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Hopf bifurcation, bistability and transient grazing in a piecewise-smooth hospital-overload model

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Abstract

Hospital overload can occur during an epidemic wave even when the endemic state is compatible with routine admission. We analyse a continuous piecewise-smooth epidemic model in which severe cases enter a waiting class and hospital admission is the minimum of a demand-driven flux and a capacity-driven bed-filling flux. Three thresholds are separated: the basic reproduction number R_0 , the endemic capacity threshold b_{crit} , and the trajectory-dependent transient threshold b_{trans} . The two one-sided Jacobians share an exact eigenpair, reducing each sixth-degree characteristic polynomial to a quintic. At the baseline parameters, the capacity-limited equilibrium undergoes a subcritical Hopf bifurcation at $b_H = 0.2263 b_{\text{crit}}$. Single-shooting continuation with Floquet multipliers connects the unstable Hopf branch to a stable piecewise-smooth periodic branch through a lower fold of cycles and identifies an upper fold beyond b_{crit} , producing two bistable capacity intervals. At $b = b_{\text{trans}}$, the normal-care outbreak has a non-degenerate quadratic tangency with the switching surface. The overload duration scales as $(b_{\text{trans}} - b)^{1/2}$, while the cumulative admission deficit and the trajectory perturbation are of order $(b_{\text{trans}} - b)^{3/2}$. A Metzler comparison system gives a rigorous global lower bound on the first switching time. Independent symbolic and fixed-step computations reproduce the Hopf coefficient and a representative attracting cycle. An NHS England stock-flow check tracks held-out COVID-19 bed occupancy but does not identify the latent waiting class or capacity thresholds. The results show that equilibrium hospital adequacy does not guarantee transient or dynamical safety.

Keywords: Epidemic model; Hospital capacity; Piecewise-smooth dynamical system; Grazing threshold; Boundary-equilibrium bifurcation; Transient overload.

MSC (2020): 92D30; 34A36; 37G15.

1 Introduction

Epidemic models are often introduced through a single question: can an infection invade? In classical compartmental theory this is answered by the basic reproduction number: below unity small introductions die out, above unity infection grows ([Kermack and McKendrick, 1927](#); [van den](#)

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Driessche and Watmough, 2002; Diekmann et al., 2010). The invasion threshold is mathematically powerful but does not answer every operational question. In particular it does not say whether the healthcare system can absorb the severe cases generated during the transient epidemic wave.

The COVID-19 pandemic made this distinction unavoidable. Many modelling studies were concerned not only with transmission but with the timing and size of hospital and intensive-care demand (Ferguson et al., 2020; Moghadas et al., 2020; Weissman et al., 2020; Grasselli et al., 2020; Booton et al., 2021). A health system may be adequate once the epidemic has settled to an endemic level while still being overwhelmed during the initial peak. Hospital overload is therefore not determined by the number of occupied beds at equilibrium: during a fast wave severe cases accumulate in a waiting state before admission, and a system may appear adequate from an endemic-equilibrium calculation while failing during the transient peak.

A related literature studies how limited treatment or medical resources alter epidemic dynamics. Early treatment-limited models showed that replacing unlimited proportional treatment by capacity-limited treatment can generate backward bifurcation, bistability and oscillations (Wang, 2006; Zhang and Liu, 2008; Cui et al., 2008). Later models incorporated hospital beds explicitly by letting recovery, hospitalisation or progression rates depend on available resources or the bed–population ratio (Shan and Zhu, 2014; Abdelrazec et al., 2016; Misra et al., 2022; Yu et al., 2023; Maurya et al., 2024; Jia et al., 2025). Most of these represent healthcare limitation through *smooth* nonlinear functions. In practice admission has a threshold-like character: while capacity is sufficient, severe cases are admitted according to clinical demand; once bed turnover or available slack becomes limiting, admission is constrained. Piecewise-smooth and Filippov formulations have been used to describe such threshold policies (Wang et al., 2018; Xiao et al., 2013), and the general theory of nonsmooth bifurcations is well developed (di Bernardo et al., 2008a; Feigin, 1994; Simpson and Meiss, 2012; Nordmark, 1991). Recent nonlinear epidemic studies have likewise combined analytical Hopf conditions, numerical continuation and parameter-region diagrams to connect local stability changes with recurrent epidemic dynamics (Zhong et al., 2026).

The present model is *continuous* piecewise-smooth rather than Filippov. The admission flux is continuous across the switching surface, because the demand-driven and capacity-driven rates agree there, but the Jacobian changes discontinuously. This places the model in the class for which boundary-equilibrium bifurcations admit the Feigin classification (Feigin, 1994; di Bernardo et al., 2008a) and for which grazing contact produces 3/2-type scaling rather than the square-root map of impacting systems (Nordmark, 1991; di Bernardo et al., 2008a). Exploiting this structure is the technical contribution of this paper.

The present work develops the nonlinear-dynamical structure generated by capacity-limited hospital admission. The first contribution concerns the endemic dynamics near and beyond the switching surface. We show that the two one-sided Jacobians possess an exact regime-independent eigenpair, which factors each sixth-degree characteristic polynomial into a demographic eigenvalue and a reduced quintic. This reduction yields explicit Routh–Hurwitz stability conditions and permits a local Feigin classification of the boundary-equilibrium transition at the equilibrium-capacity threshold. For the baseline parameter set, the capacity-limited equilibrium undergoes a Hopf bifurcation. Calculation of the first Lyapunov coefficient establishes that this bifurcation is subcritical. Numerical periodic-orbit continuation then connects the small unstable Hopf branch to a finite-amplitude attracting piecewise-smooth periodic branch through a fold of cycles, identifies a second fold, and reveals two intervals of coexistence between a stable equilibrium and a stable periodic attractor.

The second contribution concerns transient rather than endemic overload. We define the transient threshold as the maximum admission pressure generated by the normal-care outbreak trajectory and show that, under a unique non-degenerate maximum, the threshold corresponds to quadratic grazing of the switching surface. The resulting asymptotic analysis gives a square-root law for the duration of capacity-limited admission, a 3/2-power law for the cumulative admission deficit, and an $O(\varepsilon^{3/2})$ perturbation of the epidemic trajectory, where ε is the capacity shortfall

below the transient threshold. We further replace the previous early-exponential switching-time approximation by a globally valid matrix-exponential lower bound obtained through a Metzler comparison argument and quantify its numerical accuracy.

Together, these results separate three dynamically distinct questions: whether infection invades, whether the endemic state is compatible with normal admission, and whether the transient outbreak reaches the capacity-limited regime. A two-parameter bifurcation atlas, threshold sensitivities and operational capacity measures describe how these mechanisms depend on transmission and hospital parameters. The principal nonlinear calculations are reproduced using an independent symbolic and high-precision implementation, while an NHS England hospital-stock application is included as a structural illustration of the admission–occupancy mechanism rather than as a calibration of the full model.

Throughout, the model is stated once in dimensionless form. The dimensional formulation and nondimensionalisation are collected in Appendix A, while the one-sided Jacobians and polynomial-coefficient construction are provided in Appendix B.

2 The model

2.1 State variables and admission law

We work directly with the dimensionless model; the dimensional formulation, the scaling $\tau = \kappa_I t$ and the definitions of all dimensionless groups are given in Appendix A. The dimensionless state is

$$y = (s, e, i, q, h, r)^\top, \quad (1)$$

representing susceptible, exposed, infectious, severe-waiting, hospitalised and recovered fractions of the disease-free population size. The living population is $n = s + e + i + q + h + r$, and a cumulative death variable z is retained for bookkeeping only; since z does not feed back into the dynamics it is omitted from all stability and equilibrium calculations. The dimensionless hospital capacity is b .

The single modelling ingredient that generates the nonsmooth structure is the *admission law*

$$\mathcal{A}(q, b - h) = \min\{aq, c(b - h)_+\}, \quad (x)_+ = \max\{x, 0\}, \quad (2)$$

where $a > 0$ is the demand-driven admission rate and $c > 0$ the capacity-driven bed-filling rate. The term aq is the rate at which severe cases would be admitted if capacity were not limiting; $c(b - h)_+$ is the maximum rate at which available hospital slack can be filled. We define the *switching function* and *switching surface* once and for all:

$$\Psi(y) = aq - c(b - h), \quad \Sigma = \{y : \Psi(y) = 0, 0 \leq h \leq b\}. \quad (3)$$

The *normal-care regime* is $\Psi(y) < 0$, where $\mathcal{A} = aq$; the *capacity-limited regime* is $\Psi(y) \geq 0$, where $\mathcal{A} = c(b - h)$. Throughout the paper, “overload” means entry into this capacity-limited admission regime. It does not mean that the state variable h exceeds b ; Theorem 3.1 rules that out.

