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Bailouts in a Federation Revisited: Budget Pooling, Rather Than Soft Budgets, Causes Inefficiency

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Abstract

This study revisits bailouts in a federation by investigating whether soft budget constraints cause inefficient bailouts. Many studies have examined bailouts in public finance, wherein a federal government provides excessive financial support to regional governments after the need for loss compensation arises. We show that, within the framework of Goodspeed (2002, *International Tax and Public Finance*, 9(4), 409–421), inefficient bailouts arise from budget pooling rather than the lack of commitment associated with intertemporal inconsistency. Specifically, under budget pooling, the federal government balances aggregate revenues and expenditures across regional governments rather than separately balancing each regional government's budget. If the tax revenue from the regional government is spent only on the public goods in the region, no bailouts occur. As such, distinguishing between soft budget constraints and budget pooling improves our understanding of the mechanisms generating inefficient federal bailouts.

Keywords: bailouts, federal government, soft budget constraint, budget pooling, budget separation

JEL Classification: H72 , H74 , H77 , O23

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1 Introduction

This study revisits bailouts in a federation by investigating whether soft budget constraints cause inefficient bailouts. In public finance or public economics, soft budget constraints are defined as the consequence of intertemporal commitment problems, whereby the federal government cannot credibly commit not to providing ex-post bailouts. Since Goodspeed (2002) developed one of the first formal models to analyze bailouts in a federation, many studies have examined bailouts in public finances. Here, the federal government provides excessive financial support to regional governments after the need for loss compensation arises. Many studies since have referred to Goodspeed's (2002) model to emphasize the soft budget constraints as the reason for excessive borrowing. However, soft budget constraints, which are considered only one of intertemporal inconsistency problems, are not the main cause of the federal government's bailouts in the model. Instead, as we show, the bailouts are triggered not by lack of the commitment caused under soft budget constraints but by budget pooling. Under budget pooling, the federal government balances regional governments' total tax revenues and expenditures, but does not balance those of each regional government individually.

However, the problem under soft budget constraints is that when regional governments can continue operations through federal government grants, budget constraints are not working properly and excessive bailouts through grants create inefficiencies. To focus on what happens under soft budget constraints, Goodspeed (2002) explored a two-period intertemporal model, wherein after regional governments excessively borrow in the first period, the federal government bails them out by subsidizing them in the second period. The author emphasized that federal government bailouts of regional borrowing generate two effects: the opportunity cost and common pool effects. These effects respectively imply that bailouts decreases the opportunity cost of regional borrowing and that the regional government incurs an additional tax cost because a portion of the borrowing must be paid through increased common pool taxes of the federal government. If the opportunity cost effect exceeds the common pool effect, soft budget constraints cause excessive borrowing. Many subsequent articles examine the issues of federal bailouts of regional governments by extending the basic model.¹

As Kornai et al. (2003), and Qian and Roland (1998) clarified, the soft budget constraints are regarded as an intertemporal inconsistency problem caused by a lack of dynamic commitment.² Therefore, irrespective of whether multiple regional governments compete for their grants, the problem arises when the federal government has a lack of commitment not to bail out regional governments ex post. However, in the Goodspeed model and its extensions, the problem of soft budget constraints does not occur when only one regional government exists. Rather, Goodspeed (2002) and subsequent research describe a situation wherein while interactions between multiple regional governments lead to the bailouts, the bailouts are not concerned with intertemporal issues.

Among studies related to the claims made in our study, Akai and Sato (2011) explored a simple dynamic decentralized leadership model between the federal and regional governments to examine the problems under soft budget constraints. The authors noted that Goodspeed's model remains a relatively descriptive analysis, in that it fails to analyze the strategic responses

¹For a survey of recent prior studies, see Martínez-Vázquez et al. (2017). As an example of subsequent studies, Martínez-López (2022) extended Goodspeed's (2002) model by allowing the federal government to borrow to finance the grants.

²Dewatripont and Maskin (1995) formulated the soft budget constraint problem in game theory by using the concept of time inconsistency. Maskin (1999) summarized theoretical work on the soft budget constraint.

regarding regional government borrowing and the interactions between regional governments. In addition, Vigneault (2007) claimed that two conditions are necessary for soft budget constraints: 1) the federal government must find it optimal to grant additional resources or even provide a bailout to the regional government ex post; and 2) as the federal government has an incentive to provide additional resources ex post and the regional government is aware of this incentive, the regional government finds it optimal to behave strategically and selects an excessive level of expenditure ex ante. Notably, under the conditions Vigneault presented, the existence or interaction of multiple regional governments is not necessary for soft budget constraint problems to arise. Furthermore, the author argued that the theoretical literature on the soft budget constraint problem primarily focuses on three points: the federal government's lack of commitment, its resulting implications, and the strategic reactions of regional governments aware of this constraint. Notably, the common pool problem arises only when multiple regional governments exist. However, even before the common pool problem arises among multiple local governments, one needs to consider whether the federal government's ex-post bailouts drive the inefficiency.

This study aims to revisit Goodspeed's (2002) model to investigate the main factors causing bailouts in a federation. In particular, by specifying the functional form for simplicity, we explicitly derive the equilibrium and demonstrate the following: If the tax revenue from the regional government is spent only on the public goods in this region, no bailouts occur. Only when multiple regional governments exist, the budget commonly pooled by the federal government leads to regional governments competing for grants, resulting in unnecessary bailouts. A free-rider problem arises when grants are partially covered by taxation from other regional governments. Bailouts are caused by fiscal externalities or spillovers, as taxes from a region are partially spent in other regions and vice versa. Bailouts occur irrespective of whether the lack of dynamic commitment arises. Our results suggest that the distinction between soft budget constraints and budget pooling improves the understanding of the mechanisms generating inefficient federal bailouts.

The remainder of this study proceeds as follows: Section 2 describes our proposed model. Section 3 presents the results of the benchmark case, wherein the budget for the grant is separate. Section 4 presents the main results when the budget for the grant is pooled. Section 5 discusses the obtained results. Section 6 presents the concluding remarks. All the calculations are in Appendix.

2 The model

Following Goodspeed (2002), we consider a simple two-period intertemporal model with two types of players: 1) the federal government and 2) regional governments. For brevity, we consider two regions, which are indexed by $i = \{A, B\}$.³ Each region i has a population n_i , which is exogenously given. In the two-period model, the time index is denoted by $t = \{1, 2\}$. In region i , homogeneous (representative) individuals reside for two periods, without moving between regions.

Their utility in region i depends on the following four variables: $C_{i1} > 0$, $C_{i2} > 0$, $G_{i1} > 0$, and $G_{i2} > 0$, where C_{it} denotes the per capita private consumption in region i

³Even if we consider n regions, the essence of the results remains the same.

in period t and G_{it} denotes the per capita public consumption in region i in period t .⁴ Y_{it} denotes the individual's income in region i during period t . The utility function is assumed to be additively separable. Furthermore, to explicitly derive the equilibrium, we assume a log-linear utility function. The utility function of the representative individual in region i is

$$U_i = \ln C_{i1} + \alpha \ln G_{i1} + \beta(\ln C_{i2} + \alpha \ln G_{i2}), \quad (1)$$

where $\alpha > 0$ denotes the relative weight of the public goods consumption over private goods consumption and $\beta > 0$ denotes the intertemporal discount factor.⁵

Next, we describe grants from the federal government, taxation by both federal and regional governments, and intertemporal borrowing. The federal government gives region i in period t a grant g_{it} . g_{i1} is assumed to be predetermined by the federal government before the following two-period game is played. Thus, g_{i1} is exogenous and cannot be chosen in the model. Meanwhile, g_{i2} is an endogenous variable whose level is determined by the federal government. Regional government i imposes an income tax on the residents in region i . t_{it} denotes the income tax rate that regional government i imposes on its residents in period t . The federal government also imposes an income tax on all citizens of both regions in period 2. t_{c2} denotes the income tax rate that the federal government imposes on all citizens in period 2.⁶ Regional residents can borrow an amount for private and public consumption in period 1: B_{i1}^C and B_{i1}^G denote the respective per capita amount of borrowing. $r > 0$ denotes the net interest rate and is exogenously given. When individuals borrow money in the first period, they must pay back $(1+r)B_{i1}^x$, $x = \{C, G\}$ in period 2.

Regional governments aim to maximize the total utility of regional residents with respect to the four policy variables t_{i1} , t_{i2} , B_{i1}^C , and B_{i1}^G . Because individuals are homogeneous, each regional government maximizes a representative individual's utility by choosing the income tax rates in periods 1 and 2, and borrowing amounts for private and public consumption in period 1. The federal government aims to maximize national social welfare in a country, which is defined as the sum of utilities for all citizens. That is, the federal government maximizes $\sum_i n_i U_i = n_A U_A + n_B U_B$.⁷ Grants given to region i in period 2, g_{i2} , are the policy variables available to the federal government. We define the vector of grants for both regions by $\mathbf{g}_2 \equiv (g_{A2}, g_{B2})$.

The timing of the two-period game is as follows.⁸ In the first stage in the first period, regional government i non-cooperatively and simultaneously determines the income tax rate

⁴As with previous studies, we assume that public goods are not non-rival. The assumption that the variables are positive is critically important because we show that corner solutions occur in Section 4.

⁵In Goodspeed's (2002) original model, the utility function is defined in the general form by $U_i = u_i(G_{i1}) + v_i(G_{i2}) + w_i(C_{i1}) + z_i(C_{i2})$, where $u_i(\cdot)$, $v_i(\cdot)$, $w_i(\cdot)$, and $z_i(\cdot)$ are sub-utilities obtained from G_{i1} , G_{i2} , C_{i1} , and C_{i2} , respectively. Further, we assume that the first-order derivative are positive and the second-order derivatives are negative. However, because the utility function is in a general form, the equilibrium cannot be explicitly derived. Our specification corresponds to $w_i(C_{i1}) = \ln C_{i1}$, $u_i(G_{i1}) = \alpha \ln G_{i1}$, $z_i(C_{i2}) = \beta \ln C_{i2}$, and $v_i(G_{i2}) = \beta \alpha \ln G_{i2}$.

⁶Following the Goodspeed model, the income tax rate that the federal government imposes in period 1, t_{c1} , is not excluded from the model. As the federal government exogenously decides g_{i1} and t_{c1} in period 1, such a simplification does not affect the results.

