

Supplementary Information to ‘Spatiotemporal mode-locking and photonic flywheel in multimode microresonators’

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I. SUMMARY OF ADVANTAGES OF GRIN-MMF FP MICRORESONATORS

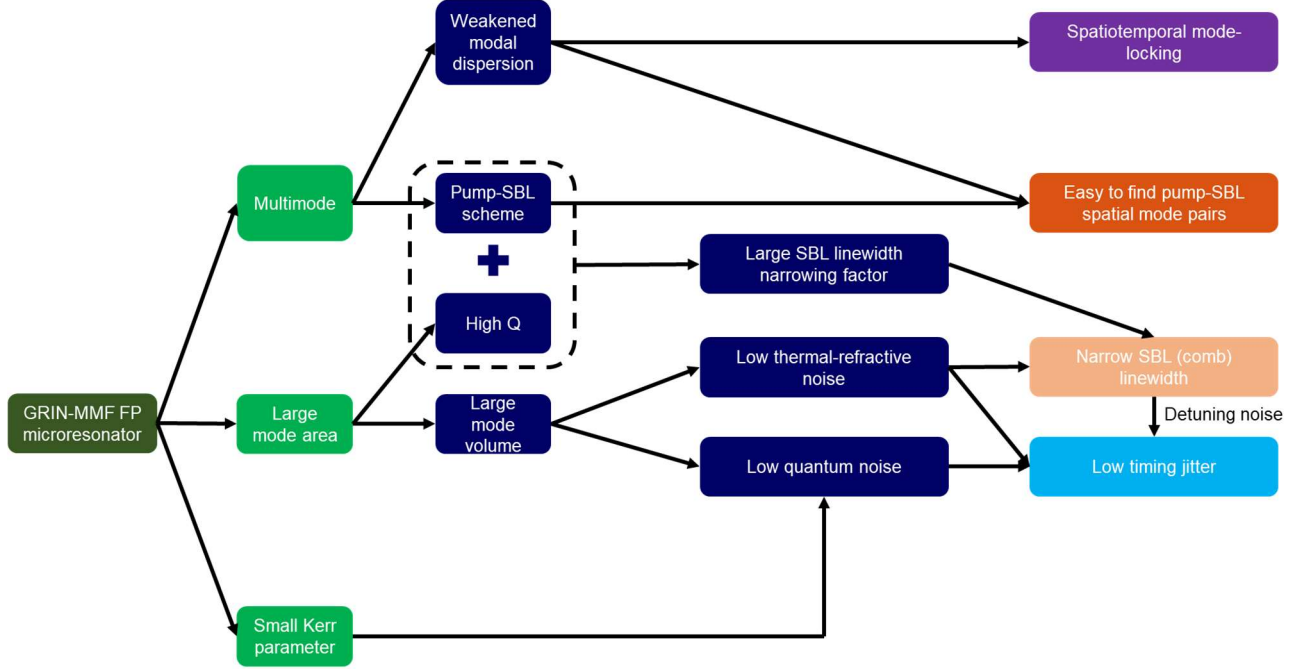


Fig. S1. Summary of advantages of GRIN-MMF FP microresonators.

II. MODE ANALYSIS OF GRADED-INDEX FIBER

We also measure the Q factors for the other three modes in Fig. 2b in the main text. As shown in Fig. S2, the Q factors are larger than 310 M for these modes, verifying the large mode area induced low diffraction loss.

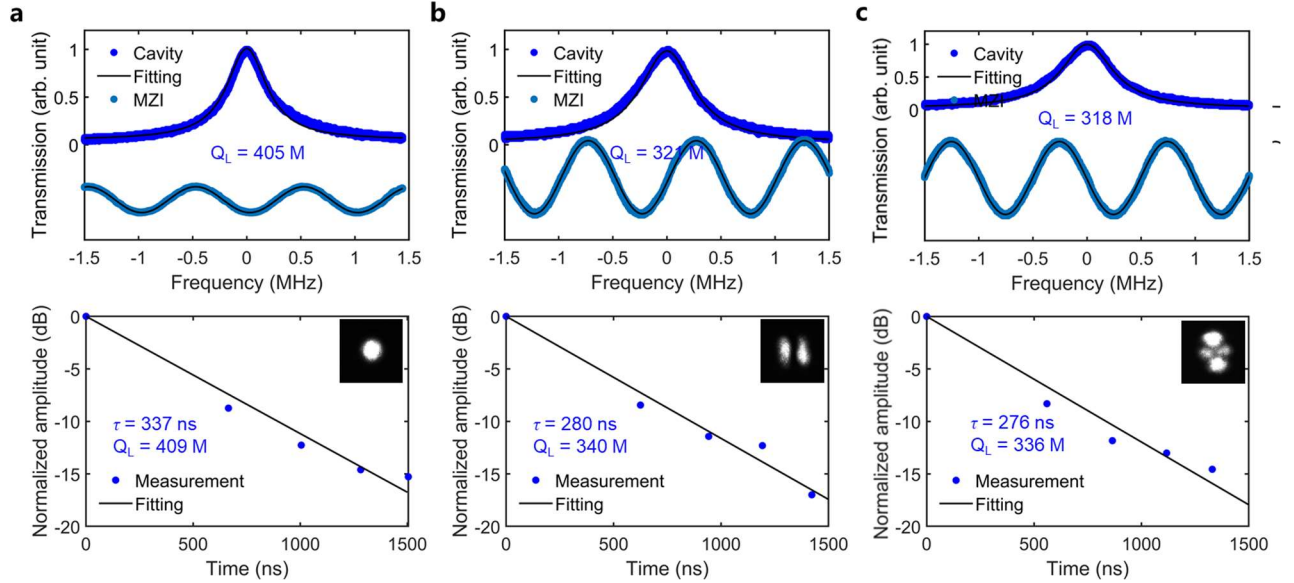


Fig. S2. Q factor measurement for the other three modes (mode profiles in the inset). Top row: transmission spectrum of the ultrahigh- Q mode and MZI (blue lines) at 1550 nm. Bottom row: ring-down traces. (“M” represents “million”)

We numerically calculated the linearly polarized (LP) mode through the code in Ref. [1]. In fact, there are hundreds of modes in GRIN-MMFs and we only show the first ten modes in Fig. S3, which are classified into four groups in the order of effective refractive index. The experimental mode profiles in Fig. 2b can be viewed as superpositions of different LP eigenmodes, especially for the second mode and fifth mode. The pump mode (SBL mode) has profiles similar to the LP₁₁ (LP₂₁) mode, but in fact they are not 100% identical.

