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Nonlinear Bending Analysis of Functionally Graded Beams with Spring Constraints and Thermal Effects

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Abstract: The nonlinear bending behavior of beams made up of functionally graded materials (FGM) is heavily influenced by boundary flexibility and thermo-mechanical coupling. However, most existing models assume conventional end conditions such as simply supported, clamped, or cantilevered beams, which may often be difficult to achieve in practice. This research presents the nonlinear static bending analysis of FGM beams with elastic boundary conditions that permit both translation and rotation, subjected to combined transverse loads and uniform heating. The governing equations are derived using von Karman geometric nonlinearity combined with first-order shear deformation theory, incorporating material gradation through a power law across the thickness. Effective properties are calculated using both the rule of mixtures and the Mori-Tanaka method. The results show how boundary stiffness, material gradation, and thermal effects influence the nonlinear load-deflection response, revealing differences from standard simply-supported and clamped boundary conditions. The comparison of the two material models illustrates variations in nonlinear behavior, especially under combined thermo-mechanical loading. Selected findings are verified through finite element simulations in ABAQUS, which agree well with the analytical results. The study underscores the importance of accurate boundary modeling and appropriate material homogenization for reliable nonlinear thermo-mechanical analysis of FGM beams.

Keywords: Functionally graded materials; Nonlinear bending; Spring constraints; Thermal effects

1. Introduction

Conventional layered composites often fail under high mechanical and thermal stress due to abrupt changes in material properties at layer interfaces, leading to stress concentrations and delamination. Functionally graded materials (FGMs) address these issues by providing a smooth, continuous variation in composition and properties throughout the thickness [1–3]. This gradual transition allows different material properties, like metal toughness and ceramic heat resistance, to be combined without creating weak interfaces. As a result, FGMs have gained considerable interest for use in structural components subjected to intense mechanical and thermal conditions, such as aerospace structures, energy systems, transportation components, and thermal protection devices.

Extensive research has examined the bending, buckling, and post-buckling behavior of FGM beams using various beam theories and material models. Although the primary focus has been on FGM beams, recent studies have extended geometrically nonlinear analysis to more complex structures, such as stiffened functionally graded doubly curved shells [4] and spherical caps [5], showcasing the versatility of FGMs in advanced structural applications. Early research on cylindrical bending and post-buckling of FGM beams [6] used first-order shear deformation theory (FSDT) and classical beam formulations with typical boundary conditions, such as simply supported or clamped-clamped edges. The methods developed by Sun et al. [7], as well as Chen et al. [8], were often adopted to address bending-extension coupling caused by unsymmetrical material gradation. Later studies incorporated cylindrical bending in Bernoulli-Euler beams and verified the analytical results with finite element solutions, emphasizing shear deformation effects and non-uniform stiffness distribution as a function of thickness.

While these studies offer a solid foundation, they primarily assume conventional boundary conditions that may be difficult to achieve in practice. Real-world structures typically include elastic supports, joints, or attachments that provide finite translational and rotational stiffness at the edges. Ignoring this boundary flexibility can lead to inaccurate predictions of nonlinear beam responses. Although elastic boundary effects have been examined for homogeneous or laminated beams, their impact on the nonlinear behavior of functionally graded beams, particularly under combined thermomechanical loading, remains underexplored. This study fills that gap by incorporating both translational and rotational spring constraints into the analysis.

Accurate representation of material behavior is also essential for FGM analysis. While the rule of mixtures is simple to apply, it only yields approximate effective properties and neglects microstructural interactions between phases. Advanced methods, such as the Mori-Tanaka scheme, account for phase interactions and are better suited to modeling particulate composites and FGMs. Volume fraction distributions following a power law, as in Javaheri and Eslami [9] are used to characterize the gradation profile across the beam thickness. Comparative studies of these material models are necessary to understand their impact on predicted nonlinear responses, particularly under thermal loading, where variations in material properties are significant.

Motivated by these considerations, this study explores the nonlinear static bending behavior of functionally graded material (FGM) beams subjected to transverse mechanical loads and uniform temperature increases, with elastic boundary conditions allowing for translation and rotation. Von Kármán geometric nonlinearity is incorporated to model large deflections, while first-order shear deformation theory accounts for transverse shear deformation. Material properties, which vary continuously through the thickness according to a power-law distribution, are evaluated using both the rule of mixtures and the Mori-Tanaka homogenization schemes. The study systematically analyzes how boundary stiffness, material gradation, beam theory, and thermal loading influence the nonlinear load-deflection response. Selected results are validated by comparing with finite element simulations in ABAQUS. This research offers a more realistic and comprehensive framework for the nonlinear thermomechanical analysis of FGBs.

2. Governing Equations and Numerical Computations

2.1. Material Modeling of the Functionally Graded Beam

The functionally graded beam is composed of ceramic and metal phases, with material properties that vary gradually through the thickness. The distribution of volume fractions follows the power-law model introduced by Javaheri and Eslami. [9], expressed as

$$V_c = \left(\frac{z}{h} + \frac{1}{2} \right)^k \quad (1)$$

where k is the power-law index, h is the beam thickness, and z is the coordinate measured from the mid-plane. The volume fractions of the ceramic and metal phases are related by:

$$V_c + V_m = 1 \quad (2)$$

The effective material properties are first evaluated using the rule of mixtures. Because the material gradation direction aligns with the loading direction, the upper-bound rule of mixtures is used. The Young's modulus is given by,

$$E(z) = E_c V_c + E_m V_m \quad (3)$$

where E_c and E_m are the Young's moduli of the ceramic and metal phases; V_c and V_m are the volume fractions of the ceramic and metal phases, respectively.

To account for the microstructural effects neglected by the rule of mixtures, the Mori-Tanaka homogenization scheme is also employed. The effective bulk modulus K and shear modulus G are given by,

$$K = K_m + \frac{V_c(K_c - K_m)}{1 + (1 - V_c) \frac{3(K_c - K_m)}{3K_m + 4G_m}} \quad ; \quad G = G_m + \frac{V_c(G_c - G_m)}{1 + (1 - V_c) \frac{(G_c - G_m)}{G_m + f_1}} \quad (4)$$

where, $f_1 = \frac{G_m(9K_m + 8G_m)}{6(K_m + 2G_m)}$

The effective Young's modulus and Poisson's ratio are obtained as,

$$E(z) = \frac{9KG}{3K + G} \quad ; \quad \nu(z) = \frac{3K - 2G}{2(3K + G)} \quad (5)$$

101 where K_m is the bulk modulus of metal, K_c is the bulk modulus of ceramic, G_c is the
102 shear modulus of the ceramic, and G_m is the shear modulus of metal.