⁷Goodspeed (2002) assumed that the federal government aims to maximize the probability that voters (citizens) re-elect the government's politicians. Specifically, it maximizes $\sum_i n_i p_i(U_i)$ by defining the utility-dependent re-election probability by $p_i(U_i)$, where $p_i' > 0$ and $p_i'' \leq 0$ are assumed. This study does not consider the federal government's objectives to be politically motivated other than being a simple social welfare maximizer. This is because it does not affect the substance of the results even if other purposes are extensively considered. Therefore, assuming $p'' = 0$, we simplify the model by setting $p_i(U_i) = U_i$. In addition, in the original study, political motivation is modeled only by introducing the function $p_i(\cdot)$. However, besides political motives, $p_i(\cdot)$ can be interpreted as the relative weight on social welfare that the federal government places on both regions. Therefore, the political motivation of winning elections is just one example of how to interpret the model.

⁸Although the game has two periods, it is a three-stage game.

in period 1, t_{i1} , and the amounts of borrowing for public consumption, B_{i1}^G , and private consumption, B_{i1}^C .⁹ Regional government i chooses t_{i1} , B_{i1}^G and B_{i1}^C , given the choices of the other regional government $j (\neq i)$. In the second stage in period 2, regional government i non-cooperatively and simultaneously determines the income tax rate in period 2, t_{i2} . Regional governments select t_{i2} given the choices of regional government j .¹⁰ Finally, in the third stage in period 2, the federal government determines the grant levels for both regions g_{i2} , given the choice variables for both regions, $(t_{i1}, B_{i1}^G, B_{i1}^C, t_{i2})$ for $i = \{A, B\}$. The solution concept is subgame perfect Nash equilibrium. We obtain the equilibrium by backward induction.

3 Benchmark

As a benchmark, we consider a situation wherein no budget pooling exists between the regions and a grant is paid separately from the tax revenue for each region. Because both regions are financially independent, this case is the same as the case wherein only one region exists. For the simplicity of notation, the index of region i is omitted in this section. Subsections 3.1 and 3.2 consider the benchmark case wherein the regional government can and cannot use the borrowing for public consumption B_1^G , respectively.

3.1 When borrowing for public consumption is possible

In the third stage in period 2, the federal government maximizes the national social welfare, $n_A U_A + n_B U_B$. However, under budget separation, no policy interaction exists between the regions. Thus, the federal government maximizes the sum of regions' social welfare, $n_i U_i$. Furthermore, as the population is exogenously given, the federal government maximizes the representative individual's utility in the region, denoted by U , with respect to the region's grant g_2 . When the budget is separated between regions, the federal government is benevolent because it has the same objectives as regional governments.

The constrained optimization problem facing the federal government is:

$$\begin{aligned} \max_{g_2} \quad & U = \ln C_1 + \alpha \ln G_1 + \beta (\ln C_2 + \alpha \ln G_2) \\ \text{s.t.} \quad & G_1 = g_1 + t_1 Y_1 + B_1^G, \\ & C_1 = (1 - t_1) Y_1 + B_1^C, \\ & G_2 = g_2 + t_2 Y_2 - (1 + r) B_1^G, \\ & C_2 = (1 - t_2 - t_{c2}) Y_2 - (1 + r) B_1^C, \\ & t_{c2} n Y_2 = n g_2 \Leftrightarrow t_{c2} = \frac{g_2}{Y_2}. \end{aligned} \tag{2}$$

The first four constraints in (2) denote the government budget constraints for public and private consumptions in periods 1 and 2. The final constraint implies that the budget for grants is

⁹Since the regional government and individual in the region have the same objective function, they maximize utility as if they were the same decision-making entity.

¹⁰If regional government i chooses t_{i2} after observing the choices of the other regional government in the first period, $(t_{j1}, B_{j1}^G, B_{j1}^C)$, the strategic considerations complicate the analysis because they have to consider $t_{i2}(t_{j1}, B_{j1}^G, B_{j1}^C)$ in the first stage. Stated differently, this model assumes that regional governments commit to t_{i2} in the first period and never change t_{i2} even after observing $(t_{j1}, B_{j1}^G, B_{j1}^C)$ in period 2.

financially independent between regions and grants are paid from tax revenues within the regions.

In the final stage, the federal government determines g_2 given the income tax rates and borrowing amounts in the region, (t_1, t_2, B_1^C, B_1^G) , determined in the previous periods. The first-order condition for the federal government to maximize national social welfare is:

$$\beta \left(\frac{\alpha}{G_2} - \frac{1}{C_2} \right) = 0 \Leftrightarrow G_2 = \alpha C_2. \quad (3)$$

(3) implies that the federal government provides the grant to equalize the marginal utility of private and public consumptions in period 2.

The optimal grant level, g_2^* , in this subgame is expressed as a function of (t_1, t_2, B_1^C, B_1^G) , which are determined in the previous stages. By substituting the government budget constraints into (3), the following equation can be obtained:

$$\begin{aligned} g_2 + t_2 Y_2 - (1+r)B_1^G &= \alpha((1-t_2-t_{c2})Y_2 - (1+r)B_1^C) \\ \Leftrightarrow g_2^* &= g_2(t_2, B_1^C, B_1^G) = \frac{(\alpha - (1+\alpha)t_2)Y_2 - (1+r)(\alpha B_1^C - B_1^G)}{1+\alpha}. \end{aligned} \quad (4)$$

Partially differentiating (4) with respect to t_1 , t_2 , B_1^C , and B_1^G , we obtain the following first-order partial derivatives:

$$\frac{\partial g_2^*}{\partial t_1} = 0, \quad \frac{\partial g_2^*}{\partial t_2} = -Y_2 < 0, \quad \frac{\partial g_2^*}{\partial B_1^C} = -\frac{\alpha(1+r)}{1+\alpha} < 0, \quad \frac{\partial g_2^*}{\partial B_1^G} = \frac{1+r}{1+\alpha} > 0. \quad (5)$$

Clearly, g_2^* is independent of t_1 . As G_2 increases when t_2 increases, g_2 decreases. The decrease in C_2 due to an increase in B_1^C requires more t_2 and reduces g_2 . As G_2 decreases, when B_1^G increases, g_2 increases.

In the first and second stages, the regional government maximizes the representative individual's utility in the region with respect to (t_1, t_2, B_1^C, B_1^G) . The constraints facing the regional government are five equations under the government budget constraints and condition for the budget for grants, which are expressed in (2), and the federal government's reaction function, (3) or (4).¹¹

The maximization problem for the regional government is:

$$\begin{aligned} \max_{\{t_1, t_2, B_1^C, B_1^G\}} U &= \ln C_1 + \alpha \ln G_1 + \beta(\ln C_2 + \alpha \ln G_2) \\ \text{s.t. } G_1 &= g_1 + t_1 Y_1 + B_1^G, \\ C_1 &= (1-t_1)Y_1 + B_1^C, \\ G_2 &= g_2^* + t_2 Y_2 - (1+r)B_1^G, \\ C_2 &= (1-t_2-t_{c2})Y_2 - (1+r)B_1^C, \end{aligned} \quad (6)$$

¹¹ Since regional governments consider the federal government's reaction function, regional governments find it impossible to play a simultaneous-move game with the federal government when deciding the income tax rate t_2 in period 2. Hence, the game comprises three stages in two periods.

$$t_{c2} = \frac{g_2^*}{Y_2},$$

$$G_2 = \alpha C_2.$$

As the first-order partial derivatives (5) are obtained from the final constraint in (6), we obtain the following first-order conditions for maximization:

$$\left(\frac{\alpha}{G_1} - \frac{1}{C_1}\right)Y_1 = 0, \quad (7)$$

$$\left(\frac{\alpha}{G_2} - \frac{1}{C_2}\right)\left(Y_2 + \frac{\partial g_2^*}{\partial t_2}\right) = \left(\frac{\alpha}{G_2} - \frac{1}{C_2}\right) \times 0 = 0, \quad (8)$$

$$\begin{aligned} \frac{1}{C_1} + \beta\left(\frac{1}{C_2}\left(-\frac{\partial g_2^*}{\partial B_1^C} - (1+r)\right) + \frac{\alpha}{G_2}\frac{\partial g_2^*}{\partial B_1^C}\right) &= \frac{1}{C_1} - \frac{\beta(1+r)}{1+\alpha}\left(\frac{1}{C_2} + \frac{\alpha^2}{G_2}\right) = 0 \\ \Rightarrow \frac{1}{C_1} - \frac{\beta(1+r)}{C_2} &= 0, \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{\alpha}{G_1} + \beta\left(\frac{\alpha}{G_2}\left(\frac{\partial g_2^*}{\partial B_1^G} - (1+r)\right) - \frac{1}{C_2}\frac{\partial g_2^*}{\partial B_1^G}\right) &= \frac{\alpha}{G_1} - \frac{\beta(1+r)}{1+\alpha}\left(\frac{\alpha^2}{G_2} + \frac{1}{C_2}\right) = 0 \\ \Rightarrow \frac{\alpha}{G_1} - \frac{\beta(1+r)}{C_2} &= 0. \end{aligned} \quad (10)$$

(7)–(10) are the first-order conditions with respect to t_1 , t_2 , B_1^C , and B_1^G , respectively. (8) shows that the federal government's reaction function renders the second-period tax rate redundant. Since any increase in tax revenue due to a higher t_2 is perfectly offset by a reduction in grants, changes in t_2 do not affect the equilibrium allocation. Consequently, the first-order condition with respect to t_2 does not yield an independent optimality condition and the policy instruments are not uniquely determined. From (7), (9), and (10), the following relationships hold.

$$G_1^* = \alpha C_1^*, \quad C_2^* = \beta(1+r)C_1^*, \quad G_1^* = \frac{\alpha}{\beta(1+r)}C_2^*. \quad (11)$$

As the third equation in (11) can be derived from the first and second equations, the third equation is not linearly independent of them. Thus, multiple maximum solutions exist.¹² From the first-order condition for the federal government (3), $G_2^* = \alpha C_2^* = \alpha\beta(1+r)C_1^*$ holds.