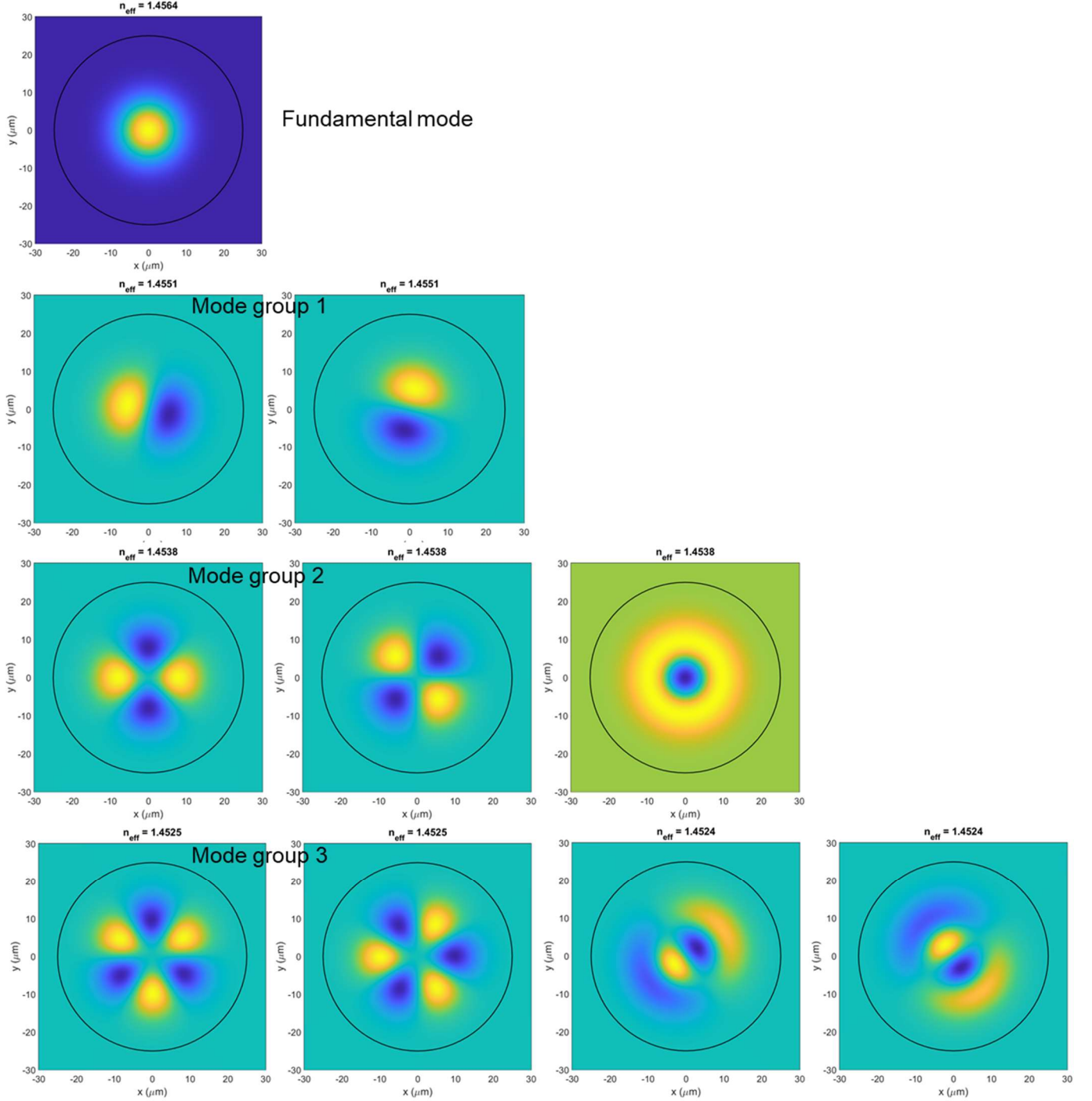


Fig. S3. LP eigenmodes of GRIN-MMFs. The black circular lines indicate the core edge of GRIN-MMF.

The simulated GVD for the ten modes in Fig. S3 is shown below in Fig. S4, indicating four mode groups. The simulated GVD is around $-28 \text{ fs}^2/\text{mm}$ for all the ten modes.

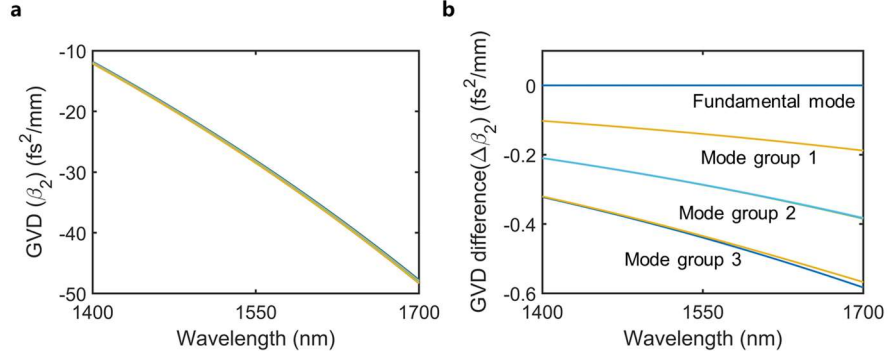


Fig. S4. Simulated GVD (a) and GVD difference (b) with respect to the fundamental mode for the modes in Fig. S3.

III. OTHER COMB STATES

We obtain other comb states for pump soliton and SBL soliton, as shown in Fig. S5.

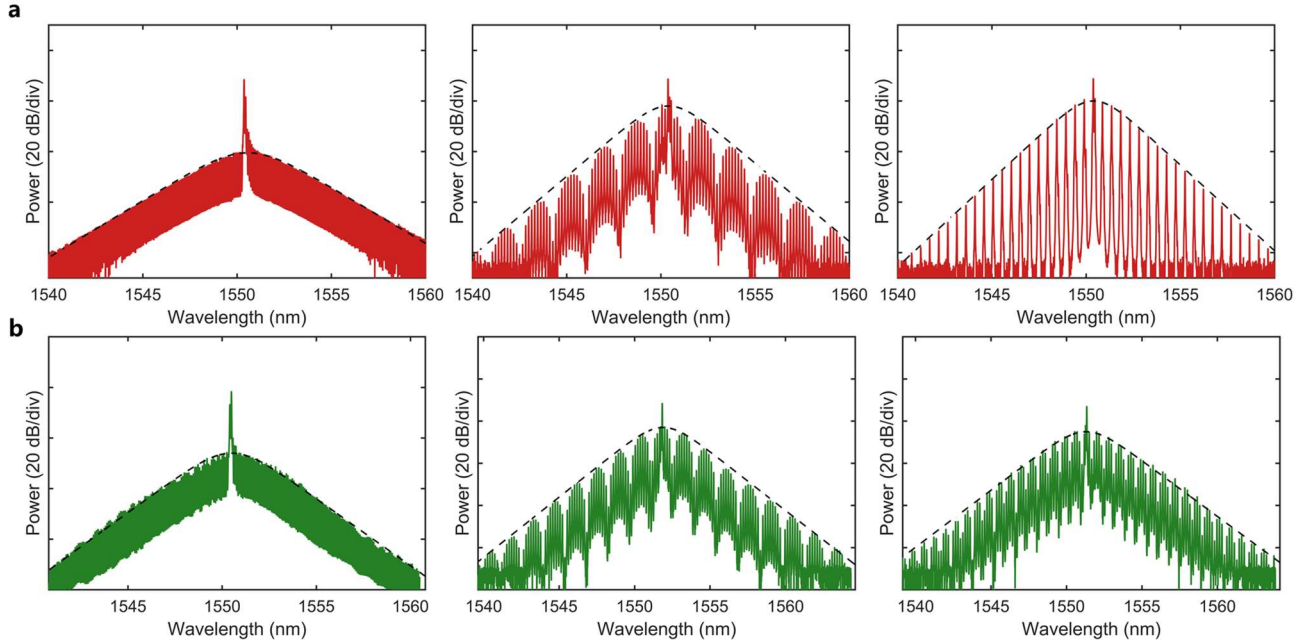


Fig. S5. Other comb states for pump soliton (a) and SBL soliton (b). Dashed lines represent the fitted spectral envelope.

IV. SOLITONS WITH OTHER GRIN-MMF FP MICRORESONATORS

We also realize the selective excitation of solitons in another high-Q GRIN-MMF FP microresonator with FSR of 10.087 GHz, which is 8 MHz larger than that the one in the main text. Despite the same mode profiles (pump mode and SBL mode) recorded by the camera, we find the SBS frequency shift of 10.3163 GHz (Fig. S6) is smaller than the measured value of 10.343 GHz in the main text, owing to the SBS process through another acoustic mode.

