

Existence analysis of nonlinear q -fractional differential inclusions with p -Laplacian operators and an aerospace vibration-modeling illustration

Vu Vu

Sejong University

Behroz Mohammadiyan

Azarbaijan Shahid Madani University

Hojjat Afshari

hojat.afshari@yahoo.com

University of Bonab

Ardashir Mohammadzadeh

Sakarya University

Research Article

Keywords: q -fractional calculus, differential inclusions, p -Laplacian operator, fixed point theorem, nonlocal boundary conditions, aerospace vibration modeling, memory-dependent damping

Posted Date: September 2nd, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-10778910/v1>

License:   This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: No competing interests reported.

Version of Record: A version of this preprint was published at Boundary Value Problems on October 6th, 2026. See the published version at <https://doi.org/10.1186/s13661-026-02372-z>.

RESEARCH

Existence analysis of nonlinear q -fractional differential inclusions with p -Laplacian operators and an aerospace vibration-modeling illustration

Mai The Vu¹, Behrooz Mohammadian², Hojjat Afshari^{3*} and Ardashir Mohammadzadeh⁴

* Correspondence:

hojat.afshari@yahoo.com

³Department of Mathematics,
Faculty of Science, University of
Bonab, Bonab, Iran
Full list of author information is
available at the end of the article

Abstract

This paper investigates a coupled system of nonlinear q -fractional differential inclusions involving nested p -Laplacian operators and nonlocal boundary conditions. The problem is formulated in a multivalued setting, where the nonlinear inclusion depends on the state vector and its Jackson q -derivative. Using tools from q -fractional calculus, Green-function representations, and multivalued fixed-point theory, we derive sufficient conditions for the existence of at least one solution in the associated product function space. The analysis combines the sequential structure of the Caputo and Riemann–Liouville q -fractional operators with compact convex multifunction values and nonlocal boundary constraints.

An aerospace vibration-modeling illustration is also presented to show how a Caputo q -fractional damping term can represent memory-dependent behavior in a base-excited inertial-measurement-unit mounting model. The numerical comparison is reported against rigid mounting and a classical viscous isolator and is interpreted as an illustration of q -fractional memory modeling. The main mathematical contribution of the paper is the existence result for the proposed nonlinear q -fractional differential inclusion system.

Keywords: q -fractional calculus; differential inclusions; p -Laplacian operator; fixed point theorem; nonlocal boundary conditions; aerospace vibration modeling; memory-dependent damping.

1 Introduction

Fractional boundary value problems with p -Laplacian operators have been studied extensively because they combine memory effects with nonlinear diffusion or nonlinear constitutive behavior [3, 10, 11, 15, 17, 19, 20]. Related work has also considered integral, multipoint, and nonlocal boundary conditions for fractional differential equations and inclusions [6–9, 14, 16, 22]. In the q -calculus setting, recent studies have addressed q -derivative boundary value problems and fractional q -difference equations with integral conditions [1, 2]. Multivalued formulations provide a natural framework for systems with set-valued nonlinearities, uncertain responses, or nonunique constitutive laws, and their existence theory commonly relies on compactness and fixed-point methods for multifunctions [4, 5, 12, 22].

The present problem differs from these studies in the simultaneous occurrence of four features. First, the system is k -dimensional and coupled through the arguments of the multifunction. Second, the operator contains a nested structure involving Caputo and Riemann–Liouville q -fractional derivatives linked by nonlinear p -Laplacian and inverse p -Laplacian transformations. Third, the multifunction depends on the full state vector and its Jackson q -derivative, not only on a single scalar state.

Fourth, the boundary conditions act at different operator layers and include non-local constraints. These features require a layerwise Green-function representation and a fixed-point formulation in a product function space. The paper develops this framework and derives sufficient conditions for the existence of at least one solution.

Throughout the paper, $\varrho \in [0, 1]$ denotes the independent variable, while ρ_i , ξ_i , and ς_i denote fractional orders. Let $k \in \mathbb{N}$, $0 < q < 1$, $0 < \vartheta < 1$, $2 < \rho_i < 3$, $3 < \xi_i < 4$, and $2 < \varsigma_i < 3$ for $i = 1, \dots, k$. Let $p > 1$ and let $z > 1$ be its conjugate exponent, so that $p + z = pz$. For $r > 1$, set $\Phi_r(s) = |s|^{r-2}s$ for $s \in \mathbb{R}$; in particular, $\Phi_z = \Phi_p^{-1}$. For a vector function

$$\mathbf{w} = (w_1, \dots, w_k) \in (V_q)^k, \quad V_q = \{w \in C([0, 1], \mathbb{R}) : D_q w \in C([0, 1], \mathbb{R})\},$$

we write

$$D_q \mathbf{w} = (D_q w_1, \dots, D_q w_k).$$

For each $i = 1, \dots, k$, define the auxiliary quantities

$$u_i(\varrho) = \Phi_z \left({}^C D_{q,0+}^{\rho_i} w_i(\varrho) \right), \quad v_i(\varrho) = D_q u_i(\varrho), \quad x_i(\varrho) = \Phi_p \left({}^C D_{q,0+}^{\xi_i} v_i(\varrho) \right).$$

The nonlinear perturbation satisfies

$$h_i \in C([0, 1] \times \mathbb{R}, \mathbb{R}),$$

and the multifunction is taken in the form

$$\mathcal{H}_i : [0, 1] \times \mathbb{R}^{2k} \rightarrow \mathcal{P}_{cp,cv}(\mathbb{R}),$$

where $\mathcal{P}_{cp,cv}(\mathbb{R})$ denotes the family of nonempty compact convex subsets of $\mathbb{R}$. The arguments of $\mathcal{H}_i$ are the state vector and its q -derivative.

The system considered in this paper is

$$D_{q,0+}^{\varsigma_i} x_i(\varrho) + \vartheta h_i(\varrho, w_i(\varrho)) \in \mathcal{H}_i(\varrho, \mathbf{w}(\varrho), D_q \mathbf{w}(\varrho)), \quad 0 < \varrho < 1, \quad i = 1, \dots, k. \quad (1.1)$$

Let $0 < \epsilon, \eta < 1$, and define

$$\mathcal{A}_q = {}^C D_{q,0+}^{5/2} (\varrho^3 + 2\varrho) \Big|_{\varrho=1}, \quad \mathcal{B}_\vartheta = \vartheta \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \vartheta.$$

The outer-layer boundary conditions are

$$x_i(0) = 0, \quad x_i(1) = \mathcal{A}_q, \quad x_i(\epsilon) = \mathcal{B}_\vartheta, \quad i = 1, \dots, k. \quad (1.2)$$

The intermediate-layer boundary conditions are

$$u_i(0) = 0, \quad D_q u_i(0) = 0, \quad D_q^2 u_i(1) = 0, \quad u_i(1) = D_q u_i(\epsilon), \quad u_i(\eta) = D_q^2 u_i(1), \quad i = 1, \dots, k.$$

$$(1.3)$$

The inner-layer boundary conditions are

$$w_i(0) = 0, \quad D_q w_i(1) = D_q^2 w_i(1), \quad w_i(1) = \int_0^\eta w_i(\varrho) d_q \varrho, \quad i = 1, \dots, k. \quad (1.4)$$

A solution of (1.1)–(1.4) is a vector function $\mathbf{w} \in (V_q)^k$ for which the auxiliary functions u_i , v_i , and x_i are well defined, the boundary conditions (1.2)–(1.4) hold, and there exist measurable selections

$$\ell_i(\varrho) \in \mathcal{H}_i(\varrho, \mathbf{w}(\varrho), D_q \mathbf{w}(\varrho)) \quad \text{for a.e. } \varrho \in [0, 1],$$

such that

$$D_{q,0+}^{\varsigma_i} x_i(\varrho) + \vartheta h_i(\varrho, w_i(\varrho)) = \ell_i(\varrho) \quad \text{for a.e. } \varrho \in [0, 1], \quad i = 1, \dots, k.$$

The purpose of the paper is to derive sufficient conditions under which this boundary value problem admits at least one solution in $(V_q)^k$.

The structure of this article is organized as follows. Section 2 recalls preliminary definitions and auxiliary results from q -fractional calculus and multivalued analysis. Section 3 presents the Green-function construction and the main existence result. Section 4 gives an aerospace vibration-modeling illustration based on a Caputo q -fractional damping term. Section 5 concludes the paper and outlines possible directions for further analysis.