The optimal solutions, $(t_1^*, t_2^*, B_1^{C*}, B_1^{G*})$, are derived as follows: Using $C_2^* = \beta(1+r)C_1^*$ and $G_2^* = \alpha\beta(1+r)C_1^*$ in (11), we express $(t_2^*, B_1^{C*}, B_1^{G*})$ as a function of t_1^* .

$$Y_2 t_2^* + (1+\beta)(1+r)B_1^{C*} = Y_2 - \beta(1+r)(1-t_1^*)Y_1 - g_2^*, \quad (12)$$

$$\alpha B_1^{C*} - B_1^{G*} = (t_1^* - \alpha(1-t_1^*))Y_1 + g_1, \quad (13)$$

$$Y_2 t_2^* - \alpha\beta(1+r)B_1^{C*} - (1+r)B_1^{G*} = \alpha\beta(1+r)(1-t_1^*)Y_1 - g_2^*. \quad (14)$$

¹²The maximum value of utility is expressed as follows: $U^* = \ln C_1^* + \alpha \ln \alpha C_1^* + \beta(\ln \beta(1+r)C_1^* + \alpha \ln \alpha\beta(1+r)C_1^*) = \ln(C_1^*)^{(1+\alpha)(1+\beta)} + \ln \alpha^{(1+\beta)} \beta^{\beta(1+\alpha)} (1+r)^{\beta(1+\alpha)}.$

Expressing (12)–(14) in matrix form, we obtain the following equation:

$$\underbrace{\begin{pmatrix} Y_2(1+\beta)(1+r) & 0 \\ 0 & \alpha & -1 \\ Y_2 & -\alpha\beta(1+r) & -(1+r) \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} t_2^* \\ B_1^{C*} \\ B_1^{G*} \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} Y_2 - \beta(1+r)(1-t_1^*)Y_1 - g_2^* \\ (t_1^* - \alpha(1-t_1^*))Y_1 + g_1 \\ \alpha\beta(1+r)(1-t_1^*)Y_1 - g_2^* \end{pmatrix}}_{\mathbf{b}}. \quad (15)$$

This equation can be rewritten as $\mathbf{Ax} = \mathbf{b}$ while defining the matrix and the column vectors as shown in (15). The determinant of $\mathbf{A}$ is $\det \mathbf{A} = -(1+\alpha)(1+\beta)(1+r)Y_2 < 0$. According to Cramer's rule, the optimal solutions are derived as follows:

$$\begin{pmatrix} t_2^* \\ B_1^{C*} \\ B_1^{G*} \end{pmatrix} = \begin{pmatrix} \frac{(1+r)(\alpha-(1+\alpha)t_1^*)Y_1 + \alpha Y_2 - (1+r)g_1 - (1+\alpha)g_2^*}{(1+\alpha)Y_2} \\ \frac{Y_2 + (1+r)[(1-(1+\alpha)(1+\beta)(1-t_1^*))Y_1 + g_1]}{(1+\alpha)(1+\beta)(1+r)} \\ \frac{(1+r)\alpha Y_1 + \alpha Y_2 - (1+\alpha)(1+\beta)(1+r)Y_1 t_1^* - (1+r)(1+\beta+\alpha\beta)g_1}{(1+\alpha)(1+\beta)(1+r)} \end{pmatrix}. \quad (16)$$

The optimal solutions are all combinations of $(t_1^*, t_2^*, B_1^{C*}, B_1^{G*})$ satisfying (16). The optimal grant, which is derived by substituting (16) into (4), is not uniquely determined.

$$g_2^* = \frac{(\alpha - (1+\alpha)t_2^*)Y_2 - (1+r)(\alpha B_1^{C*} - B_1^{G*})}{1+\alpha}. \quad (17)$$

This result can be summarized in the following proposition:

Proposition 1 *When the federal government follows budget separation between regions and regional governments can borrow for public consumption, regions can achieve the first-best welfare level by using appropriate combinations of policy variables. Furthermore, multiple combinations of policy variables exist.*

Proposition 1 suggests that the policy instruments which the federal government can implement to maximize regions' social welfare are not uniquely determined, although the allocation itself is uniquely characterized. A multiplicity of optimal solutions exists because the federal government's reaction function renders the second-period tax rate redundant. Any increase in tax revenues due to a higher t_2 is perfectly offset through a reduction in grants. The first-order condition with respect to t_2 becomes redundant, leaving one degree of freedom in the regional government's policy choice. Furthermore, in period 2, the benevolent federal government with the same objectives provides grant g_2 . Because of too many changeable policy variables compared with the number of variables to be maximized, the solution is not uniquely determined. Even if the grant is zero, that is, $g_2 = 0$, the following combinations of $(t_1^*, t_2^*, B_1^{C*}, B_1^{G*})$ achieves the first-best utility maximization:

$$\begin{pmatrix} t_2^* \\ B_1^{C*} \\ B_1^{G*} \end{pmatrix} = \begin{pmatrix} \frac{\alpha(1+r)Y_1 + \alpha Y_2 - (1+r)g_1}{(1+\alpha)Y_2} - \frac{(1+r)Y_1}{Y_2} t_1^* \\ \frac{Y_2 + (1+r)(Y_1 + g_1)}{(1+\alpha)(1+\beta)(1+r)} - (1-t_1^*)Y_1 \\ \frac{(1+r)\alpha Y_1 + \alpha Y_2 - (1+r)(1+\beta+\alpha\beta)g_1}{(1+\alpha)(1+\beta)(1+r)} - Y_1 t_1^* \end{pmatrix}. \quad (18)$$

The first-best allocation can be achieved even in the absence of ex-post grants. Therefore, ex-post grants are not essential for welfare improvement. Furthermore, depending on the regional government's policy variables $(t_1^*, t_2^*, B_1^{C*}, B_1^{G*})$, the federal government appropriately sets g_2^* in period 2 such that (17) is satisfied. Therefore, the first-best utility maximization is necessarily achieved. Goodspeed (2002) interpreted the fact that $\partial g_2^* / \partial B_1^G$ in (5) is positive as inefficient borrowing under soft budget constraints. Although federal government grants increase with regional borrowing, the federal government simultaneously adjusts grants to maintain the first-best allocation characterized by $G_2 = \alpha C_2$. Therefore, the positive response of grants to borrowing, that is, $\partial g_2^* / \partial B_1^G > 0$, does not generate any distortion in the equilibrium allocations.

The results can be explained using the following analogy: Consider the simple example of a utility maximization problem with a Cobb-Douglas utility function such as $U = (x_1 - t)^\alpha (x_2 + t)^\beta$. Or, equivalently, consider the following maximization problem for the following log-linear utility with respect to t .

$$\max_t U = \alpha \ln(x_1 - t) + \beta \ln(x_2 + t). \quad (19)$$

The first-order condition for utility maximization is:

$$\frac{\alpha}{x_1 - t} = \frac{\beta}{x_2 + t} \Leftrightarrow t^* = \frac{\beta x_1 - \alpha x_2}{\alpha + \beta}. \quad (20)$$

In the simple example given above, a single policy variable t is sufficient to satisfy the first-order condition and achieve the utility-maximizing allocation. Introducing additional policy variables does not improve welfare and only provides alternative policy combinations which achieve the same allocation. Similarly, in the aforementioned benchmark case, the federal government's reaction function makes the second-period tax rate redundant. Since any increase in a higher income tax in period 2 is fully offset by a reduction in grants, income tax rate in period 2 does not affect social welfare. Consequently, the number of independent optimality conditions is smaller than the number of policy instruments available to the regional government. Consequently, multiple combinations of policy variables can achieve the same first-best allocation. As such, ex-post grants are not required to improve welfare, although the equilibrium grant level is endogenously determined by the federal government's reaction function.

3.2 When borrowing for public consumption is impossible

Next, we consider a case wherein the regional government cannot borrow for public consumption. That is, $B_1^G = 0$.¹³ In this case, we derive the optimal solution for all three policy variables, t_1 , t_2 , and B_1^C .

In the third stage in period 2, the federal government maximizes the sum of regions' social welfare, $n_i U_i$ because no policy interaction exists between regions under budget separation. As in Subsection 3.1, the federal government maximizes the representative individual's utility in the region, U , with respect to the grant g_2 .

¹³We assume that the borrowing for private consumption is permitted. Future studies should examine what happens otherwise.

The constrained optimization problem facing the federal government is:

$$\begin{aligned}
\max_{g_2} \quad & U = \ln C_1 + \alpha \ln G_1 + \beta(\ln C_2 + \alpha \ln G_2) \\
\text{s.t.} \quad & G_1 = g_1 + t_1 Y_1, \\
& C_1 = (1 - t_1)Y_1 + B_1^C, \\
& G_2 = g_2 + t_2 Y_2, \\
& C_2 = (1 - t_2 - t_{c2})Y_2 - (1 + r)B_1^C, \\
& t_{c2}nY_2 = ng_2 \Leftrightarrow t_{c2} = \frac{g_2}{Y_2}.
\end{aligned} \tag{21}$$

The first four constraints in (21) denote the government budget constraints for public and private consumptions in periods 1 and 2. The final constraint represents budget separation between regions for the federal government.

The federal government determines g_2 given (t_1, t_2, B_1^C) . The first-order condition for the federal government to maximize social welfare is:

$$\beta \left(\frac{\alpha}{G_2} - \frac{1}{C_2} \right) = 0 \Leftrightarrow G_2 = \alpha C_2. \tag{22}$$

(22) is the same as (3), which implies marginal utility equalization for private and public consumptions in period 2. The optimal grant level, g_2^* , in the subgame is a function of (t_1, t_2, B_1^C) . (22) can be rewritten as follows:

$$\begin{aligned}
g_2 + t_2 Y_2 &= \alpha((1 - t_2 - t_{c2})Y_2 - (1 + r)B_1^C) \\
\Leftrightarrow g_2^* &= g_2(t_2, B_1^C) = \frac{(\alpha - (1 + \alpha)t_2)Y_2 - \alpha(1 + r)B_1^C}{1 + \alpha}.
\end{aligned} \tag{23}$$

Partially differentiating (23) with respect to t_1 , t_2 , and B_1^C yields the following first-order partial derivatives:

$$\frac{\partial g_2^*}{\partial t_1} = 0, \quad \frac{\partial g_2^*}{\partial t_2} = -Y_2 < 0, \quad \frac{\partial g_2^*}{\partial B_1^C} = -\frac{\alpha(1 + r)}{1 + \alpha} < 0. \tag{24}$$

Clearly, g_2^* is independent of t_1 . The signs are the same as in Subsection 3.1.

