perspective was developed through the integrated theory of whole-number and fraction development, which argues that numerical magnitude constitutes the common foundation underlying both whole-number and fraction learning (Siegler et al., 2011). This perspective was subsequently reinforced by evidence showing that fraction knowledge predicts later mathematical achievement (Ching & Li, 2024), that fraction magnitude understanding predicts both fraction arithmetic and overall mathematics performance. Building upon these findings, Braithwaite and Siegler (2021) proposed Putting Fractions Together which discusses the same generative mechanism to underlie both fraction representation and fraction arithmetic. This theoretical line was extended computationally through FARRA, demonstrating that many procedural behaviours in fraction arithmetic can be explained through common learning mechanisms (Braithwaite et al., 2022; Braithwaite & Siegler, 2024), and subsequently a unified theory of rational number arithmetic and the Unified Model of Arithmetic (UMA) (Braithwaite & Siegler, 2024), which explains whole-number, fraction, and decimal arithmetic within a single computational architecture. Although these studies originate from different theoretical traditions, they converge on a common proposition that fraction arithmetic should not be organised as a collection of independent procedures but rather through common units, common representations, and common mathematical principles that remain valid across numerical domains. Studies by Author et al. (2025) and Sujadi et al. (2026) continue this line of inquiry by proposing a Reference Epistemological Model (REM) for fractions arithmetic in which arithmetic operations are consistently interpreted through unit fractions and reduction processes.

3 The REM

In the present study, the "model" therefore refers to the REM, as we continue to take on epistemological model of fractions arithmetic by Author et al. (2025), with unit fractions as the dominant praxeology of fraction arithmetic and may be expressed in the following way.

Let $\hat{u}$ be a unit fraction. We consider the definition of a fraction by Recorde (1543) as "if a whole is divided into some equal parts, a fraction is the number of those parts". Now suppose

$\alpha_a, \alpha_s, \alpha_m, \alpha_d$ denote the binary operations of addition, subtraction, multiplication, and division respectively. We define α_a as the operation of combining quantities of unit fractions $\hat{u}$; α_s as the operation of removing one quantity of $\hat{u}$ from another; α_m as the operation of iterating or scaling quantities of $\hat{u}$; and α_d as the operation of partitioning quantities of $\hat{u}$. We then get each definition as the basis for each fraction to be processed, for which each given fraction is set. For instance, consider $\frac{1}{4} + \frac{2}{4}$ a case under α_a. In this form, we clearly have $\hat{u} = \frac{1}{4}$ or "fourth" as the unit fraction under α_a. Thus, by the definition of α_a and α_s, we obtain $\frac{1}{4} + \frac{2}{4} = u + 2u = 3u$ or "three fourth". At this point, we cannot yet have different $\hat{u}$ to perform these cases, except for α_m and α_d using some technologies. For example, given $\frac{1}{4} + \frac{2}{3}$, we need to define the $\hat{u}$ to satisfy the same form of unit. In the didactic praxeology, we have set activities on how to reduce the units by using technology θ:

- $\forall\, \hat{u} \in \mathbb{Z}\backslash\{0\}$ there exists $a, b, k \in \mathbb{Z}$ such that $a \cdot b = k \cdot \hat{u}$
- $\forall\, n \in \mathbb{Z}, d \in \mathbb{Z}\backslash\{0\}$ there exists $q, r \in \mathbb{Z}$ such that $n = q \cdot d + r \wedge 0 \leq r < |d|$
- $\forall\, q, r \in \mathbb{Z}, d \in \mathbb{Z}\backslash\{0\}$ there exists $n \in \mathbb{Z}$ such that $n = q \cdot d + r$
- $\forall\, a, c \in \mathbb{Z}, b, d \in \mathbb{Z}\backslash\{0\}$ there exists $p, q, k \in \mathbb{Z}$ such that $(a \cdot c = k \cdot p) \wedge (b \cdot d = k \cdot q) \wedge \gcd(p, q) = 1$

We can also see a and b as the factors of u. Hence, for $\frac{1}{4}$ and $\frac{1}{3}$ we can see $a = 4$ and $b = 3$ form a valid factor pair for $\hat{u} = 12$ by arithmetic. Note that the value must be preserved. It follows that both nominators are following the multiplication of their denominators. If $\hat{u} = 12$ then $\frac{2}{4} = \frac{6}{12} = 6\hat{u}$ and $\frac{1}{3} = \frac{4}{12} = 4\hat{u}$ by equivalent fraction. Hence, $6\hat{u} + 4\hat{u} = 10\hat{u} = 10\left(\frac{1}{12}\right) = \frac{10}{12}$. In that case, we have used one form of reductions, namely simplifying into one common denominator and combine into a single fraction. As the unit fractions were built as an epistemological basis, systematically the unit fractions also apply equally to multiplication and division. For instance, consider $\frac{1}{4} \times \frac{2}{3}$ as an α_m. Using the same technique τ_r (reduction) by technology θ_4 , we get $\hat{u} = \frac{1}{12}$, hence $3\hat{u} \cdot 4\hat{u} = 12\hat{u}^2 = \frac{12}{144}$. Notice that in multiplication, the result of multiplication will not always be larger than its precedents (Simon et al., 2018). The same process applies to α_d. For $\frac{8}{12} : \frac{2}{3}$, we have $\hat{u} = \frac{1}{12}$. By equivalence, $\frac{2}{3} = \frac{8}{12}$. Thus, $8\hat{u} : 8\hat{u} = 1$.

4 Research Question

Thus far, we have explained the rationale. Bearing this in mind, the goal of this study is to investigate the conditions and constraints associated with the implementation of epistemic approaches to fraction arithmetic within the Indonesian didactic system, through the following research question:

> RQ – How will Grade 6 students' response to the Situation approaching fractions arithmetic by the praxeological structure, and how didactic transposition process the model into the institutional knowledge?

To address such conditions and constraints of students, we join the discussion to seek the reality of the structure of the *knowledge to be taught* regarded as the model of fractions arithmetic. We present this form of knowledge as a model to be presented in didactic activities,

by questioning the objects in terms of institutional place, particularly in a classroom of Indonesian school. In describing such a model of subject matter, we continue the method and epistemological model issue in the next section.

5 Method

In the process of understanding the reality within a qualitative inquiry, we consider the organisation of the didactic system in the specific conditions in which it lives. This study mainly relies on the Anthropological Theory of the Didactic (ATD) (Bosch et al., 2020; Chevallard, 1985, 2019), which gradually emerged from the development of Brousseau's Theory of Didactical Situations in Mathematics (TDS) (Brousseau, 1997, 2011). The *Situation* makes it possible to describe the study process and the relations established within the classroom, and the ATD describes the mathematical activities carried out in these situations together with the discourses that explain and justify them.