Remark 2.1 (Switching is not full occupancy). The full-occupancy boundary is $h = b$, whereas Σ is the line $h = b - (a/c)q$ in the (q, h) -plane. For $q > 0$ the switching surface lies *strictly below* $h = b$. Capacity-limited admission therefore begins before the hospital is literally full: it is triggered by the joint effect of the waiting queue and the residual slack, not by occupancy alone. Operationally, c is a turnover rate — the rate at which vacated beds can actually be made available and staffed — so $c(b - h)$ is a *flow* constraint, while $h = b$ is a *stock* constraint. A hospital with many nominally empty beds but slow discharge and admission processing (small c) enters the capacity-limited regime early. This is the intuitive content of Σ .

Table 1: Dimensionless parameters, their meaning, and the baseline values used in Section 8. Derivations from dimensional quantities are in Appendix A. All parameters are collected here; none is introduced later in the text.

Symbol	Meaning	Baseline
$\mathcal{B}$	transmission parameter	3
m	demographic turnover rate	0.002
η	progression rate, exposed $\rightarrow$ infectious	0.5
π	severe-case generation rate	0.20
a	demand-driven admission rate	0.70
c	capacity-driven bed-filling (turnover) rate	2
χ_Q	total removal rate from the waiting class	0.15
χ_H	total removal rate from hospital	0.25
ρ_I	recovery rate of non-hospitalised infectious	0.788
ρ_Q	recovery rate of severe waiting individuals	0.05
ρ_H	recovery rate of hospitalised individuals	0.20
θ_Q	relative infectiousness of the waiting class	0.20
Δ_I	disease-mortality fraction in i	0.010
Δ_Q	disease-mortality fraction in q	0.098
Δ_H	disease-mortality fraction in h	0.048
b	hospital capacity (bifurcation parameter)	varied

2.2 Governing equations

With $u = i + \theta_Q q$ and force of infection $\ell = \mathcal{B}u/n$, the model is

$$s' = m(1 - s) - \ell s, \quad (4)$$

$$e' = \ell s - (\eta + m)e, \quad (5)$$

$$i' = \eta e - i, \quad (6)$$

$$q' = \pi i - \mathcal{A}(q, b - h) - \chi_Q q, \quad (7)$$

$$h' = \mathcal{A}(q, b - h) - \chi_H h, \quad (8)$$

$$r' = \rho_I i + \rho_Q q + \rho_H h - mr, \quad (9)$$

where $' = d/d\tau$. All dimensionless parameters are listed in Table 1; they satisfy the identity $\pi + \rho_I + m + \Delta_I = 1$ inherited from the scaling.

Remark 2.2 (Dimensional consistency of a and c). In dimensional variables the admission law is $\mathcal{A}(Q, B_0 - H) = \min\{\nu Q, \kappa(B_0 - H)_+\}$ with $a = \nu/\kappa_I$, $c = \kappa/\kappa_I$. Both ν and κ are *per unit time*, so a and c are dimensionless and the minimum in (2) compares two quantities of the same dimension (individuals per unit time). The two rates have different operational meanings — ν is a clinical demand rate applied to the queue, κ a throughput rate applied to slack — but the same dimensionality.

3 Well-posedness

The right-hand side of (4)–(9) is continuous and locally Lipschitz on $\{n > 0\}$, since $\min\{\cdot, \cdot\}$ and $(\cdot)_+$ are Lipschitz. Hence solutions exist and are unique in the Carathéodory sense; no Filippov convexification is required.

Theorem 3.1 (Non-negativity and capacity invariance). *Let the initial data be non-negative with $n(0) > 0$ and $0 \leq h(0) \leq b$. Then the solution of (4)–(9) satisfies $s, e, i, q, h, r \geq 0$ and $0 \leq h(\tau) \leq b$ for all $\tau > 0$.*

Proof. It is enough to check the vector field on each coordinate face. At $s = 0$, $s' = m > 0$; at $e = 0$, $e' = \ell s \geq 0$; at $i = 0$, $i' = \eta e \geq 0$; and at $q = 0$ the admission flux is $\mathcal{A}(0, b - h) = 0$, so $q' = \pi i \geq 0$. At $h = 0$, $h' = \mathcal{A}(q, b) \geq 0$, while at $r = 0$, $r' = \rho_I i + \rho_Q q + \rho_H h \geq 0$. Thus the non-negative orthant is positively invariant. On the upper face $h = b$, $(b - h)_+ = 0$ and therefore $h' = -\chi_H b < 0$, so trajectories cannot cross above capacity. $\square$

Theorem 3.2 (Boundedness). *The living population satisfies $n' = m(1 - n) - \Delta_I i - \Delta_Q q - \Delta_H h \leq m(1 - n)$, hence $n(\tau) \leq \max\{n(0), 1\}$; in particular $n(0) \leq 1$ implies $n(\tau) \leq 1$ for all $\tau \geq 0$. The region*

$$\Omega = \{y \in \mathbb{R}_+^6 : n \leq 1, 0 \leq h \leq b\} \quad (10)$$

is compact and positively invariant.

Remark 3.3. Theorem 3.1 is a statement of *non-negativity*, not strict positivity: the faces $\{e = i = q = h = 0\}$ are invariant and contain the disease-free equilibrium. We use the term non-negativity throughout.

4 Invasion threshold

The disease-free equilibrium is $\mathcal{E}_0 = (1, 0, 0, 0, 0, 0)$. Near $\mathcal{E}_0$ we have $h = 0 < b$ and q arbitrarily small, so $\mathcal{A}(q, b - h) = aq$: the capacity constraint is *inactive*, the vector field is smooth in a neighbourhood of $\mathcal{E}_0$, and the standard next-generation construction applies without modification.

With $X = (e, i, q, h)^\top$ and $X' = \mathcal{F}(X) - \mathcal{V}(X)$,

$$F = \begin{pmatrix} 0 & \mathcal{B} & \mathcal{B}\theta_Q & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \eta + m & 0 & 0 & 0 \\ -\eta & 1 & 0 & 0 \\ 0 & -\pi & a + \chi_Q & 0 \\ 0 & 0 & -a & \chi_H \end{pmatrix}. \quad (11)$$

Theorem 4.1. *The basic reproduction number is*

$$R_0 = \rho(FV^{-1}) = \frac{\mathcal{B}\eta}{\eta + m} \left(1 + \frac{\theta_Q \pi}{a + \chi_Q} \right), \quad (12)$$

and $\mathcal{E}_0$ is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. Moreover R_0 is independent of the hospital capacity b .

Proof. V is a nonsingular M -matrix and $F \geq 0$; since all new infections enter e , only the first column of V^{-1} is needed, giving $x_1 = (\eta + m)^{-1}$, $x_2 = \eta/(\eta + m)$, $x_3 = \pi\eta/[(\eta + m)(a + \chi_Q)]$. Then $R_0 = \mathcal{B}(x_2 + \theta_Q x_3)$, which is (12). Independence of b follows because the linearisation at $\mathcal{E}_0$ uses only the normal-care branch of (2). $\square$

If waiting individuals do not transmit ($\theta_Q = 0$) this reduces to $R_0 = \mathcal{B}\eta/(\eta + m)$.

Definition 4.2 (Early growth rate). The *early growth rate* is the spectral abscissa of the linearised infected block,

$$g := \max\{\operatorname{Re} \lambda : \lambda \in \sigma(F - V)\}, \quad (13)$$

which is the unique real dominant eigenvalue of the Metzler matrix $F - V$ and is positive precisely when $R_0 > 1$. For the baseline parameters, $g = 0.5162$. This quantity is used in Section 7.3.