Interestingly, the simulation [2] shows the spectral distance between adjacent acoustic resonances is ~ 27 MHz, which is perfectly with our experimental results. Besides pump-SBL mode pairs with different acoustic modes, we expect soliton generation from pump-SBL mode pairs with different optical mode profiles, which is quite feasible in our GRIN-MMF FP microresonators. With mitigated GVM and broadened intermodal SBS gain spectrum between a great variety of high-Q spatial modes and different acoustic modes, it is easy to find pump-SBL mode pairs with high-Qs and apply the scheme presented here for feedback-free soliton microcombs generation, which will definitely lower the fabrication requirement for cavity length.

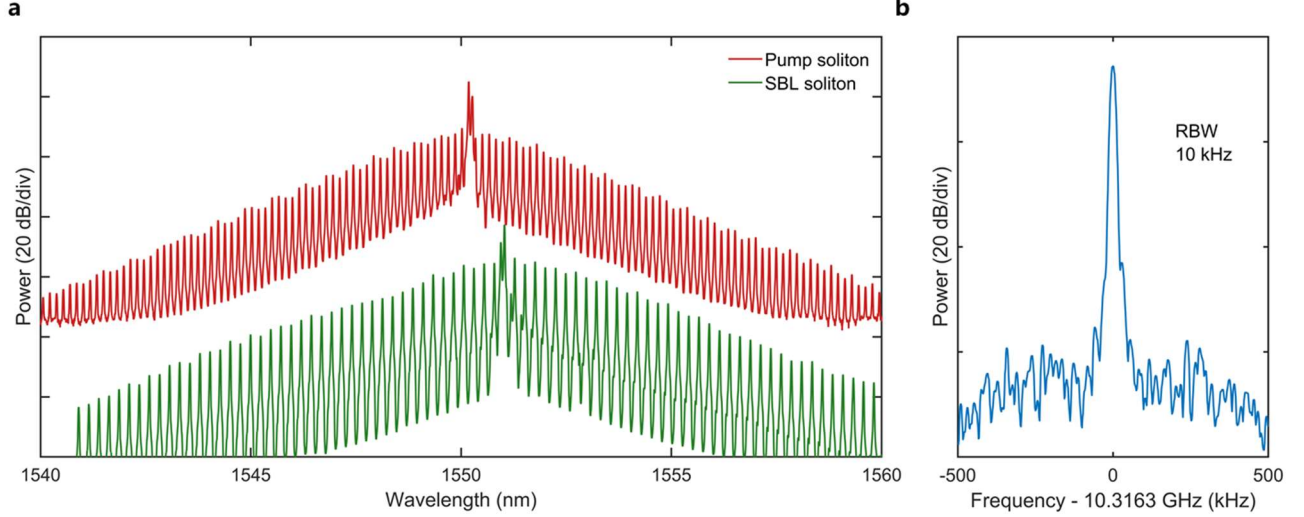


Fig. S6. Pump soliton and SBL soliton generated in another GRIN-MMF FP microresonator with FSR of 10.087 GHz. (a) spectra; (b) Measured SBS frequency shift when pump soliton is generated.

V. LINEWIDTH MEASUREMENT

Our frequency discriminator is first tested by a known reference laser from NIST at Boulder, as shown in Fig. S7. The measured fundamental linewidth is 0.3 Hz and verify the reliability of the measured comb linewidths at sub-hertz level. The noise floor tested by our SBL laser indicates the minimum fundamental linewidth we can measure is about 3 mHz, which is limited by both the test laser source [3] and the shot noise of the BPD.

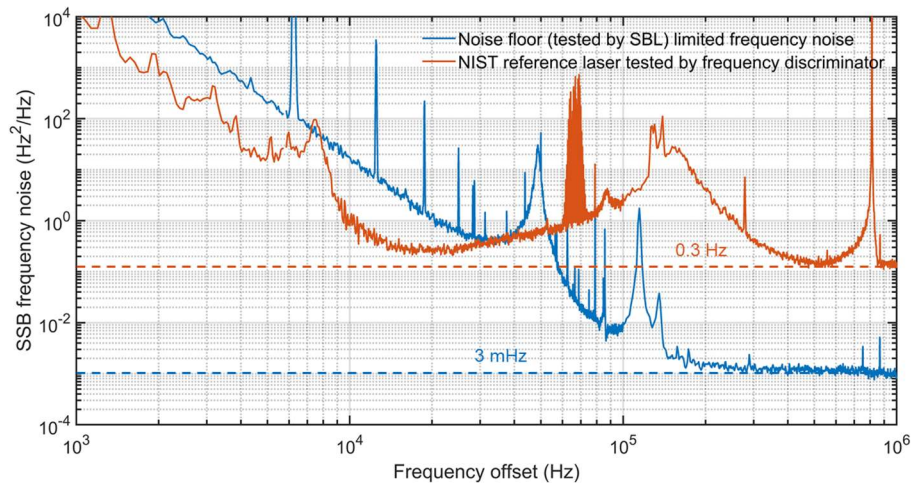


Fig. S7. Optical frequency discriminator characterization with NIST reference laser.

VI. SBL SOLITON REPETITION RATE PHASE NOISE ANALYSIS

A. Detuning effect of microresonator output on RIN

We will first study the detuning effect on relative intensity noise (RIN) (intracavity power fluctuations) for the output of our high-Q microresonator, which helps to later distinguish the origin of soliton repetition rate phase noise (PN), whether from the SBL RIN or soliton RIN. As shown in Fig. S8, we record the output RIN PSD of two modes (fundamental mode at low input power and pump mode at high input power) by tuning the ECDL into the resonance when SBS is below the threshold. The intracavity RIN is inherently attributed to the frequency noise of the resonant laser. In addition, the peak position on the Fourier frequency axis is related to both the microresonator linewidth and laser detuning, given by [4]

$$S_{RIN}(\Omega) \propto \frac{\eta^2 \Delta^2 \kappa^2 \left[(1-\eta)^2 \kappa^2 + \Omega^2 \right]}{\left[\Delta^2 + (\kappa/2)^2 \right] \left[(\Delta - \Omega)^2 + (\kappa/2)^2 \right] \left[(\Delta + \Omega)^2 + (\kappa/2)^2 \right]} S_\phi(\Omega), \quad (1)$$

where Δ is the laser detuning from the cavity resonance, $\kappa/2\pi$ is the cavity FWHM linewidth and Ω is the Fourier frequency. As shown for both modes with different input powers, clear RIN peak shifts with blue-detuning variation. Therefore, the RIN peak is a convenient indicator to monitor the laser detuning variation.