In the first and second stages, the regional government maximizes U with respect to t_1 , t_2 , and B_1^C . The maximization problem for the regional government is:

$$\begin{aligned}
\max_{\{t_1, t_2, B_1^C\}} \quad & U = \ln C_1 + \alpha \ln G_1 + \beta(\ln C_2 + \alpha \ln G_2) \\
\text{s.t.} \quad & G_1 = g_1 + t_1 Y_1, \\
& C_1 = (1 - t_1)Y_1 + B_1^C, \\
& G_2 = g_2^* + t_2 Y_2, \\
& C_2 = (1 - t_2 - t_{c2})Y_2 - (1 + r)B_1^C,
\end{aligned} \tag{25}$$

$$t_{c2} = \frac{g_2^*}{Y_2},$$

$$G_2 = \alpha C_2.$$

As (24) holds for the first-order partial derivatives, we obtain the following first-order conditions for maximization with respect to t_1 , t_2 , and B_1^C , (26)–(28), respectively.

$$\left(\frac{\alpha}{G_1} - \frac{1}{C_1}\right)Y_1 = 0, \quad (26)$$

$$\left(\frac{\alpha}{G_2} - \frac{1}{C_2}\right)\left(Y_2 + \frac{\partial g_2^*}{\partial t_2}\right) = \left(\frac{\alpha}{G_2} - \frac{1}{C_2}\right) \times 0 = 0, \quad (27)$$

$$\begin{aligned} \frac{1}{C_1} + \beta\left(\frac{1}{C_2}\left(-\frac{\partial g_2^*}{\partial B_1^C} - (1+r)\right) + \frac{\alpha}{G_2}\frac{\partial g_2^*}{\partial B_1^C}\right) &= \frac{1}{C_1} - \frac{\beta(1+r)}{1+\alpha}\left(\frac{1}{C_2} + \frac{\alpha^2}{G_2}\right) = 0 \\ \Rightarrow \frac{1}{C_1} - \frac{\beta(1+r)}{C_2} &= 0. \end{aligned} \quad (28)$$

As in Subsection 3.1, (27) does not yield an independent optimality condition because the federal government's reaction function renders the second-period tax rate redundant. Therefore, the optimal allocation is characterized using (26) and (28) as follows:

$$G_1^* = \alpha C_1^*, \quad C_2^* = \beta(1+r)C_1^*. \quad (29)$$

From (22), $G_2^* = \alpha C_2^* = \alpha\beta(1+r)C_1^*$ is derived. When $B_1^G = 0$, only two policy variables are sufficient to satisfy the previously described first-order conditions (29). Furthermore, another policy variable is required to satisfy (22). Unlike the case with four control variables, three policy variables must be adjusted and the optimal solution can be uniquely determined.¹⁴

By substituting (t_1^*, t_2^*, B_1^{C*}) into (29), we derive the optimal solutions.

$$\beta(1+r)Y_1t_1^* - Y_2t_2^* - (1+\beta)(1+r)B_1^{C*} = \beta(1+r)Y_1 - Y_2 + g_2^*, \quad (30)$$

$$(1+\alpha)Y_1t_1^* - \alpha B_1^{C*} = \alpha Y_1 - g_1, \quad (31)$$

$$\alpha\beta(1+r)Y_1t_1^* + Y_2t_2^* - \alpha\beta(1+r)B_1^{C*} = \alpha\beta(1+r)Y_1 - g_2^*. \quad (32)$$

By rewriting (30)–(32) using matrix calculation, we obtain the following:

$$\begin{pmatrix} \beta(1+r)Y_1 & -Y_2 & -(1+\beta)(1+r) \\ (1+\alpha)Y_1 & 0 & -\alpha \\ \alpha\beta(1+r)Y_1 & Y_2 & -\alpha\beta(1+r) \end{pmatrix} \begin{pmatrix} t_1^* \\ t_2^* \\ B_1^{C*} \end{pmatrix} = \begin{pmatrix} \beta(1+r)Y_1 - Y_2 + g_2^* \\ \alpha Y_1 - g_1 \\ \alpha\beta(1+r)Y_1 - g_2^* \end{pmatrix}. \quad (33)$$

According to Cramer's rule, the optimal solutions are derived as follows:

$$\begin{pmatrix} t_1^* \\ t_2^* \\ B_1^{C*} \end{pmatrix} = \begin{pmatrix} \frac{\alpha(1+r)Y_1 + \alpha Y_2 - (1+r)(1+\beta+\alpha\beta)g_1}{(1+\alpha)(1+\beta)(1+r)Y_1} \\ \frac{\alpha\beta(1+r)Y_1 + \alpha\beta Y_2 + \alpha\beta(1+r)g_1 - (1+\alpha)(1+\beta)g_2^*}{(1+\alpha)(1+\beta)Y_2} \\ \frac{-\beta(1+r)Y_1 + Y_2 - \beta(1+r)g_1}{(1+\beta)(1+r)} \end{pmatrix}. \quad (34)$$

¹⁴The maximum utility value is the same as that when the borrowing for public consumption is possible.

Given g_2^* , the optimal solution satisfying (34) is uniquely determined. In particular, (t_1^*, B_1^{C*}) is uniquely determined regardless of the value of g_2^* . As the solution is uniquely determined, this policy variable cannot be eliminated to maximize welfare.

Meanwhile, the optimal t_2^* changes depending on g_2^* . In the third state in period 2, a benevolent federal government gives grants g_2 to the region. Thus, the optimal t_2^* decreases with g_2 . The relationship between the optimal grant and tax in period 2 satisfies the following:

$$g_2^* = \frac{\alpha\beta(1+r)Y_1 + \alpha\beta Y_2 + \alpha\beta(1+r)g_1}{(1+\alpha)(1+\beta)} - Y_2 t_2^*. \quad (35)$$

(35) shows the origin of the multiplicity of optimal solutions. The federal government's reaction function implies a trade-off between the second-period income tax rate and grants. Since a higher t_2 is exactly offset by a lower g_2 , the equilibrium allocation depends only on their combined fiscal effect rather than on their individual levels. Consequently, the first-best allocation can be implemented using a continuum of (t_2^*, g_2^*) combinations, implying that the tax and grant are functionally equivalent policy instruments in equilibrium. For example, even when the income tax rate or grant is zero, welfare maximization can occur by choosing the appropriate tax or grant level, respectively, as follows:

$$(t_2^*, g_2^*) = \left(\frac{\alpha\beta(1+r)Y_1 + \alpha\beta Y_2 + \alpha\beta(1+r)g_1}{(1+\alpha)(1+\beta)Y_2}, 0 \right) \text{ or } \left(0, \frac{\alpha\beta(1+r)Y_1 + \alpha\beta Y_2 + \alpha\beta(1+r)g_1}{(1+\alpha)(1+\beta)} \right). \quad (36)$$

This result is summarized in the following proposition:

Proposition 2 *When the federal government follows budget separation between regions and regional governments cannot borrow for public consumption, the regions can achieve the first-best welfare level by using appropriate combinations of policy variables. The optimal income tax rate and amount of borrowing for private consumption in period 1, (t_1^*, B_1^{C*}) , are uniquely determined, whereas multiple optimal solutions of the income tax rate and grant in period 2, (t_2^*, g_2^*) , exist.*

Thus, even when the regional government cannot borrow for public consumption, three policy variables are sufficient to maximize utility. Furthermore, as the benevolent federal government subsidizes the region, too many policy variables exist. Thus, regardless of whether the regional government can or cannot borrow, the same optimal utility is achieved. Hence, borrowing for public consumption by regional governments may not generate inefficiencies under budget separation. Although the federal government gives grants to regions ex post, excessive borrowing by regional governments is not induced under budget separation between regions.

4 Budget pooling between regions

Next, we consider a situation in which the federal government pools the tax revenue for grants between regions. Grants are funded by tax revenues from both regions and neither region

is financially independent. Subsections 4.1 and 4.2 present the case in which the regional government can and cannot use the borrowing for public consumption B_1^G , respectively.

4.1 When borrowing for public consumption is possible

In the third stage in period 2, the federal government maximizes the national social welfare, $n_A U_A + n_B U_B$ with respect to grants to both regions, $\mathbf{g}_2 \equiv (g_{A2}, g_{B2})$. It faces the following constrained optimization problem:

$$\max_{\mathbf{g}_2} \sum_{i=\{A,B\}} n_i U_i = \sum_i n_i \{ \ln C_{i1} + \alpha \ln G_{i1} + \beta (\ln C_{i2} + \alpha \ln G_{i2}) \} \quad (37)$$

$$\text{s.t. } G_{i1} = g_{i1} + t_{i1} Y_{i1} + B_{i1}^G, \quad (38)$$

$$C_{i1} = (1 - t_{i1}) Y_{i1} + B_{i1}^C, \quad (39)$$

$$G_{i2} = g_{i2} + t_{i2} Y_{i2} - (1 + r) B_{i1}^G, \quad (40)$$

$$C_{i2} = (1 - t_{i2} - t_{c2}) Y_{i2} - (1 + r) B_{i1}^C \quad \forall i = \{A, B\}, \quad (41)$$

$$t_{c2}(n_A Y_{A2} + n_B Y_{B2}) = n_A g_{A2} + n_B g_{B2}. \quad (42)$$

The first four constraints, (38)–(41), denote the government budget constraints for public and private consumptions in periods 1 and 2. The final constraint (42) is the pooling budget constraint under which the budget for grants is pooled between regions.

In the final stage, the federal government determines $\mathbf{g}_2$ given the income tax rates and borrowing amounts in the region, $(t_{i1}, t_{i2}, B_{i1}^C, B_{i1}^G)$ for all $i = \{A, B\}$, already determined in the previous periods. The federal government's first-order condition to maximize social welfare is:

$$n_i \beta \left(\frac{\alpha}{G_{i2}} - \frac{Y_{i2}}{C_{i2}} \frac{n_i}{n_A Y_{A2} + n_B Y_{B2}} \right) - n_j \beta \frac{Y_{j2}}{C_{j2}} \frac{n_i}{n_A Y_{A2} + n_B Y_{B2}} = 0 \quad \forall i \quad (43)$$

$$\Leftrightarrow \frac{\alpha}{G_{i2}} - \frac{\sum_i \frac{n_i Y_{i2}}{C_{i2}}}{n_A Y_{A2} + n_B Y_{B2}} = 0 \Leftrightarrow G_{i2} = \frac{\alpha(n_A Y_{A2} + n_B Y_{B2})}{\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}} \quad \forall i. \quad (44)$$

(44) implies that the federal government provides the grant to equalize the marginal utility of increasing a grant to a region to the marginal cost of increasing taxes paid in two regions. Assuming an interior solution, we obtain the following equation from (44):

$$G_{A2} = G_{B2}. \quad (45)$$

(45) implies that to equalize the marginal utility between regions, the federal government equalizes public consumption in period 2 across regions. Both regions are guaranteed the same amount of public consumption. The optimal grants $\mathbf{g}_2^* \equiv (g_{A2}^*, g_{B2}^*)$ in this subgame are a function of $(t_{i1}, t_{i2}, B_i^C, B_i^G)_{i=\{A,B\}}$. We derive the first-order partial derivatives of g_{i2} with respect to $(t_{i1}, t_{i2}, B_i^C, B_i^G)_{i=\{A,B\}}$ in Appendix A, because the derivation is complex.¹⁵

¹⁵In Appendix A, $\partial g_{i2}/\partial t_{i1}$ and $\partial g_{i2}/\partial t_{j1}$ are derived in (A16), and optimal grants do not depend on t_{i1} . $\partial g_{i2}/\partial t_{i2}$ and $\partial g_{i2}/\partial t_{j2}$ are derived in (A21)–(A24). $\partial g_{i2}/\partial B_i^C$ and $\partial g_{i2}/\partial B_j^C$ are derived in (A29)–(A30). $\partial g_{i2}/\partial B_i^G$ and $\partial g_{i2}/\partial B_j^G$ are derived in (A35)–(A38).