5 Equilibria and the capacity threshold

Write $L = (\eta + m)/(m\eta)$ so that $s = 1 - Li$ at any endemic state, and set

$$\begin{aligned} K_Q &= \frac{\pi}{a + \chi_Q}, & K_H &= \frac{a\pi}{(a + \chi_Q)\chi_H}, \\ K_R &= \frac{\rho_I + \rho_Q K_Q + \rho_H K_H}{m}, & G &= \frac{1}{\eta} + 1 + K_Q + K_H + K_R. \end{aligned} \quad (14)$$

Theorem 5.1 (Normal-care endemic equilibrium). *If $R_0 > 1$ the normal-care subsystem has a unique positive equilibrium $\mathcal{E}_N^*$ with*

$$i_N^* = \frac{R_0 - 1}{G + L(R_0 - 1)}, \quad e_N^* = \frac{i_N^*}{\eta}, \quad q_N^* = K_Q i_N^*, \quad h_N^* = K_H i_N^*, \quad r_N^* = K_R i_N^*. \quad (15)$$

It is admissible for the full piecewise-smooth model if and only if $aq_N^* < c(b - h_N^*)$, i.e. if and only if $b > b_{\text{crit}}$, where

$$b_{\text{crit}} = h_N^* + \frac{a}{c} q_N^* = \frac{aK_Q + cK_H}{c} i_N^* \quad (16)$$

If $R_0 \leq 1$ no positive normal-care endemic equilibrium exists.

Theorem 5.2 (Capacity-limited endemic equilibria). *In the capacity-limited regime,*

$$h_C^* = \frac{c}{c + \chi_H} b, \quad A_C = \frac{c\chi_H}{c + \chi_H} b, \quad q_C(i) = \frac{\pi i - A_C}{\chi_Q}.$$

Let $s(i) = 1 - Li$,

$$r_C(i) = \frac{\rho_I i + \rho_Q q_C(i) + \rho_H h_C^*}{m}, \quad n_C(i) = s(i) + \frac{i}{\eta} + i + q_C(i) + h_C^* + r_C(i),$$

and define

$$F_C(i) = \mathcal{B}s(i)[i + \theta_Q q_C(i)] - \frac{\eta + m}{\eta} i n_C(i). \quad (17)$$

Each root of $F_C(i) = 0$ in $i \in [i_\Sigma^{(C)}, 1/L)$, where $i_\Sigma^{(C)} = A_C(a + \chi_Q)/(a\pi)$, gives an admissible capacity-limited equilibrium, and every such equilibrium arises this way. This statement is a parameterisation; it does not assert global existence or uniqueness for arbitrary parameter values. For the baseline set, numerical root isolation gives one admissible root for $0 < b < b_{\text{crit}}$ and the boundary root at $b = b_{\text{crit}}$.

Proposition 5.3 (Continuity of the equilibrium branch). *At $b = b_{\text{crit}}$ the two equilibrium formulae agree: $h_C^* = h_N^*$ and $q_C^* = q_N^*$. Consequently the endemic equilibrium branch is continuous across Σ , and $\mathcal{E}^*(b_{\text{crit}}) \in \Sigma$ is a boundary equilibrium.*

Proof. From $aK_Q = \chi_H K_H$ and (16), $b_{\text{crit}} = (c + \chi_H)K_H i_N^*/c$, hence $h_C^* = \frac{c}{c + \chi_H} b_{\text{crit}} = K_H i_N^* = h_N^*$. Substituting $A_C = c\chi_H b_{\text{crit}}/(c + \chi_H) = \chi_H K_H i_N^* = aK_Q i_N^*$ into $q_C = (\pi i_N^* - A_C)/\chi_Q$ gives $q_C^* = K_Q i_N^* = q_N^*$. $\square$

Proposition 5.3 is the reason the analysis of Section 6 is possible: it places the model in the boundary-equilibrium setting of Feigin (1994) and di Bernardo et al. (2008a, Ch. 5) rather than in a discontinuous-branch setting.

6 Local stability and the boundary equilibrium at $b = b_{\text{crit}}$

The living subsystem is continuous across Σ but not differentiable there. The smooth extensions on the two sides have Jacobians J_N and J_C listed in Appendix B.

6.1 An exact, regime-independent eigenpair

Theorem 6.1 (Shared eigenpair). *Let $\mathcal{E}^* = (s^*, c^*, i^*, q^*, h^*, r^*)$ be any equilibrium with $n^* > 0$. Then*

$$v = (s^*, 0, 0, 0, 0, n^* - s^*)^\top$$

satisfies $J_N v = -mv$ and $J_C v = -mv$. Hence

$$P_\bullet(\lambda) = \det(\lambda I - J_\bullet) = (\lambda + m)\tilde{P}_\bullet(\lambda), \quad \deg \tilde{P}_\bullet = 5, \quad (18)$$

for $\bullet \in \{N, C\}$.

Proof. Write $T = \mathcal{B}su/n$, $u = i + \theta_Q q$, and denote its derivatives by $\hat{T}_j = \partial T / \partial x_j$. They are

$$\hat{T}_s = \mathcal{B} \frac{u(n-s)}{n^2}, \quad \hat{T}_e = -\mathcal{B} \frac{su}{n^2}, \quad \hat{T}_i = \mathcal{B} \frac{s(n-u)}{n^2}, \quad \hat{T}_q = \mathcal{B} \frac{s(\theta_Q n - u)}{n^2}, \quad \hat{T}_h = \hat{T}_r = -\mathcal{B} \frac{su}{n^2}. \quad (19)$$

The entries in which J_N and J_C differ multiply the zero q - and h -components of v . Rows 3–5 therefore give zero, and row 6 gives $-m(n^* - s^*)$. In rows 1–2 the incidence contribution cancels because $\hat{T}_s s^* + \hat{T}_r (n^* - s^*) = 0$. The remaining contribution is $-ms^*$ in row 1 and zero in row 2. Thus $J_\bullet v = -mv$. $\square$

Corollary 6.2. *If $\tilde{\alpha}_\bullet$ is the spectral abscissa of $\tilde{P}_\bullet$, then $\alpha_\bullet = \max\{-m, \tilde{\alpha}_\bullet\}$.*

6.2 Routh–Hurwitz test for the reduced quintic

Write $\tilde{P}_\bullet(\lambda) = \lambda^5 + c_1 \lambda^4 + c_2 \lambda^3 + c_3 \lambda^2 + c_4 \lambda + c_5$.

Proposition 6.3 (Routh–Hurwitz). *An admissible endemic equilibrium is locally asymptotically stable if and only if*

$$\begin{aligned} H_1 &= c_1 > 0, & H_2 &= c_1 c_2 - c_3 > 0, \\ H_3 &= c_1 c_2 c_3 - c_1^2 c_4 - c_3^2 + c_1 c_5 > 0, \\ H_4 &= \det \begin{pmatrix} c_1 & c_3 & c_5 & 0 \\ 1 & c_2 & c_4 & 0 \\ 0 & c_1 & c_3 & c_5 \\ 0 & 1 & c_2 & c_4 \end{pmatrix} > 0, & c_5 &> 0. \end{aligned} \quad (20)$$

The coefficients are computed exactly from the explicit Jacobian by the Faddeev–LeVerrier and synthetic-division recursions in Appendix B.

6.3 Spectral crossing and Hopf criticality

For the baseline parameters, J_N is independent of b and $\alpha_N = -m = -0.002$. On the admissible capacity-limited branch a conjugate pair crosses the imaginary axis at

$$b_H = 2.32155208 \times 10^{-4} = 0.2263478 b_{\text{crit}}, \quad \omega_H = 0.03629529, \quad (21)$$

with $d \operatorname{Re} \lambda / db = 20.7516$.

Proposition 6.4 (Subcritical Hopf point). *At $b = b_H$ the capacity-limited smooth extension has a simple pair $\pm i\omega_H$, all other eigenvalues have negative real part, and the transversality condition is non-zero. Using the standard multilinear-form normalisation of Kuznetsov (2023), the first Lyapunov coefficient is*

$$\ell_1 = 42.44 > 0. \quad (22)$$

Therefore the Hopf bifurcation is subcritical: the small local periodic orbit lies on the stable-equilibrium side and is unstable. Numerical continuation in Section 6.4 connects this local branch to a finite-amplitude attracting piecewise-smooth branch through a fold of periodic orbits; thus the large cycle is not born as a stable orbit at the Hopf point.

Table 2: Baseline nonlinear-dynamics landmarks. Fold locations and periodic orbit quantities are numerical single-shooting results; the Hopf point and its criticality were independently reproduced.

Event	b	b/b_{crit}	$\max q - \min q$	largest nontrivial $ \mu_F $
Lower fold LPC^-	8.6241×10^{-5}	0.08408	9.70×10^{-3}	≈ 1
Hopf point H	2.32155×10^{-4}	0.22635	0	1
Boundary equilibrium	1.02566×10^{-3}	1	–	–
Upper fold LPC^+	2.09249×10^{-3}	2.04015	4.44×10^{-3}	≈ 1
Transient grazing	5.48817×10^{-2}	53.5088	–	–

At $b = b_{\text{crit}}$ the one-sided spectra give $\alpha_N = -0.002$ and $\alpha_C = 0.020628$. Thus the equilibrium is stable on the normal-care side and unstable under the capacity-limited smooth extension.