103 **2.2. Classical Theory**

104 In classical beam theory, transverse shear deformation is neglected. The nonlinear governing
105 equations are derived using von Karman's geometric nonlinearity. The in-plane force N_x ,
106 transverse shear force Q_x , and bending moment resultants M_x acting on the geometric mid-
107 surface are defined as,

$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x dz \quad (6)$$

$$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zx} dz \quad (7)$$

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z dz \quad (8)$$

108 The equilibrium equations are derived to be

$$\frac{\partial N_x}{\partial x} = 0 \quad (9)$$

$$\frac{\partial M}{\partial x} - Q_x = 0 \quad (10)$$

$$N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial Q_{xz}}{\partial x} + P(x) = 0 \quad (11)$$

109 where $w(x)$ is the transverse deflection and $P(x)$ is the transverse distributed load.

110 The resulting nonlinear governing equation is

$$\frac{\partial^4 w}{\partial x^4} - \frac{N_0}{\left(\frac{E_0 E_2 - E_1^2}{E_0}\right)} \frac{\partial^2 w}{\partial x^2} - \frac{P(x)}{\left(\frac{E_0 E_2 - E_1^2}{E_0}\right)} = 0 \quad (12)$$

111 where, $E_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) dz$; $E_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} (E(z) * z) dz$; $E_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} (E(z) * z^2) dz$ if $\beta = \left(\frac{E_0 E_2 - E_1^2}{E_0}\right)$;

112 The transverse deflection is expressed as,

$$w(x) = C_1 + C_2 x + C_3 \cosh(\zeta x) + C_4 \sinh(\zeta x) - \frac{\varphi}{2\zeta^2} x^2 \quad (13)$$

113 **2.3. First Order Shear Deformation Theory**

Shear deformation terms don't appear in the Bernoulli-Euler theory. It can thus be concluded that the Bernoulli-Euler theory works well for thin beams but fails for thick beams, as the shear deformation effect must not be ignored in thick beams. Timoshenko[10] was the first to propose incorporating these two terms for thick beams. The governing differential equations in terms of displacements for a uniform cross-section of a functionally graded beam undergoing large deflections and subjected to a uniformly distributed transverse load are derived, including transverse shear effects.

The governing differential equation is found to be

$$\frac{\partial^4 w}{\partial x^4} - \frac{N_0}{\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \left(1 + \frac{N_0}{k_s G_0}\right)} \frac{\partial^2 w}{\partial x^2} - \frac{P(x)}{\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \left(1 + \frac{N_0}{k_s G_0}\right)} = 0 \quad (14)$$

If $\alpha^2 = \frac{N_0}{\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \left(1 + \frac{N_0}{k_s G_0}\right)}$; $\eta = \frac{P(x)}{\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \left(1 + \frac{N_0}{k_s G_0}\right)}$; $G_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} G(z) dz$ is the shear

modulus and k_s is the shear correction factor.

Transverse deflection is thus,

$$w(x) = C_1 + C_2 x + C_3 \cosh(\alpha x) + C_4 \sinh(\alpha x) - \frac{\eta}{2\alpha^2} x^2 \quad (15)$$

where C_1, C_2, C_3 and C_4 depend not only on the boundary conditions, but on N_0 as well.

2.4. Numerical Computations

In the case of a simply-supported beam, when classical theory is used, the constants are found to be:

$$C_3 = \frac{\frac{P}{N_0} \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) + N_0 \frac{E_1}{E_0}}{\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \zeta^2 \cosh(\zeta a)}; C_4 = 0; C_1 = \frac{\varphi}{2\zeta^2} a^2 - C_3 \cosh(\zeta a); C_2 = 0 \quad (16)$$

$$N_0 = \frac{E_0}{4a} \int_{-a}^a \left(C_3 \zeta \sinh(\zeta x) - \frac{\varphi}{\zeta^2} x \right)^2 dx - \frac{E_1}{2a} \int_{-a}^a \left(C_3 \zeta^2 \cosh(\zeta x) - \frac{\varphi}{\zeta^2} \right) dx \quad (17)$$

Clearly, deriving a closed-form expression for in relation to P is highly complex. Therefore, it is calculated for specific P values through an iterative approach. The method employed here is the Newton-Raphson Method. For a simply supported beam, the constants based on first-order shear deformation theory are determined as

$$C_1 = \frac{Pa^2}{2N_0} - C_3 \cosh(\alpha a); C_2 = 0; C_3 = \frac{\frac{\beta P}{N_0} + \frac{E_1}{E_0} N_0}{N_0 \cosh(\alpha a)} ; C_4 = 0 \quad (18)$$

These constants are used to determine the transverse deflection. The in-plane stress resultant, N_0 , is then evaluated using the following integral equation

$$N_0 = \frac{E_0}{4a} \int_{-a}^a \left(C_3 \alpha \sinh(\alpha x) - \frac{\eta}{\alpha^2} x \right)^2 dx - \frac{E_1}{2a} * \left(1 + \frac{N_0}{k_s G_0} \right) * \int_{-a}^a \left(C_3 \alpha^2 \cosh(\alpha x) - \frac{\eta}{\alpha^2} \right) dx - E_1 * \frac{P}{k_s G_0} \quad (19)$$

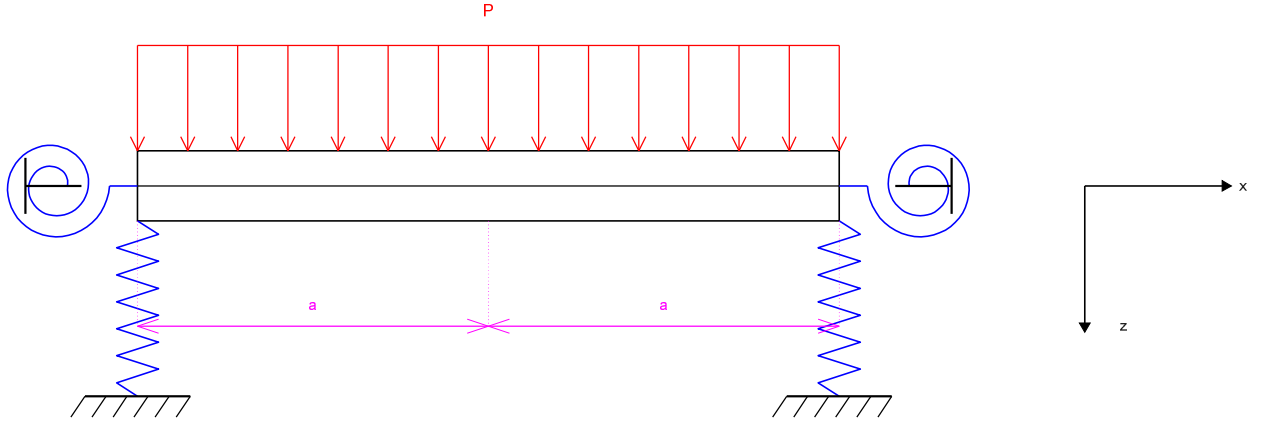
135 In the case of a clamped-clamped beam, the constants when classical theory is used are found
136 to be:

$$C_1 = \frac{Pa^2}{2N_0} - \frac{P}{\zeta N_0} a \coth(\zeta a); C_2 = 0; C_3 = \frac{\frac{\eta}{\zeta^2} a}{\zeta \sinh(\zeta a)} = \frac{\frac{P}{N_0} a}{\zeta \sinh(\zeta a)}; C_4 = 0 \quad (20)$$

137 In the case of a clamped-clamped beam, the constants when the first-order shear deformation
138 theory are found to be:

$$C_1 = \frac{Pa^2}{2N_0} - \frac{P}{\alpha N_0} a \coth(\alpha a); C_2 = 0; C_3 = \frac{\frac{\eta}{\alpha^2} a}{\alpha \sinh(\alpha a)} = \frac{\frac{P}{N_0} a}{\alpha \sinh(\alpha a)}; C_4 = 0 \quad (21)$$

139 In the case of elastic support boundary conditions,



140
141 **Figure 1.** Spring boundary conditions are considered in this paper.