In the first and second stages, regional government i maximizes a representative individual's utility in the region, U_i , with respect to t_{i1} , t_{i2} , B_{i1}^C , and B_{i1}^G , under six constraints, (38)–(42), and the federal government's reaction function, (44).

The regional government i 's maximization problem is:

$$\begin{aligned} \max_{\{t_{i1}, t_{i2}, B_{i1}^C, B_{i1}^G\}} U_i &= \ln C_{i1} + \alpha \ln G_{i1} + \beta(\ln C_{i2} + \alpha \ln G_{i2}) \\ \text{s.t. } G_{i1} &= g_{i1} + t_{i1}Y_{i1} + B_{i1}^G, \\ C_{i1} &= (1 - t_{i1})Y_{i1} + B_{i1}^C, \\ G_{i2} &= g_{i2} + t_{i2}Y_{i2} - (1 + r)B_{i1}^G, \\ C_{i2} &= (1 - t_{i2} - t_{c2})Y_{i2} - (1 + r)B_{i1}^C, \\ t_{c2} &= \frac{n_A g_{A2} + n_B g_{B2}}{n_A Y_{A2} + n_B Y_{B2}}, \\ G_{i2} &= \frac{\alpha(n_A Y_{A2} + n_B Y_{B2})}{\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}}. \end{aligned} \quad (46)$$

If the optimal solution is interior, the following first-order conditions for maximization are satisfied:

$$\left(\frac{\alpha}{G_{i1}} - \frac{1}{C_{i1}}\right)Y_{i1} = 0, \quad (47)$$

$$\frac{\alpha}{G_{i2}}\left(\frac{\partial g_{i2}}{\partial t_{i2}} + Y_{i2}\right) - \frac{1}{C_{i2}}\left(1 + \frac{\partial t_{c2}}{\partial t_{i2}}\right)Y_{i2} = 0, \quad (48)$$

$$\frac{1}{C_{i1}} + \beta\left(\frac{1}{C_{i2}}\left(-\frac{\partial t_{c2}}{\partial B_{i1}^C}Y_{i2} - (1 + r)\right) + \frac{\alpha}{G_{i2}}\frac{\partial g_{i2}}{\partial B_{i1}^C}\right) = 0, \quad (49)$$

$$\frac{\alpha}{G_{i1}} + \beta\left(\frac{1}{C_{i2}}\left(-\frac{\partial t_{c2}}{\partial B_{i1}^G}Y_{i2}\right) + \frac{\alpha}{G_{i2}}\left(\frac{\partial g_{i2}}{\partial B_{i1}^G} - (1 + r)\right)\right) = 0. \quad (50)$$

From (47), $G_{i1} = \alpha C_{i1}$ is immediately obtained, which is equivalent to the first equation of (11). However, as the first-order partial derivatives are derived in Appendix A, we obtain the following equation:¹⁶

$$C_{i2} = \beta \frac{(1 + r)C_{i2}^2 Y_{j2}}{C_{i2}^2 Y_{j2} - C_{j2}^2 Y_{i2}} C_{i1} \Leftrightarrow C_{i2} = \frac{C_{i2}^2 Y_{j2} - C_{j2}^2 Y_{i2}}{\beta(1 + r)C_{i1} Y_{j2}}. \quad (51)$$

However, both regions cannot simultaneously satisfy (51) If the optimal consumption in region i , C_{i2} , is positive, $C_{i2}^2 Y_{j2} > C_{j2}^2 Y_{i2}$ holds. Conversely, the optimal consumption in region j , C_{j2} , must be negative because $C_{j2}^2 Y_{i2} < C_{i2}^2 Y_{j2}$ holds. This contradicts the fact that the optimal private consumptions in both regions satisfy the first-order conditions. Therefore, the optimal solution cannot satisfy the first-order conditions and must be a corner solution wherein the positive constraint on public consumption is binding. In any case, unlike the case

¹⁶For the derivation of (51), see Appendix B.

of the separating budget, $C_{i2} = \beta(1+r)C_{i1}$ never occurs and the first-best interior solution cannot be achieved.¹⁷

This result is summarized in the following proposition:

Proposition 3 *When the federal government follows budget pooling between regions and the regional governments can borrow for public consumption, the first-best welfare level cannot be achieved.*

Proposition 3 holds because of a simple reason. When the federal government establishes budget pooling between regions, regional governments must determine the levels of policy variables considering the fiscal spillovers on other regions. For example, an increase in the income tax rate in period 2 in a region decreases the federal government grant to the region and increases that to the other region. These strategic interactions between regions lead to inefficient policy choices. However, borrowing for public consumption does not create additional distortion because, if the federal government chooses the budget separation, the first-best result occurs. Next, we examine the case wherein the regional governments cannot borrow for public consumption under budget separation.

4.2 When borrowing for public consumption is impossible

In the third stage in period 2, the federal government maximizes the national social welfare, with respect to grants to both regions, g_2 . The constrained optimization problem facing the federal government is:

$$\begin{aligned} \max_{g_2} \quad & \sum_{i=\{A,B\}} n_i U_i = n_A U_A + n_B U_B = \sum_i n_i \{ \ln C_{i1} + \alpha \ln G_{i1} + \beta (\ln C_{i2} + \alpha \ln G_{i2}) \} \\ \text{s.t.} \quad & G_{i1} = g_{i1} + t_{i1} Y_{i1}, \\ & C_{i1} = (1 - t_{i1}) Y_{i1} + B_{i1}^C, \\ & G_{i2} = g_{i2} + t_{i2} Y_{i2}, \\ & C_{i2} = (1 - t_{i2} - t_{c2}) Y_{i2} - (1 + r) B_{i1}^C \quad \forall i = \{A, B\}, \\ & t_{c2}(n_A Y_{A2} + n_B Y_{B2}) = n_A g_{A2} + n_B g_{B2}. \end{aligned} \tag{52}$$

All constraints are the same as those in (38)–(42), except that B_{i1}^G does not exist.

In the final stage, the federal government determines g_2 given the income tax rates and borrowing amounts in the region, (t_{i1}, t_{i2}, B_i^C) for all $i = \{A, B\}$, determined in the previous periods. The first-order condition for the federal government to maximize social welfare is as follows:

$$G_{i2} = \frac{\alpha(n_A Y_{A2} + n_B Y_{B2})}{\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}} \quad \forall i. \tag{53}$$

¹⁷Unfortunately, in this case, the optimal solution cannot be derived because it is very complicated to determine which constraints are binding and the optimal solution may be an open solution.

The first-order condition (53) is identical to that in (44). Thus, the federal government provides the grant to equalize the marginal utility of increasing a grant to a region to the marginal cost of increasing taxes paid in the two regions. Assuming an interior solution, we obtain:

$$G_{A2} = G_{B2}. \quad (54)$$

The optimal grants g_2^* in this subgame are a function of $(t_{i1}, t_{i2}, B_i^C)_{i=\{A,B\}}$.¹⁸

In the first and second stages, regional government i maximizes U_i with respect to $(t_{i1}, t_{i2}, B_{i1}^C)$, under six constraints. The regional government i 's maximization problem is:

$$\begin{aligned} \max_{\{t_{i1}, t_{i2}, B_{i1}^C\}} U_i &= \ln C_{i1} + \alpha \ln G_{i1} + \beta (\ln C_{i2} + \alpha \ln G_{i2}) \\ \text{s.t. } G_{i1} &= g_{i1} + t_{i1} Y_{i1}, \\ C_{i1} &= (1 - t_{i1}) Y_{i1} + B_{i1}^C, \\ G_{i2} &= g_{i2} + t_{i2} Y_{i2}, \\ C_{i2} &= (1 - t_{i2} - t_{c2}) Y_{i2} - (1 + r) B_{i1}^C, \\ t_{c2} &= \frac{n_A g_{A2} + n_B g_{B2}}{n_A Y_{A2} + n_B Y_{B2}}, \\ G_{i2} &= \frac{\alpha (n_A Y_{A2} + n_B Y_{B2})}{\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}}. \end{aligned} \quad (55)$$

If the optimal solution is interior, the following first-order conditions for maximization are satisfied:

$$\left(\frac{\alpha}{G_{i1}} - \frac{1}{C_{i1}} \right) Y_{i1} = 0, \quad (56)$$

$$\frac{\alpha}{G_{i2}} \left(\frac{\partial g_{i2}}{\partial t_{i2}} + Y_{i2} \right) - \frac{1}{C_{i2}} \left(1 + \frac{\partial t_{c2}}{\partial t_{i2}} \right) Y_{i2} = 0, \quad (57)$$

$$\frac{1}{C_{i1}} + \beta \left(\frac{1}{C_{i2}} \left(-\frac{\partial t_{c2}}{\partial B_{i1}^C} Y_{i2} - (1 + r) \right) + \frac{\alpha}{G_{i2}} \frac{\partial g_{i2}}{\partial B_{i1}^C} \right) = 0. \quad (58)$$

From (56), $G_{i1} = \alpha C_{i1}$ is immediately obtained. However, using the same procedure as in the previous subsection, we obtain the following:

$$C_{i2} = \beta \frac{(1 + r) C_{i2}^2 Y_{j2}}{C_{i2}^2 Y_{j2} - C_{j2}^2 Y_{i2}} C_{i1} \Leftrightarrow C_{i2} = \frac{C_{i2}^2 Y_{j2} - C_{j2}^2 Y_{i2}}{\beta (1 + r) C_{i1} Y_{j2}}. \quad (59)$$

This equation is the same as (51). As shown in Subsection 4.1, in either region, the private consumption must be negative. Therefore, unlike the budget separation case, the optimal solution does not satisfy the first-order conditions. Consequently, the first-best interior solution cannot be achieved.

This result can be summarized in the following proposition:

¹⁸The first-order partial derivatives of g_{i2} with respect to $(t_{i1}, t_{i2}, B_i^C)_{i=\{A,B\}}$ are the same as those in Subsection 4.1, which are derived in Appendix A.