This context of continuous study is recalled that mathematical knowledge Immanuel Kant's synthetic a priori relies on principles that make rational knowledge possible prior to experience (Kant, 1781). Yet the didactic problem does not concern these principles alone, but also the conditions under which such knowledge may appear, be studied, and be justified within institutions. In this sense, the present study proceeds from the rational requirements of mathematical knowledge to the empirical conditions in which this knowledge is encountered in the classroom. The reference epistemological model previously proposed by Author et al. (2025), and further described in the previous section constitutes such a rational reference. In the present study, the questions are presented in RQ to brought into classroom conditions and examined through what Brousseau called a Situation, as explained in the next section.

5.1 ATD – TDS Institutional Framework

The present study starts from the notion of *Situation* introduced within TDS as the "home ground theory" of mathematics education, as a model of the didactic system between students, a teacher, and a mathematical *milieu*. The following description thus refer to TDS (Brousseau, 1997) and ATD (Chevallard, 2019) and , In this perspective, the object is rather the system of relations in which mathematical activity takes place, than taking into account the individuality of each student. Students interact with a *milieu* and the interactions that take place during the study process. The *milieu* reacts to the students' actions, sometimes in a collaborative way and sometimes in an antagonistic way. Through these actions and reactions, students explore different ways of doing and progressively produce some knowing or "*savoir*".

The study process may then be described through successive phases of 1) action, 2) formulation, and 3) validation. Students interact with the *milieu* and progressively elaborate ways of doing and ways of speaking about what is done. During the successive phases of action, formulation, and validation, the knowledge at stake appears first as a means of acting upon the *milieu*, then as a means of expressing and communicating, and as an object of discussion and justification. In the didactic situation, the teacher comes into interverence the situation for reconciliation to the institution of where the knowledge comes from their scholar, namely bringing back to the scholarly mathematics through the process of decontextualisation, depersonalisation, and detemporalisation. The description of these phases allows one to follow the evolution of a *Situation* and the evolution of the students' activity within. At this point, the development of the TDS toward the ATD becomes particularly important. The activities carried out by the students may then be described in terms of the tasks to be accomplished and the ways of accomplishing them. The corresponding discourses that explain and justify these ways of doing may also be identified. In this way, the study of situations naturally leads to the study of praxeologies.

5.2 Praxeological and Didactic Analysis

Any action rarely exists without some discourse that explains, justifies, or makes them intelligible. From the perspective of the ATD, any human activity can be described as a praxeology or as an amalgamation of praxeologies of different sizes. Furthermore, the anthropological principle states that every human activity contains a praxis component together with a logos component (Bosch & Gascón, 2014; Chevallard, 2019). The praxis refers to the doing and to the corresponding know-how, while the logos refers to the discourses that describe, explain, and justify this doing.

The praxis block Π is made up of types of tasks T and techniques τ used to carry out these tasks. The logos block Λ contains the technology θ, understood as a discourse about the technique, together with the theory Θ, which provides the principles and notions upon which the technological discourse relies. The four elements T, τ, θ, and Θ form what the ATD calls a praxeology, the two blocks written as

$$\Pi = [\mathrm{T} \,/\, \tau] \tag{1}$$

$$\Lambda = [\theta \,/\, \Theta] \tag{2}$$

$$\Pi \oplus \Lambda = [\mathrm{T} \,/\, \tau] \oplus [\theta \,/\, \Theta] = [\mathrm{T} \,/\, \tau \,/\, \theta \,/\, \Theta] \tag{3}$$

The quartet $[\wp = \mathrm{T} / \tau / \theta / \Theta]$ being ['capital tau' / 'tau' / 'theta' / 'capital theta'], thus referred to as a praxeology of which Π and Λ identify the pinpoint praxeology $\Pi \oplus \Lambda$.

5.3 Participants and Settings

The study took place during the first semester of the school year participating sixth graders of 21 individuals. The class was not intentionally selected. The study was rather carried out under classroom conditions as part of the regular mathematics lessons. The students had already encountered fractions in previous grades. In particular, they had studied the representation of fractions, equivalent fractions, and the addition and subtraction of fractions with different denominators. The multiplication and division of fractions, however, had not yet been introduced in the classroom. Consequently, the students entered the study with a set of previous organizations concerning fractions and their arithmetic.

The study process extended over seven sessions distributed over approximately two months. The sessions were not organized as independent lessons devoted to isolated topics. Questions introduced during one session frequently reappeared in subsequent sessions, sometimes under different conditions and sometimes through different tasks. Likewise, techniques developed in one situation could be mobilized again in later situations. The seven sessions therefore formed a continuous study process rather than a succession of complete lessons (see Fig 1 for details).

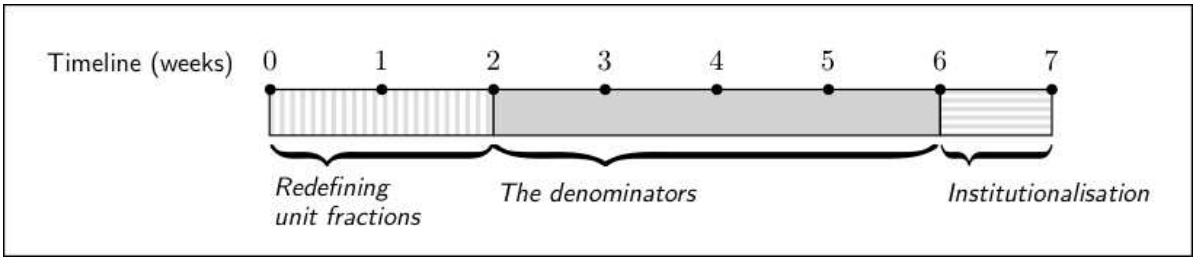


Fig. 1 Phases of the didactic sessions

- Session 1 and 2 (redefining unit fractions) explores and redefines unit fractions through partitioning and representation;
- Session 3 to 6 (the denominators) investigates denominators through arithmetic operations and fraction equivalence;
- Session 7 (institutionalisation) discuss the mathematical organisation developed throughout the previous sessions back into mathematics.

The researcher was also the classroom teacher and participated in the study as an interventionist. This position made possible the management of the situations as they evolved and allowed decisions to be taken according to the students' activity and the constraints emerging during the study process. The teacher's interventions therefore formed part of the didactic system itself and were considered as part of the observed phenomena.