142 If K_r is the rotational spring constant and k_t is the translational spring constant, we have
143 the constants of transverse deflection in the case of classical theory as follows,

$$C_1 = \frac{C_3}{k_t} \left(\left(\frac{E_0 E_2 - E_1^2}{E_0} \right) \zeta^3 \sinh(\zeta a) - k_t \cosh(\zeta a) \right) + \frac{Pa^2}{2N_0}; C_2 = 0; C_3 = \frac{\frac{P}{N_0} \left(\left(\frac{E_0 E_2 - E_1^2}{E_0} \right) + K_r a \right) + N_0 \frac{E_0}{E_1}}{\left(\frac{E_0 E_2 - E_1^2}{E_0} \right) \zeta^2 \cosh(\zeta a) + K_r \zeta \sinh(\zeta a)}; C_4 = 0 \quad (22)$$

144 When K_r is set to 0 and k_t is set to infinity, the maximum deflection results are the same as
145 those for simply supported beams. Similarly, when K_r and k_t are set to infinity, the results
146 match those for clamped-clamped beams. The same process is repeated for the First-order
147 shear deformation theory, and the constants are determined.

148 3. Results

149 This section presents the results of nonlinear bending analyses of FGM beams subjected to a
150 uniformly distributed transverse load. Both CBT and FSDT are applied to simply supported,
151 clamped-clamped, and elastically supported beams. Two material models: the rule of

mixtures and the Mori-Tanaka homogenization scheme, are used to characterize the beam material. All numerical computations are performed in MATLAB.

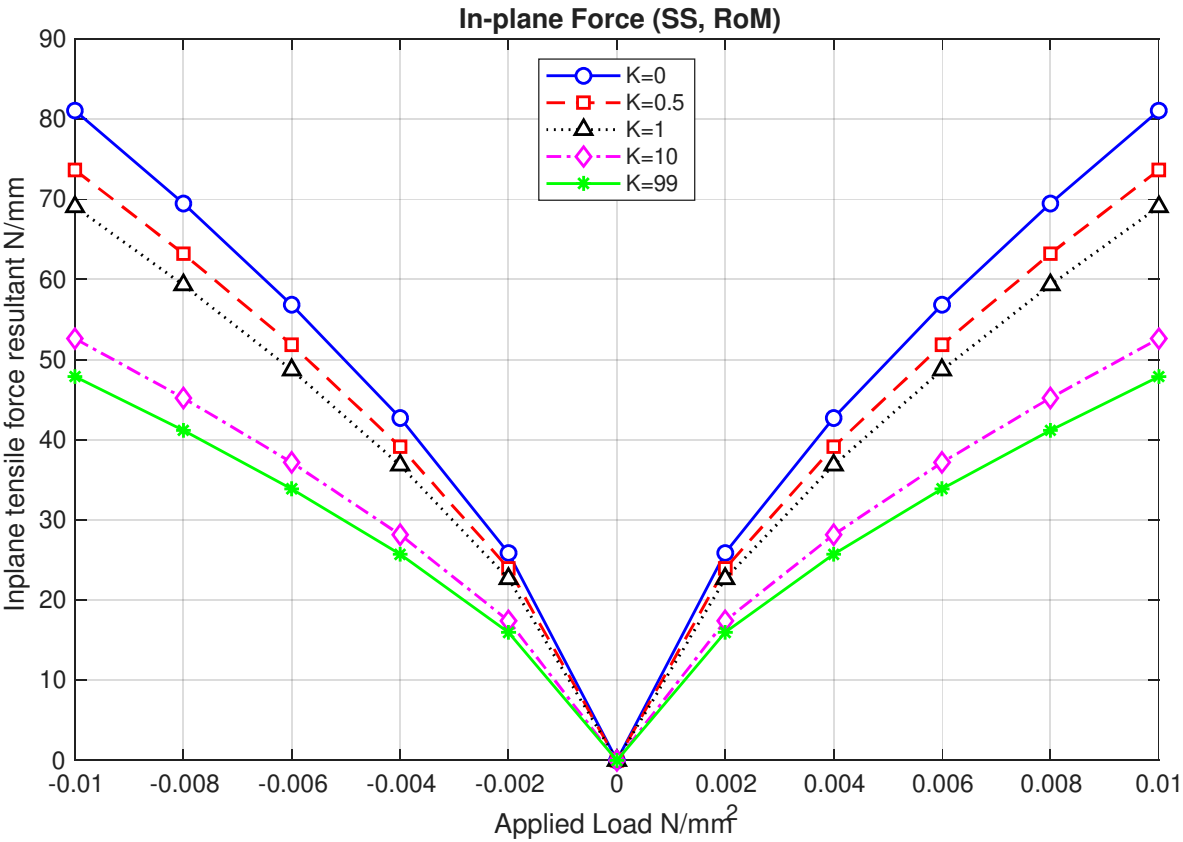


Figure 2. In-plane stress resultant for different values of k of an SS beam using the rule of mixtures.