Proposition 4 *When the federal government follows budget pooling between regions and the regional governments cannot borrow for public consumption, the first-best welfare level cannot be achieved.*

Similar to Proposition 3, Proposition 4 holds because the budget pooling between regions set by the federal government. Irrespective of whether the regional government can borrow for public consumption, the first-best welfare cannot be achieved. However, borrowing for public consumption does not exacerbate inefficient policy choices. This model has too many policy variables to achieve welfare maximization. In the benchmark case where the federal government sets budget separation, the first-best welfare level can be derived with three of the four policy variables (t_1, t_2, B_1^C, B_1^G) . Even with the three policy variables (t_1, t_2, B_1^C) , the first-best welfare can be achieved by combining them with the federal government grant g_2 . Meanwhile, under the budget pooling by the federal government, the three policy variables cannot achieve the first-best welfare. Even if public borrowing is possible, the first-order conditions for welfare maximization are not satisfied and the optimal solution is a corner solution which depends on the binding constraints. Therefore, welfare level remains below the first-best level. In any case, intertemporal borrowing does not cause the soft budget problem which requires excessive federal government grants to bail out regional governments. Overall, rather than soft budget constraints, pooling budget leads to inefficient decisions regarding policy variables by regional governments through strategic interactions between regions.

5 Discussion

5.1 Simplifying the model

Following Goodspeed's (2002) model, we specify the model's functional form and attempt to derive the equilibrium. However, even in the simplified model, the calculations are still complicated, and the equilibrium cannot be derived under budget pooling. To further simplify the model, we must reduce the number of variables. The problem of excessive variables falls into the following two categories: the existence of exogenous variables unnecessary for the analysis; and excessive policy variables. Regarding the former issue, for example, the grant given to region i in period 1, g_{i1} , which is an exogenous variable, can be omitted from the model. The model in which social welfare depends on four variables, $(C_{i1}, G_{i1}, C_{i2}, G_{i2})$, complicates the analysis. Even when private consumption in period 1, C_{i1} , is excluded, governments continue to face intertemporal choice problems. One may analyze the effects of borrowing for public consumption on grants. Regarding the latter issue, Propositions 1 and 2 show that the multiplicity of the optimal solutions arises because certain policy instruments become redundant when the federal government's reaction function is incorporated into regional governments' decision making. In particular, the second period income tax rate and grants are alternative instruments for implementing the same allocation. A potential solution is to reduce the number of policy variables by considering situations wherein feasible policies are limited. For example, when an income tax cannot be levied in either period or borrowing for private consumption is not possible, the first-best outcome may not be possible even under budget separation. Future research can consider a model where the first-best outcome cannot occur under budget separation to examine how the results change under budget pooling.

5.2 Improving inefficiencies

How can the federal government achieve the first-best welfare? The first and obvious solution is for the federal government to adopt budget separation rather than pooling. In the model, it has perfect and complete information on the regional governments' policy variables chosen until the third stage. Thus, the federal government correctly recognizes income tax rates and borrowing amounts in both regions, and the grant amount given to regions. Therefore, by separating rather than pooling budgets, which deliberately creates distortions in regional governments' decision-making, the federal government can eliminate this inefficiency. Stated differently, the federal government can simply set the budget rule, $t_{c2}Y_{i2} = g_{i2}$. In this framework, no additional policy instruments are required; the first-best is achieved simply by replacing the budget pooling with budget separation.¹⁹

Second, the federal government can give no grants to regional governments at all. As ex-post grants under budget pooling distort the ex-ante decision-making of regional governments, committing not to provide any ex-post grants yields the first-best outcome. This implies a commitment by the federal government to not bailout regional governments.²⁰ However, because this commitment leads to first-best welfare level, the federal government has no incentive to break a commitment. That is, the time inconsistency problem does not arise in this setting.

Third, the decision-making timing between the federal and regional governments can be changed. If the federal government is the first mover and regional governments are the second movers, rather than the original model's timing, the regional governments, as followers, appropriately choose their policy variables to maximize their region's social welfare in the subgame. Then, correctly predicting the choices made by regional governments, the federal government appropriately determines the grant levels. In the game with a reverse timing, no strategic interactions between the regions exist and the regional governments face no distortions in decision-making. Specifically, in the second stage, the regional governments choose $\{t_{i1}, t_{i2}, B_{i1}^C, B_{i1}^G\}_{i=\{A,B\}}$ to satisfy the first-order conditions for the first-best welfare maximization. In the first stage, the federal government sets grants (g_{A2}, g_{B2}) to satisfy $G_{2A} = G_{2B}$, under which the national social welfare is maximized. As the first-best result is achieved, the federal government does not need to deviate from the commitment in the first stage. Therefore, even without deriving the equilibrium, we can see that the first-best outcome can be achieved.

Another possible solution for the federal government is introducing other policy variables besides grants. However, whether this can actually solve the problem remains unclear since deriving the equilibrium in a more complicated model is more difficult. For example, Martinez-Lopez (2021) introduced federal government borrowing into the model. However, unless the distortions caused by strategic interactions between regional governments can be resolved, first-best welfare is unlikely to be achieved even if federal borrowing is introduced.

¹⁹While budget separation may not be possible, due to political motivations or administrative constraints, such as legal prohibition, economic reasons do not hold.

²⁰Therefore, when exploring a model, we need to clarify the purpose of the federal government giving grants to regional governments. In the benchmark case, the grants have no clear purpose because welfare maximization can be achieved without grants.

6 Concluding remarks

We investigated whether soft budget constraints as an intertemporal inconsistency problem cause inefficient bailouts by revisiting bailouts in a federation. We showed that rather than soft budget constraints, the main source of inefficient bailout outcomes is budget pooling, whereby the federal government balances the total tax revenues and expenditures of regional governments, but does not balance those of each regional government individually. We demonstrated that under budget separation, the inefficient outcome does not occur. Rather, the model describes a situation wherein interactions between multiple regional governments lead to bailouts, which are not concerned with intertemporal issues. Unnecessary bailouts are caused by fiscal externalities or spillovers, in a sense that taxes from a region are partially spent in other regions and vice versa, irrespective of whether the lack of dynamic commitment arises under soft budget constraints. This suggests that explaining soft budget constraints as an intertemporal inconsistency requires an alternative approach which extends beyond the framework of Goodspeed's (2002) seminal study on bailouts. Moreover, our analysis highlights the importance of distinguishing intertemporal inconsistency-driven inefficiencies from budget pooling-driven ones. This distinction may provide a more appropriate foundation for future theoretical research on federal bailouts.

Finally, we discuss possible future research extensions. First, by specifying the utility function, we explicitly derive the equilibrium variables and the comparative static analysis is possible. Further extensions can be performed using the same model with a log-linear utility function. For example, to simplify the analysis, we exclude political motivations and assume that public goods are non-rivalry, the regional government and individuals have the same objective, and regional governments commit to choose the income tax rate in the second period. Scholars can investigate what happens when these assumptions are changed. Another possible extension is the change in the timing between the federal and regional governments. Our model involves a three-stage game wherein the federal government determines the grant levels in the second period after observing the income tax rates in both regions. However, games with different timings can be considered. If the federal and regional governments simultaneously and independently choose the grants and income tax rates in the second period, the timing of the game changes into a two-stage game wherein all governments engage in a simultaneous-move game in the second period. In addition, future studies can investigate whether the first- or second-mover receives greater grants and/or welfare.

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Appendix A The first-order partial derivatives of g_{i2}

By substituting into (44) into (40) and substituting (42) into (41), we can rewrite g_{A2} , g_{B2} , C_{A2} , and C_{B2} as follows:

$$g_{A2} = \frac{\alpha(n_A Y_{A2} + n_B Y_{B2})}{\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}} - t_{A2} Y_{A2} + (1+r) B_{A1}^G, \quad (A1)$$

$$g_{B2} = \frac{\alpha(n_A Y_{A2} + n_B Y_{B2})}{\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}} - t_{B2} Y_{B2} + (1+r) B_{B1}^G, \quad (A2)$$

$$C_{A2} = \left(1 - t_{A2} - \frac{n_A g_{A2} + n_B g_{B2}}{n_A Y_{A2} + n_B Y_{B2}}\right) Y_{A2} - (1+r) B_{A1}^C, \quad (A3)$$

$$C_{B2} = \left(1 - t_{B2} - \frac{n_A g_{A2} + n_B g_{B2}}{n_A Y_{A2} + n_B Y_{B2}}\right) Y_{B2} - (1+r) B_{B1}^C, \quad (A4)$$

Partially differentiating g_{A2} and g_{B2} with respect to C_{A2} and C_{B2} in (A1) and (A2) yields the following:

$$\frac{\partial g_{A2}}{\partial C_{A2}} = \frac{\partial g_{B2}}{\partial C_{A2}} = \frac{\alpha(n_A Y_{A2} + n_B Y_{B2})}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2} \frac{n_A Y_{A2}}{C_{A2}^2} > 0, \quad (A5)$$

$$\frac{\partial g_{B2}}{\partial C_{B2}} = \frac{\partial g_{A2}}{\partial C_{B2}} = \frac{\alpha(n_A Y_{A2} + n_B Y_{B2})}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2} \frac{n_B Y_{B2}}{C_{B2}^2} > 0. \quad (A6)$$