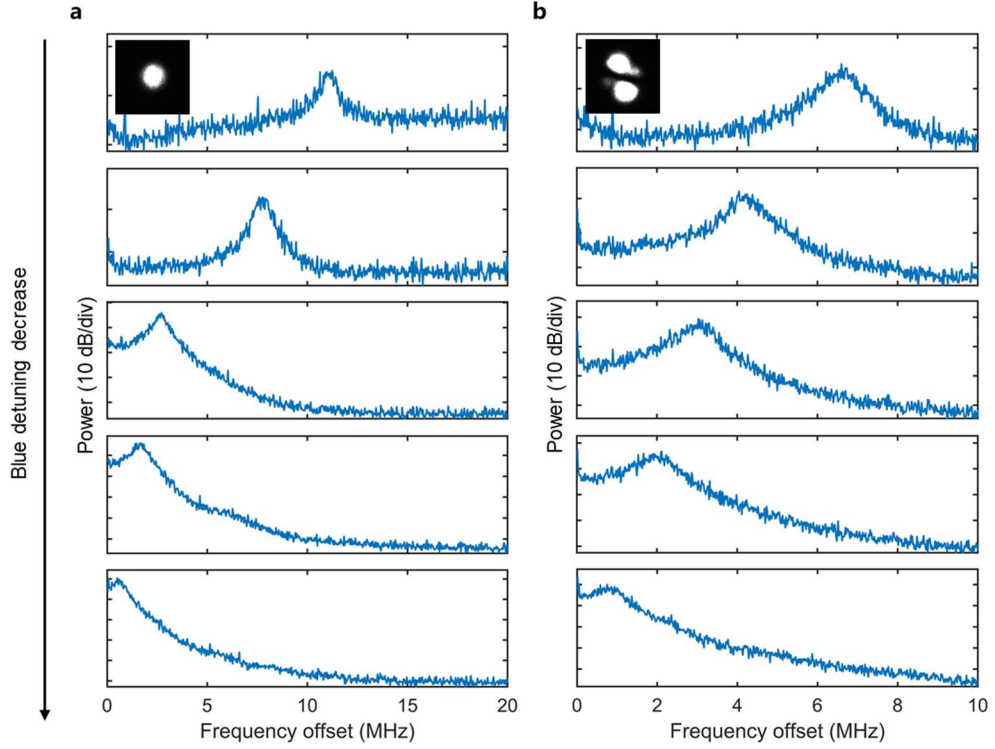


Fig. S8. Detuning effect of microresonator output on intensity noise. With decreased blue detuning, the peak of intensity noise shifts towards zero offset frequency for both the fundamental mode at low power (a) and pump mode at high power (b).

B. SBL RIN, soliton RIN and influence on timing jitter

In this section, we precisely measure the RINs of the pump, the SBL before and after SBL soliton generation and the SBL soliton RIN, based on which we determine the dominant factor influencing the SBL soliton repetition rate PN.

First, we measure the total RIN of pump plus generated SBL before SBL soliton generation (Fig. S9) with different SBL intensities. As the intensity of SBL increases, the total RIN decreases, indicating the SBL RIN is smaller than the pump RIN. When the intensity of SBL is 10 dB stronger than that of the pump, the total RIN, which is dominated by the SBL RIN, is ~ 10 dB smaller than the pump RIN before SBL generation. According to Ref. [5], we calculate the transduction coefficient from pump RIN to SBL RIN (inset of Fig. S9, details see Appendix), with transduction coefficient of -10 dB at low offset frequencies. The transduced SBL RIN from pump RIN is also overlaid in Fig. S9, which agrees well with the measured total RIN dominated by SBL RIN. Of note, the deviation at the low offset frequencies is attributed to the technical noise.

By fine tuning the wavelength of ECDL, we obtain the 2-FSR SBL microcombs in Fig. 2b (main text). In this case, SBL intensity is ~ 7 dB stronger than pump's intensity. By simultaneously monitoring the spectrum, the total RIN, the soliton RIN, the comb repetition rate PN and the comb fundamental linewidth, we find all the measured results strongly depends on the wavelength of ECDL. As shown in Fig. S10, when we tune the wavelength of ECDL towards red-detuned side, (i) the total RIN decreases by 4 dB (Fig. S10a), (ii) the soliton RIN decreases by 5 dB (Fig. S10b), (iii) comb repetition rate PN decreases by ~ 5 dB (Fig. S10c) and (iv) the SBL fundamental linewidth narrows from 6.1 Hz to 0.4 Hz thus decreases by 12 dB (Fig. S10d). Of note, the total RIN is measured by filtering the pump, SBL and some comb lines together with a 2-nm bandpass filter, leading to the dip at 1.2 MHz resulting from the interference between pump, SBL and comb lines. In this case, the intensity of SBL is ~ 7 dB larger than the pump's, which means the total RIN is dominated by SBL RIN. Interestingly, the total RIN peak shifts as shown by the blue arrow in Fig. S10a, indicating the reduced value of red detuning for SBL according to detuning effect discussed above in Section VI A. Therefore, SBL linewidth narrowing in Fig. S10d can be explained due to the smaller detuning induced better phase matching during SBS process [6].

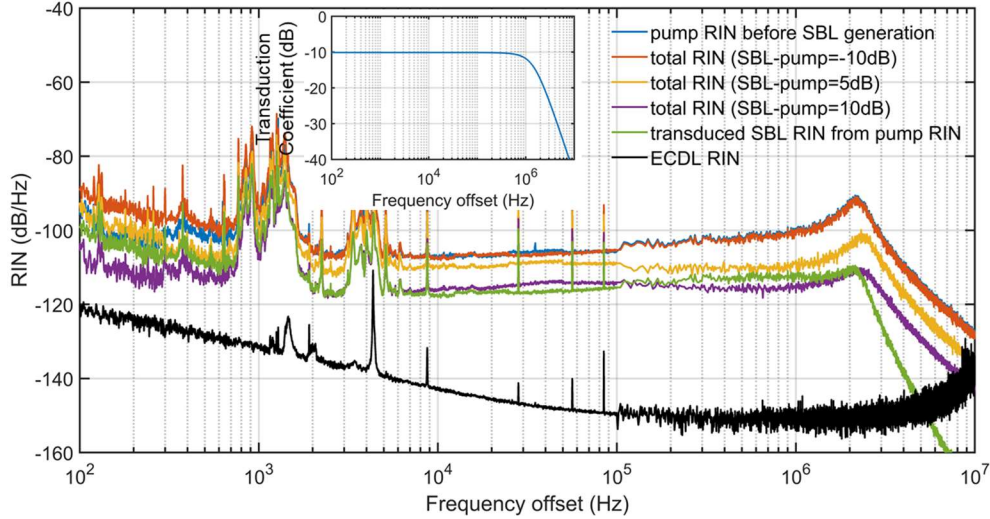


Fig. S9. RIN for pump before SBL generation, after SBL generation with different SBL intensities, transduction from pump, and the ECDL. Inset: transduction coefficient from pump RIN to SBL RIN.

Soliton RIN does not dominate the SBL soliton timing jitter. In Fig. S10b, the intensity of soliton RIN peak at 1.2 MHz sharply decreases by ~ 25 dB with decreased SBL laser detuning. As for the comb repetition rate PN in Fig. S10c, there is a small peak at 1.2 MHz at large SBL detuning while it disappears at small SBL detuning. We attribute the small peak to the soliton RIN transduced soliton repetition rate PN, indicated by the blue arrows in Fig. S10b and S10c. However, the soliton RIN is not the dominant origin for the soliton PN due to the inconsistent variation between 5 dB (Fig. S10c) for soliton PN and ~ 25 dB (Fig. S10b) for soliton RIN peak at 1.2 MHz. Furthermore, the relatively small comb bandwidth and thus weak self-steepening effect will ensure small transduction coefficient from soliton RIN to soliton PN.

Detuning noise from SBL does not dominate the SBL soliton timing jitter. When decreasing the SBL detuning, the SBL phase noise at high offset frequencies decreases by 12 dB (Fig. S10d) while the repetition rate PN only decreases by 5 dB (Fig. S10c). The inconsistent intensity changes suggest that the soliton repetition rate PN is not determined by the SBL phase noise.