2 Preliminaries

At the beginning of this section, we recall some basic concepts and auxiliary results from q -fractional calculus and multivalued analysis which will be used throughout the paper [23, 25]. In what follows, we assume that: (S1) the function $h : [0, 1] \times (0, \infty) \rightarrow (0, \infty)$ defined by $h(\varrho, \mathbf{w}(\varrho))$ is continuous; (S2) the functions Φ_p and Φ_z satisfy the relation $p + z = pz$, and Φ_z is the inverse function of Φ_p . Let $q \in (0, 1)$, $\theta > 0$, and suppose that $m - 1 < \theta < m$, where $m = [\theta] + 1$. We first recall the left-sided Riemann–Liouville q -fractional integral of order θ for a function $w : [0, 1] \rightarrow \mathbb{R}$:

$$(I_{q,0+}^\theta w)(\varrho) = \frac{1}{\Gamma_q(\theta)} \int_0^\varrho (\varrho - qt)_q^{\theta-1} w(t) d_q t,$$

provided that the above q -integral exists, where $\Gamma_q(\theta)$ denotes the q -gamma function. Next, if $w \in BC_{\mathbb{R}}^{(m)}([0, 1])$, then the Caputo q -fractional derivative of order θ is defined by

$$({}^C D_{q,0+}^\theta w)(\varrho) = (I_{q,0+}^{m-\theta} D_q^m w)(\varrho) = \frac{1}{\Gamma_q(m-\theta)} \int_0^\varrho (\varrho - qt)_q^{m-\theta-1} (D_q^m w)(t) d_q t,$$

whenever the above expression is well defined and finite. On the other hand, the Riemann–Liouville q -fractional derivative of order θ is given by

$$(D_{q,0+}^{\theta} w)(\varrho) = D_q^m (I_{q,0+}^{m-\theta} w)(\varrho),$$

where $m = [\theta] + 1$. In addition, for sufficiently smooth functions, the sequential q -fractional derivative can be written as

$$D_{q,0+}^{\theta} w(\varrho) = \left(D_{q,0+}^{\theta_1} D_{q,0+}^{\theta_2} \cdots D_{q,0+}^{\theta_m} \right) w(\varrho),$$

where $\theta = (\theta_1, \theta_2, \dots, \theta_m)$ is a multi-index [21]. In this paper, we work mainly with Caputo q -fractional derivatives of different orders. It is well known that, for $m - 1 < \theta < m$, the Caputo q -fractional derivative may also be expressed in the operator form

$${}^C D_{q,0+}^{\theta} w(\varrho) = I_{q,0+}^{m-\theta} (D_q^m w)(\varrho),$$

while the negative-order operator is interpreted as the q -fractional integral, namely

$$D_{q,0+}^{-(m-\theta)} w(\varrho) = I_{q,0+}^{m-\theta} w(\varrho).$$

Moreover, the general solution of the homogeneous Caputo q -fractional equation

$${}^C D_{q,0+}^{\theta} w(\varrho) = 0$$

is given by

$$I_{q,0+}^{\theta} \left({}^C D_{q,0+}^{\theta} w(\varrho) \right) = w(\varrho) + \sum_{n=0}^{m-1} \tilde{d}_n \varrho^n = w(\varrho) + \tilde{d}_0 + \tilde{d}_1 \varrho + \tilde{d}_2 \varrho^2 + \cdots + \tilde{d}_{m-1} \varrho^{m-1}, \quad (2.1)$$

where $\tilde{d}_0, \tilde{d}_1, \dots, \tilde{d}_{m-1} \in \mathbb{R}$. Similarly, for the homogeneous Riemann–Liouville q -fractional equation

$$D_{q,0+}^{\theta} w(\varrho) = 0,$$

we have

$$I_{q,0+}^{\theta} \left(D_{q,0+}^{\theta} w(\varrho) \right) = w(\varrho) + \sum_{n=1}^m \tilde{d}_n \varrho^{\theta-n}, \quad (2.2)$$

where $\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_m \in \mathbb{R}$ and $m = [\theta] + 1$ [21]. We now recall some notions from multivalued analysis. Let (E, d_E) be a metric space. We define the Pompeiu–Hausdorff metric

$$P.H_{d_E} : \mathbb{B}_{c.l.s}(E) \times \mathbb{B}_{c.l.s}(E) \rightarrow \mathbb{R} \cup \{\infty\}$$

by

$$\text{P.H}_{d_E}(C_1, C_2) = \max \left\{ \sup_{c_1 \in C_1} d_E(c_1, C_2), \sup_{c_2 \in C_2} d_E(C_1, c_2) \right\},$$

where

$$d_E(c_1, C_2) = \inf_{c_2 \in C_2} d_E(c_1, c_2), \quad d_E(C_1, c_2) = \inf_{c_1 \in C_1} d_E(c_1, c_2).$$

For a normed space $(E, \|\cdot\|_E)$, we use the notations $\mathbb{B}_{c.l.s}(E)$, $\mathbb{B}_{c.m.p}(E)$, $\mathbb{B}_{b.n.d}(E)$ and $\mathbb{B}_{c.v.x}(E)$ to denote, respectively, the families of all closed, compact, bounded and convex subsets of E . A point $l^* \in E$ is called a fixed point of a multifunction $\mathcal{G} : E \rightarrow \mathbb{B}(E)$ if

$$l^* \in \mathcal{G}(l^*).$$

The set of all fixed points of $\mathcal{G}$ is denoted by $\text{FIX}(\mathcal{G})$. Also, a multifunction $\mathcal{G} : E \rightarrow \mathbb{B}_{c.l.s}(E)$ is called Lipschitzian with constant $\hat{\theta} > 0$ if

$$\text{P.H}_{d_E}(\mathcal{G}(l'), \mathcal{G}(l'')) \leq \hat{\theta} d_E(l', l'')$$

for all $l', l'' \in E$. If $0 < \hat{\theta} < 1$, then $\mathcal{G}$ is a contraction. A multifunction $\mathcal{K}$ is said to be completely continuous if $\mathcal{K}(Q)$ is relatively compact for every $Q \in \mathbb{B}_{b.n.d}(E)$. Moreover, $\mathcal{K} : [0, 1] \rightarrow \mathbb{B}_{c.l.s}(\mathbb{R})$ is said to be measurable if, for each $v \in \mathbb{R}$, the function

$$z \mapsto d_E(v, \mathcal{K}(z)) = \inf \{|v - x| : x \in \mathcal{K}(z)\}$$

is measurable [4, 12]. Furthermore, $\mathcal{K}$ is upper semi-continuous if for every $l^* \in E$ and every open set $\mathcal{Y}$ containing $\mathcal{K}(l^*)$, there exists a neighborhood $\mathcal{T}_0^*$ of l^* such that

$$\mathcal{K}(\mathcal{T}_0^*) \subseteq \mathcal{Y}.$$

The graph of $\mathcal{K} : E \rightarrow \mathbb{B}_{c.l.s}(\mathcal{Y})$ is defined by

$$\text{Graph}(\mathcal{K}) = \{(l, y) \in E \times \mathcal{Y} : y \in \mathcal{K}(l)\}.$$

We say that $\text{Graph}(\mathcal{K})$ is closed if, whenever $l_m \rightarrow l_0$ and $x_m \rightarrow x_0$ with $x_m \in \mathcal{K}(l_m)$ for all m , then $x_0 \in \mathcal{K}(l_0)$. It is well known that every upper semi-continuous compact multifunction with closed values has a closed graph. For $l \in C_{\mathbb{R}}([0, 1])$, the set of measurable selections of a multifunction $\mathcal{K}$ is denoted by

$$(\mathcal{Z}.\mathcal{X}.\mathcal{J})_{\mathcal{K}, l} := \left\{ \hat{l} \in L_{\mathbb{R}}^1([0, 1]) : \hat{l}(z) \in \mathcal{K}(z, l(z)) \text{ for a.e. } z \in [0, 1] \right\}.$$

If $\mathcal{K}$ is a multifunction, then $(\mathcal{Z}.\mathcal{X}.\mathcal{J})_{\mathcal{K}, l} \neq \emptyset$ for every $l \in C_E([0, 1])$ whenever $\dim(E) < \infty$ [24]. A multifunction $\mathcal{K} : [0, 1] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is called Carathéodory if

$t \mapsto \mathcal{K}(t, v)$ is measurable for each $l \in \mathbb{R}$ and $l \mapsto \mathcal{K}(t, l)$ is upper semi-continuous for almost every $t \in [0, 1]$. It is called L^1 -Carathéodory if for each $\varphi > 0$ there exists $\vartheta_\varphi \in L^1_{\mathbb{R}^+}([0, 1])$ such that

$$\|\mathcal{K}(t, l)\| = \sup\{|x| : x \in \mathcal{K}(t, l)\} \leq \vartheta_\varphi(t)$$

for all $|l| \leq \varphi$ and for almost every $t \in [0, 1]$ [4, 12]. We now state the main fixed point tools which will be used in the sequel.

Theorem 2.1 ([18]) *Suppose that E is a separable Banach space, and let $\mathcal{D} : [0, 1] \times E \rightarrow \mathbb{B}_{c.m.p.c.v.x}(E)$ denote an L^1 -Carathéodory multifunction. Additionally, assume that $\zeta : L^1_E([0, 1]) \rightarrow C_E([0, 1])$ is a continuous linear operator. Under these assumptions, the composite mapping*

$$\zeta \circ (\mathcal{Z}.\mathcal{X}.\mathcal{J})_{\mathcal{D}} : C_E([0, 1]) \rightarrow \mathbb{B}_{c.m.p.c.v.x}(C_E([0, 1])),$$

which is characterized by

$$l \mapsto (\zeta \circ (\mathcal{Z}.\mathcal{X}.\mathcal{J})_{\mathcal{D}})(l) = \zeta((\mathcal{Z}.\mathcal{X}.\mathcal{J})_{\mathcal{D}, l}),$$

possesses a closed graph.