6.4 Global continuation of periodic orbits

The local Hopf calculation determines only the behaviour arbitrarily close to b_H . To determine the origin of the finite-amplitude attractor, periodic solutions of the full continuous piecewise-smooth system were continued by a single-shooting formulation. For each orbit, the periodicity equations were supplemented by a phase condition; the variational equations, using the one-sided Jacobian selected by the active regime, supplied the monodromy matrix and Floquet multipliers. Continuation was performed in orbit amplitude near the subcritical Hopf branch and in b away from it. All reported shooting residuals are below 1.4×10^{-10} . The branch was traversed using fixed- b , equilibrium-relative amplitude and prescribed q_{max} parameterisations; the same topology was recovered wherever the parameterisations overlap. The two folds are identified by a nontrivial Floquet multiplier crossing $+1$. These checks reduce, but do not eliminate, the numerical uncertainty associated with single shooting.

The unstable periodic branch emerging on the stable-equilibrium side of the Hopf point initially remains entirely in the capacity-limited smooth region. As its amplitude grows it intersects the switching surface and joins a stable piecewise-smooth periodic branch at a lower fold of cycles

$$b_{\text{LPC}}^- \approx 8.624 \times 10^{-5} = 0.3715 b_H. \quad (23)$$

The stable branch then passes through b_H and b_{crit} , so stable periodic motion coexists with the equilibrium on both sides of these local thresholds. At an upper fold

$$b_{\text{LPC}}^+ \approx 2.09249 \times 10^{-3} = 2.0402 b_{\text{crit}}, \quad (24)$$

the branch turns and becomes unstable. The unstable return branch approaches the boundary equilibrium as $b \downarrow b_{\text{crit}}$. These are numerical global bifurcation results, not consequences of the local Feigin classification. They show two distinct bistable windows: stable equilibrium versus stable cycle for $b_{\text{LPC}}^- < b < b_H$, and again for $b_{\text{crit}} < b < b_{\text{LPC}}^+$. The unstable periodic branch is a natural candidate for part of the basin boundary, but its global stable manifold and the complete basin geometry are not resolved here.

6.5 Local persistence classification at $b = b_{\text{crit}}$

The two smooth extensions satisfy $f_C(y, b) - f_N(y, b) = \Psi(y, b)(0, 0, 0, 1, -1, 0)^\top$. Consequently this is a continuous piecewise-smooth boundary-equilibrium transition, not a Filippov sliding equilibrium. Under the usual non-singularity and transversality assumptions, the Feigin-type local classification for piecewise-smooth continuous flows distinguishes persistence from a nonsmooth fold by the determinant parity (di Bernardo et al., 2008b; Simpson and Meiss, 2012).

Theorem 6.5 (Baseline transition is locally persistent). *At the baseline boundary equilibrium,*

$$\det J_N = 9.50855 \times 10^{-7}, \quad \det J_C = 1.33074 \times 10^{-6}.$$

Their product is positive; equivalently the numbers of unstable eigenvalues are $\sigma_N = 0$ and $\sigma_C = 2$, whose sum is even. The transition at $b = b_{\text{crit}}$ is therefore of persistence type, rather than a local nonsmooth fold. This is a local statement: it excludes a fold generated at this boundary equilibrium but does not by itself prove global uniqueness, absence of remote bistability, or absence of other periodic attractors.

7 Transient overload and grazing

7.1 Admission-pressure functional

Let $y_N(\tau)$ denote the normal-care solution obtained by taking $\mathcal{A} = aq$ from the same initial data, and define

$$\mathcal{P}(\tau) = h_N(\tau) + \frac{a}{c}q_N(\tau), \quad b_{\text{trans}} = \sup_{\tau \geq 0} \mathcal{P}(\tau). \quad (25)$$

Because $\Psi_N = c(\mathcal{P} - b)$, $\mathcal{P}(\tau)$ is the instantaneous capacity needed to keep the normal-care trajectory strictly below the switching surface.

Theorem 7.1 (Limit, supremum and threshold property). *Assume $R_0 > 1$, the initial state is strictly in normal care, and $y_N(\tau) \rightarrow \mathcal{E}_N^*$ as $\tau \rightarrow \infty$. Then $\mathcal{P}$ is continuous and bounded, $\mathcal{P}(\tau) \rightarrow b_{\text{crit}}$, and therefore $b_{\text{trans}} \geq b_{\text{crit}}$. If $b_{\text{trans}} > b_{\text{crit}}$, the supremum is attained at a finite time. If $b_{\text{trans}} = b_{\text{crit}}$, the supremum need not be attained; it may be approached only asymptotically.*

If $b > b_{\text{trans}}$, the full solution remains in normal care for all time and coincides with y_N . If $b < b_{\text{trans}}$, it reaches Σ at a finite first time

$$\tau_c = \inf\{\tau > 0 : \mathcal{P}(\tau) = b\},$$

and a transverse first contact leads immediately into the capacity-limited regime.

Proof. Continuity and boundedness follow from Theorem 3.2; convergence to the normal-care equilibrium gives $\mathcal{P} \rightarrow h_N^* + (a/c)q_N^* = b_{\text{crit}}$. If the supremum strictly exceeds this limiting value, the tail is separated from the supremum and the maximum is attained on a compact interval. The threshold property follows because the full and normal-care vector fields coincide until first contact and because $b < \sup \mathcal{P}$ implies $\mathcal{P}(\tau) > b$ for some finite τ . $\square$

7.2 Quadratic grazing at $b = b_{\text{trans}}$

The next result requires a genuine finite-time overshoot, not merely $\mathcal{P}(\tau) \rightarrow b_{\text{crit}}$.

Theorem 7.2 (Grazing scaling laws). *Assume $b_{\text{trans}} > b_{\text{crit}}$ and that $\mathcal{P}$ has a unique strict global maximiser τ^* satisfying*

$$\mathcal{P}'(\tau^*) = 0, \quad k = -\mathcal{P}''(\tau^*) > 0.$$

Let $\varepsilon = b_{\text{trans}} - b > 0$. Then, for sufficiently small ε :

1. *the capacity-limited episode associated with the grazing event is a single interval and*

$$\Delta\tau = 2\sqrt{\frac{2\varepsilon}{k}}(1 + O(\varepsilon^{1/2})); \quad (26)$$

2. *the cumulative admission deficit is*

$$\int [aq - c(b - h)]_+ d\tau = \frac{4\sqrt{2}}{3} \frac{c}{\sqrt{k}} \varepsilon^{3/2} + o(\varepsilon^{3/2}); \quad (27)$$

3. on every fixed time interval containing the grazing episode,

$$\sup_{\tau} \|y(\tau; b) - y_N(\tau)\| = O(\varepsilon^{3/2}). \quad (28)$$

Proof. A Taylor expansion gives $\Psi_N(\tau) = c[\varepsilon - k(\tau - \tau^*)^2/2 + O(|\tau - \tau^*|^3)]$. The strict isolated maximum supplies a positive gap away from a neighbourhood of τ^* , so no other contact is possible for sufficiently small ε . The positive interval has the length in (26). The two vector fields differ by $[aq - c(b - h)]_+$ in the q - and h -components, with opposite signs. Its amplitude is $O(\varepsilon)$ and its support has length $O(\varepsilon^{1/2})$; variation of constants and Gronwall's inequality give (28). Integrating the leading parabolic cap gives (27). The argument establishes the order in (28); it does not establish a non-zero asymptotic constant for every norm and parameter set. $\square$

7.3 A rigorous lower bound on first switching time

Let $X = (e, i, q, h)^\top$, let F, V be as in (11), set $M = F - V$, and let $w = (0, 0, a/c, 1)^\top$, so $\mathcal{P} = w^\top X$.

Theorem 7.3 (Metzler comparison). *Along the normal-care trajectory,*

$$X'(\tau) \leq MX(\tau), \quad X(\tau) \leq e^{M\tau} X(0) \quad (29)$$

componentwise. Hence

$$\mathcal{P}(\tau) \leq \mathcal{P}^+(\tau) := w^\top e^{M\tau} X(0)$$

and the first switching time satisfies the globally valid lower bound

$$\tau_c \geq \tau_c^+ := \inf\{\tau > 0 : \mathcal{P}^+(\tau) = b\}. \quad (30)$$

More generally, if a number $\underline{\theta}$ is independently known to satisfy $s(\tau)/n(\tau) \geq \underline{\theta}$ on $[0, T]$, then $X' \geq (\underline{\theta}F - V)X$ on that interval. If the corresponding lower comparison trajectory reaches b at a time $\tau_c^- \leq T$, then $\tau_c \leq \tau_c^-$. Taking $\underline{\theta} = \min_{[0, \tau_c]} s/n$ gives a valid a-posteriori bracket, but it is circular as an a-priori forecasting formula and is not used as such.