SBL RIN dominates SBL soliton timing jitter. Considering the change of 5 dB (Fig. S10c). for repetition rate PN during the frequency tuning process, the possible reason mainly affecting the repetition rate PN is the SBL RIN. Here we calculate the theoretical transduction coefficient from soliton pump RIN to repetition rate PN. The power fluctuations of the soliton pump (here the SBL) will induce timing jitter of the repetition rate since the FSR depends on the Kerr shift related to the intra-cavity circulating optical power. Assuming a laser on resonance, the SPM induced FSR shift follows [7]

$$\begin{aligned} \frac{\delta D_1(f)}{2\pi} &= \alpha \delta P(f) \\ \alpha &= \frac{D_1}{2\pi} \frac{4\eta c n_2}{\kappa V_{eff} n_0^2}, \end{aligned} \quad (2)$$

where $\kappa/2\pi = 0.5$ MHz is the cavity energy decay rate, $\eta = \kappa_{ext}/\kappa = 0.45$ is the coupling impedance of the resonator (κ_{ext} is the coupling rate), $V_{eff} = 5 \times 10^{-12}$ m³ is the mode volume, $n_2 = 3.6 \times 10^{-20}$ m²/W is the Kerr nonlinear index and $n_0 = 1.4682$ is the refractive index. These values yield a conversion coefficient $\alpha = 5.8$ kHz/W. The relationship between SBL RIN $S_{RIN}(f)$ and the associated induced soliton phase noise $S_\phi(f)$ can be estimated by

$$S_\phi(f) = \left(\frac{\alpha}{f} P \right)^2 S_{RIN}(f). \quad (3)$$

The equivalent SBL power is estimated to be $P \approx 800$ mW. The transduced repetition rate PN is displayed in Fig. 3b (blue line) in the main text, remarkably matching the measured repetition rate PN over the almost whole range of 4 kHz–1 MHz. Some difference (<6 dB) for the low offset frequencies is due to the effect of the servo in the ARMI setup and the EDFA induced SBL RIN. Therefore, we claim that the SBL RIN is the dominant factor affecting soliton timing jitter.

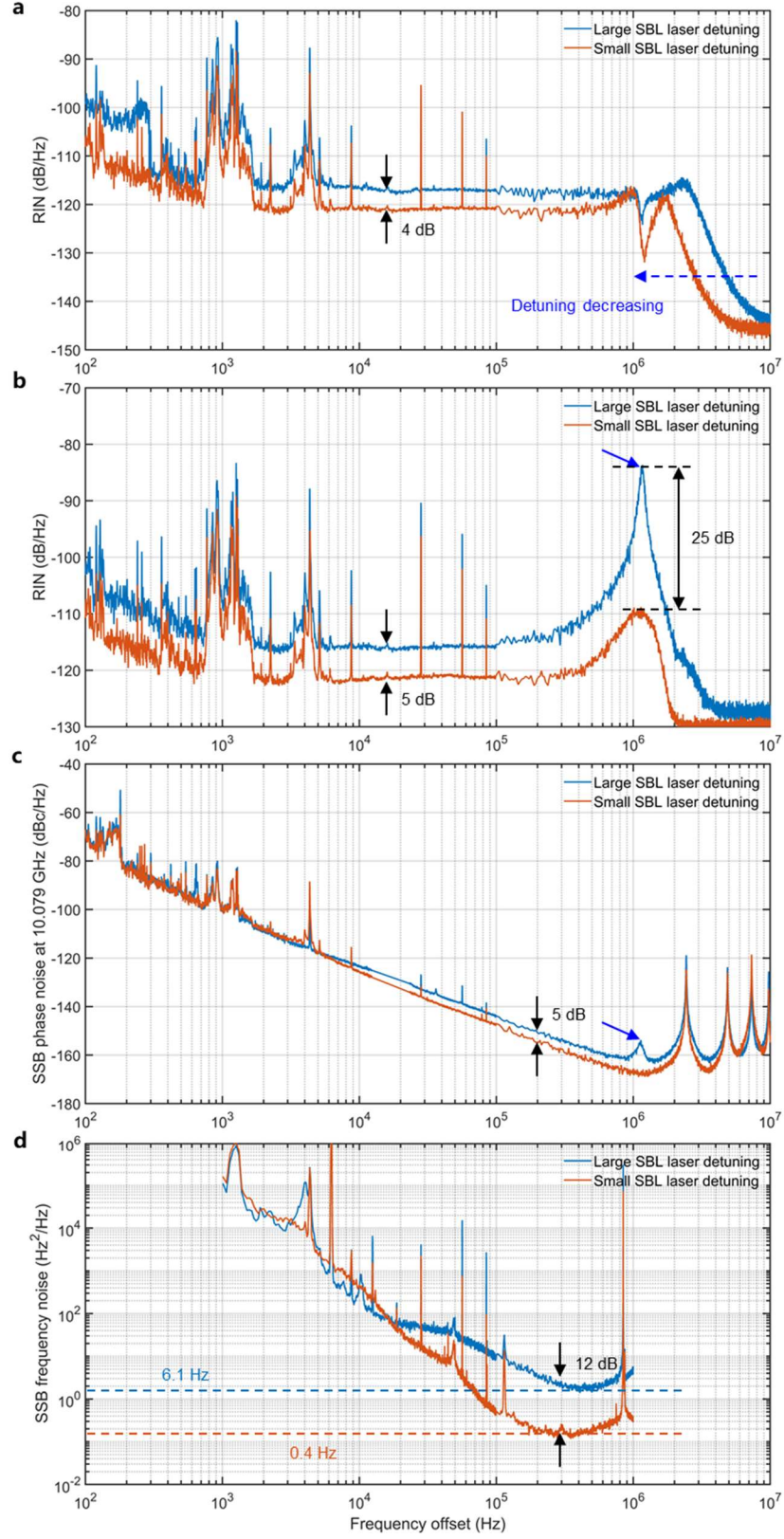


Fig. S10. Soliton performance improvement with decreased SBL laser detuning. (a) Total RIN; (b) Soliton RIN; (c) SSB phase noise of SBL soliton repetition rate; (d) SSB frequency noise of SBL.

C. Detuning noise from SBL

The detuning fluctuations will influence the stability of comb repetition rate through nonlinear effects, such as the Raman self-frequency shift and soliton recoil corresponding to dispersive wave emission [7, 8]. This PM2PM transduction process can be resulted from either soliton pump phase noise (here SBL phase noise) or cavity resonance phase noise.

However, we find that the comb repetition rate PN is not dominated by the soliton pump phase noise. There are two evidences. (i) *SBL soliton PN follows $1/f^2$ trend, while the SBL PN does not.* According to the inset of Fig. 3a in the main text, SBL frequency noise follows pump frequency noise in the offset frequencies below 10 kHz, which is also found in other on-chip SBS microlasers [9]. At high frequencies, the SBL improves from the pump noise by 25 dB. Since the pump phase noise deviates from $1/f^2$ trend (Fig. S11) by over 20 dB at low offset frequencies, the SBL phase noise should deviate from $1/f^2$ trend by over $(20+25)=45$ dB at low offset frequencies (both the pump and SBL phase noise follows $1/f^2$ trend at high offset frequencies). Assuming the transduction from SBL phase noise to the comb repetition rate phase, we can not obtain the experimental repetition rate PN roughly following $1/f^2$ trend at low offset frequencies. (ii) *The variation inconsistency between the SBL soliton PN and SBL phase noise (frequency noise).* According to the discussion in Section B, when decreasing the SBL detuning, the SBL phase noise at high offset frequencies decreases by 12 dB while the repetition rate PN only decreases by 5 dB. The inconsistent intensity changes suggest that the soliton repetition rate PN is not determined by the SBL phase noise.