Theorem 2.2 ([13]) *Let $\mathcal{T}$ represent a closed convex subset within a Banach space E . Suppose that $\mathcal{U}$ is an open subset of $\mathcal{T}$ containing the origin ($0 \in \mathcal{U}$). If*

$$H : \overline{\mathcal{U}} \rightarrow \mathbb{B}_{c.m.p.c.v.x}(\mathcal{T})$$

is a compact, upper semi-continuous multifunction, then one of the following statements holds:

- (i) *The multifunction κ has at least one fixed point in $\overline{\mathcal{U}}$;*
- (ii) *There exist an element $m \in \partial\mathcal{U}$ and a parameter $\beta \in (0, 1)$ fulfilling the condition $m \in \beta\kappa(u)$.*

Theorem 2.3 (Bohnenblust–Karlin fixed point theorem) *Let Q be a nonempty, closed, bounded, and convex subset of a Banach space E . If $\mathcal{T} : Q \rightarrow \mathbb{B}_{c.m.p.c.v.x}(Q)$ is upper semicontinuous and $\mathcal{T}(Q)$ is relatively compact, then $\mathcal{T}$ has at least one fixed point in Q .*

3 Main Results

3.1 Layerwise decomposition and construction of Green's functions

For each $i = 1, \dots, k$, let

$$r_i(\varrho) = \ell_i(\varrho) - \vartheta h_i(\varrho, w_i(\varrho)), \quad \ell_i(\varrho) \in \mathcal{H}_i(\varrho, \mathbf{w}(\varrho), D_q \mathbf{w}(\varrho)).$$

Then (1.1) can be written in the form

$$D_{q,0+}^{\varsigma_i} x_i(\varrho) = r_i(\varrho), \quad 0 < \varrho < 1.$$

The auxiliary functions introduced in (1.1) are used in the following order:

$$u_i(\varrho) = \Phi_z \left({}^C D_{q,0+}^{\rho_i} w_i(\varrho) \right), \quad (3.1)$$

$$v_i(\varrho) = D_q u_i(\varrho), \quad (3.2)$$

$$x_i(\varrho) = \Phi_p \left({}^C D_{q,0+}^{\xi_i} v_i(\varrho) \right). \quad (3.3)$$

Since $\Phi_z = \Phi_p^{-1}$, the three layers are solved successively as follows. First, x_i is obtained from the outer fractional equation. Second, v_i is obtained from ${}^C D_{q,0+}^{\xi_i} v_i = \Phi_z(x_i)$. Third, u_i is recovered from $D_q u_i = v_i$ and $u_i(0) = 0$, and w_i is obtained from ${}^C D_{q,0+}^{\rho_i} w_i = \Phi_p(u_i)$.

The outer layer is

$$D_{q,0+}^{\varsigma_i} x_i(\varrho) = r_i(\varrho), \quad 2 < \varsigma_i < 3, \quad 0 < \varrho < 1, \quad (3.4)$$

with

$$x_i(0) = 0, \quad x_i(1) = \mathcal{A}_q, \quad x_i(\epsilon) = \mathcal{B}_\vartheta. \quad (3.5)$$

Set

$$\Delta_1 = \epsilon^2 - \epsilon.$$

Since $0 < \epsilon < 1$, one has $\Delta_1 \neq 0$.

Lemma 3.1 *For every $r_i \in L_q^1([0, 1], \mathbb{R})$, the boundary value problem (3.4)–(3.5) has the representation*

$$x_i(\varrho) = \int_0^1 G_i^{(1)}(\varrho, t) r_i(t) d_q t + \Psi_i(\varrho),$$

where

$$\Psi_i(\varrho) = \frac{\varrho(\epsilon^2 \mathcal{A}_q - \mathcal{B}_\vartheta) + \varrho^2(\mathcal{B}_\vartheta - \epsilon \mathcal{A}_q)}{\Delta_1}$$

and

$$G_i^{(1)}(\varrho, t) = \frac{1}{\Gamma_q(\varsigma_i)} \left[\chi_{[0, \varrho]}(t) (\varrho - qt)_q^{\varsigma_i-1} + \frac{\varrho}{\Delta_1} \left(\chi_{[0, \epsilon]}(t) (\epsilon - qt)_q^{\varsigma_i-1} - \epsilon^2 (1 - qt)_q^{\varsigma_i-1} \right) \right. \\ \left. + \frac{\varrho^2}{\Delta_1} \left(\epsilon (1 - qt)_q^{\varsigma_i-1} - \chi_{[0, \epsilon]}(t) (\epsilon - qt)_q^{\varsigma_i-1} \right) \right].$$

Here $\chi_{[0, a]}$ denotes the characteristic function of $[0, a]$.

After x_i is obtained, the relation

$$x_i(\varrho) = \Phi_p \left({}^C D_{q,0+}^{\xi_i} v_i(\varrho) \right)$$

yields the second-layer equation

$${}^C D_{q,0+}^{\xi_i} v_i(\varrho) = \Phi_z(x_i(\varrho)), \quad 3 < \xi_i < 4, \quad 0 < \varrho < 1. \quad (3.6)$$

The boundary conditions on u_i imply boundary and integral constraints for v_i . Since $v_i = D_q u_i$ and $u_i(0) = 0$,

$$u_i(\varrho) = \int_0^\varrho v_i(s) d_q s.$$

Consequently, (1.3) is equivalent to

$$v_i(0) = 0, \quad D_q v_i(1) = 0, \quad \int_0^1 v_i(s) d_q s = v_i(\epsilon), \quad \int_0^\eta v_i(s) d_q s = 0. \quad (3.7)$$

For $F_i \in L_q^1([0, 1], \mathbb{R})$, define

$$\mathcal{J}_{i,2}[F_i](\varrho) = \frac{1}{\Gamma_q(\xi_i)} \int_0^\varrho (\varrho - qt)^{\xi_i-1} F_i(t) d_q t.$$

Let

$$\mathcal{M}_2 = \begin{pmatrix} 1 & [2]_q & [3]_q \\ \frac{1}{[2]_q} - \epsilon & \frac{1}{[3]_q} - \epsilon^2 & \frac{1}{[4]_q} - \epsilon^3 \\ \frac{\eta^2}{[2]_q} & \frac{\eta^3}{[3]_q} & \frac{\eta^4}{[4]_q} \end{pmatrix}, \quad \det(\mathcal{M}_2) \neq 0.$$

Lemma 3.2 *Let $F_i(\varrho) = \Phi_z(x_i(\varrho))$. The boundary value problem (3.6)–(3.7) has the representation*

$$v_i(\varrho) = \mathcal{J}_{i,2}[F_i](\varrho) + a_{i,1}\varrho + a_{i,2}\varrho^2 + a_{i,3}\varrho^3,$$

where

$$\begin{pmatrix} a_{i,1} \\ a_{i,2} \\ a_{i,3} \end{pmatrix} = \mathcal{M}_2^{-1} \begin{pmatrix} -D_q \mathcal{J}_{i,2}[F_i](1) \\ \mathcal{J}_{i,2}[F_i](\epsilon) - \int_0^1 \mathcal{J}_{i,2}[F_i](s) d_q s \\ - \int_0^\eta \mathcal{J}_{i,2}[F_i](s) d_q s \end{pmatrix}.$$

Equivalently, there is a Green kernel $G_i^{(2)}$ determined by the preceding matrix relation such that

$$v_i(\varrho) = \int_0^1 G_i^{(2)}(\varrho, t) \Phi_z(x_i(t)) d_q t.$$

The function u_i is then recovered by

$$u_i(\varrho) = \int_0^\varrho v_i(s) d_q s.$$

Thus the innermost equation is

$${}^C D_{q,0+}^{\rho_i} w_i(\varrho) = \Phi_p(u_i(\varrho)), \quad 2 < \rho_i < 3, \quad 0 < \varrho < 1, \quad (3.8)$$

subject to

$$w_i(0) = 0, \quad D_q w_i(1) = D_q^2 w_i(1), \quad w_i(1) = \int_0^1 w_i(\varrho) d_q \varrho. \quad (3.9)$$

For $Y_i \in L_q^1([0, 1], \mathbb{R})$, define

$$\mathcal{J}_{i,3}[Y_i](\varrho) = \frac{1}{\Gamma_q(\rho_i)} \int_0^\varrho (\varrho - qt)_q^{\rho_i-1} Y_i(t) d_q t.$$

Let

$$\mathcal{M}_3 = \begin{pmatrix} 1 & 0 \\ 1 - \frac{\eta^2}{[2]_q} & 1 - \frac{\eta^3}{[3]_q} \end{pmatrix}, \quad \det(\mathcal{M}_3) = 1 - \frac{\eta^3}{[3]_q} \neq 0.$$