5.4 Data Collection and Analysis

The collected materials included 1) video recordings of all sessions, 2) field notes collected during the classroom observations, 3) transcripts of selected discussions, 4) students' written productions, and 5) the teacher's notes prepared during the study process. These materials made it possible to reconstruct both the devolution of the tasks and the progress of students' mathematical activity.

As described earlier, the didactic and praxeological analysis is conducted from the perspective of TDS and ATD. Particular attention was given to the phases of action, formulation, validation, and institutionalization that appeared during the study process. The tasks proposed in the situations had previously been examined through their corresponding praxeologies within the reference epistemological model. The collected materials therefore allowed the identification of the tasks carried out by the students, the techniques employed, and the technological and theoretical elements mobilized during the study.

The next section presents the situations together with the tasks associated with them. The corresponding praxeologies are first described, after which the evolution of the situations and of the students' mathematical activity is analysed.

6 Findings

We began with the following Situation in which students explored fractions through the construction and manipulation of unit fractions, beginning with partitioning a unit strip and progressively moving toward number-line representations and arithmetic operations. The sequence was organised into three phases: Sessions 1–2 focused on redefining unit fractions, Sessions 3–6 on denominators, equivalence, reduction, and arithmetic operations, and Session 7 on institutionalisation.

By the time the tasks were implemented, students had already encountered fractions and basic fraction arithmetic through conventional representations and procedures. The proposed Situation therefore did not introduce fractions as a completely new mathematical topic, but reorganised students' existing knowledge through the unit-fraction approach, with the teacher and the milieu supporting the successive situations of action, formulation, validation, and institutionalisation.

6.1 Session 1 and 2: Redefining unit fractions

The didactic praxeologies developed during Sessions 1 and 2 were designed to develop the knowledge of the unit fraction $\hat{u}$ as the fundamental object of fractions. The objective was to construct $\hat{u} = \frac{1}{n}$ as the elementary unit from which subsequent arithmetic operations $\alpha_a, \alpha_s, \alpha_m, \alpha_d$ would later emerge. The task therefore initiated a situation of action in which students *X* would construct the mathematical meaning of fractions through interaction with the milieu, while teacher *Y* performed the devolution of responsibilities.

The initial didactic praxeology may be represented as $\mathrm{T}_{cut}, \tau_{cut}, \theta_u, \Theta$, where the task T_{cut} required students grouped by $n \in \{3, 4, 5, 6, 7\}$ to partition a ribbon of unit length into n equal pieces (Fig 2). The *milieu* consisted of ribbons, rulers, pencils, and scissors that provided immediate antagonist feedback to the students' actions.

Fig 2. Students' relation with *milieu* by the type of task T_{cut}.



We observed S_{11} from Group 4 hesitate before making his cuts and ask teacher *Y*, "*Pak, kalau enam berarti enam garis?*" [Sir[1], if there are six, does that mean there are six lines?] indicating an attempt to decode the didactical contract by asking whether the procedure itself was correct. Teacher *Y* intentionally refrained from answering in order to preserve the adidactical situation. S_{11} subsequently drew six cutting lines and produced seven pieces. The contradiction immediately emerged because the ribbon itself invalidated his technique. Through adaptation, S_{11} reformulated his understanding by stating, *"Oh, enam potong ternyata bukan enam garis"* [Uh, six pieces actually do not mean six lines]. The adaptation generated the technique τ_{cut}, namely that the number of pieces rather than the number of cuts determines the partition.

The subsequent task T_{name} shifted the situation from action toward formulation. This task was intentionally designed to confront the students' previous praxeology $\mathrm{T}_{part-whole}$; $\tau_{notation}$, which interpreted fractions as static symbolic expressions $\frac{k}{n}$. Several students immediately wrote expressions such as $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$ when asked to identify a single piece. We observed S_2 from Group 2 write "$\frac{1}{4}$" and state, "*Ini satu per empat*" [This is one-fourth]. Teacher *Y* then asked, "That is the fraction, but what is the name of the piece?" S_2 remained silent for several moments. His existing technology interpreted $\frac{1}{4}$ merely as notation rather than as an independent mathematical object.

The worksheet subsequently provided the example *"*perdua*,"* [half] allowing S_2 and his group to formulate the expression "perempat" [fourth] (Fig 3). This adaptation generated the

[1] Indonesian utterances are presented in the original language, followed by English translations in square brackets. The address term "Pak," commonly used by Indonesian students when addressing a male teacher, is translated as "Sir."

technology θ_u, namely that a fraction $\frac{k}{n}$ is the number k of a unit $\hat{u} = \frac{1}{n}$. The denominator therefore ceased to indicate merely the number of partitions and instead became the definition of the unit itself. The subsequent task T_{count} immediately validated this technology. S_2 pasted two of his perempat strips and verbalised "*dua perempat*," [two fourths] or equivalently $2\hat{u}_A$, where $\hat{u}_A = \frac{1}{4}$. The notation $\frac{2}{4}$ therefore emerged as the consequence of counting units rather than partitioning wholes. Fig 3 presents examples of students' written responses concerning the construction and naming of the unit fractions. The figure illustrates the transition from symbolic notation toward the interpretation of fractions as quantities of units.

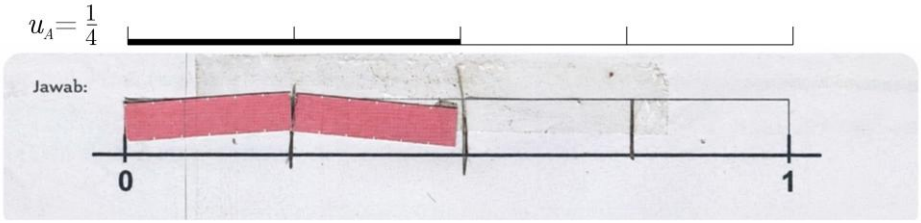


Fig 3. Students' work on the construction and counting of unit fractions.

Following the construction of the unit fractions, we introduced the task T_{map}, which required students to place their constructions $k\hat{u}$ onto a number line scaled from 0 to 1. This new milieu represented an important didactical shift because the ribbon strips no longer served as the principal reference object. Instead, the number line became the validating environment through which fractions were interpreted as magnitudes. We observed S_2 confidently place his construction $2\hat{u}_A$, or 2 $\frac{1}{4}$s, onto the number line. Moments later, S_5 from another group positioned $1\hat{u}_B$, where $\hat{u}_B = \frac{1}{2}$, at exactly the same location.