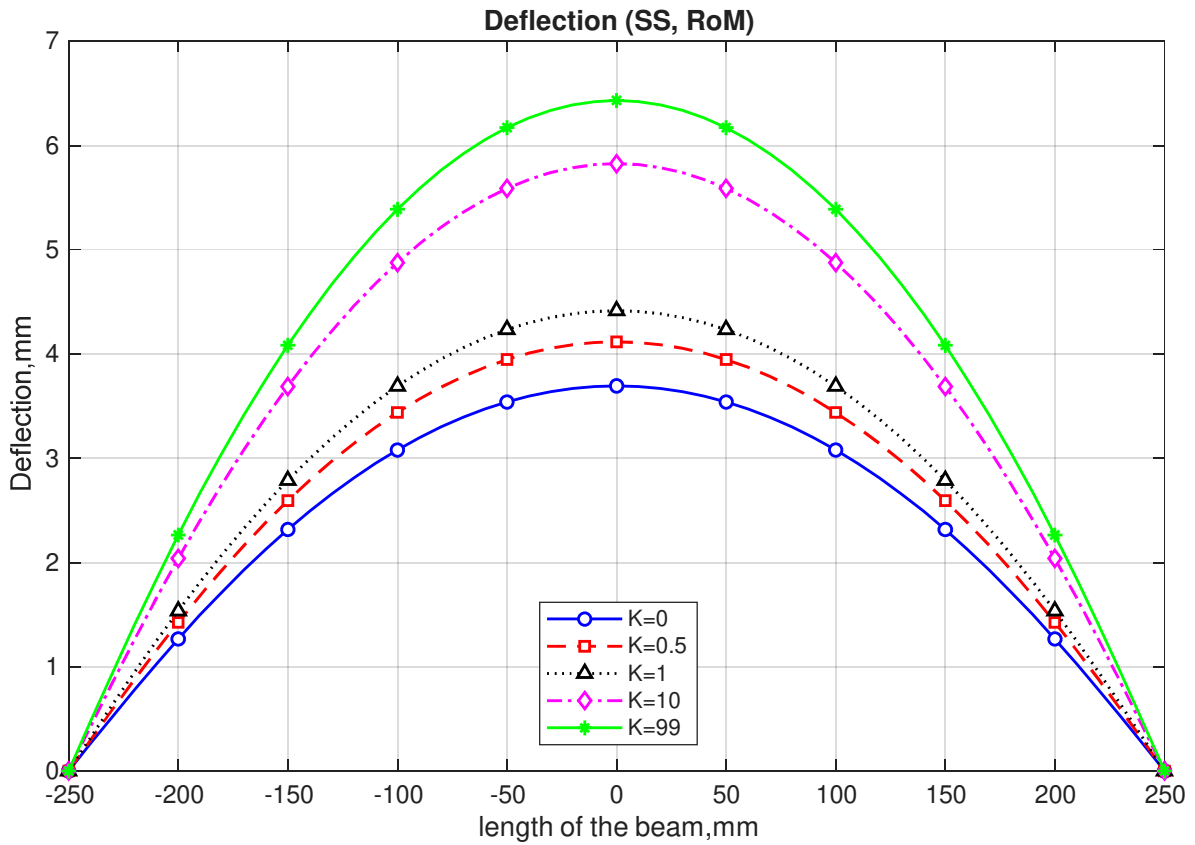


Figure 3. Deflections of an SS beam for different values of k using the rule of mixtures and classical theory under 0.01 uniformly distributed pressure.

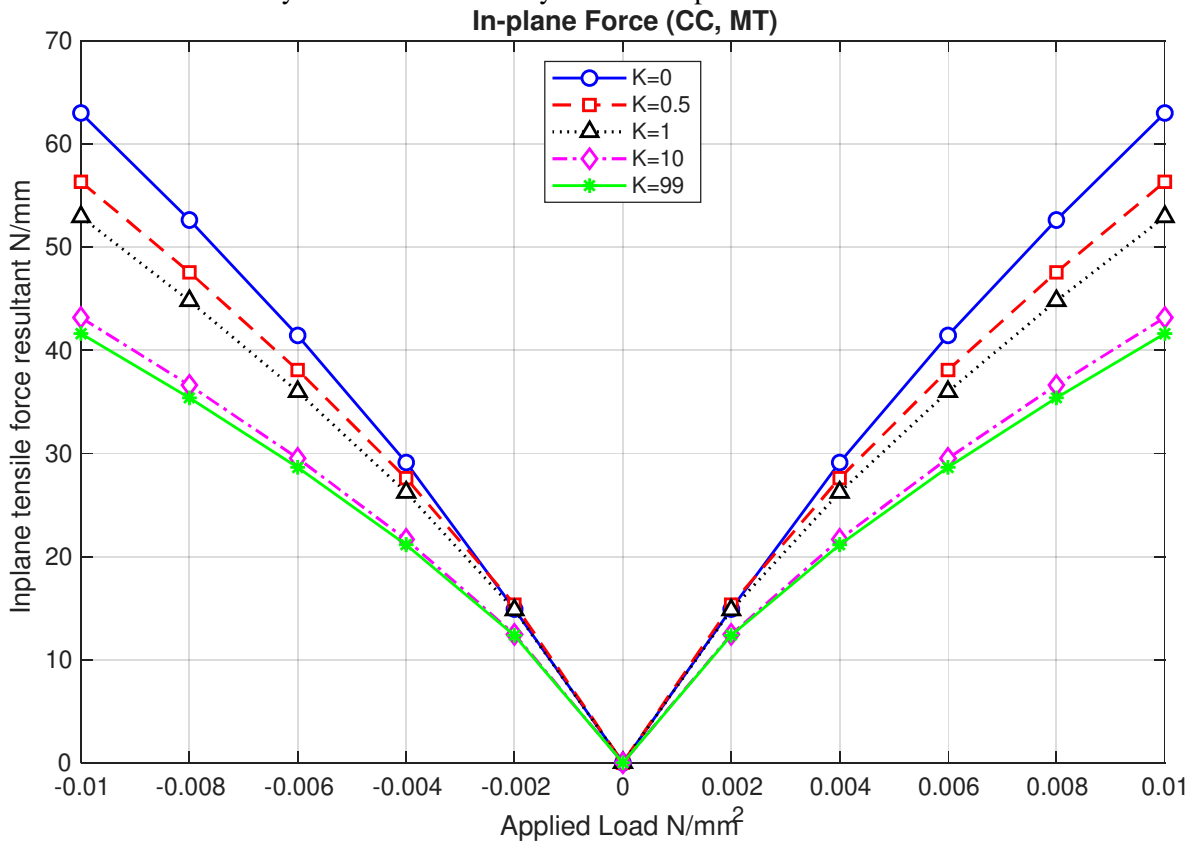


Figure 4. In-plane stress resultant for different values of k of a CC beam using Mori-Tanaka.