Proof. Only the exposed equation is nonlinear in the infected block and $s/n \leq 1$, so $e' \leq \mathcal{B}(i + \theta_Q q) - (\eta + m)e$; the remaining equations are linear. The matrix M is Metzler, and monotone-system comparison gives (29) (Smith, 1995). The hitting-time inequality follows from $w \geq 0$ and continuity. The reversed comparison is identical once an independent lower bound on s/n is supplied. $\square$

Let g be the dominant real eigenvalue of M , and let right and left eigenvectors ϕ, ψ be normalised by $\psi^\top \phi = 1$. Then

$$\mathcal{P}^+(\tau) \sim C_+ e^{g\tau}, \quad C_+ = (w^\top \phi)(\psi^\top X(0)). \quad (31)$$

Thus $(1/g) \log(b/C_+)$ is the leading approximation to the exact lower-bound hitting time τ_c^+ . The earlier formula written solely in terms of an “initial amplitude” i_0 is valid only when i_0 denotes the coefficient of the dominant eigenmode, not for an arbitrary coordinate initial condition.

8 Numerical results

All computations use Table 1 with $i(0) = 10^{-6}$, $s(0) = 1 - i(0)$ and the other components zero. The primary reproduction script uses SciPy adaptive solvers. A separate validation script uses SymPy-generated derivatives with 80-digit mpmath arithmetic for the Hopf coefficient and a fixed-step classical Runge–Kutta method for the global cycle. The transient threshold depends on the stated initial condition.

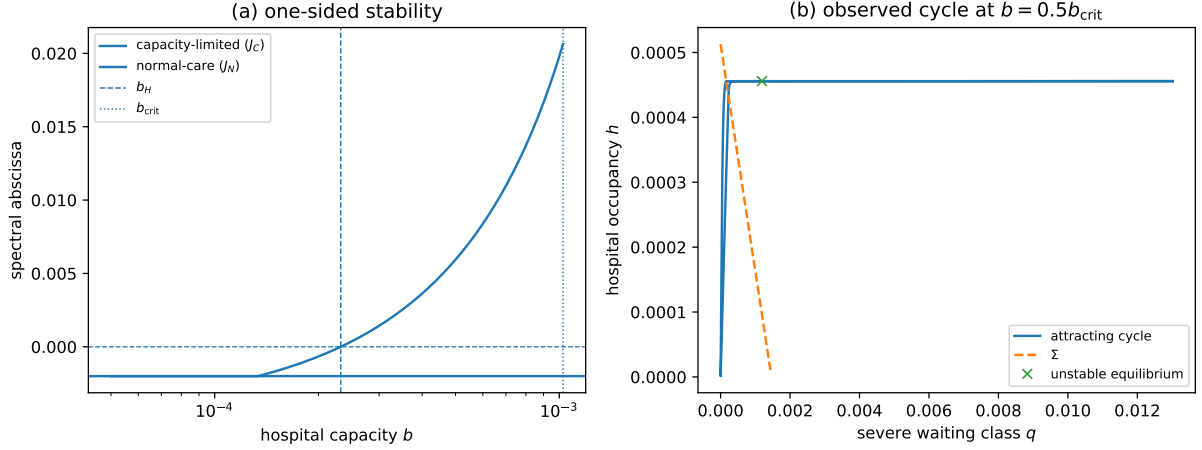


Figure 1: (a) One-sided spectral abscissa. The capacity-limited branch crosses zero at b_H , while the normal-care branch has spectral abscissa $-m$. (b) A numerically observed attracting piecewise-smooth cycle at $b = 0.5 b_{\text{crit}}$; the dashed line is Σ .

For the baseline set,

$$R_0 = 3.128662, \quad b_{\text{crit}} = 1.025657 \times 10^{-3}, \quad b_{\text{trans}} = 5.488167 \times 10^{-2}, \quad (32)$$

so $b_{\text{trans}}/b_{\text{crit}} = 53.51$. These dimensionless values illustrate the threshold separation; they are not calibrated bed ratios.

8.1 One-sided stability and global periodic-orbit structure

Figure 1a reproduces the capacity-branch spectral crossing. At b_H the critical eigenvalues are $\pm 0.0362953i$, the transversality is 20.7516, and the first Lyapunov coefficient is $\ell_1 = 42.44 > 0$. Hence the local Hopf is subcritical.

Figure 2 gives the global continuation result. The unstable small-amplitude Hopf branch and the stable large-amplitude branch meet at $b_{\text{LPC}}^- \approx 8.624 \times 10^{-5}$, where a nontrivial Floquet multiplier crosses +1. The stable branch extends through b_H and b_{crit} and turns at $b_{\text{LPC}}^+ \approx 2.09249 \times 10^{-3}$. The return branch is unstable and approaches the boundary equilibrium near b_{crit} . At $b = 0.5 b_{\text{crit}}$, direct integration converges to the orbit in Figure 1b. Its period is 247.6, it crosses Σ twice, and it spends 29.4% of each cycle in the capacity-limited regime.

Independent reproduction. The second implementation does not call the primary verification code. It constructs the second- and third-order derivative tensors symbolically, evaluates the Kuznetsov coefficient using 80-digit arithmetic, and obtains $\ell_1 = 42.4417862821$. It then integrates the full piecewise-smooth system by classical RK4 with step 0.05 to $\tau = 12000$, obtaining period 247.600, two crossings of Σ per period and a capacity-limited fraction 0.29443. These values agree with the adaptive-solver calculation to the reported digits. The independent implementation verifies the local Hopf calculation and one finite-amplitude attractor; it does not independently continue the two folds. Accordingly, b_{LPC}^- and b_{LPC}^+ are reported as baseline numerical approximations rather than theorem-level thresholds.

8.2 Verification of the grazing constants

For the baseline trajectory, $\tau^* = 30.83253$ and $k = 2.519206 \times 10^{-3}$. The predicted compensated limits are

$$2\sqrt{2/k} = 56.3525, \quad \frac{4\sqrt{2}c}{3\sqrt{k}} = 75.1367.$$

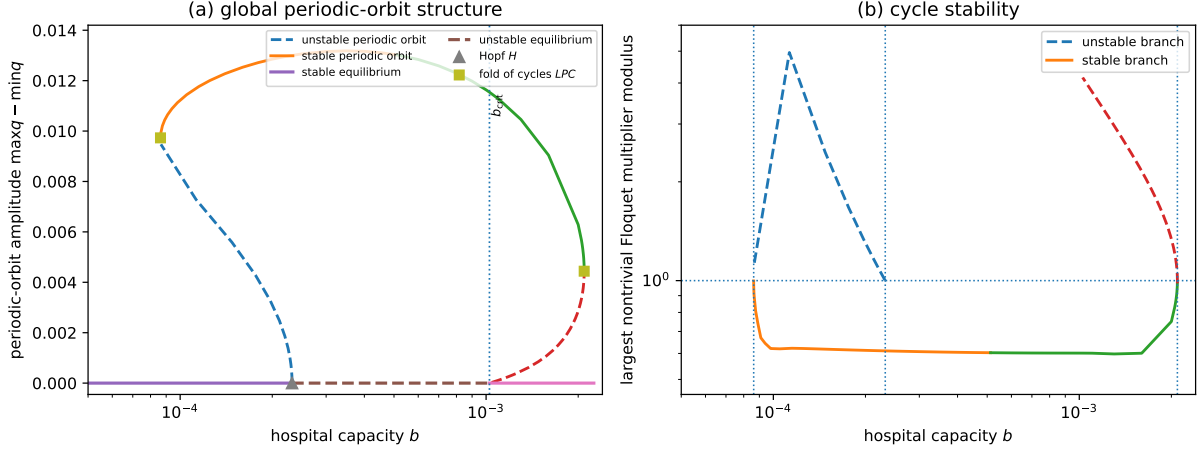


Figure 2: Global periodic-orbit continuation. (a) Waiting-class amplitude along stable and unstable periodic branches, together with the equilibrium branch. H denotes the subcritical Hopf point and LPC the two numerical folds of cycles. (b) Largest nontrivial Floquet multiplier modulus; crossings of unity confirm the changes of cycle stability.