This situation generated a cognitive conflict for the entire class. We observed S_{17} point toward the board and exclaim, "*Kok sama tempatnya*? Punyanya S_2 kan $\frac{2}{4}$, punya S_5 itu $\frac{1}{2}$" [Why do they have the same position? S_2's is $\frac{2}{4}$, while S_5's is $\frac{1}{2}$]. His previous technology that different symbolic expressions necessarily represent different quantities was falsified by the milieu. Unlike the ribbon strips, which remained local to each group, the number line became a common reference system that validated magnitude independently of notation.

The disequilibrium experienced by S_{17} subsequently provided the *raison d'être* for the planned math congress. Teacher Y devolved the validation task back to the class by asking, "*Jadi, siapa yang benar?*" [So, who is correct?]. Rather than resolving the contradiction, teacher Y returned it to the students. We observed S_8 from Group 3 approach the front and place his construction $3\hat{u}_C$, where $\hat{u}_C = \frac{1}{6}$, at the same location as $2\hat{u}_A$ and $1\hat{u}_B$. He then stated, "*Punyaku juga sama*" [Mine is the same too]. The students therefore collectively formulated $2\hat{u}_A = 1\hat{u}_B = 3\hat{u}_C$ or equivalently $\frac{2}{4} = \frac{1}{2} = \frac{3}{6}$. This adaptation generated the technology θ_{equiv}, namely that different units may nevertheless produce the same magnitude. Teacher Y subsequently institutionalised this student-generated knowledge by stating, *"Ini yang kita sebut pecahan senilai"* [This is what we call equivalent fractions]. The technology θ_{equiv} later becomes the epistemological basis for the reduction techniques developed in the subsequent sessions. With the technologies θ_u and θ_{equiv} established, we introduced the operation of addition α_a through tasks involving common denominators. Since each group possessed only one type of unit, the students naturally interpreted expressions such as $\frac{1}{5} + \frac{2}{5}$ and $\frac{4}{6} - \frac{1}{6}$ as

operations on identical units. Nevertheless, several students initially reverted to whole-number reasoning. We observed S_{17} conjecture that the operation involved both numerators and denominators. Teacher Y again refrained from intervention. Instead, S_8 physically placed one perenam strip together with two additional perenam strips and stated, "*Potongannya sama*" [The pieces are the same]. He subsequently counted the pieces and concluded, *"Jadi tiga"* [Becomes three].

This validation generated the technique τ_{add_u}, namely that $k_1\hat{u} + k_2\hat{u} = (k_1 + k_2)\hat{u}$. The technology θ_a consequently emerged, which may be verbalised as: arithmetic operations are performed on identical units. We later observed S_{17} adapt his previous conjecture and state, "*Ditambah potongnya karena potongannya sama*" [We add the pieces because the pieces are the same]. The local praxeology [$T_{add-like}$;τ_{add_u};θ_a;Θ] was therefore constructed through the falsification of a previous technique. The same technology was subsequently validated through subtraction α_s. Students interpreted $\frac{6}{7} - \frac{1}{7}$ as removing one unit from a collection of identical units. We observed S_{11} remove one strip from six sevenths and state, "*Sisanya lima*" [The remaining is five]. The operation $6\hat{u}_D - 1\hat{u}_D - 5\hat{u}_D$, where $\hat{u}_D = \frac{1}{7}$, emerged naturally as the action of removing identical units.

The final situation of Session 2 extended the counting process beyond one whole. Students from the thirds group observed that $3\hat{u} = 1$. Teacher Y subsequently asked them to construct $4\hat{u}$. Several students initially rejected the task because their previous conception assumed that fractions satisfy $k < n$. We observed S_{15} state, "*Gak bisa karena lebih dari satu*" [You can't, because there is more than one]. However, S_5 added another third strip beyond the unit interval and stated, "*Masih ada satu sepertiga*" [There is still one-third left]. The number line immediately validated this construction because the additional third occupied a position beyond the point 1. The class consequently formulated $\frac{4}{3} = 1 + \frac{1}{3}$. Similar situations occurred for $\frac{5}{4}$ and $\frac{7}{6}$, where students repeatedly counted the same unit beyond one whole. The number line again functioned as the validating milieu because these quantities could be located beyond the position 1. This adaptation generated the technology θ_{mixed}, namely that the operation $k\hat{u}$ remains valid even when $k > n$. Improper fractions and mixed numbers therefore emerged not as symbolic procedures but as equivalent descriptions of the same accumulation of units.

The didactic contract established at the end of Sessions 1 and 2 may therefore be summarised by the theory Θ that all fractions are counts of unit fractions. Students no longer interpreted fractions as static parts of a whole but instead understood them as quantities of identical units located on a number line. The denominator became the definition of the unit, the numerator represented the number of those units, and arithmetic operations became actions on identical units. Moreover, the validation situations involving the number line established the technology θ_{equiv}, while the extension beyond one whole legitimised improper fractions and mixed numbers. Nevertheless, the praxeology remained intentionally incomplete. The institutionalised technologies θ_u, θ_{equiv}, and θ_a subsequently create the epistemological need for the following sessions, where operations involving different units break the existing didactical contract and motivate the reduction technology.

6.2 Session 3 to 6: The denominators

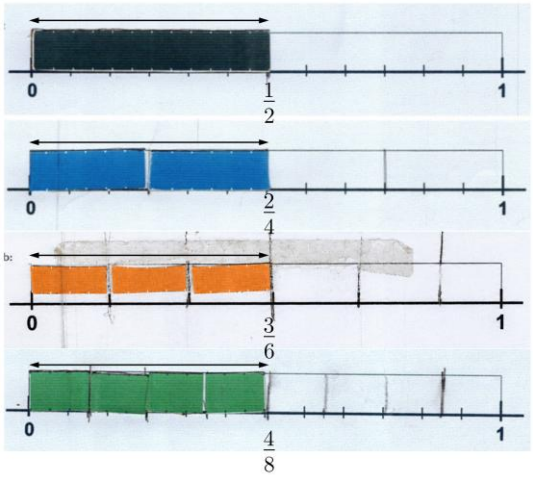
The institutionalisation of the praxeologies and [$T_{add-like}$,τ_{add_u},θ_a,Θ] and [$T_{sub-like}$,τ_s,θ_a,Θ] at the end of Session 2 established an important didactical contract: arithmetic operations are performed on identical units. Students had successfully adapted to the technology θ_a, namely that $k_1\hat{u} + k_2\hat{u} = (k_1 + k_2)\hat{u}$ whenever the units are identical. However, this local praxeology was intentionally incomplete. We subsequently introduced

situations that systematically violated this contract in order to generate the epistemological necessity of the denominator.

