

Online Resource 1

Supplementary numerical validation and reproducibility record

For “Hopf bifurcation, bistability and transient grazing in a piecewise-smooth
hospital-overload model”

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Permanent reproducibility record. The code, processed data, continuation outputs and figure-generation scripts are archived at Zenodo: [10.5281/zenodo.21807502](https://zenodo.org/record/21807502). The development repository is github.com/Manirmaths/hospital-overload-bifurcations.

1 Purpose and evidential status

This document records the computational evidence supporting the nonlinear-dynamics claims in the accompanying manuscript. Its purpose is not to duplicate the analytical development, but to state precisely which results are analytical, which were reproduced independently, and which rely on the primary numerical continuation workflow.

Table 1: Status of the principal claims. “Independent” means that a separate implementation using different differentiation and integration routes reproduced the reported quantity.

Result	Status	Evidence used in the manuscript
R_0 , b_{crit} , exact shared eigenpair and local characteristic reduction	Analytical, with numerical evaluation	Closed-form formulae and direct eigenvalue checks
Hopf point, critical frequency, transversality and first Lyapunov coefficient	Independently reproduced	Primary adaptive implementation and independent SymPy/mpmath calculation
Representative attracting switching cycle at $b = 0.5b_{\text{crit}}$	Independently reproduced	Primary adaptive integration and independent fixed-step RK4 integration
Lower and upper folds of periodic orbits; global branch topology	Primary numerical continuation	Single shooting, overlapping continuation parameterisations, residual checks and Floquet multipliers
Grazing constants and switching-time bound	Analytical asymptotics plus numerical verification	Compensated limits and exact matrix-exponential comparison
NHS England hospital-stock illustration	Empirical consistency check	Public aggregate data; not a calibration of the full model

2 Independent local Hopf calculation

For the baseline dimensionless parameter set,

$$R_0 = 3.1286618233, \quad b_{\text{crit}} = 1.0256571259 \times 10^{-3},$$

and the capacity-limited equilibrium loses stability at

$$b_H = 2.3215520815 \times 10^{-4} = 0.2263477748 b_{\text{crit}}.$$

At $b = b_H$, the critical pair is

$$\lambda_{1,2} = \pm 0.0362952881 i,$$

and the crossing speed is

$$\left. \frac{d}{db} \operatorname{Re} \lambda(b) \right|_{b=b_H} = 20.7516074 > 0.$$

Using the standard Kuznetsov multilinear-form convention, the first Lyapunov coefficient is

$$\ell_1 = 42.4417863 > 0,$$

so the local Hopf bifurcation is subcritical under that convention.

The primary calculation uses the model Jacobian together with automatic differentiation of the quadratic and cubic terms. The independent calculation was written separately, generated the derivative tensors with SymPy, evaluated the multilinear expression with 80-digit `mpmath` arithmetic, and used no functions imported from the primary verification script.

Table 2: Agreement between the primary and independent Hopf calculations.

Quantity	Primary	Independent	Absolute difference
b_H	$2.3215520815078 \times 10^{-4}$	$2.3215520815068 \times 10^{-4}$	9.9×10^{-17}
ω_H	0.0362952881	0.0362952881	$< 10^{-10}$
$d \operatorname{Re} \lambda / db$	20.7516064	20.7516074	1.0×10^{-6}
ℓ_1	42.4417862817	42.4417862821	3.8×10^{-10}

3 Periodic-orbit computation and global branch

Periodic solutions were formulated as a shooting problem. If $\Phi(T; y_0, b)$ denotes the flow after one period, the unknown initial state y_0 , period T and continuation parameter satisfy

$$\Phi(T; y_0, b) - y_0 = 0,$$

together with a phase condition. The variational equations produce the monodromy matrix

$$M = D_{y_0} \Phi(T; y_0, b),$$

whose eigenvalues are the Floquet multipliers. The neutral multiplier at one is excluded when classifying cycle stability.

The branch was traversed using three overlapping parameterisations: fixed hospital capacity b , an equilibrium-relative amplitude, and prescribed $q_{\max}$. Agreement on overlapping sections is an internal continuation check, but it is not equivalent to an independent collocation computation.