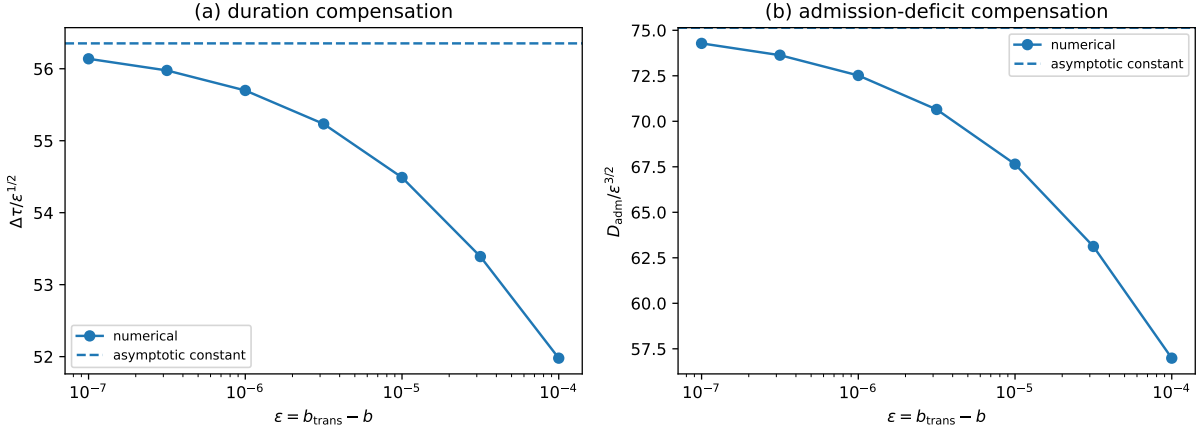


Figure 3: Compensated grazing quantities. Convergence to a horizontal constant is the numerical signature of the $\varepsilon^{1/2}$ duration law and the $\varepsilon^{3/2}$ admission-deficit law.

Figure 3 plots the compensated numerical quantities rather than fitting slopes over a finite range. As $\varepsilon \downarrow 0$, both approach the independently predicted constants, supporting the $1/2$ and $3/2$ laws.

8.3 Sharpness of the rigorous switching-time lower bound

Figure 4 compares the true first contact with the exact matrix-exponential lower bound (30). The relative gap is about 0.2% near $1.2 b_{\text{crit}}$ and increases to about 11% by $0.95 b_{\text{trans}}$. Thus the bound is useful but is not uniformly within 4% over the full interval, contrary to the earlier draft.

8.4 One-at-a-time dimensionless sensitivities

Elasticities were computed by the centred logarithmic difference

$$E_X^Y \approx \frac{\log Y(X(1+h)) - \log Y(X(1-h))}{\log(1+h) - \log(1-h)}, \quad h = 10^{-4},$$

holding the other dimensionless groups and the initial condition fixed. The calculation is numerically stable when h is changed by one order of magnitude. This is a local dimensionless sensitivity

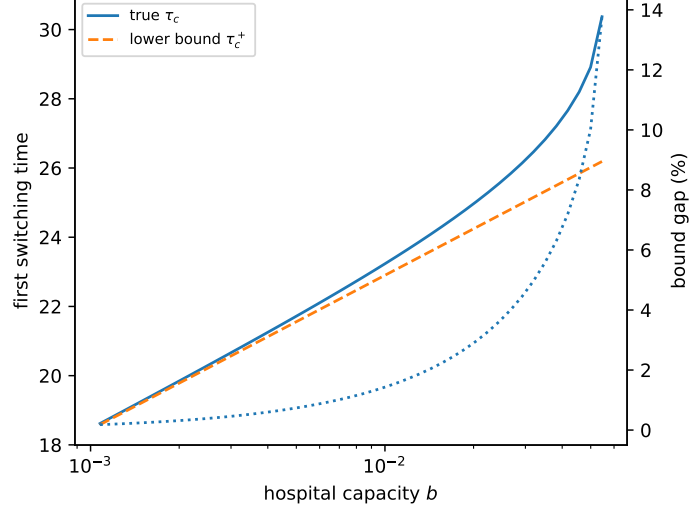


Figure 4: True switching time, the rigorous lower bound τ_c^+ , and their relative gap.

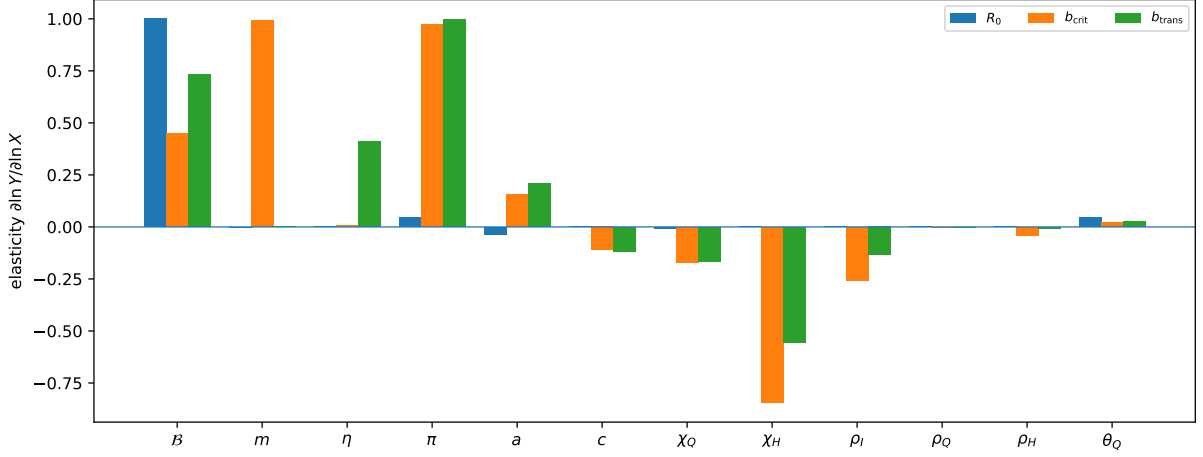


Figure 5: One-at-a-time elasticities of R_0 , b_{crit} and b_{trans} at the baseline dimensionless parameter set. The profiles differ substantially but are not disjoint.

analysis; perturbing underlying dimensional rates while enforcing all algebraic dependencies would give a different experiment.

Figure 5 shows distinct but overlapping profiles. In particular, R_0 has unit elasticity to $\mathcal{B}$; b_{crit} is most sensitive to m (0.993), π (0.973), χ_H (-0.846) and also $\mathcal{B}$ (0.449); while b_{trans} is most sensitive to π (0.996), $\mathcal{B}$ (0.732), χ_H (-0.555) and η (0.411). The sharpest separation is demographic: the elasticity to m is 0.993 for b_{crit} but only 0.00145 for b_{trans} .

8.5 Two-parameter threshold and stability atlas

Figure 6 augments the threshold diagram with the smooth Hopf curve $b_H(\mathcal{B})$ obtained from the capacity-limited equilibrium and its one-sided Jacobian. The band $b_H < b < b_{\text{crit}}$ contains an unstable capacity-limited equilibrium, while $b_{\text{crit}} < b < b_{\text{trans}}$ contains a normal-care endemic equilibrium but transient contact with the switching surface. The periodic-orbit fold curves have only been continued at the baseline $\mathcal{B} = 3$ and are therefore not drawn as two-parameter boundaries.

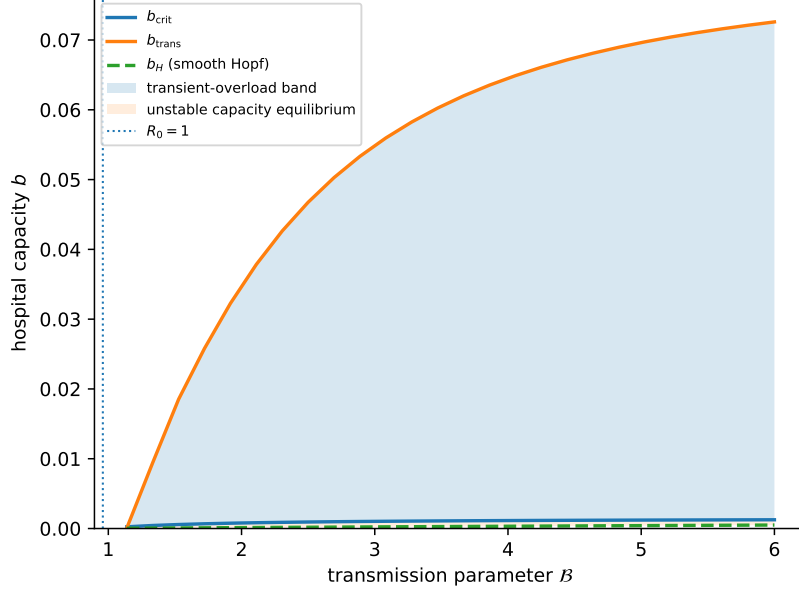


Figure 6: Two-parameter atlas in the $(\mathcal{B}, b)$ -plane. The curves show invasion ($R_0 = 1$), the smooth capacity-branch Hopf threshold b_H , the equilibrium capacity threshold b_{crit} , and the transient threshold b_{trans} . Shading separates the unstable-capacity-equilibrium band and the transient-overload band.