Lemma 3.3 *Let $Y_i(\varrho) = \Phi_p(u_i(\varrho))$. The boundary value problem (3.8)–(3.9) has the representation*

$$w_i(\varrho) = \mathcal{J}_{i,3}[Y_i](\varrho) + c_{i,1}\varrho + c_{i,2}\varrho^2,$$

where

$$\begin{pmatrix} c_{i,1} \\ c_{i,2} \end{pmatrix} = \mathcal{M}_3^{-1} \begin{pmatrix} D_q^2 \mathcal{J}_{i,3}[Y_i](1) - D_q \mathcal{J}_{i,3}[Y_i](1) \\ \int_0^1 \mathcal{J}_{i,3}[Y_i](s) d_q s - \mathcal{J}_{i,3}[Y_i](1) \end{pmatrix}.$$

Equivalently, there is a Green kernel $G_i^{(3)}$ determined by the preceding matrix relation such that

$$w_i(\varrho) = \int_0^1 G_i^{(3)}(\varrho, t) \Phi_p(u_i(t)) d_q t.$$

3.2 Integral formulation and fixed-point framework

This subsection converts the layerwise representation into a multivalued fixed-point problem on $(V_q)^k$. For $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{b} = (b_1, \dots, b_k)$ in $\mathbb{R}^k$, define the shifted multifunction

$$\mathcal{R}_i(\varrho, \mathbf{a}, \mathbf{b}) = \{\xi - \vartheta h_i(\varrho, a_i) : \xi \in \mathcal{H}_i(\varrho, \mathbf{a}, \mathbf{b})\}, \quad i = 1, \dots, k.$$

For $\mathbf{w} \in (V_q)^k$, let

$$S_{\mathcal{R}_i, \mathbf{w}} = \{r_i \in L_q^1([0, 1], \mathbb{R}) : r_i(\varrho) \in \mathcal{R}_i(\varrho, \mathbf{w}(\varrho), D_q \mathbf{w}(\varrho)) \text{ for a.e. } \varrho \in [0, 1]\}.$$

For each $r_i \in S_{\mathcal{R}_i, \mathbf{w}}$, define

$$x_i(\varrho) = \int_0^1 G_i^{(1)}(\varrho, t) r_i(t) d_q t + \Psi_i(\varrho), \quad (3.10)$$

$$v_i(\varrho) = \int_0^1 G_i^{(2)}(\varrho, t) \Phi_z(x_i(t)) d_q t, \quad (3.11)$$

$$u_i(\varrho) = \int_0^\varrho v_i(s) d_q s, \quad (3.12)$$

and

$$y_i(\varrho) = \int_0^1 G_i^{(3)}(\varrho, t) \Phi_p(u_i(t)) d_q t. \quad (3.13)$$

Thus the multivalued operator $\mathcal{L} : (V_q)^k \rightarrow \mathcal{P}((V_q)^k)$ is defined by

$$\mathcal{L}(\mathbf{w}) = \{\mathbf{y} = (y_1, \dots, y_k) \in (V_q)^k : \text{for each } i, y_i \text{ is generated by some } r_i \in S_{\mathcal{R}_i, \mathbf{w}} \text{ through (3.10)–(3.13)}\}. \quad (3.14)$$

A fixed point $\mathbf{w} \in \mathcal{L}(\mathbf{w})$ determines measurable functions $r_i \in S_{\mathcal{R}_i, \mathbf{w}}$. By the definition of $\mathcal{R}_i$, there exist measurable selections

$$\ell_i(\varrho) \in \mathcal{H}_i(\varrho, \mathbf{w}(\varrho), D_q \mathbf{w}(\varrho))$$

such that $r_i(\varrho) = \ell_i(\varrho) - \vartheta h_i(\varrho, w_i(\varrho))$ for a.e. $\varrho \in [0, 1]$. Hence the fixed points of $\mathcal{L}$ are precisely the solutions of (1.1)–(1.4).

The space $(V_q)^k$ is equipped with the norm

$$\|\mathbf{w}\|_{E^q} = \sum_{i=1}^k \|w_i\|_{V_q}, \quad \|w_i\|_{V_q} = \|w_i\|_\infty + \|D_q w_i\|_\infty. \quad (3.15)$$

The following hypotheses are imposed.

- (H1)** For each $i = 1, \dots, k$, the multifunction $\mathcal{H}_i : [0, 1] \times \mathbb{R}^{2k} \rightarrow \mathcal{P}_{cp, cv}(\mathbb{R})$ is L^1 -Carathéodory, and $h_i \in C([0, 1] \times \mathbb{R}, \mathbb{R})$.
- (H2)** For each $i = 1, \dots, k$, there exist functions $\alpha_i, \beta_i \in L_q^1([0, 1], \mathbb{R}^+)$ such that

$$\|\mathcal{R}_i(\varrho, \mathbf{a}, \mathbf{b})\| \leq \alpha_i(\varrho) + \beta_i(\varrho) \left(\sum_{j=1}^k |a_j| + \sum_{j=1}^k |b_j| \right)$$

for a.e. $\varrho \in [0, 1]$ and all $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^k \times \mathbb{R}^k$, where

$$\|\mathcal{R}_i(\varrho, \mathbf{a}, \mathbf{b})\| = \sup\{|s| : s \in \mathcal{R}_i(\varrho, \mathbf{a}, \mathbf{b})\}.$$

- (H3)** For every $\mathbf{w} \in (V_q)^k$, the value $\mathcal{L}(\mathbf{w})$ defined by (3.14) is convex.

(H4) There exists $R > 0$ such that

$$\sum_{i=1}^k \Theta_i(R) \leq R,$$

where the functions Θ_i are defined in (3.18) below.

Since the Green kernels obtained in Lemmas 3.1–3.3 are continuous on compact domains, there exist finite constants

$$M_{i1} = \sup_{(\varrho, t) \in [0,1]^2} |G_i^{(1)}(\varrho, t)|, \quad M_{i2} = \sup_{(\varrho, t) \in [0,1]^2} |G_i^{(2)}(\varrho, t)|, \quad M_{i3} = \sup_{(\varrho, t) \in [0,1]^2} |G_i^{(3)}(\varrho, t)|, \quad (3.16)$$

and

$$\widehat{M}_{i3} = \sup_{(\varrho, t) \in [0,1]^2} |D_{\varrho, \varrho} G_i^{(3)}(\varrho, t)|.$$

For $R > 0$, set

$$\mu_i(R) = \|\alpha_i + R\beta_i\|_{L_q^1}, \quad A_i(R) = M_{i1}\mu_i(R) + \|\Psi_i\|_\infty, \quad (3.17)$$

and define

$$\Theta_i(R) = (M_{i3} + \widehat{M}_{i3}) (M_{i2} [A_i(R)]^{z-1})^{p-1}. \quad (3.18)$$

Theorem 3.4 *Assume that (H1)–(H4) hold. Then the boundary value problem (1.1)–(1.4) has at least one solution in $(V_q)^k$.*

Proof Let

$$B_R = \{\mathbf{w} \in (V_q)^k : \|\mathbf{w}\|_{E^q} \leq R\}.$$

Then B_R is nonempty, closed, bounded, and convex. By (H1) and the measurable selection theorem recalled in the preliminaries, $S_{\mathcal{R}_i, \mathbf{w}}$ is nonempty for every $\mathbf{w} \in B_R$. Therefore, $\mathcal{L}(\mathbf{w})$ is nonempty. By (H3), the value $\mathcal{L}(\mathbf{w})$ is convex.

Let $\mathbf{w} \in B_R$ and $\mathbf{y} \in \mathcal{L}(\mathbf{w})$. For each i , choose $r_i \in S_{\mathcal{R}_i, \mathbf{w}}$ generating y_i . Since $\|\mathbf{w}\|_{E^q} \leq R$, hypothesis (H2) gives

$$|r_i(\varrho)| \leq \alpha_i(\varrho) + R\beta_i(\varrho) \quad \text{for a.e. } \varrho \in [0, 1].$$

Thus (3.10) implies

$$\|x_i\|_\infty \leq M_{i1}\mu_i(R) + \|\Psi_i\|_\infty = A_i(R).$$

Using $|\Phi_z(s)| = |s|^{z-1}$, (3.11) yields

$$\|v_i\|_\infty \leq M_{i2}[A_i(R)]^{z-1}.$$

Since $u_i(\varrho) = \int_0^\varrho v_i(s) d_qs$, one has

$$\|u_i\|_\infty \leq M_{i2}[A_i(R)]^{z-1}.$$

Using $|\Phi_p(s)| = |s|^{p-1}$ in (3.13), we obtain

$$\|y_i\|_\infty \leq M_{i3} (M_{i2}[A_i(R)]^{z-1})^{p-1}.$$

Applying D_q to (3.13) gives

$$\|D_q y_i\|_\infty \leq \widehat{M}_{i3} (M_{i2}[A_i(R)]^{z-1})^{p-1}.$$

Consequently,

$$\|y_i\|_{V_q} \leq \Theta_i(R), \quad \|\mathbf{y}\|_{\mathbb{E}^q} \leq \sum_{i=1}^k \Theta_i(R) \leq R,$$

by (H4). Hence $\mathcal{L}(B_R) \subset B_R$.