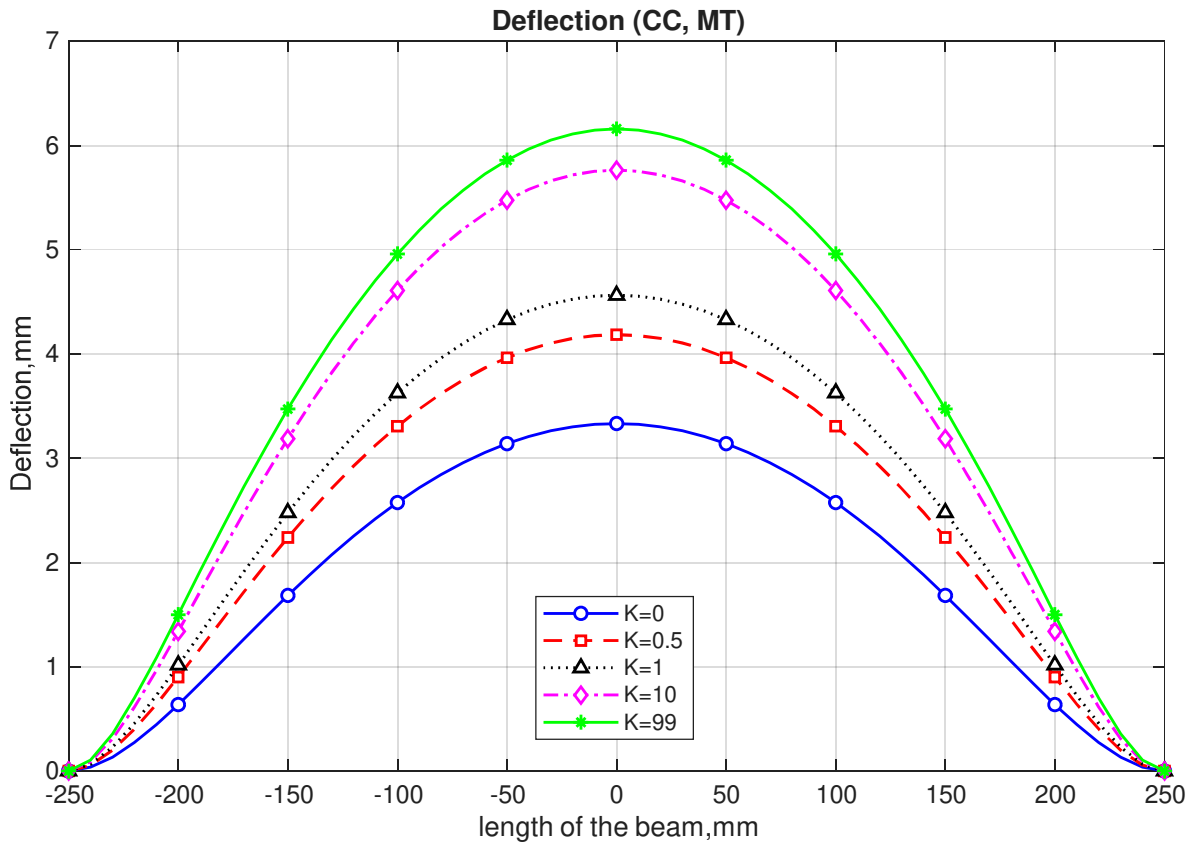


Figure 5. Deflections of the CC beam for different values of k using Mori-Tanaka and classical theory under 0.01 uniformly distributed pressure.

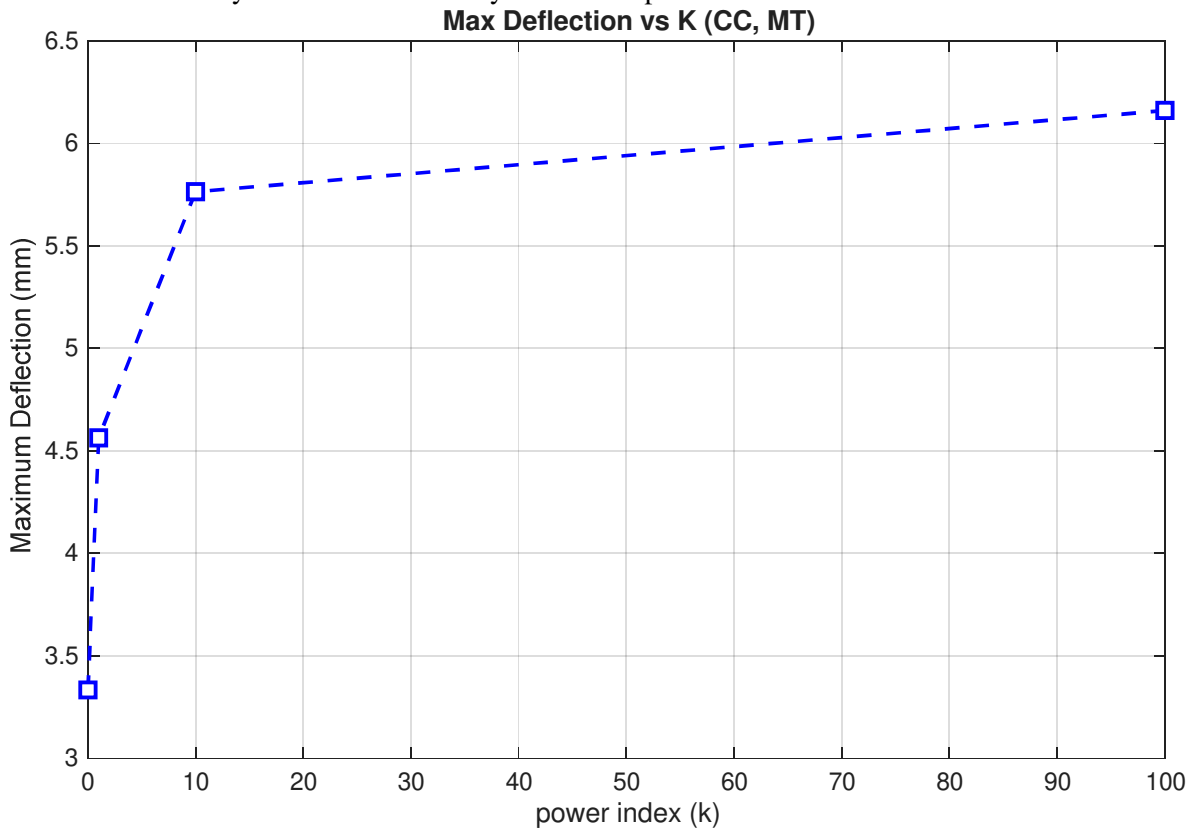


Figure 6. Maximum deflection values vs. gradation parameter k for an FG beam modeled using the Mori-Tanaka method and with spring boundary conditions that replicate clamped-clamped beams.