9 Empirical hospital-stock check: England, winter 2020/21

This section provides a deliberately limited real-data application. NHS England publishes daily management information on COVID-19 admissions and beds occupied by confirmed COVID-19 patients. We use the national series from 1 August 2020 to 6 April 2021: “total reported admissions to hospital and diagnoses in hospital” as a proxy A_t for entries into recorded COVID-19 occupancy, and occupied COVID-19 beds at 08:00 as the stock H_t (NHS England, 2021). Because reporting has a strong day-of-week pattern, both series are replaced by seven-day moving means. NHS England cautions that these rapidly collected situation-report data are management information, so the exercise is a structural check rather than a clinical length-of-stay estimate.

The hospital equation of the dimensional model implies the discrete balance

$$H_{t+1} - H_t = A_t - \gamma_H H_t + \varepsilon_t, \quad (33)$$

when the reporting interval is one day and γ_H is interpreted as an effective daily turnover rate. We estimate the single coefficient in (33), without an intercept, using data through 30 November 2020. The estimate is

$$\hat{\gamma}_H = 0.09841 \text{ day}^{-1}, \quad \hat{\gamma}_H^{-1} = 10.16 \text{ days}.$$

Holding this estimate fixed, we initialise the predicted stock at its observed value on 1 December 2020 and propagate (33) using only the subsequent observed inflow. Over the held-out period to 6 April 2021, the predicted occupied-bed series has $R^2 = 0.988$ and $\text{RMSE} = 1027$ beds (Figure 7). Thus the elementary stock–flow relation used for h tracks the held-out winter occupancy trajectory with one coefficient estimated before the wave peak. The high R^2 is not interpreted as identification of the full epidemic model or as a clinical length-of-stay estimate.

The reported inflow proxy reached 4134 patients on 12 January 2021, while occupied beds reached 34336 on 18 January, a six-day lag. The occupancy peak was 10.74 times the mean occupancy during the final fourteen days of the series. This is a direct observational example of a

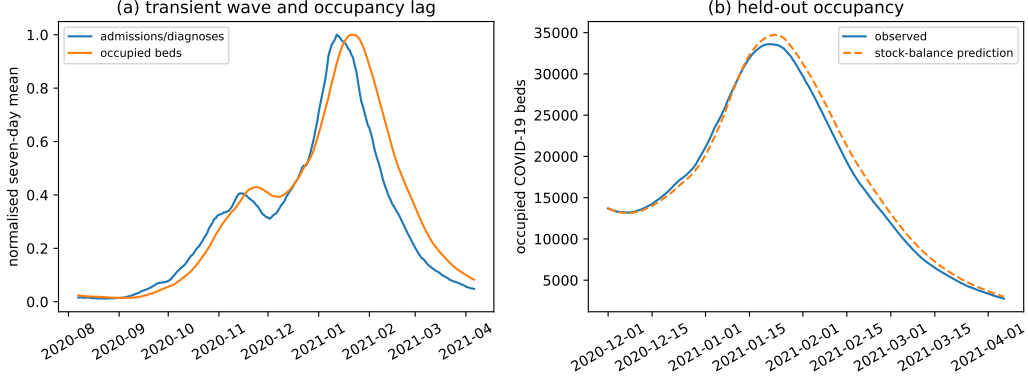


Figure 7: NHS England COVID-19 hospital data. (a) Seven-day mean admissions and occupied beds, each normalised by its own peak, showing the delay between the reported inflow proxy and occupancy. (b) Held-out occupancy from 1 December 2020 and the prediction of the one-parameter balance (33); the turnover coefficient was fitted only through 30 November 2020.

large transient hospital burden followed by a much lower post-wave load, which is the distinction motivating b_{trans} versus b_{crit} .

This application validates only the hospital-stock balance and the transient peak/lag mechanism. The severe waiting class Q was not reported in the NHS series, and effective total capacity, staffing constraints and unmet admission demand are not observed. Consequently a , c , B_0 , b_{crit} and b_{trans} cannot be identified from these two series alone. Estimating the full threshold structure requires linked admissions, occupancy, discharge, staffed-capacity and waiting-time data. This limitation is explicit rather than absorbed into an apparently well-fitting but non-identifiable calibration.

10 Operational interpretation

Let $N_* = \Lambda/\mu$, so a calibrated dimensionless capacity b corresponds to $B_0 = bN_*$ staffed admission places.

Routine requirement. $B_{\text{crit}} = b_{\text{crit}}N_*$ is the capacity that places the endemic equilibrium on the normal-care side of the switching surface.

Transient requirement and surge difference. For a specified outbreak initial condition, $B_{\text{trans}} = b_{\text{trans}}N_*$ avoids capacity-limited admission along the entire normal-care outbreak, and

$$B_{\text{surge}} = (b_{\text{trans}} - b_{\text{crit}})N_* \quad (34)$$

is the difference between the transient and endemic requirements. These become bed-planning quantities only after the model, the effective capacity definition and the time scale have been calibrated.

Certified lead time. The exact matrix-exponential quantity τ_c^+ is a conservative horizon before first capacity limitation. In dimensional time it is $t_c^+ = \tau_c^+/\kappa_I$.

Near-threshold marginal effect. Equation (26) implies a singular local derivative of overload duration with respect to capacity as $b \uparrow b_{\text{trans}}$. This is a local asymptotic statement, not a universal economic valuation of a bed.

Calibration caveat. The baseline value $b = 0.5 b_{\text{crit}}$ used to expose the oscillatory attractor is approximately 0.513 places per thousand units of N_* . It is deliberately an illustrative low-capacity value and should not be interpreted as representative of a particular country. The England application in Section 9 checks the hospital-stock equation but does not identify the latent queue or the capacity parameters. No policy recommendation should be based on the baseline threshold numbers without linked admissions, occupancy, discharge, staffed-capacity and waiting-list data.

11 Discussion

The model separates invasion, endemic admission adequacy, recurrent endemic dynamics and transient capacity limitation. The exact shared eigenpair reduces the one-sided stability calculation to quintics and exposes a genuine spectral crossing on the capacity-limited branch. The positive first Lyapunov coefficient shows that this Hopf bifurcation is subcritical: the local periodic orbit lies on the stable-equilibrium side and is unstable.

The global continuation resolves the relationship between that local orbit and the finite-amplitude attractor. The unstable Hopf branch grows, intersects the switching surface and meets an attracting piecewise-smooth branch at the lower fold b_{LPC}^- . The attracting branch persists through b_H and b_{crit} and terminates at the upper fold b_{LPC}^+ . Consequently, stable equilibria coexist with a stable periodic attractor on

$$b_{\text{LPC}}^- < b < b_H \quad \text{and} \quad b_{\text{crit}} < b < b_{\text{LPC}}^+.$$

This coexistence is dynamically more consequential than the local Hopf point alone: capacity changes within either interval need not determine the observed long-time state without information about initial conditions and perturbations. The fold locations are, however, numerical single-shooting results. Multiple parameterisations, small periodicity residuals and Floquet crossings support the branch topology, while an independent implementation reproduces the Hopf coefficient and a representative attracting cycle. Independent collocation continuation and a full basin computation remain desirable checks.

The boundary-equilibrium calculation and the periodic-orbit continuation answer different questions. The positive determinant product classifies the baseline transition at b_{crit} locally as persistence; it neither excludes remote attractors nor contradicts the bistability created by folds of periodic orbits. Likewise, the two-parameter atlas contains the local Hopf, b_{crit} and b_{trans} curves, but not two-parameter continuations of the fold-of-cycle curves. It should therefore be read as a threshold and local-stability atlas, not as a complete global bifurcation map.