Partially differentiating C_{A2} with respect to t_{i1} , t_{i2} , B_i^C , and B_i^G , $i = \{A, B\}$ in (A3) yields the following:

$$\frac{\partial C_{A2}}{\partial t_{A1}} = -\frac{n_A \frac{\partial g_{A2}}{\partial t_{A1}} + n_B \frac{\partial g_{B2}}{\partial t_{A1}}}{n_A Y_{A2} + n_B Y_{B2}} Y_{A2}, \quad \frac{\partial C_{A2}}{\partial t_{B1}} = -\frac{n_A \frac{\partial g_{A2}}{\partial t_{B1}} + n_B \frac{\partial g_{B2}}{\partial t_{B1}}}{n_A Y_{A2} + n_B Y_{B2}} Y_{A2}, \quad (A7)$$

$$\frac{\partial C_{A2}}{\partial t_{A2}} = -\left(1 + \frac{n_A \frac{\partial g_{A2}}{\partial t_{A2}} + n_B \frac{\partial g_{B2}}{\partial t_{A2}}}{n_A Y_{A2} + n_B Y_{B2}}\right) Y_{A2}, \quad \frac{\partial C_{A2}}{\partial t_{B2}} = -\frac{n_A \frac{\partial g_{A2}}{\partial t_{B2}} + n_B \frac{\partial g_{B2}}{\partial t_{B2}}}{n_A Y_{A2} + n_B Y_{B2}} Y_{A2}, \quad (A8)$$

$$\frac{\partial C_{A2}}{\partial B_{A1}^C} = -\frac{n_A \frac{\partial g_{A2}}{\partial B_{A1}^C} + n_B \frac{\partial g_{B2}}{\partial B_{A1}^C}}{n_A Y_{A2} + n_B Y_{B2}} Y_{A2} - (1+r), \quad \frac{\partial C_{A2}}{\partial B_{B1}^C} = -\frac{n_A \frac{\partial g_{A2}}{\partial B_{B1}^C} + n_B \frac{\partial g_{B2}}{\partial B_{B1}^C}}{n_A Y_{A2} + n_B Y_{B2}} Y_{A2}, \quad (A9)$$

$$\frac{\partial C_{A2}}{\partial B_{A1}^G} = -\frac{n_A \frac{\partial g_{A2}}{\partial B_{A1}^G} + n_B \frac{\partial g_{B2}}{\partial B_{A1}^G}}{n_A Y_{A2} + n_B Y_{B2}} Y_{A2}, \quad \frac{\partial C_{A2}}{\partial B_{B1}^G} = -\frac{n_A \frac{\partial g_{A2}}{\partial B_{B1}^G} + n_B \frac{\partial g_{B2}}{\partial B_{B1}^G}}{n_A Y_{A2} + n_B Y_{B2}} Y_{A2}. \quad (A10)$$

The similar partial derivatives are derived when we partially differentiate C_{B2} with respect to t_{i1} , t_{i2} , B_i^C , and B_i^G , $i = \{A, B\}$ in (A4).

First, using (A5)–(A7), we obtain $\partial g_{i2}/\partial t_{j1}$ as follows:

$$\frac{\partial g_{A2}}{\partial t_{A1}} = \frac{\partial g_{A2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial t_{A1}} + \frac{\partial g_{A2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial t_{A1}} = -\frac{\alpha\left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2} \left(n_A \frac{\partial g_{A2}}{\partial t_{A1}} + n_B \frac{\partial g_{B2}}{\partial t_{A1}}\right), \quad (A11)$$

$$\frac{\partial g_{A2}}{\partial t_{B1}} = \frac{\partial g_{A2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial t_{B1}} + \frac{\partial g_{A2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial t_{B1}} = -\frac{\alpha\left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2} \left(n_A \frac{\partial g_{A2}}{\partial t_{B1}} + n_B \frac{\partial g_{B2}}{\partial t_{B1}}\right), \quad (A12)$$

$$\frac{\partial g_{B2}}{\partial t_{A1}} = \frac{\partial g_{B2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial t_{A1}} + \frac{\partial g_{B2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial t_{A1}} = -\frac{\alpha\left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2} \left(n_A \frac{\partial g_{A2}}{\partial t_{A1}} + n_B \frac{\partial g_{B2}}{\partial t_{A1}}\right), \quad (A13)$$

$$\frac{\partial g_{B2}}{\partial t_{B1}} = \frac{\partial g_{B2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial t_{B1}} + \frac{\partial g_{B2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial t_{B1}} = -\frac{\alpha(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2} \left(n_A \frac{\partial g_{A2}}{\partial t_{B1}} + n_B \frac{\partial g_{B2}}{\partial t_{B1}} \right). \quad (\text{A14})$$

From (A11)–(A14), $\partial g_{A2}/\partial t_{A1} = \partial g_{B2}/\partial t_{A1}$ and $\partial g_{A2}/\partial t_{B1} = \partial g_{B2}/\partial t_{B1}$ must be satisfied. Substituting them, for example, into (A11), the following equation must be satisfied:

$$\frac{\partial g_{A2}}{\partial t_{A1}} = -\frac{\alpha(n_A + n_B)(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2} \frac{\partial g_{A2}}{\partial t_{A1}}. \quad (\text{A15})$$

As the coefficient on the right-hand side in (A15) is negative and cannot be unity, all partial derivatives are zero. That is,

$$\frac{\partial g_{A2}}{\partial t_{A1}} = \frac{\partial g_{B2}}{\partial t_{A1}} = 0, \quad \frac{\partial g_{A2}}{\partial t_{B1}} = \frac{\partial g_{B2}}{\partial t_{B1}} = 0. \quad (\text{A16})$$

Second, using (A5), (A6), and (A8), we obtain the following equations:

$$\begin{aligned} \frac{\partial g_{A2}}{\partial t_{A2}} &= \frac{\partial g_{A2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial t_{A2}} + \frac{\partial g_{A2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial t_{A2}} - Y_{A2} \\ &= -\frac{\alpha \left[\frac{n_A(n_A Y_{A2} + n_B Y_{B2}) Y_{A2}^2}{C_{A2}^2} + \left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2} \right) (n_A \frac{\partial g_{A2}}{\partial t_{A2}} + n_B \frac{\partial g_{B2}}{\partial t_{A2}}) \right]}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2} - Y_{A2}, \end{aligned} \quad (\text{A17})$$

$$\begin{aligned} \frac{\partial g_{A2}}{\partial t_{B2}} &= \frac{\partial g_{A2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial t_{B2}} + \frac{\partial g_{A2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial t_{B2}} \\ &= -\frac{\alpha \left[\frac{n_B(n_A Y_{A2} + n_B Y_{B2}) Y_{B2}^2}{C_{B2}^2} + \left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2} \right) (n_A \frac{\partial g_{A2}}{\partial t_{B2}} + n_B \frac{\partial g_{B2}}{\partial t_{B2}}) \right]}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2}, \end{aligned} \quad (\text{A18})$$

$$\begin{aligned} \frac{\partial g_{B2}}{\partial t_{A2}} &= \frac{\partial g_{B2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial t_{A2}} + \frac{\partial g_{B2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial t_{A2}} \\ &= -\frac{\alpha \left[\frac{n_A(n_A Y_{A2} + n_B Y_{B2}) Y_{A2}^2}{C_{A2}^2} + \left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2} \right) (n_A \frac{\partial g_{A2}}{\partial t_{A2}} + n_B \frac{\partial g_{B2}}{\partial t_{A2}}) \right]}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2}, \end{aligned} \quad (\text{A19})$$

$$\begin{aligned} \frac{\partial g_{B2}}{\partial t_{B2}} &= \frac{\partial g_{B2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial t_{B2}} + \frac{\partial g_{B2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial t_{B2}} - Y_{B2} \\ &= -\frac{\alpha \left[\frac{n_B(n_A Y_{A2} + n_B Y_{B2}) Y_{B2}^2}{C_{B2}^2} + \left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2} \right) (n_A \frac{\partial g_{A2}}{\partial t_{B2}} + n_B \frac{\partial g_{B2}}{\partial t_{B2}}) \right]}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2} - Y_{B2}. \end{aligned} \quad (\text{A20})$$

From (A17)–(A20), $\partial g_{A2}/\partial t_{A2} = (\partial g_{B2}/\partial t_{A2}) - Y_{A2}$ and $\partial g_{B2}/\partial t_{B2} = (\partial g_{A2}/\partial t_{B2}) - Y_{B2}$ hold. From (A17) and (A18), we obtain $\partial g_{i2}/\partial t_{j2}$ as follows:²¹

$$\frac{\partial g_{A2}}{\partial t_{A2}} = -\frac{Y_{A2} \left[(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2 + \alpha n_B (\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}) + \frac{\alpha n_A Y_{A2} (n_A Y_{A2} + n_B Y_{B2})}{C_{A2}^2} \right]}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2 + \alpha(n_A + n_B)(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})} < 0, \quad (\text{A21})$$

²¹ If $Y_{A2}/C_{A2}^2 \geq Y_{B2}/C_{B2}^2$, $\partial g_{B2}/\partial t_{A2} \leq 0$ and $\partial g_{A2}/\partial t_{B2} \geq 0$ hold.

$$\frac{\partial g_{B2}}{\partial t_{A2}} = -\frac{\alpha n_A n_B Y_{A2} Y_{B2} (\frac{Y_{A2}}{C_{A2}^2} - \frac{Y_{B2}}{C_{B2}^2})}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2 + \alpha(n_A + n_B)(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})}, \quad (\text{A22})$$

$$\frac{\partial g_{A2}}{\partial t_{B2}} = -\frac{\alpha n_A n_B Y_{A2} Y_{B2} (\frac{Y_{B2}}{C_{B2}^2} - \frac{Y_{A2}}{C_{A2}^2})}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2 + \alpha(n_A + n_B)(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})}, \quad (\text{A23})$$

$$\frac{\partial g_{B2}}{\partial t_{B2}} = -\frac{Y_{B2} \left[(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2 + \alpha n_A (\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}) + \frac{\alpha n_B Y_{B2} (n_A Y_{A2} + n_B Y_{B2})}{C_{B2}^2} \right]}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2 + \alpha(n_A + n_B)(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})} < 0. \quad (\text{A24})$$

Third, using (A5), (A6), and (A9), we obtain the following equations:

$$\begin{aligned} \frac{\partial g_{A2}}{\partial B_{A1}^C} &= \frac{\partial g_{A2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial B_{A1}^C} + \frac{\partial g_{A2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial B_{A1}^C} \\ &= -\frac{\alpha \left[\frac{n_A Y_{A2} (1+r)(n_A Y_{A2} + n_B Y_{B2})}{C_{A2}^2} + (\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})(n_A \frac{\partial g_{A2}}{\partial B_{A1}^C} + n_B \frac{\partial g_{B2}}{\partial B_{A1}^C}) \right]}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2}, \end{aligned} \quad (\text{A25})$$

$$\begin{aligned} \frac{\partial g_{A2}}{\partial B_{B1}^C} &= \frac{\partial g_{A2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial B_{B1}^C} + \frac{\partial g_{A2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial B_{B1}^C} \\ &= -\frac{\alpha \left[\frac{n_B Y_{B2} (1+r)(n_A Y_{A2} + n_B Y_{B2})}{C_{B2}^2} + (\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})(n_A \frac{\partial g_{A2}}{\partial B_{B1}^C} + n_B \frac{\partial g_{B2}}{\partial B_{B1}^C}) \right]}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2}, \end{aligned} \quad (\text{A26})$$

$$\begin{aligned} \frac{\partial g_{B2}}{\partial B_{A1}^C} &= \frac{\partial g_{B2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial B_{A1}^C} + \frac{\partial g_{B2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial B_{A1}^C} \\ &= -\frac{\alpha \left[\frac{n_A Y_{A2} (1+r)(n_A Y_{A2} + n_B Y_{B2})}{C_{A2}^2} + (\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})(n_A \frac{\partial g_{A2}}{\partial B_{A1}^C} + n_B \frac{\partial g_{B2}}{\partial B_{A1}^C}) \right]}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2}, \end{aligned} \quad (\text{A27})$$

$$\begin{aligned} \frac{\partial g_{B2}}{\partial B_{B1}^C} &= \frac{\partial g_{B2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial B_{B1}^C} + \frac{\partial g_{B2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial B_{B1}^C} \\ &= -\frac{\alpha \left[\frac{n_B Y_{B2} (1+r)(n_A Y_{A2} + n_B Y_{B2})}{C_{B2}^2} + (\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})(n_A \frac{\partial g_{A2}}{\partial B_{B1}^C} + n_B \frac{\partial g_{B2}}{\partial B_{B1}^C}) \right]}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2}. \end{aligned} \quad (\text{A28})$$