Near the lower fold, archived points on opposite sides of the stability change have largest nontrivial multiplier moduli 0.9976 and 1.0060, with shooting residual norms below 8×10^{-13} at the closest unstable-side point. Near the upper fold, neighbouring points have multiplier moduli 0.9940 and 1.0021, with residual norms below 5×10^{-11} at the reported turning point. These data support, but do not analytically prove, the two fold locations

$$b_{\text{LPC}}^- \approx 8.6241 \times 10^{-5}, \quad b_{\text{LPC}}^+ \approx 2.09249 \times 10^{-3}.$$

Table 3: Numerical landmarks from the archived continuation data. The periodic-orbit amplitude is $\max q - \min q$.

Event	b	b/b_{crit}	Period	q amplitude
Lower fold LPC^-	8.6241×10^{-5}	0.084083	≈ 215.8	$\approx 9.70 \times 10^{-3}$
Hopf point H	2.32155×10^{-4}	0.226348	≈ 173.1	0
Representative attracting cycle	5.12829×10^{-4}	0.5	247.6	1.3018×10^{-2}
Upper fold LPC^+	2.09249×10^{-3}	2.040146	≈ 201.2	$\approx 4.44 \times 10^{-3}$

Consequently, the manuscript reports the fold values as numerical single-shooting approximations. The continuation supports two intervals of coexistence between a stable equilibrium and a stable periodic orbit:

$$b_{\text{LPC}}^- < b < b_H, \quad b_{\text{crit}} < b < b_{\text{LPC}}^+.$$

This establishes bistability within the computed branch structure, but the complete basin boundary was not calculated. The unstable periodic branch is therefore described as a candidate component of that boundary, not as a proven complete separator.

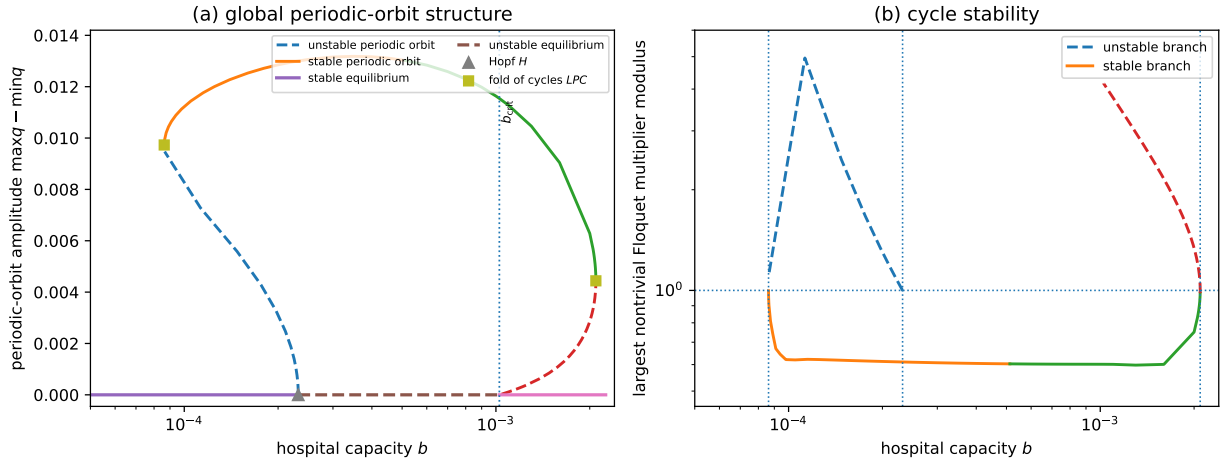


Figure 1: Primary single-shooting continuation. Panel (a) shows the equilibrium and periodic-orbit branches. Panel (b) shows the largest nontrivial Floquet-multiplier modulus. The local unstable Hopf branch passes through a lower fold and becomes attracting; the attracting branch persists across b_{crit} and terminates at an upper fold. Fold locations are numerical approximations.

4 Independent attracting-cycle check

At

$$b = 0.5b_{\text{crit}} = 5.1282856295 \times 10^{-4},$$

the primary shooting solution has period 247.6015, largest nontrivial Floquet-multiplier modulus 0.6031, two switching-surface crossings per period, and capacity-limited fraction 0.2947.