The transient threshold has separate dynamical content. When the normal-care admission pressure has an isolated non-degenerate maximum, $b = b_{\text{trans}}$ is a quadratic grazing event. This yields the square-root overload-duration law and the $3/2$ admission-deficit law. The trajectory estimate is an order statement, not a universal asymptotic equality. The Metzler comparison result is global and rigorous, but the usefulness of its hitting-time bound decreases as susceptible depletion becomes substantial.

The severe waiting class is essential throughout. Occupancy can remain below the stock limit while admission is throughput-constrained and the queue grows; $\mathcal{P} = h + (a/c)q$ combines these effects. The NHS England analysis supports the stock-flow and peak-lag mechanism but does not observe the latent queue, unmet demand, staffed effective capacity or the admission-throughput parameters. It is therefore a structural illustration rather than a calibration of b_{crit} or b_{trans} .

Limitations and next steps. The system assumes homogeneous mixing, constant effective capacity, fixed progression rates and one aggregate hospital. The sensitivity analysis varies dimensionless groups independently rather than perturbing a fully constrained dimensional parameterisation. The global cycle continuation has been performed only for the baseline transmission parameter, and the basin geometry has not been mapped. The most valuable

extensions are independent collocation continuation of both folds, two-parameter continuation of $LPC^\pm$, and calibration to linked admissions, occupancy, discharge, staffed-capacity and waiting-time data across multiple regions.

12 Conclusion

A continuous piecewise-smooth hospital-overload model can exhibit four distinct threshold mechanisms. The basic reproduction number governs invasion; b_{crit} locates the endemic equilibrium relative to the admission switch; a subcritical Hopf point and two folds of periodic orbits organise recurrent capacity-limited dynamics; and b_{trans} governs quadratic grazing of the switching surface by the transient outbreak.

At the baseline parameters, numerical continuation connects the unstable local Hopf branch to an attracting piecewise-smooth periodic branch at $b_{\text{LPC}}^- \approx 8.6241 \times 10^{-5}$ and finds a second fold at $b_{\text{LPC}}^+ \approx 2.09249 \times 10^{-3}$. This creates two intervals in which a stable equilibrium and a stable periodic attractor coexist. The local Hopf coefficient and a representative attracting cycle are independently reproduced; the fold locations remain numerical single-shooting estimates. Under a non-degenerate transient maximum, overload duration scales as $(b_{\text{trans}} - b)^{1/2}$ and cumulative admission deficit as $(b_{\text{trans}} - b)^{3/2}$, while a Metzler comparison gives a rigorous lower bound on first switching time. The analysis therefore shows that steady-state bed adequacy is insufficient: hospital safety depends on transient admission pressure, nonlinear attractor selection and available surge capacity.

A Dimensional model and nondimensionalisation

The dimensional model, with S, E, I, Q, H, R, D and total living population $N = S + E + I + Q + H + R$, is

$$\begin{aligned} \dot{S} &= \Lambda - \lambda S - \mu S, & \dot{E} &= \lambda S - (\sigma + \mu)E, \\ \dot{I} &= \sigma E - (p\alpha + \delta + \mu + d_I)I, & \dot{Q} &= p\alpha I - \mathcal{A}(Q, B_0 - H) - (\gamma_Q + d_Q + \mu)Q, \\ \dot{H} &= \mathcal{A}(Q, B_0 - H) - (\gamma_H + d_H + \mu)H, & \dot{R} &= \delta I + \gamma_Q Q + \gamma_H H - \mu R, \\ \dot{D} &= d_I I + d_Q Q + d_H H, & \lambda &= \beta \frac{I + \theta_Q Q}{N}, \end{aligned} \quad (35)$$

with $\mathcal{A}(Q, B_0 - H) = \min\{\nu Q, \kappa(B_0 - H)_+\}$. Setting $N_* = \Lambda/\mu$, $\kappa_I = p\alpha + \delta + \mu + d_I$ and $\tau = \kappa_I t$, the dimensionless variables are $s = S/N_*$, $e = E/N_*$, $i = I/N_*$, $q = Q/N_*$, $h = H/N_*$, $r = R/N_*$, $z = D/N_*$, $b = B_0/N_*$, and the groups are

$$\mathcal{B} = \frac{\beta}{\kappa_I}, \quad m = \frac{\mu}{\kappa_I}, \quad \eta = \frac{\sigma}{\kappa_I}, \quad \pi = \frac{p\alpha}{\kappa_I}, \quad a = \frac{\nu}{\kappa_I}, \quad c = \frac{\kappa}{\kappa_I}, \quad (36)$$

$$\chi_Q = \frac{\gamma_Q + d_Q + \mu}{\kappa_I}, \quad \chi_H = \frac{\gamma_H + d_H + \mu}{\kappa_I}, \quad \rho_\bullet = \frac{\gamma_\bullet}{\kappa_I}, \quad \Delta_\bullet = \frac{d_\bullet}{\kappa_I}, \quad (37)$$

with $\rho_I = \delta/\kappa_I$ and the identity $\pi + \rho_I + m + \Delta_I = 1$. Substitution yields (4)–(9). No further restatement of the model occurs anywhere in the paper.

B One-sided Jacobians

With the incidence derivatives $\hat{T}_j$ defined in (19), the one-sided Jacobians of (4)–(9) are

$$J_N = \begin{pmatrix} -m - \hat{T}_s & -\hat{T}_e & -\hat{T}_i & -\hat{T}_q & -\hat{T}_h & -\hat{T}_r \\ \hat{T}_s & \hat{T}_e - (\eta + m) & \hat{T}_i & \hat{T}_q & \hat{T}_h & \hat{T}_r \\ 0 & \eta & -1 & 0 & 0 & 0 \\ 0 & 0 & \pi & -(a + \chi_Q) & 0 & 0 \\ 0 & 0 & 0 & a & -\chi_H & 0 \\ 0 & 0 & \rho_I & \rho_Q & \rho_H & -m \end{pmatrix}, \quad (38)$$

$$J_C = \begin{pmatrix} -m - \hat{T}_s & -\hat{T}_e & -\hat{T}_i & -\hat{T}_q & -\hat{T}_h & -\hat{T}_r \\ \hat{T}_s & \hat{T}_e - (\eta + m) & \hat{T}_i & \hat{T}_q & \hat{T}_h & \hat{T}_r \\ 0 & \eta & -1 & 0 & 0 & 0 \\ 0 & 0 & \pi & -\chi_Q & c & 0 \\ 0 & 0 & 0 & 0 & -(c + \chi_H) & 0 \\ 0 & 0 & \rho_I & \rho_Q & \rho_H & -m \end{pmatrix}. \quad (39)$$

The two matrices differ only in the q - and h -rows. By Theorem 6.1, both possess the factor $(\lambda + m)$.

For reproducible evaluation of Proposition 6.3, write

$$P_\bullet(\lambda) = \lambda^6 + p_1\lambda^5 + p_2\lambda^4 + p_3\lambda^3 + p_4\lambda^2 + p_5\lambda + p_6.$$

The full coefficients are obtained without symbolic expansion by the Faddeev–LeVerrier recursion

$$p_0 = 1, \quad p_k = -\frac{1}{k} \sum_{j=1}^k p_{k-j} \operatorname{tr}(J_\bullet^j), \quad k = 1, \dots, 6. \quad (40)$$

Synthetic division by the exact factor $(\lambda + m)$ then gives

$$c_1 = p_1 - m, \quad c_2 = p_2 - mc_1, \quad c_3 = p_3 - mc_2, \quad c_4 = p_4 - mc_3, \quad c_5 = p_5 - mc_4, \quad (41)$$

with the consistency check $p_6 = mc_5$. Equations (40)–(41), together with the displayed Jacobians, are explicit parameter-and-equilibrium formulas for all coefficients used in the Routh–Hurwitz test.

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Data availability. The processed NHS England hospital-activity data used in the empirical illustration are included in the archived reproducibility materials under DOI 10.5281/zenodo.21807502. The underlying aggregate hospital-activity data are publicly available from NHS England. No patient-level or personally identifiable data were used.

Code availability. The Python code, processed data, numerical continuation outputs, and figure-generation scripts supporting this study are publicly available on GitHub and are permanently archived in Zenodo, version 1.0.0, under DOI 10.5281/zenodo.21807502. The archive includes instructions and dependency information for reproducing the principal analytical and numerical results.

Ethics approval and consent. Not applicable; the study uses a deterministic mathematical model and publicly available aggregate hospital statistics.

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