From (A25)–(A28), $\partial g_{A2}/\partial B_{A1}^C = \partial g_{B2}/\partial B_{A1}^C$ and $\partial g_{A2}/\partial B_{B1}^C = \partial g_{B2}/\partial B_{B1}^C$ hold. From (A25) and (A26), we obtain $\partial g_{i2}/\partial B_{j1}^C$ as follows:

$$\begin{aligned} \frac{\partial g_{A2}}{\partial B_{A1}^C} &= \frac{\partial g_{B2}}{\partial B_{A1}^C} = -\frac{\alpha n_A Y_{A2} (1+r)(n_A Y_{A2} + n_B Y_{B2})}{C_{A2}^2 \left[(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2 + \alpha(n_A + n_B)(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}) \right]} < 0, \quad (\text{A29}) \\ \frac{\partial g_{A2}}{\partial B_{B1}^C} &= \frac{\partial g_{B2}}{\partial B_{B1}^C} = -\frac{\alpha n_B Y_{B2} (1+r)(n_A Y_{A2} + n_B Y_{B2})}{C_{B2}^2 \left[(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2 + \alpha(n_A + n_B)(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}) \right]} < 0. \end{aligned} \quad (\text{A30})$$

Fourthly, using (A5), (A6), and (A10), we obtain the following equations:

$$\begin{aligned}\frac{\partial g_{A2}}{\partial B_{A1}^G} &= \frac{\partial g_{A2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial B_{A1}^G} + \frac{\partial g_{A2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial B_{A1}^G} + (1+r) \\ &= -\frac{\alpha(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})(n_A \frac{\partial g_{A2}}{\partial B_{A1}^G} + n_B \frac{\partial g_{B2}}{\partial B_{A1}^G})}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2} + (1+r),\end{aligned}\quad (\text{A31})$$

$$\frac{\partial g_{A2}}{\partial B_{B1}^G} = \frac{\partial g_{A2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial B_{B1}^G} + \frac{\partial g_{A2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial B_{B1}^G} = -\frac{\alpha(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})(n_A \frac{\partial g_{A2}}{\partial B_{A1}^G} + n_B \frac{\partial g_{B2}}{\partial B_{A1}^G})}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2}, \quad (\text{A32})$$

$$\frac{\partial g_{B2}}{\partial B_{A1}^G} = \frac{\partial g_{B2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial B_{A1}^G} + \frac{\partial g_{B2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial B_{A1}^G} = -\frac{\alpha(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})(n_A \frac{\partial g_{A2}}{\partial B_{A1}^G} + n_B \frac{\partial g_{B2}}{\partial B_{A1}^G})}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2}, \quad (\text{A33})$$

$$\begin{aligned}\frac{\partial g_{B2}}{\partial B_{B1}^G} &= \frac{\partial g_{B2}}{\partial C_{A2}} \frac{\partial C_{A2}}{\partial B_{B1}^G} + \frac{\partial g_{B2}}{\partial C_{B2}} \frac{\partial C_{B2}}{\partial B_{B1}^G} + (1+r) \\ &= -\frac{\alpha(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2})(n_A \frac{\partial g_{A2}}{\partial B_{B1}^G} + n_B \frac{\partial g_{B2}}{\partial B_{B1}^G})}{(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}})^2} + (1+r).\end{aligned}\quad (\text{A34})$$

From (A35)–(A38), $\partial g_{B2}/\partial B_{A1}^G = (\partial g_{A2}/\partial B_{A1}^G) - (1+r)$ and $\partial g_{A2}/\partial B_{B1}^G = (\partial g_{B2}/\partial B_{B1}^G) - (1+r)$ hold. From (A35)–(A38), we obtain $\partial g_{i2}/\partial B_{j1}^G$ as follows:

$$\frac{\partial g_{A2}}{\partial B_{A1}^G} = \frac{(1+r)\left[\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2 + \alpha n_B \left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)\right]}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2 + \alpha(n_A + n_B)\left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)} > 0, \quad (\text{A35})$$

$$\frac{\partial g_{B2}}{\partial B_{A1}^G} = -\frac{\alpha n_A (1+r)\left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2 + \alpha(n_A + n_B)\left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)} < 0, \quad (\text{A36})$$

$$\frac{\partial g_{A2}}{\partial B_{B1}^G} = -\frac{\alpha n_B (1+r)\left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2 + \alpha(n_A + n_B)\left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)} < 0, \quad (\text{A37})$$

$$\frac{\partial g_{B2}}{\partial B_{B1}^G} = \frac{(1+r)\left[\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2 + \alpha n_A \left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)\right]}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2 + \alpha(n_A + n_B)\left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right)} > 0. \quad (\text{A38})$$

Appendix B The derivation of (51)

Partially differentiating the budget pooling constraint (42) with respect to t_{i2} and B_{i1}^C , we obtain:

$$\begin{aligned}\frac{\partial t_{c2}}{\partial t_{i2}} &= \frac{n_i \frac{\partial g_{i2}}{\partial t_{i2}} + n_j \frac{\partial g_{j2}}{\partial t_{i2}}}{n_A Y_{A2} + n_B Y_{B2}} \\ &= -\frac{n_i Y_{i2} \left[\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2 + \frac{\alpha(n_A + n_B) Y_{i2} (n_A Y_{A2} + n_B Y_{B2})}{C_{i2}^2} \right]}{(n_A Y_{A2} + n_B Y_{B2}) \left[\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}}\right)^2 + \alpha(n_A + n_B) \left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2}\right) \right]} < 0,\end{aligned}\quad (\text{B39})$$

$$\begin{aligned}
\frac{\partial t_{c2}}{\partial B_{i1}^C} &= \frac{n_i \frac{\partial g_{i2}}{\partial B_{i1}^C} + n_j \frac{\partial g_{j2}}{\partial B_{i1}^C}}{n_A Y_{A2} + n_B Y_{B2}} \\
&= - \frac{\alpha n_i (n_A + n_B) Y_{i2} (1+r)}{C_{i2}^2 \left[\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}} \right)^2 + \alpha (n_A + n_B) \left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2} \right) \right]} < 0.
\end{aligned} \tag{B40}$$

From (B39), we have:

$$\frac{\partial g_{i2}}{\partial t_{i2}} + Y_{i2} = \frac{\alpha n_A n_B Y_{A2} Y_{B2} \left(\frac{Y_{j2}^2}{C_{j2}^2} - \frac{Y_{i2}^2}{C_{i2}^2} \right)}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}} \right)^2 + \alpha (n_A + n_B) \left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2} \right)}, \tag{B41}$$

$$\left(1 + \frac{\partial t_{c2}}{\partial t_{i2}} \right) Y_{i2} = \frac{n_j Y_{A2} Y_{B2} \left[\frac{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}} \right)^2}{n_A Y_{A2} + n_B Y_{B2}} + \frac{\alpha (n_A + n_B) Y_{j2}}{C_{j2}^2} \right]}{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}} \right)^2 + \alpha (n_A + n_B) \left(\frac{n_A Y_{A2}^2}{C_{A2}^2} + \frac{n_B Y_{B2}^2}{C_{B2}^2} \right)} > 0. \tag{B42}$$

Substituting (B41) and (B42) into (48), we obtain the following equation:²²

$$G_{i2} = \frac{\alpha^2 n_i \left(\frac{Y_{j2}^2}{C_{j2}^2} - \frac{Y_{i2}^2}{C_{i2}^2} \right)}{\frac{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}} \right)^2}{n_A Y_{A2} + n_B Y_{B2}} + \frac{\alpha (n_A + n_B) Y_{j2}}{C_{j2}^2}} C_{i2}. \tag{B43}$$

By substituting (B43) into (49) and solving the equation for C_{i2} , we obtain:

$$C_{i2} = \beta \left(\frac{\partial t_{c2}}{\partial B_{i1}^C} Y_{i2} + (1+r) - \frac{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}} \right)^2}{n_A Y_{A2} + n_B Y_{B2}} + \frac{\alpha (n_A + n_B) Y_{j2}}{C_{j2}^2} \frac{\partial g_{i2}}{\partial B_{i1}^C} \right) C_{i1}. \tag{B44}$$

Substituting $\partial g_{i2}/\partial B_{i1}^C$ obtained from (A25) and (A28), and $\partial t_{c2}/\partial B_{i1}^C$ obtained from (B42) into (B44), we obtain:

$$\frac{\partial t_{c2}}{\partial B_{i1}^C} Y_{i2} + (1+r) - \frac{\left(\frac{n_A Y_{A2}}{C_{A2}} + \frac{n_B Y_{B2}}{C_{B2}} \right)^2}{n_A Y_{A2} + n_B Y_{B2}} + \frac{\alpha (n_A + n_B) Y_{j2}}{C_{j2}^2} \frac{\partial g_{i2}}{\partial B_{i1}^C} = \frac{(1+r) Y_{j2} C_{i2}^2}{C_{i2}^2 Y_{j2} - C_{j2}^2 Y_{i2}}. \tag{B45}$$

Therefore, (B44) can be rewritten as follows:

$$C_{i2} = \beta \frac{(1+r) C_{i2}^2 Y_{j2}}{C_{i2}^2 Y_{j2} - C_{j2}^2 Y_{i2}} C_{i1} \Leftrightarrow C_{i2} = \frac{C_{i2}^2 Y_{j2} - C_{j2}^2 Y_{i2}}{\beta (1+r) C_{i1} Y_{j2}}. \tag{B46}$$

²²The condition $G_{i2} = \alpha C_{i2}$ does not necessarily hold. When $C_{i2}^2 Y_{j2} - C_{j2}^2 Y_{i2} < 0$ and $C_{i2} > 0$ hold, G_{i2} must be negative.