The same estimates show that $\mathcal{L}(B_R)$ is uniformly bounded in $(V_q)^k$. Moreover, the kernels $G_i^{(3)}$ and $D_{q,\varrho}G_i^{(3)}$ are uniformly continuous with respect to ϱ on $[0, 1]^2$. The preceding bounds on $\Phi_p(u_i)$ then imply that the families $\{y_i : y_i \in \mathcal{L}_i(B_R)\}$ and $\{D_q y_i : y_i \in \mathcal{L}_i(B_R)\}$ are equicontinuous on $[0, 1]$. By the Arzelà–Ascoli theorem, $\mathcal{L}(B_R)$ is relatively compact in $(V_q)^k$.

It remains to verify upper semicontinuity. Let $\mathbf{w}_n \rightarrow \mathbf{w}$ in $(V_q)^k$, let $\mathbf{y}_n \in \mathcal{L}(\mathbf{w}_n)$, and suppose that $\mathbf{y}_n \rightarrow \mathbf{y}$ in $(V_q)^k$. For every n and i , choose $r_{i,n} \in S_{\mathcal{R}_i, \mathbf{w}_n}$ that generates $y_{i,n}$. By (H1) and the closed-graph property of the selection operator stated in Theorem 2.1, together with the continuity of the layerwise integral operators in (3.10)–(3.13), there exists $r_i \in S_{\mathcal{R}_i, \mathbf{w}}$ that generates y_i . Hence $\mathbf{y} \in \mathcal{L}(\mathbf{w})$, so the graph of $\mathcal{L}$ is closed. Since $\mathcal{L}$ has compact values on B_R , the closed-graph property implies upper semicontinuity.

The operator $\mathcal{L} : B_R \rightarrow \mathbb{B}_{c.m.p.c.v.x}(B_R)$ is therefore upper semicontinuous and has relatively compact range. By Theorem 2.3, $\mathcal{L}$ has a fixed point in B_R . This fixed point is a solution of (1.1)–(1.4). $\square$

3.3 Illustrative example

Example 3.5 We consider the scalar case $k = 1$ and choose

$$q = \frac{1}{2}, \quad p = z = 2, \quad \varsigma_1 = \frac{5}{2}, \quad \xi_1 = \frac{7}{2}, \quad \rho_1 = \frac{5}{2},$$

$$\epsilon = \frac{1}{3}, \quad \eta = \frac{1}{2}, \quad \vartheta = \frac{1}{4}.$$

For $p = z = 2$, one has $\Phi_p(s) = \Phi_z(s) = s$. Define

$$\mathcal{H}_1(\varrho, a, b) = \left[0, \frac{\varrho + 1}{20(3 - \varrho)} \left(\frac{|a|}{1 + |a|} \right) \left(\frac{|b|}{1 + |b|} \right) + \frac{1}{40} \right], \quad (\varrho, a, b) \in [0, 1] \times \mathbb{R}^2,$$

and let

$$h_1(\varrho, w) = 0, \quad (\varrho, w) \in [0, 1] \times \mathbb{R}.$$

Then the shifted multifunction used in the fixed-point formulation satisfies

$$\mathcal{R}_1(\varrho, a, b) = \mathcal{H}_1(\varrho, a, b).$$

The corresponding scalar problem is

$$D_{q,0+}^{5/2} x_1(\varrho) \in \mathcal{H}_1(\varrho, w_1(\varrho), D_q w_1(\varrho)), \quad 0 < \varrho < 1, \quad (3.19)$$

where

$$x_1(\varrho) = {}^C D_{q,0+}^{7/2} v_1(\varrho), \quad v_1(\varrho) = D_q u_1(\varrho), \quad u_1(\varrho) = {}^C D_{q,0+}^{5/2} w_1(\varrho).$$

The outer boundary conditions are

$$x_1(0) = 0, \quad x_1(1) = \mathcal{A}_q = \frac{\Gamma_q(4)}{\Gamma_q(3/2)}, \quad x_1\left(\frac{1}{3}\right) = \mathcal{B}_\vartheta = \frac{1}{4}.$$

The intermediate layer is

$${}^C D_{q,0+}^{7/2} v_1(\varrho) = x_1(\varrho), \quad (3.20)$$

with

$$v_1(0) = 0, \quad D_q v_1(1) = 0, \quad \int_0^1 v_1(s) d_q s = v_1\left(\frac{1}{3}\right), \quad \int_0^{1/2} v_1(s) d_q s = 0.$$

The inner layer is

$${}^C D_{q,0+}^{5/2} w_1(\varrho) = u_1(\varrho), \quad u_1(\varrho) = \int_0^\varrho v_1(s) d_q s, \quad (3.21)$$

subject to

$$w_1(0) = 0, \quad D_q w_1(1) = D_q^2 w_1(1), \quad w_1(1) = \int_0^{1/2} w_1(\varrho) d_q \varrho.$$

The parameter values and the corresponding nonresonance quantities are summarized in Table 1.

Table 1 Parameter values and nonresonance quantities for Example 3.5

Quantity	Value
q	$1/2$
$p = z$	2
ς_1	$5/2$
ξ_1	$7/2$
ρ_1	$5/2$
ϵ	$1/3$
η	$1/2$
ϑ	$1/4$
$[2]_q$	$3/2$
$[3]_q$	$7/4$
$[4]_q$	$15/8$
$\Delta_1 = \epsilon^2 - \epsilon$	$-2/9$
$\det(\mathcal{M}_2)$	$-1/189$
$\det(\mathcal{M}_3)$	$13/14$

Since $0 < \epsilon < 1$ and the determinants in Table 1 are nonzero, the Green kernels $G_1^{(1)}$, $G_1^{(2)}$, and $G_1^{(3)}$ are well defined by Lemmas 3.1–3.3.

We verify the hypotheses of Theorem 3.4. The multifunction $\mathcal{H}_1$ has nonempty compact convex values. For each fixed $(a, b) \in \mathbb{R}^2$, the map $\varrho \mapsto \mathcal{H}_1(\varrho, a, b)$ is measurable, and for each fixed $\varrho \in [0, 1]$, the map $(a, b) \mapsto \mathcal{H}_1(\varrho, a, b)$ is continuous in the Hausdorff metric. Thus $\mathcal{H}_1$ is an L^1 -Carathéodory multifunction with compact convex values.

Moreover,

$$0 \leq \frac{|a|}{1 + |a|} \leq 1, \quad 0 \leq \frac{|b|}{1 + |b|} \leq 1, \quad 0 \leq \varrho \leq 1,$$

and hence

$$\|\mathcal{R}_1(\varrho, a, b)\| = \|\mathcal{H}_1(\varrho, a, b)\| \leq \frac{\varrho + 1}{20(3 - \varrho)} + \frac{1}{40} \leq \frac{3}{40}.$$

Therefore hypothesis (H2) holds with

$$\alpha_1(\varrho) = \frac{3}{40}, \quad \beta_1(\varrho) = 0, \quad \varrho \in [0, 1].$$

The value $\mathcal{H}_1(\varrho, a, b)$ is an interval for every (ϱ, a, b) ; consequently, the corresponding values of the multivalued operator $\mathcal{L}$ are convex.

Let M_{11} , M_{12} , M_{13} , and $\widehat{M}_{13}$ be the finite kernel bounds associated with $G_1^{(1)}$, $G_1^{(2)}$, $G_1^{(3)}$, and $D_{q,\varrho}G_1^{(3)}$, respectively, as defined in (3.16). Since $\int_0^1 1 \, d_q \varrho = 1$, one has

$$\mu_1(R) = \|\alpha_1 + R\beta_1\|_{L_q^1} = \frac{3}{40}.$$

Thus

$$A_1 = M_{11} \frac{3}{40} + \|\Psi_1\|_\infty.$$

Because $p = z = 2$, the radius function in (3.18) reduces to

$$\Theta_1 = (M_{13} + \widehat{M}_{13})M_{12}A_1.$$

This number is finite. Choosing any $R > \Theta_1$ gives $\Theta_1 \leq R$, and hence hypothesis (H4) is satisfied. Therefore, by Theorem 3.4, the scalar boundary value problem (3.19)–(3.21) admits at least one solution in V_q .