The first task T_{equiv} revisited the equivalence situations previously encountered during Session 2. Students were asked to compare $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{4}{8}$ by placing the corresponding ribbon constructions on a common number line. Although the equality $\frac{2}{4} = \frac{1}{2}$ had previously emerged as a local observation, the present task required students to justify why different units could nevertheless represent identical magnitudes. We observed S_{17} revisit his earlier conjecture and state, *"Kalau angkanya beda kenapa panjangnya sama?"* [If the numbers are different, why are their lengths the same?]. His question demonstrates that the symbolic difference between fractions remained an epistemological obstacle despite the validation situations of Session 2.

Teacher Y devolved the validation task back to the students. We observed S_8 place $2\hat{u}_A$, $1\hat{u}_B$, and $3\hat{u}_C$, where $\hat{u}_A = \frac{1}{4}$, $\hat{u}_B = \frac{1}{2}$, $\hat{u}_C = \frac{1}{6}$, and $\hat{u}_D = \frac{1}{8}$ at the same location on the number line. He subsequently stated, *"Potongannya beda, tapi panjangnya sama"* [The cuts are different, but their lengths are the same] (Fig 4). The adaptation strengthened the technology θ_{equiv}, namely that different units may represent the same quantity. Teacher Y institutionalised this observation by introducing the terminology of equivalent fractions. The denominator therefore acquired a second meaning: besides defining the unit, it also determines how a quantity may be represented through equivalent units.

Fig 4. Students perform on the same number line to observe positions



The development of θ_{equiv} subsequently created the need for the task $T_{add-unlike}$. Expressions such as $\frac{1}{2}+\frac{1}{4}$, $\frac{1}{3}+\frac{1}{6}$, $\frac{1}{4}+\frac{1}{6}$, were deliberately chosen to violate the existing technology θ_a. Unlike the previous situations, the units were no longer identical. We observed S_{17}, who had confidently applied τ_{add_u} during Session 2, stop and state, *"Ini...[terjeda] satu unit sama satu unit, tapi unitnya beda"* [This is… [pause] one unit and one unit, but the units are different]. This formulation itself demonstrates the institutionalisation of the previous sessions. Students no longer interpreted fractions as symbols but as quantities of units. Nevertheless, the operation became impossible because the units were different.

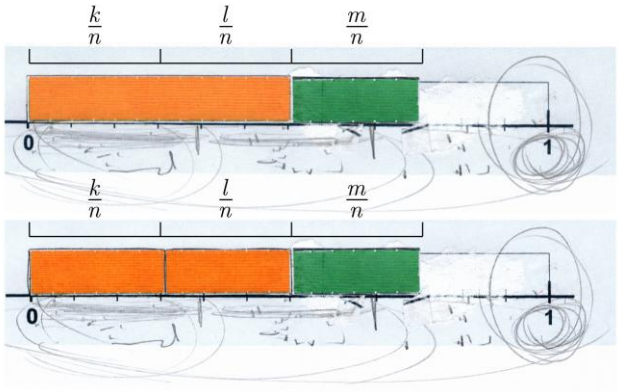
Several students initially attempted to extend their previous procedures. One conjectured that $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$, another student proposed $\frac{1}{2}+\frac{1}{4}=\frac{3}{8}$. Teacher Y intentionally refrained from evaluating these answers. Instead, he asked, *"Kalau satu pensil ditambah satu spidol hasilnya apa?"* [What do you get if you add one pencil and one marker?]. The class immediately responded negatively. One student answered, *"Gakbisa lah"* [You can't do that]. Another student added, *"Pensil tambah pensil baru bisa"* [You can only add a pencil to a pencil]. The

analogy became an important technology because it extended θ_a beyond fractions and established the necessity of identical units for arithmetic operations. The contradiction generated the task T_{find-u}, whose objective was to determine a common unit. Teacher *Y* reformulated the problem:

> *"Bagaimana supaya setengah dan seperempat menjadi satuan yang sama?"* [How can we make one-half and one-fourth have the same unit?]
> We observed S_5 place a half strip beside two fourth strips and state,
> *"Ini sama"* [These are the same], Another student suggested,
> *"Setengahnya dipecah lagi"* [The half needs to be divided again].

The class therefore reconstructed $\frac{1}{2} = 2\left(\frac{1}{4}\right)$ (see Fig 5). The original expression $\frac{1}{2} + \frac{1}{4}$ was subsequently reformulated as $2\hat{u}_{new} + 1\hat{u}_{new}$, $\hat{u}_{new} = \frac{1}{4}$. The adaptation generated the reduction technique τ_r, namely that unlike units may be transformed into a common unit before arithmetic operations are performed. The common denominator therefore emerged not as an algorithm but as a consequence of restoring identical units.

Fig 5. Reconstruction of units preserving the fractions

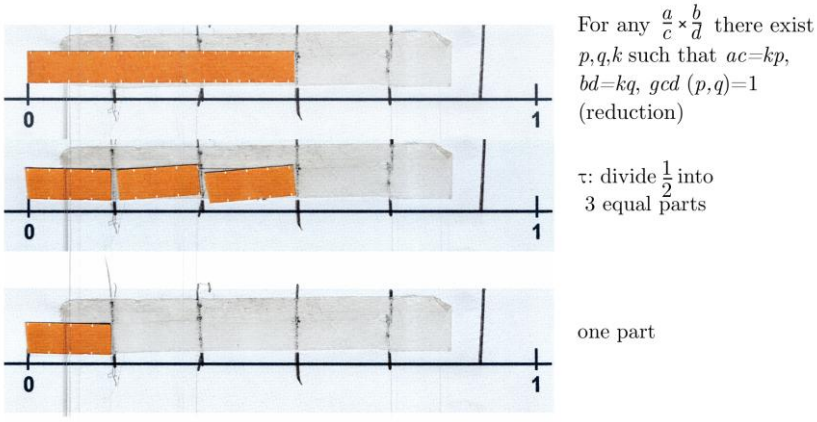


The same difficulty emerged for $\frac{1}{3} + \frac{1}{4}$. Several students initially attempted to use sixths as the common unit because they had previously constructed $\frac{1}{6}$. We observed S_8 place sixth strips beside the original pieces and state, "*Nggak pas ini*" [This doesn't fit]. The failure of the measurement was important because the milieu itself invalidated the proposed unit. Teacher *Y* subsequently asked, *"Kalau satuannya belum sama, apa yang harus dilakukan?"* [If the units are not yet the same, what should we do?]. We observed S_2 divide his third strip into four pieces while S_{11} divided his fourth strip into three pieces. The resulting pieces had equal lengths. One student immediately stated, *"Sekarang sama"* [Now they are the same]. The class therefore constructed $\hat{u}_{new} = \frac{1}{2}$, which leads to $\frac{1}{3} + \frac{1}{4} = 4\hat{u}_{new} + 3\hat{u}_{new}$ and subsequently became $7\hat{u}_{new}$ or $\frac{7}{12}$. The technology θ_r therefore emerged: arithmetic operations require the restoration of identical units.