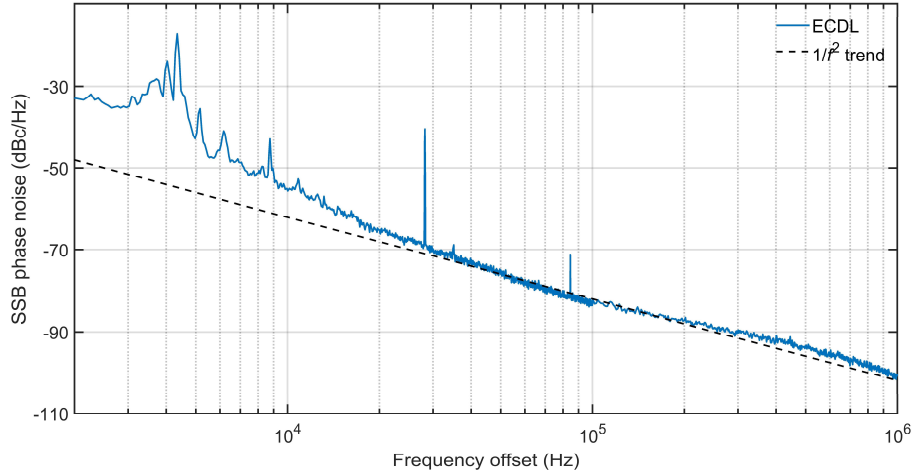


Fig. S11. SSB phase noise of ECDL by beating the ECDL with a referenced narrow-linewidth laser. The pump phase noises strongly deviate from $1/f^2$ trend at low offset frequencies.

In our GRIN-MMF FP microresonator, nonlinear effects induced timing jitter is weak, since (i) SBL soliton spectra do not exhibit obvious dispersion wave emission; (ii) SBL soliton pulse duration is ~ 1 ps which sharply reduces the Raman self-frequency shift.

We will discuss the cavity resonance fluctuations due to thermal noise in Section D.

D. Resonator thermal noise limit

There are three spontaneous processes can cause the phase fluctuations of the light propagating in fibers, the thermal expansion effect, thermo-optic effect and thermal mechanical effect. Usually, the thermo-optic effect is the dominant factor for the light phase fluctuations. The fundamental thermo-dynamic fluctuations within the optical mode volume of the microresonator result from the heat exchange between the microresonator and its ambient and can lead to refractive index change through thermo-optic effect, resulting in the thermo-refractive noise (TRN) for the resonant frequencies. In general, the thermodynamic fluctuations of temperature, whose variance is given by [10]

$$\langle \delta T^2 \rangle = \frac{k_B T^2}{\rho C V} \quad (4)$$

where T is the temperature of the heat bath, k_B the Boltzmann constant, ρ the density, C the specific heat, and V the mode volume. Owing to the large mode volume of our GRIN-MMF FP microresonator, we can achieve notably smaller TRN than FP microresonator made of HNLF.

The relationship between frequency noise of the resonant frequency $S_\nu(\omega)$ and temperature fluctuations $S_{\delta T}(\omega)$ is given by [10]

$$S_\nu(\omega) = \left(\nu_0 \frac{dn/dT}{n_0} \right)^2 S_{\delta T}(\omega), \quad (5)$$

where $dn/dT = 9.5 \times 10^{-6} K^{-1}$ is the thermo-optic coefficient, $n_0 = 1.4682$ is the refractive index, $\nu_0 = 193$ THz is the resonant frequency, ω is the angular Fourier frequency.

The SSB temperature noise power spectral density for a single-mode fiber is given by [11, 12]

$$S_{\delta T}(\omega) = \frac{k_B T^2}{4\pi^2 \kappa_T l} \ln \left[\frac{k_{\max}^4 + \left(\frac{\omega}{D}\right)^2}{k_{\min}^4 + \left(\frac{\omega}{D}\right)^2} \right], \quad (6)$$

where $\kappa_T = 1.37 Wm^{-1}K^{-1}$ is thermal conductivity, $k_{\max} = 2/w_0$ where $w_0 = 9 \mu m$ is the fiber mode field radius, $k_{\min} = 2.405/a_f$ is insulating boundary conditions, where $2a_f = 125 \mu m$ is the fiber outer diameter, $D = 0.82 \times 10^{-6} m^2/s$ is the thermal diffusivity.

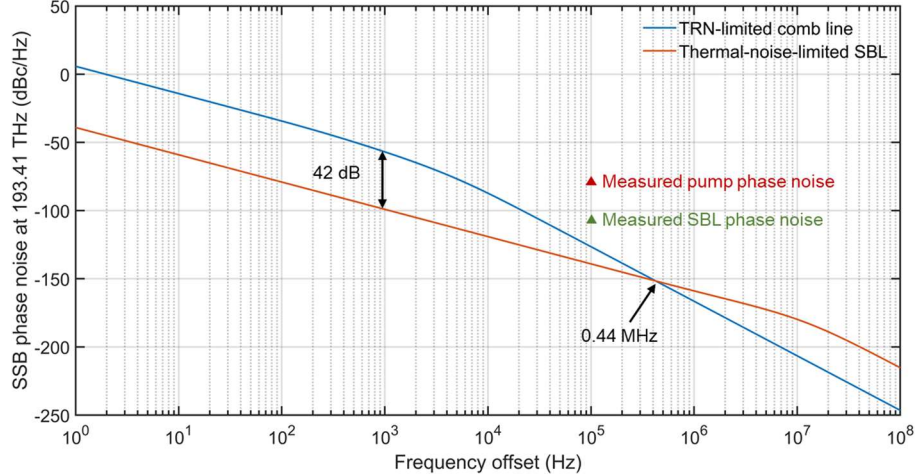


Fig. S12. Comparison of TRN-limited phase noise for comb line and thermal-noise-limited phase noise for SBL. The measured pump phase noise (Fig. S11) as well as the SBL phase noise at 100 kHz are marked in red and green triangles.

Although Eq. (6) is designed for a single mode fiber, we can still use it to estimate the TRN since it follows the same rule by Eq. (4): increased mode volume will lead to reduced temperature fluctuations. First, we study the TRN-limited phase noise of the comb line, as well as the thermal-noise-limited phase noise of SBL [5]. With the aid of SBS process, the TRN-limited phase noise can be further suppressed, similar to a laser cooling process for the selected optical frequency via a phonon mode. Therefore, we expect the TRN-limited phase noise of SBL *comb line* (generated through Kerr effect) is larger than the thermal-noise-limited phase noise of SBL (generated through SBS process). As shown in Fig. S12, the TRN-limited phase noise of SBL *comb line* is 42 dB larger than the thermal-noise-limited phase noise of SBL at low offset frequencies, which is close to the theoretical narrowing factor of 36

dB. The detailed calculation parameters can be found in Appendix. In addition, the intersection at 0.44 MHz is quite close to the cavity linewidth of 0.5 MHz, above which the TRN response is quite weak. These results quantitatively verify the effectiveness of Eq. (6).

We then study how TRN influence the stability of comb repetition rate. As TRN-limited phase noise is much stronger than the other one in the concerned Fourier frequency range < 1 MHz (Fig. S12), we will only focus on the TRN-limited phase noise. The thermal fluctuations will influence the repetition rate through two channels owing to the cavity resonance fluctuation: (i) directly change the resonator FSR (minified by mode index=193 THz/10.079 GHz from cavity resonance), which is the fundamental limit imposed by TRN; (ii) detuning noise (PM2PM). In Fig. S13, we overlay the measured repetition rate PN, the quantum limit, the fundamental limit imposed by TRN, the detuning noise imposed by TRN. It can be inferred that the experimental phase noise of SBL comb repetition rate is not caused by the above-mentioned factors. Furthermore, at high offset frequencies quantum limit is thought to be the ultimate limit of the repetition rate PN in our GRIN-MMF FP microresonator.