As shown in Figure 1, the function $\alpha(\rho)$ remains nearly constant over the interval $\rho \in [0, 1]$. The computed growth bound is $\alpha(\rho) = 3.40$, indicating that no noticeable variation occurs across the considered range of ρ . Figure 3 compares three represen-

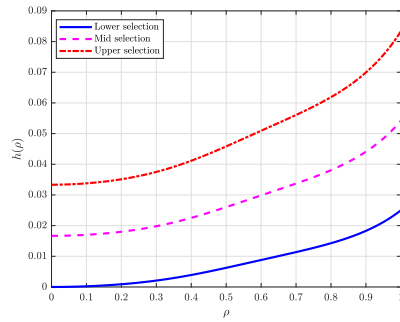


Figure 2 Lower, mid, and upper selections of the multifunction $\mathcal{H}$ corresponding to the test profiles; see Table 2 for the associated numerical values.

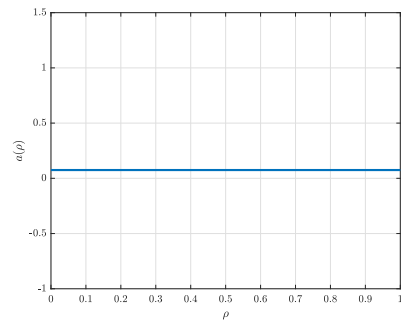


Figure 1 Plot of $\alpha(\rho)$ for $\rho \in [0, 1]$, indicating a growth bound of 3.40 and an approximately constant profile across the domain.

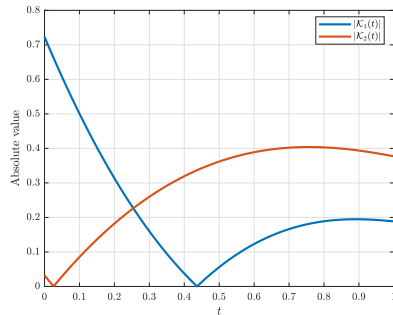


Figure 4 Comparison of $\mathcal{K}_1(t)$ and $\mathcal{K}_2(t)$ in terms of their absolute values over $t \in [0, 1]$.

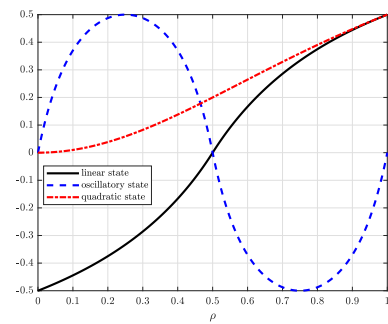


Figure 3 Comparison of the system states versus the normalized parameter $\rho \in [0, 1]$: linear state (solid black), oscillatory state (dashed blue), and quadratic state (dash-dotted red).

tative state profiles as functions of the normalized parameter $\rho \in [0, 1]$. The linear state (solid black) exhibits a monotonic increase across the domain, whereas the oscillatory state (dashed blue) shows a sign-changing periodic-like behavior with a

peak and a trough within the interval. In contrast, the quadratic state (dash-dotted red) increases smoothly and remains positive, providing an intermediate trend between the other two profiles. As shown in Figure 4, the absolute values of $\mathcal{K}_1(t)$ and $\mathcal{K}_2(t)$ exhibit distinct trends over the time interval $t \in [0, 1]$. While $|\mathcal{K}_1(t)|$ decreases sharply at the beginning and then gradually increases, $|\mathcal{K}_2(t)|$ grows from zero and reaches a higher level in the second half of the interval, indicating a different contribution of the two kernels over time. Figure 2 displays the lower, mid,

Table 2 Lower/mid/upper selections of $\mathcal{H}$ for test profiles with step size 0.05

rho	u_test	v_test	LowerSelection	MidSelection	UpperSelection
0.00000	0.00000	1.00000	0.00000	0.01667	0.03333
0.05000	0.15643	0.97553	0.00006	0.01676	0.03345
0.10000	0.30902	0.90451	0.00023	0.01701	0.03379
0.15000	0.45399	0.79389	0.00050	0.01742	0.03433
0.20000	0.58779	0.65451	0.00089	0.01800	0.03512
0.25000	0.70711	0.50000	0.00141	0.01878	0.03614
0.30000	0.80902	0.34549	0.00209	0.01980	0.03750
0.35000	0.89101	0.20611	0.00292	0.02105	0.03917
0.40000	0.95106	0.09549	0.00389	0.02250	0.04112
0.45000	0.98769	0.02447	0.00503	0.02421	0.04339
0.50000	1.00000	0.00000	0.00625	0.02604	0.04583
0.55000	0.98769	0.02447	0.00755	0.02799	0.04842
0.60000	0.95106	0.09549	0.00880	0.02986	0.05093
0.65000	0.89101	0.20611	0.01007	0.03177	0.05347
0.70000	0.80902	0.34549	0.01137	0.03372	0.05608
0.75000	0.70711	0.50000	0.01286	0.03595	0.05905
0.80000	0.58779	0.65451	0.01424	0.03802	0.06181
0.85000	0.45399	0.79389	0.01606	0.04076	0.06545
0.90000	0.30902	0.90451	0.01830	0.04412	0.06993
0.95000	0.15643	0.97553	0.02117	0.04843	0.07569
1.00000	0.00000	1.00000	0.02500	0.05417	0.08333

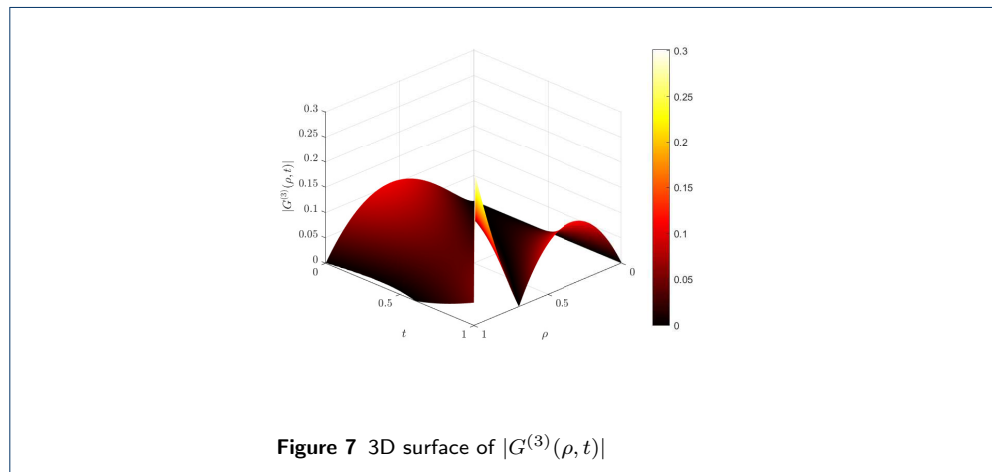
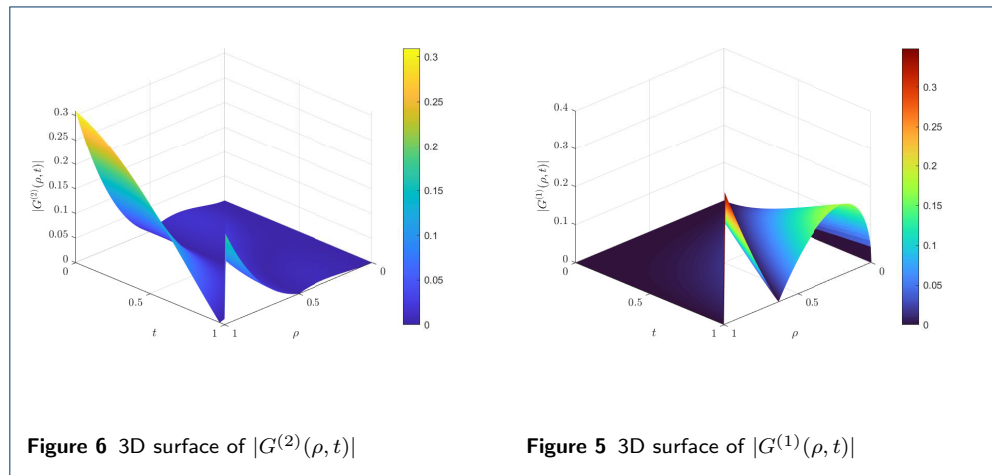
and upper selections of the set $\mathcal{H}$, denoted by $h(\rho)$, over the parameter interval $\rho \in [0, 1]$. All three selection profiles exhibit a strictly monotonic, upward trend as ρ increases from 0 to 1. The lower selection (solid blue line) starts from approximately 0 and reaches about 0.025 at the boundary, while the mid (dashed magenta line) and upper (dash-dotted red line) selections maintain a consistent hierarchical order throughout the entire domain, terminating at approximately 0.054 and 0.083, respectively. Table 3 provides a comprehensive statistical summary of the three kernels $|G^{(1)}|$, $|G^{(2)}|$, and $|G^{(3)}|$. While all kernels share a minimum value of 0.00000, $|G^{(1)}|$ exhibits the highest peak with a maximum value of 0.34785 and the largest L^2 norm (0.07250). In contrast, $|G^{(3)}|$ shows the highest mean (0.04517) and median (0.03942), suggesting a more uniform distribution of values compared to $|G^{(1)}|$ and $|G^{(2)}|$, despite having the smallest L^2 norm among the three.