The same praxeology was immediately validated through subtraction. Students were asked to solve $\frac{1}{2} - \frac{1}{3}$. Several students initially attempted to subtract numerators and denominators directly. We observed S_{15} propose $\frac{1}{2} - \frac{1}{3} = \frac{-1}{-1}$. The absurdity of the result produced immediate discussion. S_8 subsequently stated, *"Satuannya dulu"* [The units first]. The students reconstructed $\hat{u}_{new} = \frac{1}{6}$ allowing $\frac{1}{2} - \frac{1}{3} = 3\hat{u}_{new} - 2\hat{u}_{new}$. The operation therefore became $1\hat{u}_{new}$ or $\frac{1}{6}$. The reduction technique τ_r thus functioned equally for addition and subtraction. We subsequently extended these situations to improper fractions and mixed numbers. Tasks such as $\frac{5}{4} + \frac{1}{6}$, $1\frac{1}{3} - \frac{1}{4}$, $\frac{7}{6} + \frac{1}{3}$ were intentionally designed to determine whether

students would revert to symbolic procedures. However, we observed that the students continued to search for common units. S_2 explicitly stated, *"Yang dicari tetap satuannya"* [What we are looking for is still the unit]. For example, $\frac{5}{4}+\frac{1}{6}$ was transformed into $1+\frac{1}{4}+\frac{1}{6}$.

Students subsequently constructed $\hat{u}_{new}=\frac{1}{2}$ and resulting $1+3\hat{u}_{new}+2\hat{u}_{new}$. The operation therefore became independent of the symbolic form of the fractions. Proper fractions, improper fractions, and mixed numbers were all treated as collections of units. This adaptation generated the technology $\theta_{general}$, namely that arithmetic operations should remain invariant regardless of the representation of the fractions. The operation itself remains identical; only the units change.



a b

Fig 6. Multiplication as scaling as a) paper stripes and b) interpretation

The introduction of multiplication α_m challenged another existing technology. Many students initially associated multiplication with repeated addition. We therefore reformulated the task as "one third of a half" (Fig 6) rather than $\frac{1}{3}\times\frac{1}{2}$. We observed S_5 take his third strip and fold it into two equal pieces. The resulting piece immediately generated the question, *"Ini namanya apa?"* [what is this called?]. The question resembled the epistemological obstacle encountered in Session 1. Students had produced a new quantity but lacked an appropriate unit. S_8 subsequently compared the new piece with previously constructed units and stated, *"Kalau setengah ini dibagi jadi tiga berarti..."* [if this half is divided into three, then...]. The class therefore constructed $\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$ (Fig 5). Teacher (Y) asked,

"Mengapa hasilnya lebih kecil?" [Why is the result smaller?], one student responded, *"Satuannya berubah."* [The unit has changed].

The multiplication technology θ_m therefore emerged: multiplication acts as a scaling operation that generates a new unit. The same reasoning was extended to $\frac{2}{3}\times\frac{4}{5}$. Students first considered $\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$. This new unit subsequently became $\hat{u}_{new}=\frac{1}{15}$. The operation was then reformulated as $2\hat{u}_{new}\times 4$ which produced $8\hat{u}_{new}$. The resulting quantity therefore became $\frac{8}{15}$. The multiplication algorithm emerged as a consequence of unit multiplication rather than symbolic manipulation.

The final extension concerned division α_d rejected the procedural rule of "invert and multiply" and instead introduced division as measurement. The task $\frac{1}{2}\div\frac{1}{6}$ was reformulated as: *"Berapa banyak seperenam yang muat dalam setengah?"* [How many one-sixths fit into one-half?]. We observed S_8 place sixth strips on top of a half strip and count: *"Satu, dua, tiga."* He

subsequently concluded, *"tiga"* [three]. Several students immediately objected because the answer was greater than one. We observed S_{17} ask, *"Kenapa tiga? Bukannya setengah lebih kecil?"* [Why three? Isn't one-half smaller?], teacher Y returned the question to the class. S_5 responded, "*Karena kita menghitung banyak seperenamnya"* [Because we are counting how many one-sixths there are].

This adaptation generated the technology θ_d, namely that division determines how many units fit inside another quantity. The same difficulty emerged for $\frac{1}{3} \div \frac{1}{4}$. Students initially failed because the units differed. We observed S_2, who had previously developed τ_r, state, "*Samakan satuannya dulu."* [Make the units the same first]. The students therefore constructed $\hat{u}_{new} = \frac{1}{12}$. The original task became $4\hat{u}_{new} \div 3\hat{u}_{new}$. One student subsequently observed, *"Ini tinggal empat dibagi tiga."* [Now, we just divide four by three]. The result $\frac{4}{3}$ therefore emerged through measurement rather than inversion.

The end of Sessions 3 to 6 may therefore be summarised by the theory Θ that arithmetic operations must remain invariant regardless of the symbolic form of the fractions involved. Proper fractions, improper fractions, and mixed numbers all represent quantities of units, while the four operations $\alpha_a, \alpha_s, \alpha_m, \alpha_d$ operate on these units according to the same principles. The reduction technique τ_r restores common units for addition and subtraction, multiplication generates new units through scaling, and division measures the number of units contained within a quantity. The procedural algorithms commonly taught in school therefore remained intentionally absent, as the REM sought first to establish the conceptual legitimacy of the four arithmetic operations before their institutionalisation in Session 7.

6.3 Session 7: Institutionalisation

The institutionalisation process may further be interpreted through the notions of decontextualisation, depersonalisation, and detemporalisation. Throughout Sessions 1–6, students' knowledge was inseparable from ribbons, number lines, physical units, and classroom discussions. Fractions existed as actions upon units, and arithmetic emerged through the manipulation of material objects. During Session 7, these contextual supports gradually disappeared. The ribbon strips, physical pieces, and number lines ceased to function as legitimising milieus, and the symbolic expressions themselves became sufficient representations of the mathematical objects. The reduction $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$ therefore became detached from the physical reconstruction of units. This decontextualisation transformed the students' situated knowings into abstract mathematical objects.