The independent implementation used classical fixed-step RK4 with $\Delta\tau = 0.05$ up to $\tau = 12000$. It reproduced a period of 247.6000, two crossings per period, capacity-limited fraction 0.2944, and

$$q_{\min} = 4.224 \times 10^{-7}, \quad q_{\max} = 1.30182 \times 10^{-2}.$$

The close agreement confirms the existence and principal characteristics of this representative attracting cycle without reusing the primary integrator. It does not independently reproduce the full continuation branch or either fold.

5 Grazing and first-switching-time checks

For the baseline normal-care trajectory, the admission-pressure maximum occurs at

$$\tau^* = 30.8325323, \quad k = -P''(\tau^*) = 2.5192060 \times 10^{-3}.$$

The predicted compensated constants are

$$2\sqrt{\frac{2}{k}} = 56.3525, \quad \frac{4\sqrt{2}c}{3\sqrt{k}} = 75.1367.$$

The archived computations approach these values as $b_{\text{trans}} - b \downarrow 0$, supporting the square-root overload-duration law and the three-halves cumulative-deficit law. The trajectory-deviation estimate is stated as $O((b_{\text{trans}} - b)^{3/2})$, not as an asymptotic equality.

The exact matrix-exponential comparison gives a rigorous global lower bound on the first switching time. Across the range plotted in the manuscript, the relative gap is approximately 0.2% near $1.2b_{\text{crit}}$ and increases to approximately 11% near $0.95b_{\text{trans}}$. No claim of uniform four-percent accuracy is made.

6 Empirical stock-flow check

The processed NHS England series covers 1 August 2020 to 6 April 2021. Fitting the aggregate stock-flow relation on data through 30 November 2020 gives an effective turnover rate of 0.09841 day^{-1} , corresponding to an effective turnover time of 10.16 days. On the held-out period from 1 December 2020 to 6 April 2021, the occupancy-level prediction gives $R^2 = 0.98835$ and $\text{RMSE} = 1026.6$ beds.

This is a structural consistency check for the hospital stock equation. It does not identify the latent severe waiting class, effective staffed capacity, the switching coefficients, b_{crit} , or b_{trans} . The manuscript therefore does not describe it as a full epidemiological or operational calibration.

7 Reproducibility workflow

The Zenodo record contains the exact archived snapshot used for the manuscript. The main entry points are:

- `verify_revision.py`: equilibrium, stability, Hopf, grazing, sensitivity and switching-time checks;
- `independent_validation.py`: independent Hopf and representative-cycle reproduction;
- `fixed_b_cycle.py`, `amplitude_custom.py`, `qmax_continuation.py`: overlapping periodic-orbit continuation parameterisations;
- `make_nd_figures.py`: regeneration of the global branch and two-parameter atlas from archived branch data;
- `england_empirical_analysis.py`: empirical stock-flow calculation;
- JSON files containing branch points, shooting residuals, Floquet multipliers and summary quantities.

Recommended environment setup and the principal checks are:

```
python -m pip install -r requirements.txt
```

```
python verify_revision.py
python independent_validation.py
python england_empirical_analysis.py
python make_nd_figures.py
```

Regenerating the figures from archived branch data is substantially faster than recomputing the entire continuation branch.

8 Interpretive limits

The following limitations are part of the claims, not qualifications added after the analysis:

1. The Feigin classification is local to the boundary equilibrium and does not exclude remote periodic attractors.
2. The first Lyapunov coefficient determines the local Hopf criticality under the stated convention; it does not by itself explain the remote attracting cycle.
3. The lower and upper fold locations are primary single-shooting approximations. They have not been independently collocated.
4. Bistability is supported by coexistence of stable equilibria and stable periodic solutions, but the full basin geometry has not been mapped.
5. The two-parameter atlas contains $R_0 = 1$, b_H , b_{crit} and b_{trans} curves, but not two-parameter continuations of b_{LPC}^- and b_{LPC}^+ .
6. The empirical analysis uses aggregate public hospital data and is not a calibration of unobserved waiting demand or staffed capacity.

Software citation

Ibrahim, M., Alhassan B., Ibrahim S., Ibrahim S. (2026) *Hospital-Overload Bifurcations: Reproducible Code and Data for a Capacity-Limited Piecewise-Smooth Epidemic Model*. Zenodo. <https://doi.org/10.5281/zenodo.21807502>.