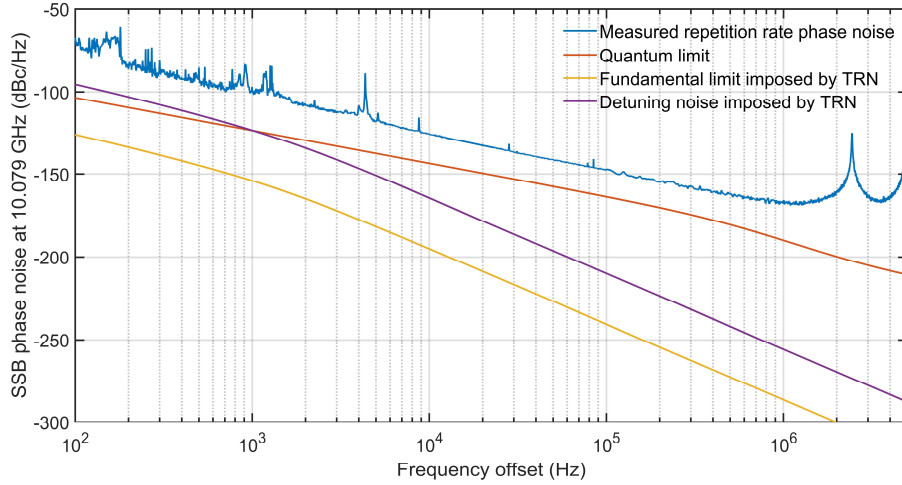


Fig. S13. Comparison of the measured repetition rate PN, quantum limit of repetition rate PN, fundamental limit of repetition rate imposed by TRN and detuning noise of repetition rate imposed by TRN. PM2PM transduction coefficient is set to be 55 dB [13], which is the value for on-chip Si_3N_4 microresonator with broad-bandwidth combs. In theory, the PM2PM transduction coefficient for our GRIN-MMF FP microresonators should be smaller than that in Si_3N_4 microresonators.

E. SBL RIN limit

According to the analysis above, the SBL soliton repetition rate PN is limited by the SBL RIN, which is transduced from the pump RIN and is much larger than the thermal-noise limited SBL RIN (purple line in Fig. S14, details see Appendix). Therefore, how to lower the intra-cavity pump RIN is the key to approach the quantum limit of timing jitter. In Fig. S14, we compare the output laser RIN (without any active control) from the GRIN-MMF FP microresonator and another step-index single-mode fiber (SMF) FP microresonator (cavity linewidth of ~ 9 MHz). Both the microresonators are directly pumped by the same ECDL without an amplifier. It can be clearly seen that both microresonators impose extra RIN to the output laser. What's more, the GRIN-MMF FP microresonator imposes 30 dB [$\sim 20 \cdot \log(12/0.5) = 27$ dB] RIN larger than the SMF FP microresonator, which can be attributed to the difference of Q factors: RIN will be boosted much more in higher Q microresonators. Therefore, compromise between Q factor, pump threshold, RIN and repetition rate PN should be made. However, it is still feasible to reduce the pump RIN thus the SBL RIN by active control [9, 14] in the experiment.

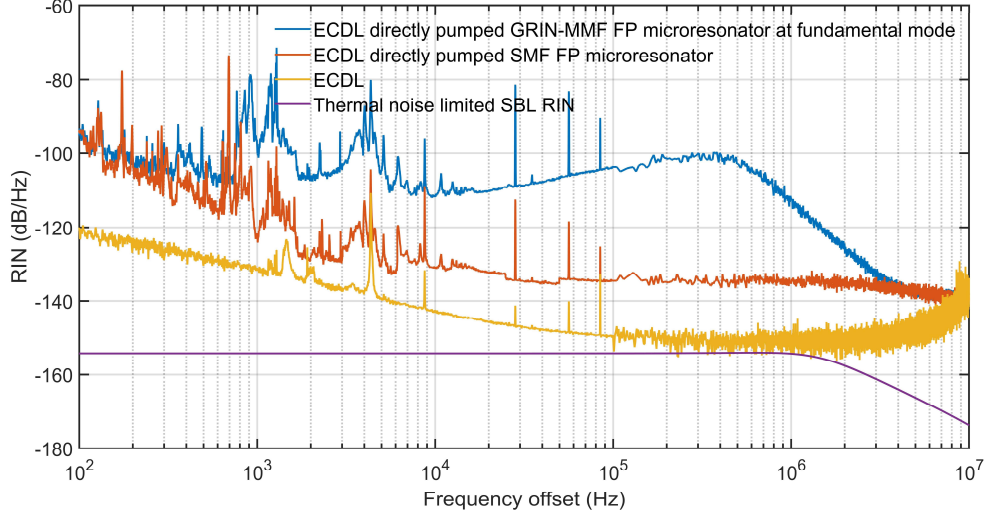


Fig. S14. RIN measurement, including output laser RIN from the GRIN-MMF FP microresonator, SMF FP microresonator and calculated thermal-noise-limited RIN for SBL without considering the cavity detuning effect.

VII. TIMING JITTER MEASUREMENT SETUP

The detailed schematic of ARMI timing jitter measurement setup can be found in Ref. [15]. The method can be summarized briefly below. For each comb line, its optical frequency can be written as $f_k = kf_{rep} + f_{ceo}$ ($k=1,2,3\dots$), where f_{rep} is the repetition rate and f_{ceo} is the carrier-envelope-offset frequency. Two frequency modes (f_m and f_n) are used to extract phase noise $(m-n)f_{rep}$ by subtracting the noise of common mode f_{ceo} .

In our experiment, two filters (1545 nm with 2-nm FWHM bandwidth and 1555 nm with 2-nm FWHM bandwidth) are employed to filter out the two main frequency modes (f_m and f_n). An unbalanced Michelson interferometer with an 82-m-long fiber delay line (time delay of $\tau = 410$ ns) is used for converting phase noise into intensity noise. The delay line is implemented by using an 80-m-long fiber PZT stretcher and 2-m-long extra fiber. The length of the fiber delay line is chosen to make a compromise between the timing detection sensitivity (-174 dBc/Hz at 10.079 GHz) proportional to τ and measurement bandwidth scaling with $1/\tau$ (2.5 MHz). A delay control unit (DCU) consisting of two electrically-driven delay lines is implemented to compensate the group velocity mismatch between the two frequency modes and maximize the interferometer output voltage. An acousto-optic frequency shifter (AOFS) with frequency shift of 50 MHz is inserted in the fiber delay line to avoid background noise in the baseband by synchronous detection. The interferometer output then contains the frequency noise of each frequency mode weighted by the delay time at center frequency of $f_{AOM} = 100$ MHz, in the form of phase noise as $\delta[(f_m + f_{AOM})\tau]$ and $\delta[(f_n + f_{AOM})\tau]$. We then filter the photo-detected signal of each mode at $f_{AOM} = 100$ MHz and mix them by an RF mixer to reject the common-mode f_{ceo} noise. This down-converted RF mixer output contains the repetition rate frequency noise in the form of phase noise (timing jitter) as $\delta[(m-n)f_{rep}\tau]$.