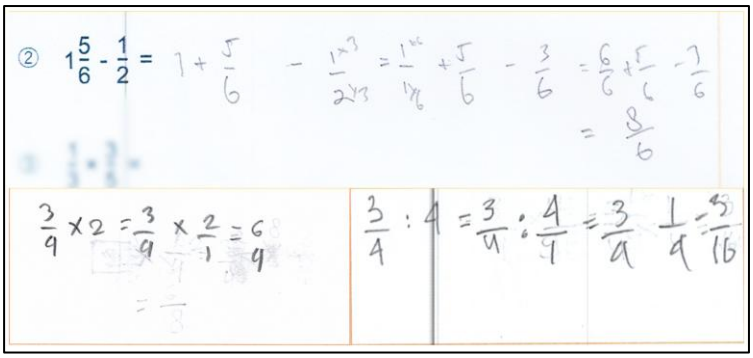


Fig 7. Subtraction, multiplication and division in formal forms

The institutionalisation also produced depersonalisation. During the previous sessions, the technologies θ_u , θ_r, θ_m, and θ_d emerged through the conjectures of particular students.

Statements such as *"potongannya harus sama"* [the pieces have to be the same] or *"yang dicari satuannya"* [what we are looking for is the unit] belonged to S_8, S_{17}, or other members of the classroom community. During Session 7, these personal formulations were transformed into institutional mathematical statements. The common denominator became a mathematical rule, multiplication became the product of numerators and denominators, and division became multiplication by the reciprocal. Knowledge was therefore no longer owned by individual students but became part of the mathematical institution itself, as shown by the works in Fig 7.

The long sequence of conjectures, failures, adaptations, and validations that characterised Sessions 1–6 disappeared from the final symbolic formulations, as a form of detemporalisation. Expressions such as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \tag{1}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} \tag{2}$$

appeared as stable mathematical truths independent of their historical production. The temporal process through which students constructed these ideas was removed, and the resulting knowledge acquired the permanence and justification associated with institutional mathematics. The students' personal, contextual, and temporal experiences with unit fractions were transformed into decontextualised, depersonalised, and detemporalised mathematical knowledge. The final institutional praxeology therefore no longer belonged to the classroom community but to the mathematical institution itself. Consequently, Session 7 completed the transition from knowing to knowledge.

Having described in detail the didactical situations, tasks, techniques, conjectures, and validation processes developed throughout Sessions 1–7, we now summarise their corresponding praxeological components. The complete set of types of tasks, techniques, technologies, and the underlying theory identified across the sessions is presented in Table 1.

Table 1. Pinpoint praxeology in the Sessions

Session	T type of task	t task example	τ technique	θ technology	Θ theory
1-2	T_u: Constructing unit fractions	Partition a unit strip into n equal pieces; name and count the pieces; represent k pieces	τ_u: partition (1) into (n) equal units and iterate k units, $\hat{ku} = \frac{k}{n}$	θ_u: $\hat{u} = \frac{1}{n}$; the denominator defines the unit and the numerator counts the units	Θ: Arithmetic
	T_{eq}: Establishing equivalence	Compare $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}$ using strips and the number line	τ_{eq}: map units onto a common length, compare their positions	θ_{eq}: different units may represent the same quantity, $\hat{u}_A \neq \hat{u}_B$ but $k_A\hat{u}_A = k_B\hat{u}_B$	
	$T_{add/sub-like}$: Operating on identical units	Add or subtract fractions with the same denominator	τ_a, τ_s: combine or remove the number of identical units	θ_a: $k_1\hat{u} \pm k_2\hat{u} = (k_1 + k_2)\hat{u}$	
3-6	$T_{add/sub-unlike}$: Operating on unlike units	Add or subtract fractions with different denominators	$\tau_r \rightarrow \tau_a, \tau_s$: reconstruct a common unit, then combine or remove units	θ_r: equivalent quantities can be expressed through a common $\hat{u}_{\text{new}}$	
	$T_{improper/mixed}$: Extending unit counting	Construct fractions beyond (1), e.g. $\frac{4}{3}, \frac{5}{4}$, and express them as mixed numbers	τ_{mix}: continue counting $\hat{u}$ beyond one whole and regroup into whole units	θ_{mix}: $k\hat{u}$ remains valid for $k > n$; improper and mixed forms describe the same quantity	
	T_m: Multiplication as scaling	Determine "$\frac{1}{2}$ of $\frac{1}{3}$" and extend to products of quantities	τ_m: scale one unit fraction by another and construct the resulting smaller unit	θ_m multiplication generates a new unit, $\hat{u}_A \times \hat{u}_B = \hat{u}_C$	
	T_d: Division as measurement	Determine how many $\frac{1}{6}$-units fit into $\frac{1}{2}$; extend to unlike fractional units	τ_d: repeatedly place the divisor unit and count how many units fit; use τ_r when units differ	θ_d: division determines the number of units contained in a quantity	
7	T_{inst}: Institutionalising fraction arithmetic	Formalise the relations through strips, number lines, and classroom validation into symbolic procedures	τ_{inst}: replace contextual techniques with conventional symbolic algorithms	θ_{inst}: the institutional rules express the previously constructed unit-based technologies	

7 General Discussion

The first and perhaps most important suggestion from the findings is that fraction arithmetic requires a single mathematical definition before it can support multiple interpretations. School mathematics has traditionally introduced fractions simultaneously as parts of a whole, quotients, ratios, operators, measures, and numerical values, despite these interpretations originating from different mathematical contexts (Wu, 2002, 2011). Furthermore, Wu (2011) also argued that this institutional practice leaves students without a clear definition of fraction lead to a situation in which learners manipulate symbols whose mathematical status remains ambiguous.

Considering the prior praxeological organisation of school fractions, this obstacle appears rather natural. Fractions often introduced as symbolic pairs of whole numbers. However, the technological discourse concerning the unit remains implicit. Consequently, students encounter numerators and denominators as independent numbers rather than as a quantity–unit relationship, which had previously discussed by Author et al. (2025) with how Indonesian interpretation of "per" as the bar followed by denominator, e.g. "per three" instead of "thirds". Siegler, Thompson, and Schneider (2011), who argue that fraction magnitude knowledge occupies a position in overall fraction understanding and explains substantial variation in arithmetic performance. Likewise, Siegler and Pyke (2013) report that differences in fraction magnitude knowledge predict both fraction arithmetic achievement and overall mathematical achievement, even after controlling for executive functioning and reading ability. The present findings provide a possible didactical explanation for these relationships. If the denominator does not function as the definition of the unit, then fraction magnitudes themselves become difficult to construct