Figures 5, 6, 7 illustrate the three-dimensional surface profiles of the kernels $|G^{(1)}|$, $|G^{(2)}|$, and $|G^{(3)}|$ within the domain $(\rho, t) \in [0, 1] \times [0, 1]$. The plots reveal distinct

Table 3 Summary statistics for $|G^{(1)}|$, $|G^{(2)}|$, and $|G^{(3)}|$

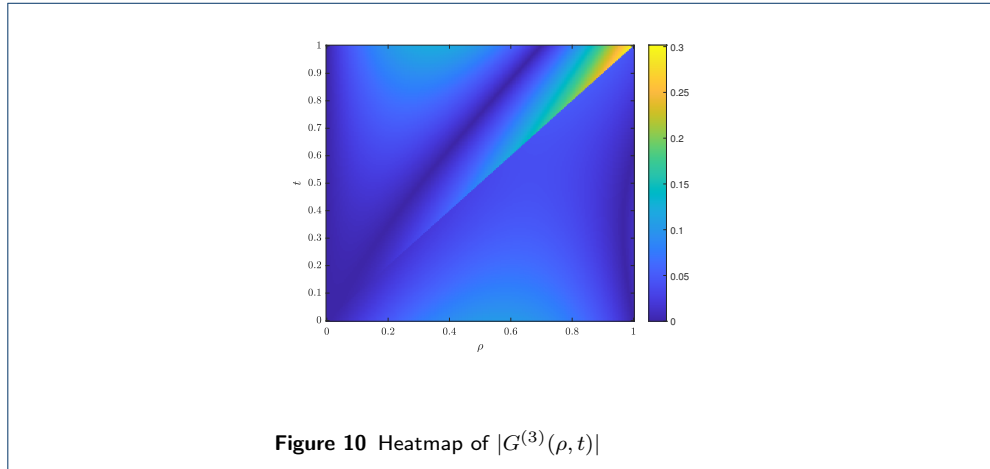
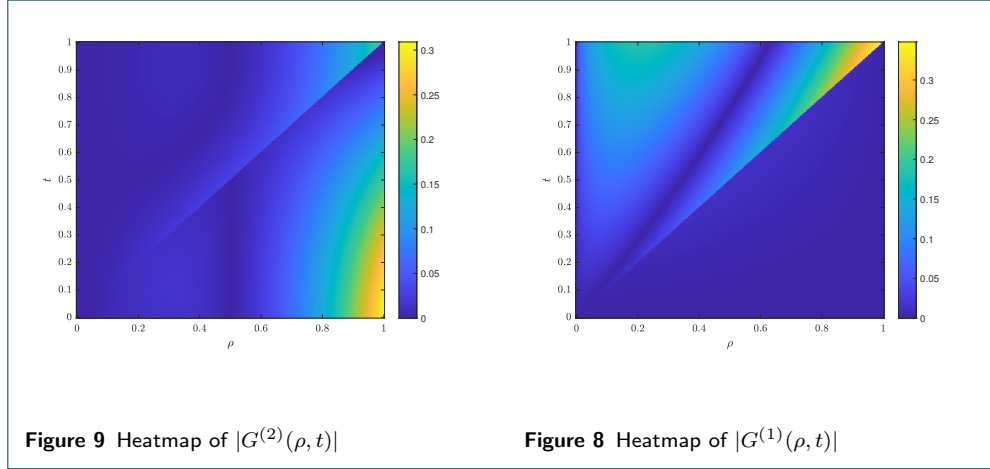
Kernel	Min	Max	Mean	Median	L2norm
$ G^{(1)} $	0.00000	0.34785	0.04285	0.00999	0.07250
$ G^{(2)} $	0.00000	0.30907	0.03923	0.01327	0.06932
$ G^{(3)} $	0.00000	0.30160	0.04517	0.03942	0.05845

characteristics for each kernel: while $|G^{(1)}|$ and $|G^{(2)}|$ exhibit sharp, concentrated peaks, indicating localized influence or potential singularities near specific regions, $|G^{(3)}|$ displays a more distributed and smoother surface profile. This visual comparison highlights the varying sensitivity and spatial-temporal dependencies inherent in each kernel formulation.



Figures 8, 9, 10 provide a top-down visualization of the kernel magnitudes, offering complementary insights to the 3D surface plots discussed previously. The heatmaps clearly delineate the regions of high and low intensity across the spatial-temporal domain $(\rho, t) \in [0, 1] \times [0, 1]$. It is observed that while $|G^{(1)}|$ (a) and $|G^{(3)}|$ (c) demonstrate a similar diagonal structural dependency, indicating a strong coupling along the line $\rho \approx t$, $|G^{(2)}|$ (b) displays a markedly different behavior with high-intensity concentrations localized near the boundary $\rho = 1$. These variations in

kernel structure confirm that the different formulations of G exert distinct influences on the system dynamics, particularly in how they weigh the spatial and temporal components.



To further investigate the temporal dynamics of the kernels, Figures 11, 12, 13 display cross-sections of the absolute kernel values at three distinct spatial locations: $\rho = 0.25$, $\rho = 0.50$, and $\rho = 0.75$. At each cross-section, the kernels exhibit non-smooth transitions, likely corresponding to the intersection of the spatial and temporal boundaries. As ρ increases from 0.25 to 0.75, a clear shift in the relative dominance of the kernels is observed. Specifically, while $|G^{(1)}|$ (solid blue) tends to gain intensity in the later stages of the temporal interval, $|G^{(2)}|$ (dashed red) shows a more muted response, and $|G^{(3)}|$ (dash-dotted black) maintains significant activity near the discontinuity points. This detailed slice-by-slice comparison highlights the time-dependent evolution of each kernel and clarifies how spatial parameter variations affect their temporal signatures.

4 Aerospace vibration-modeling illustration

A practical motivation for memory-dependent fractional modeling appears in vibration isolation for multirotor unmanned aerial vehicles. Rotor imbalance, motor

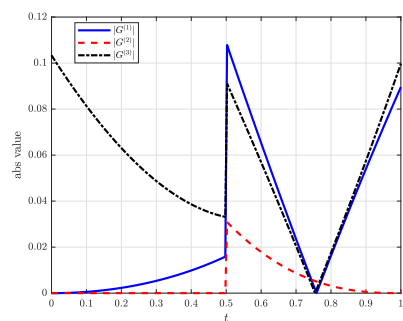


Figure 12 kernel comparison at $\rho = 0.50$

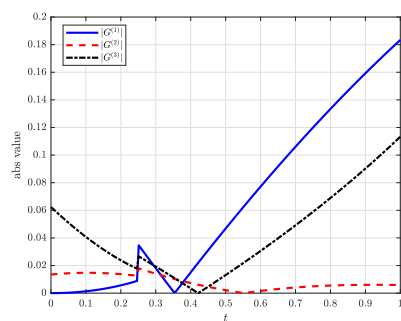


Figure 11 kernel comparison at $\rho = 0.25$

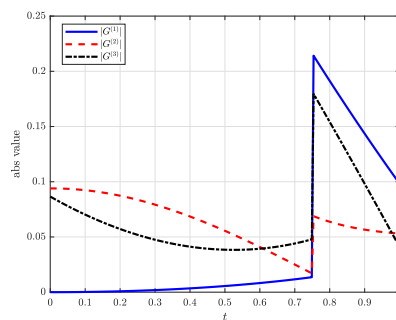


Figure 13 kernel comparison at $\rho = 0.75$

torque ripple, and aerodynamic disturbances transmit structural vibration from the arms and frame to the inertial measurement unit (IMU). When this vibration reaches the sensor board, it increases the effective noise level in gyroscope and accelerometer signals and can degrade attitude estimation. A compact isolation layer between the airframe and the IMU board is therefore a common engineering mechanism for reducing transmitted acceleration.

Viscoelastic and polymeric mounting layers often exhibit frequency-dependent and history-dependent behavior. A purely viscous term may not represent this response over a wide frequency range. This section uses a base-excited single-degree-of-freedom model to illustrate how a Caputo q -fractional damping term can describe such memory-dependent behavior. The illustration is used to connect the analytical framework with an engineering modeling context; the existence result itself is established by the assumptions and fixed-point arguments developed in the previous sections.

Let $x_b(t)$ denote the base displacement of the UAV frame at the mounting point and let $x_s(t)$ denote the displacement of the IMU support platform. The relative isolator deformation is

$$y(t) = x_s(t) - x_b(t).$$

The q -fractional damping model is written as

$$m\ddot{y}(t) + ky(t) + c {}^C D_{q,0+}^\alpha y(t) = -m\ddot{x}_b(t), \quad 0 < \alpha < 1, \quad 0 < q < 1,$$

where m is the effective mass of the IMU support, k is the isolator stiffness, c is the damping coefficient, and ${}^C D_{q,0+}^\alpha$ is the Caputo q -fractional derivative. The transmitted acceleration at the IMU platform is

$$\ddot{x}_s(t) = \ddot{y}(t) + \ddot{x}_b(t).$$

The RMS value of $\ddot{x}_s(t)$ is used as a compact time-domain measure of vibration transmission.