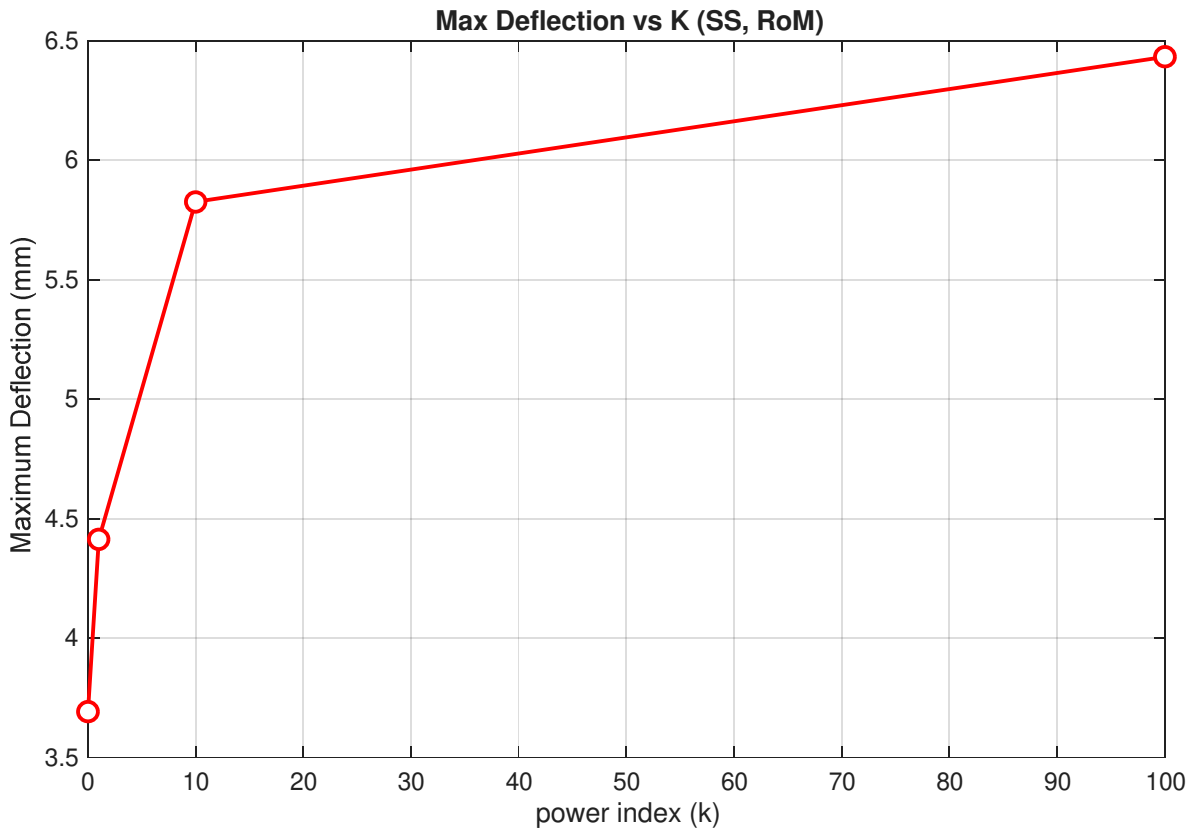


Figure 7. Maximum deflection values vs. gradation parameter k for an FG beam modeled using the rule of mixtures method for a simply-supported beam.

Table 1. Comparison of Analytical (Rule of Mixtures) and FEM Results for Simply-Supported FG Beams

Gradation Power Index k	Analytical $w_{\max}$ (mm)	FE Results $w_{\max}$ (mm)	Percentage Difference (%)
$k=0$	3.693	3.694	0.027
$k=1$	4.254	4.255	0.001
$k=\infty$	6.527	6.527	0

4. Thermal Effects

This section examines the nonlinear bending response of functionally graded beams under a combination of uniformly distributed transverse mechanical loads and a uniform temperature rise. Clamped-clamped boundary conditions with immovable ends are considered. Both the rule of mixtures and the Mori-Tanaka material schemes are applied, and analyses are performed using CBT and FSDT.

4.1. Calculation of critical buckling temperature and Nonlinear bending analysis for Classical theory

To ensure stable solutions, the beam's critical buckling temperature is first calculated. If the applied temperature exceeds the critical value, the beam enters the post-buckling regime, which is outside the scope of this study. The mechanical strain is expressed as:

$$\varepsilon_x^M = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} - \alpha(z)[T - T_0] \quad (23)$$

191 The material stiffness parameters are evaluated as:

$$E_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) dz \quad (24)$$

$$E_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} (E(z)z) dz \quad (25)$$

192 The thermal axial force is:

$$N^T = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\alpha(z)\Delta T(z) dz \quad (26)$$

193 The in-plane force and bending moment resultants are:

$$N_x = E_0 \left(\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right) - E_1 \frac{\partial^2 w}{\partial x^2} - N^T \quad (27)$$

$$M_x = E_1 \left(\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right) - E_2 \frac{\partial^2 w}{\partial x^2} - M^T \quad (28)$$

194 The differential equation governing bending under combined thermal and mechanical loads
195 is:

$$\frac{\partial^4 w}{\partial x^4} + \frac{N_0}{\left(\frac{-E_2 E_0 + E_1^2}{E_0} \right)} \frac{\partial^2 w}{\partial x^2} = 0 \quad (29)$$

196 The critical buckling temperature, ΔT_{cr} , is obtained following the methodology of Thivend,
197 Eslami, and Zhao [11]:

$$\Delta T_{cr} = \frac{\frac{4\pi^2}{L^2} \left[\frac{E_2 E_0 - E_1^2}{E_0} \right]}{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\alpha(z) dz} \quad (30)$$

198 Calculating this value is a prerequisite to ensuring stable solutions. If the applied temperature
199 exceeds this critical threshold, the beam enters a post-buckling regime that is beyond the
200 scope of this specific study.

201 The final governing differential equation is found to be:

$$\frac{\partial^4 w}{\partial x^4} - \frac{N_0}{\beta} \frac{\partial^2 w}{\partial x^2} = \frac{P(x)}{\beta} \quad (31)$$

202 Where,

$$\int_{-a}^a N_0 dx = E_0 \int_{-a}^a \frac{\partial u_0}{\partial x} dx + \frac{E_0}{2} \int_{-a}^a \left(\frac{\partial w}{\partial x} \right)^2 dx - E_1 \int_{-a}^a \frac{\partial^2 w}{\partial x^2} dx - N_T(2a) \quad (32)$$

In which N_0 is evaluated for discrete values of P using the Newton-Raphson method [9,11].

The constants for simply-supported or clamped beams C_3 and C_1 are found to be:

$$C_3 = \frac{\frac{\varphi}{\zeta^2} a}{\zeta \sinh(\zeta a)} = \frac{\frac{P}{N_0} a}{\zeta \sinh(\zeta a)} \quad (33)$$

$$C_1 = \frac{Pa^2}{2N_0} - C_3 \cosh(\zeta a) \quad (34)$$

Some of the results are given below:

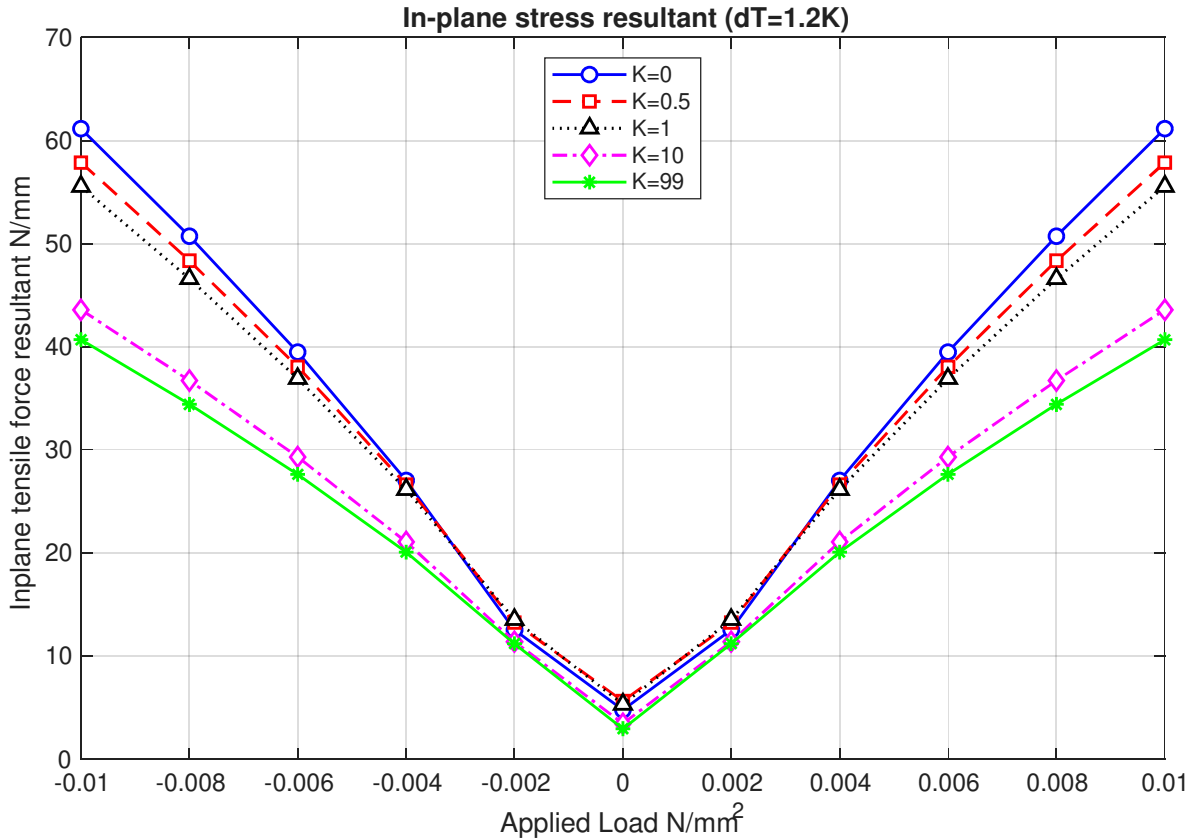


Figure 8. In-plane stress resultant for different values of k of a CC beam using the rule of mixtures theory under 0.01 pressure and 1.2K temperature difference.

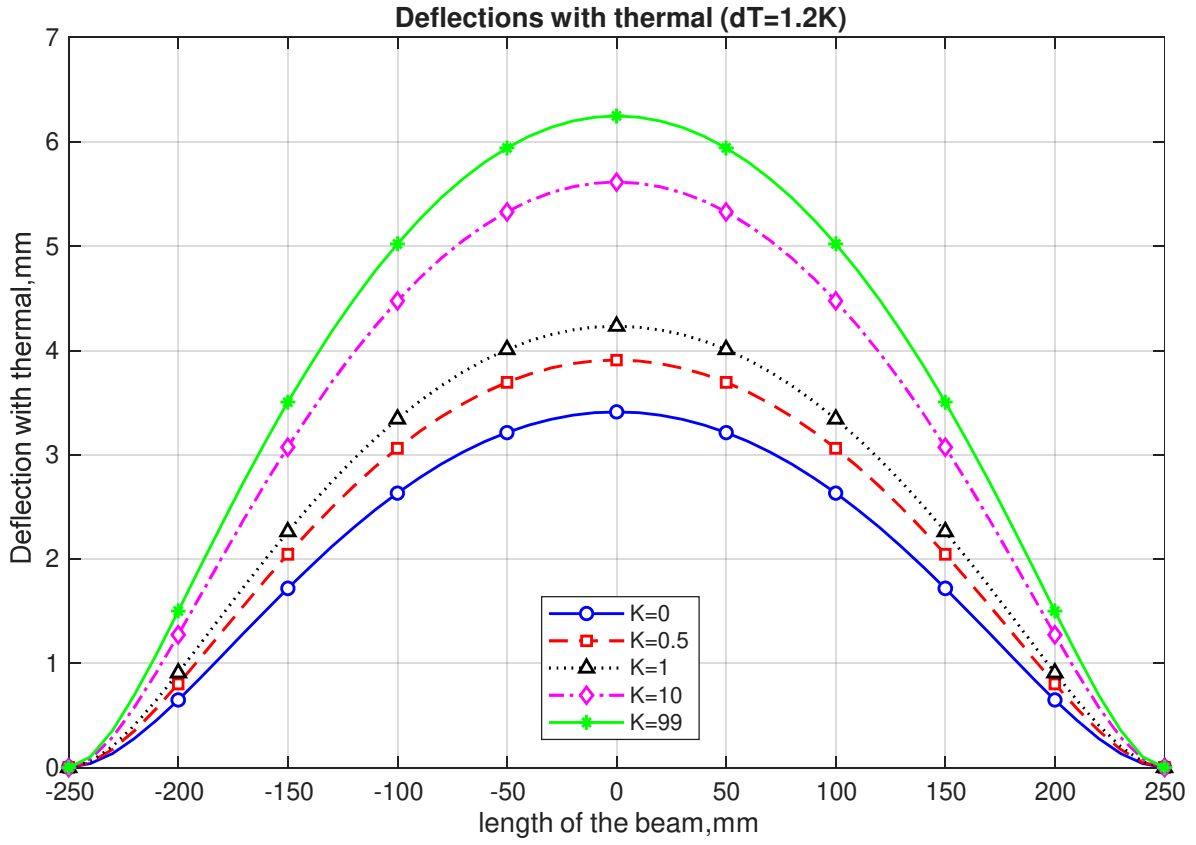


Figure 9. Deflections of CC beam for different values of k using the rule of mixtures and classical theory under 0.01 pressure and 1.2K temperature difference.