Fraction arithmetic becomes a collection of procedures detached from the mathematical object they supposedly operate upon. Our findings suggest that constructing the unit fraction as the primary ontological object fundamentally changes this situation. Students no longer encountered fractions as two integers separated by a bar, but as quantities of identical units from which symbolic representations were subsequently derived. Thus, quotient, ratio, operator, part-whole, and measure ceased to function as competing definitions and instead became different institutional manifestations of the same mathematical object. This interpretation strengthens Wu's (2002) proposal that all meanings of fractions should be deduced from a single definition rather than introduced independently, while simultaneously aligning with recent movements toward integrated and unified conceptions of rational numbers (Braithwaite et al., 2022; Braithwaite & Siegler, 2024; Sujadi et al., 2026; Wu, 2002). The part-whole interpretation should no longer be regarded as the conceptual foundation of fraction arithmetic but rather as an epistemological obstacle once arithmetic operations become the object of study. Wu (2011) criticised this approach because it prioritises contextual complexity at the expense of mathematical simplicity, while developmental studies likewise report that students who rely exclusively on part-whole reasoning struggle to generalise fractions as numerical magnitudes (DeWolf et al., 2014; Resnick et al., 2023; Schneider & Siegler, 2010).

From the findings, our results provide an alternative explanation. The transition observed from partitions to the number line validated the mathematical *milieu*. The emergence of unit fractions also offers a different interpretation of the whole-number bias. Previous studies consistently conclude that children's difficulty with fraction and rational numbers is associated with their whole-number knowledge, including the whole-number bias (Ni & Zhou, 2005; Yuan & Chen, 2023). Rather than interpreting fractions as single numerical quantities, students frequently process numerators and denominators independently, overgeneralising whole-number procedures to fraction situations (Lortie-Forgues et al., 2015). Recent unified models similarly explain many procedural errors as consequences of overgeneralisation and

differential learning experiences rather than entirely new cognitive mechanisms (Braithwaite & Siegler, 2024). Our findings do not contradict these accounts but suggest a complementary explanation from the perspective of didactic transposition. The whole-number bias appears not merely because children possess prior whole-number knowledge but because the institutional object initially presented to them is itself organised as two whole numbers. When students first encounter the notation $\frac{a}{b}$, their existing whole-number schemas naturally dominate because no alternative ontological object has yet been established. In contrast, beginning with quantities of unit fractions enables students to reuse previous whole-number knowledge productively through counting, iteration, partitioning, and accumulation of identical units. Thus, whole-number knowledge becomes the epistemological resource from which fraction arithmetic develops rather than the source of interference that must later be suppressed. Such an interpretation directly addresses Ni and Zhou's (2005) question concerning how fraction learning might be organised to capitalise on existing whole-number knowledge, while extending the integrated developmental framework proposed by Siegler et al. (2011) and the unified arithmetic mechanisms formalised in UMA (Braithwaite & Siegler, 2024).

The addition, subtraction, multiplication, and division as different actions performed on the same mathematical object of unit fractions. Conventional instruction treats each operation as governed by separate rules creating the impression that fraction arithmetic is a collection of algorithms, on which Wu (2002) described as “ad hoc basis”. The present study however shows that addition combines, subtraction removes, multiplication scales, and division measures quantities of unit fractions, while the underlying object remains unchanged. Furthermore, we also learn from the institutionalisation that throughout the study, the mathematical technologies associated with unit fractions, equivalence, reduction, multiplication, and division emerged through interactions between students and the milieu. During institutionalisation, these locally validated conjectures underwent decontextualisation, depersonalisation, and detemporalisation before becoming recognised mathematical knowledge. From TDS, the experience is essential as mathematical knowledge exists institutionally only after its dependence upon particular students, materials, and moments of construction has been removed (Brousseau, 1997), and simultaneously completed the transition from students' situated knowings toward classroom institution.

The coherence of fraction arithmetic depends on the coherence of its praxeology. Addition, subtraction, multiplication, and division presented as four independent procedures governed by different rules. Such fragmentation was criticised by Wu (1999, 2001), who argued that the rules of fraction arithmetic appear to be made up on an ad hoc basis and lack mathematical justification. Our observation strengthens the growing body of literature (Braithwaite et al., 2022; Braithwaite & Siegler, 2021, 2024; DeWolf et al., 2014; Saxe et al., 2013; Siegler et al., 2011, 2013; Sujadi et al., 2026; Wu, 2002) in the discussion that fraction arithmetic should be understood through common units, common representations, and common learning mechanisms rather than isolated computational procedures.

Unit fractions functioned as the dominant praxeology from which reduction, equivalence, addition, subtraction, multiplication, and division were derived. The fundamental concerns raised in the introduction regarding multiple definitions of fractions, the absence of mathematical explanations, operation-specific algorithms, and the conceptual discontinuity between whole numbers and rational numbers.

8 Concluding Remark

This study provides empirical support for an emerging international movement toward integrated and unified perspectives, starting from the contribution to the existing discussion in

this relative institution. In this concluding section, we summarise the principal findings and discuss their implications and limitations for future research and practice.

As a summary, the findings revealed that students progressively reconstructed fractions from symbolic expressions into quantities of units. The denominator gradually changed from a symbolic component of notation into the definition of the unit itself, while arithmetic operations became organised around common units, scaling, and measurement. The instructional sequence showed that equivalent fractions, reduction, multiplication, and division were not introduced as isolated procedures but emerged from a common technological system $\Theta = \theta_u, \theta_a, \theta_r, \theta_m, \theta_d$, which respectively describe the unit, arithmetic operations on identical units, reduction to common units, scaling, and measurement. This technological organisation suggests that the four arithmetic operations can be understood as different manifestations of a common arithmetic structure.

The findings further suggest that several difficulties commonly associated with fractions may originate from the absence of an explicit fraction unit in school mathematics. The distinction between the unit and the quantity of units appeared essential for the development of fraction meanings, equivalent fractions, reduction, and arithmetic operations. Consequently, the denominator functioned not merely as a symbolic component of notation but as the definition of the unit itself. The study therefore suggests that the fraction unit constitutes an important mathematical object that is often underrepresented in conventional instruction. This study also demonstrates the usefulness of combining TDS and ATD for investigating conceptual development in mathematics classrooms. The analysis of situations made it possible to describe students' adaptations, conjectures, and validations, while the praxeological perspective provided an explanation of the technological organisation underlying fraction arithmetic. The institutionalisation phase further clarified how students' contextual and personal knowings became decontextualised, depersonalised, and detemporalised as institutional mathematical knowledge. Furthermore, the higher levels of co-determinacy, including curriculum, textbooks, and broader institutional conditions, were not explicitly examined and may therefore investigate for future study.

9 References

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