During the measurement, a delay-locked loop (DLL) is introduced for locking the delay to the comb under test and preventing optical frequency drift. Served as error signal, the mixer output is fed to the servo, whose output is applied to the PZT stretcher in the fiber delay line. It is important to keep the locking bandwidth as low as possible to ensure broadband characterization because the free-running comb jitter can be analyzed only outside the DLL locking bandwidth. In this work, we used a 100 Hz locking bandwidth. As a result, the jitter PSD from 100 Hz to 2.5 MHz can be measured. The measured voltage PSD (intensity noise) is converted to the frequency noise by the transfer function $T(f)$ resulting from the fiber delay line [15-17]

$$T(f) = V_p \frac{|1 - e^{-i2\pi f\tau}|}{|i \times f|} (\text{V/Hz}) \quad (7)$$

where V_p is half of the peak-to-peak voltage from the mixer output (within Fourier frequency range $< f_{\text{AOM}}$). Equivalent timing jitter or repetition rate phase noise can then be converted from the frequency noise. Of note, the transduction from amplitude to timing jitter in the ARMI setup is extremely low. The soliton RIN as well as other common noise for the two filtered frequency modes do not influence the accuracy of the measured soliton jitter [16].

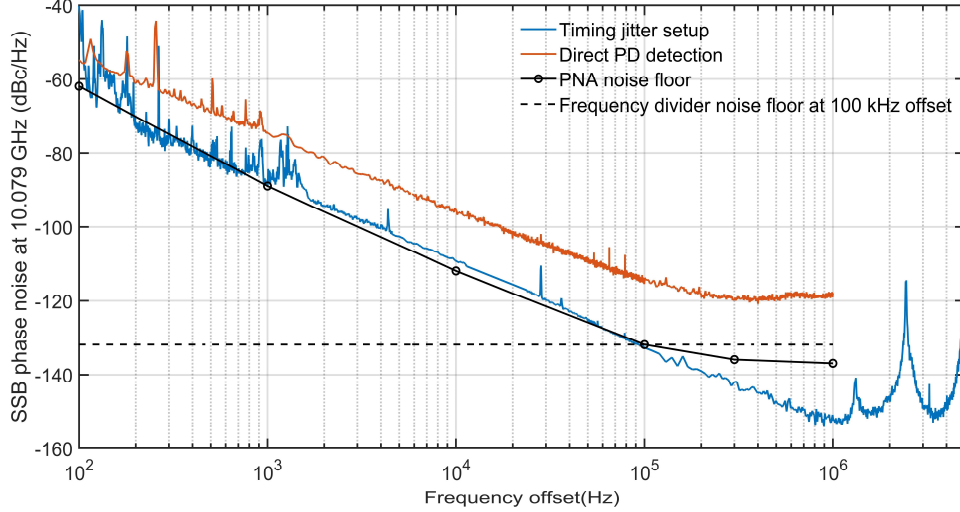


Fig. S15. Comparison of SSB phase noise of locked soliton repetition rate with different methods, the timing jitter setup and the direct PD detection. For the latter method, we inject the comb lines into the fast PD, electrically frequency divide the 10.079 GHz RF signal by 8 times (with a low noise floor) and then analyze the divided signal with a PNA. The noise floor of the frequency divider has been included in the plot of ‘PNA noise floor’.

The timing jitter measurement setup is designed for low repetition rate PN, i. e., small $\delta[(m-n)f_{\text{rep}}\tau]$ thus low peak-to-peak voltage with DLL. Large phase noise will lead to high peak-to-peak voltage with DLL and invalidate the linear approximation around the locking point, which locates at the center of the sinusoidal voltage function. Therefore, the measured voltage PSD is incorrect and smaller than the correct one. We show in Fig. S15 the comparison of measured results for the STML soliton, with both the ARMI timing jitter setup and the conventional direct method using a fast PD and a phase noise analyzer. According to Fig. S15, the result from timing jitter setup is incorrect and smaller than the other one (the correct one) due to the large phase noise.

Apart from the single SBL soliton, we also measure the repetition rate PN of the 2-FSR SBL soliton comb as shown in Fig. S16.

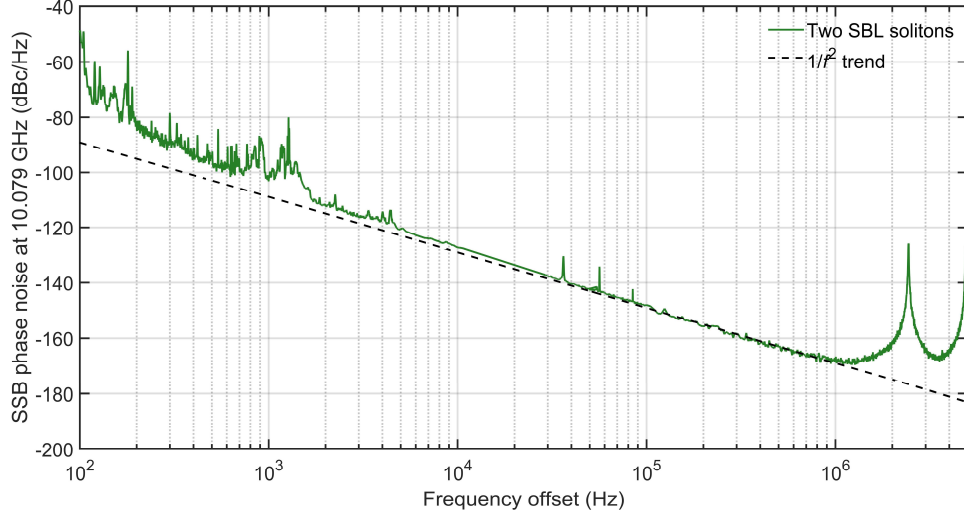


Fig. S16. SSB repetition rate PN of the 2-FSR SBL soliton.

APPENDIX: SBL CALCULATION

This Section provides the key equations and parameters to calculate (i) the transduction coefficient from pump RIN to SBL RIN in Fig. S9; (ii) thermal-noise-limited phase noise for SBL in Fig. S12; and (iii) thermal-noise-limited RIN for SBL in Fig. S14. The complete equation derivations can be found in Ref. [5]. The SBS laser parameters and their definitions along with representing symbols here follows the ones in Ref. [5].

Table 1. Simulation parameters for SBL

Parameter	Value	Parameter	Value
τ_F	$1/(2\pi \times 0.5 \times 10^6) s$	ρ_0	$2200 kg/m^3$
τ_B	$1/(2\pi \times 0.5 \times 10^6) s$	S	$4.7 \times 10^{10} \sqrt{W/m^2 F}$
τ_{ext}	$1/(2\pi \times 0.4 \times 10^6) s$	Γ_b	$2\pi \times 31 \times 10^6 rad/s$
Λ_F	0.5	Ω_b	$2\pi \times 10.343 \times 10^9 rad/s$
Λ_B	0.5	Ω	$2\pi \times 10.079 \times 10^9 rad/s$
Λ_ρ	0.5	l_ρ/R	$1.22 \times 10^7 1/m$
γ_e	1.5	V	$5 \times 10^{-12} m^3$
ω_l	$1.2153 \times 10^{15} rad/s$	V_{ph}	$5 \times 10^{-12} m^3$
n_0	1.4682		

- (i) the transduction coefficient from pump RIN to SBL RIN in Fig. S9 is calculated by Eq. (30) in Ref. [5].
- (ii) thermal-noise-limited phase noise for SBL in Fig. S12 is calculated by Eq. (36) in Ref. [5].
- (i) thermal-noise-limited RIN for SBL in Fig. S14 is calculated by Eq. (32) and Eq. (43) in Ref. [5].

Supplementary References

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