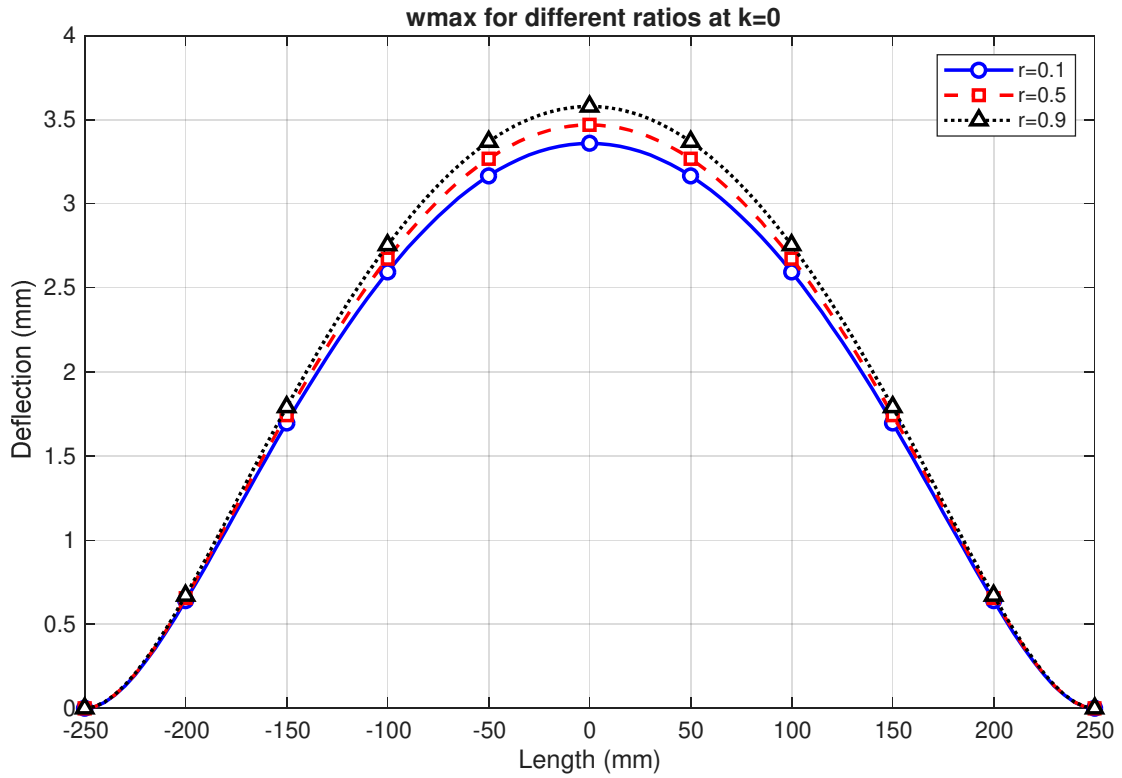


Figure 10. Deflection graph for different $\Delta T / \Delta T_{cr}$ ratios at gradation parameter $k = 0$ for rule of mixtures and classical theory.

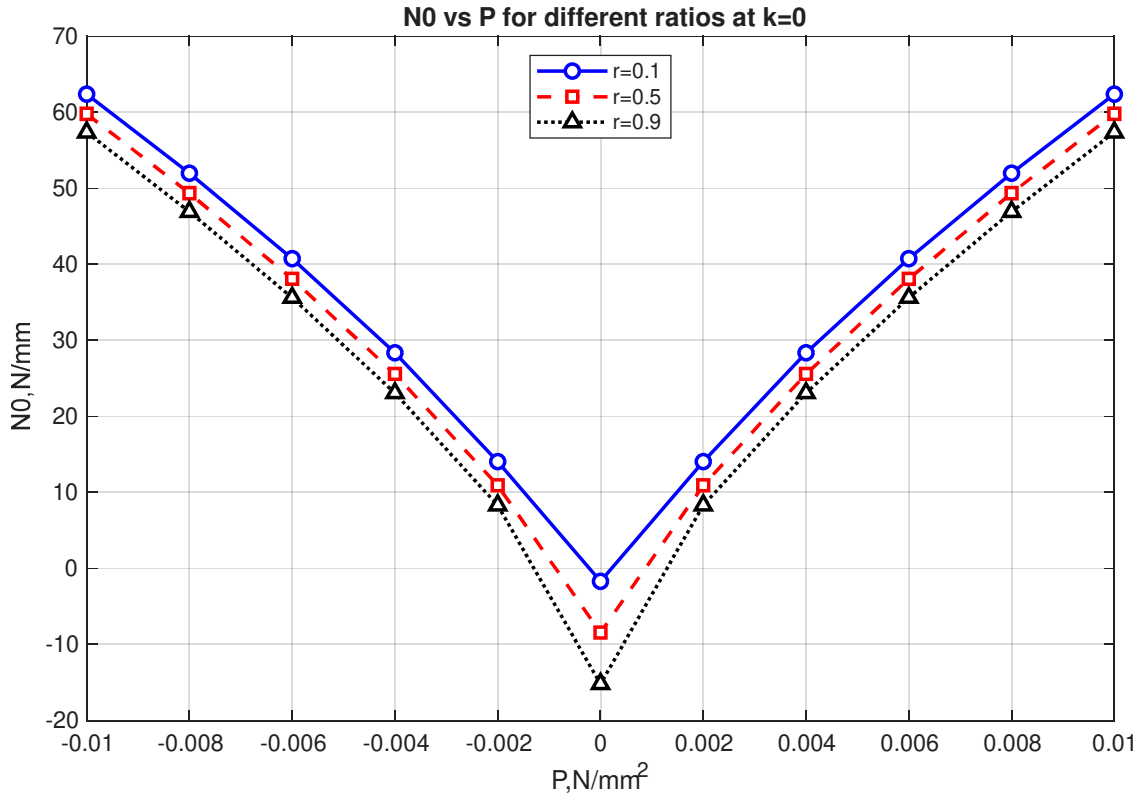


Figure 11. N_0 vs. pressure P graphs for different $\Delta T / \Delta T_{cr}$ ratios at gradation parameter $k = 0$ for the rule of mixture.

5. Conclusion

This study presents a detailed nonlinear bending analysis of functionally graded beams subjected to uniform mechanical and thermal loads, using Classical Beam Theory (CBT) and First-Order Shear Deformation Theory (FSDT). Beams were modeled using the Rule of Mixtures and Mori–Tanaka homogenization. Results reveal significant nonlinear effects due to asymmetric material gradation, necessitating von Kármán geometric nonlinearity. For beams with $L/h = 333.33$, shear deformation was minimal, and the deflections predicted by CBT and FSDT were nearly identical. Elastic spring supports effectively simulated typical boundary conditions, and analytical results closely aligned with finite element simulations, with near-zero percentage differences. As the gradation parameter k approaches 0 or infinity, the maximum deflection behavior tends toward that of isotropic beams. Thermal analysis of clamped-clamped beams highlighted the importance of evaluating the buckling temperature before applying thermal loads to avoid post-buckling conditions and ensure the equations accurately describe the coupled thermo-mechanical response.

Future work can extend the current methodology to more complex structural elements, including plates, shells, and stiffened structures [4]. Thermal post-buckling analysis can be performed under various boundary conditions, including spring constraints. Linear and nonlinear vibration studies of functionally graded beams, plates, and shells can be conducted. Experimental validation using 3D-printed FGMs is recommended, and this approach can be

expanded to plates and shells. Future developments could include multi-axial material gradation in both the thickness and axial directions, spatial or temporal variations in thermal loads, and three-dimensional temperature distributions, thereby enabling more accurate modeling of engineering scenarios.

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