For consistency with Section 2, the Caputo q -fractional derivative used in this illustration is

$${}^C D_{q,0+}^\alpha f(t) = I_{q,0+}^{1-\alpha} D_q f(t) = \frac{1}{\Gamma_q(1-\alpha)} \int_0^t (t-q\tau)_q^{-\alpha} D_q f(\tau) d_q \tau, \quad 0 < \alpha < 1.$$

Thus, the damping force depends on a weighted history of the Jackson q -rate of the isolator deformation. This representation provides a tunable memory term while remaining consistent with the q -fractional calculus used in the theoretical development.

For a representative numerical comparison, the base acceleration is modeled as a dominant rotor-frequency component and its second harmonic,

$$\ddot{x}_b(t) = A \sin(2\pi f t) + 0.2A \sin(2\pi(2f)t),$$

with $A = 5 \text{ m/s}^2$ and $f = 150 \text{ Hz}$. Three mounting cases are compared: rigid mounting, a classical viscous isolator, and the q -fractional isolator. The classical viscous isolator is described by

$$m\ddot{y}(t) + c_v\dot{y}(t) + ky(t) = -m\ddot{x}_b(t),$$

whereas the q -fractional isolator is governed by the preceding Caputo q -fractional damping model. The comparison is intended to illustrate the modeling effect of the fractional memory term under one parameter setting, not to claim optimality of the selected fractional parameters.

Table 4 RMS transmitted acceleration for the three vibration-isolation cases.

Case	RMS acceleration	Reduction relative to rigid mounting (%)
Rigid mounting	3.605326	0.00000
Classical viscous isolator	0.94004	73.92636
q -fractional isolator	2.072711	42.50973

Table 4 shows that both isolation models reduce the transmitted acceleration relative to rigid mounting. For the reported parameter set, the classical viscous isolator gives the lowest RMS acceleration, while the q -fractional isolator provides an intermediate reduction. The comparison therefore supports the use of the q -fractional term as a memory-dependent modeling element, rather than as a claim of superior attenuation for the selected parameters.

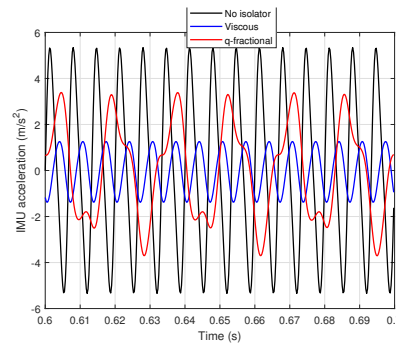


Figure 14 Zoomed time-history of transmitted IMU-platform acceleration $\ddot{x}_s(t)$ for three mounting configurations: rigid mounting (No isolator), classical viscous isolator, and the proposed q -fractional isolator, shown over $t \in [0.6, 0.7] \text{ s}$. The zoomed view highlights the reduction in oscillation amplitude achieved by the isolation layers, with the viscous isolator producing the smallest response and the q -fractional model yielding an intermediate attenuation relative to the rigid case.

Figures 15 and 14 present the transmitted acceleration at the IMU platform, $\ddot{x}_s(t)$, under the same base excitation for three mounting cases. Figure 15 provides the overall response over $0 \leq t \leq 2 \text{ s}$ to demonstrate the steady-state vibration envelope,

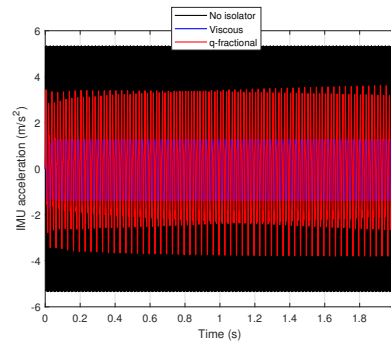


Figure 15 Full time-history of transmitted IMU-platform acceleration $\ddot{x}_s(t)$ over $t \in [0, 2]$ s for the same three mounting configurations. While the rigid mounting exhibits the largest steady-state vibration envelope, both isolator models reduce the transmitted acceleration throughout the entire interval. The viscous isolator provides the strongest attenuation, and the proposed q -fractional isolator maintains a consistent reduction compared to the no-isolator case.

whereas Figure 14 zooms into $t \in [0.6, 0.7]$ s to resolve cycle-by-cycle differences in peak amplitudes. In both views, the isolation layers reduce the transmitted acceleration relative to rigid mounting. The viscous isolator achieves the largest attenuation, while the proposed q -fractional isolator delivers a clear intermediate reduction, consistent with the RMS-based performance comparison reported in Table 4.

Figure 16 shows the frequency-domain comparison via the PSD of the transmitted acceleration at the IMU platform. The no-isolator case exhibits pronounced spectral peaks at the dominant excitation frequency and its harmonics, indicating strong vibration transmission from the airframe to the sensor mount. Both the viscous and q -fractional isolators suppress the spectral magnitude around these components, which is desirable for flight-control sensing because it reduces narrowband vibration contamination in IMU measurements. In line with the time-domain envelopes (Figures 15–14) and the RMS metrics (Table 4), the viscous isolator achieves the greatest reduction, whereas the q -fractional isolator provides a clear but smaller attenuation relative to the rigid mounting baseline.

5 Conclusion

In this work, we studied a coupled system of nonlinear q -fractional differential inclusions involving nested p -Laplacian operators and nonlocal boundary conditions. After reformulating the problem through a layerwise Green-function representation, we established sufficient conditions for the existence of at least one solution by using multivalued fixed-point theory. The analysis was developed in the framework of q -fractional calculus and incorporated compact convex multifunction values, nonlinear p -Laplacian terms, and nonlocal boundary constraints.

The illustrative example verified the applicability of the existence theorem under an explicit set of admissible parameters. An aerospace vibration-modeling illus-

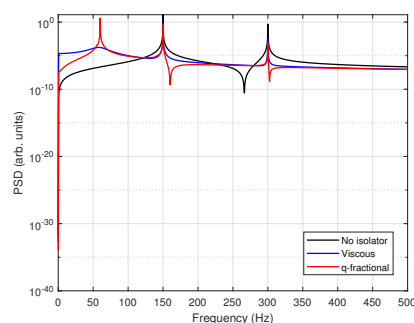


Figure 16 Power spectral density (PSD) of the transmitted IMU-platform acceleration for three mounting configurations (No isolator, viscous isolator, and q -fractional isolator). The spectrum reveals dominant peaks near the excitation frequencies (fundamental and harmonics), and demonstrates that both isolation strategies reduce spectral energy around these components relative to rigid mounting. The viscous isolator provides the strongest broadband attenuation, while the proposed q -fractional isolator achieves an intermediate reduction consistent with the time-domain results.

tration was also included to show how a Caputo q -fractional damping term can represent memory-dependent behavior in an IMU isolation model. In the reported numerical comparison, both isolation models reduced transmitted acceleration relative to rigid mounting, while the classical viscous isolator produced the lowest RMS response for the selected parameter set. Future work may address uniqueness, stability, numerical approximation, parameter identification, and control-oriented extensions of the proposed q -fractional inclusion framework.

Declarations

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

Funding

Not applicable.

Authors' contributions

MT. V.: Formal analysis, investigation. **B. M.:** Conceptualization, Writing – original draft. **H. A.:** Formal analysis, Conceptualization. **A. M.:** Writing – review & editing.

Author details

¹Department of Artificial Intelligence and Robotics, Sejong University, Seoul 05006, Republic of Korea.

²Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran. ³Department of Mathematics, Faculty of Science, University of Bonab, Bonab, Iran. ⁴Faculty of Engineering, Department of Electrical and Electronics Engineering, Sakarya University, Sakarya, Türkiye.

References

1. H. Afshari and A. Ahmadvanlu. A discussion on some new boundary value problems involving the q -derivative operator via the existence of positive solutions. *Optical and Quantum Electronics*, 56(8):1312, 2024.
2. A. Ahmadvanlu, H. Afshari, and J. Alzabut. A new fixed point approach for solutions of a p -laplacian fractional q -difference boundary value problem with an integral boundary condition. *AIMS Mathematics*, 9(9):23770–23785, 2024.
3. W.F. Alfwzan, H. Khan, and J. Alzabut. Stability analysis for a fractional coupled hybrid pantograph system with p -laplacian operator. *Results in Control and Optimization*, 14:100333, 2024.
4. J. Aubin and A. Cellina. *Differential Inclusions, Set-Valued Maps and Viability Theory*. Springer-Verlag, Berlin, 1984.
5. S. M. Aydogan, F. Muge Sakar, M. Fatehi, S. Rezapour, and H. Parvaneh Masiha. Two hybrid and non-hybrid k -dimensional inclusion systems via sequential fractional derivatives. *Advances in Difference Equations*, 2021(1):449, 2021.
6. A. Berhail, J. Alzabut, and M.E. Samei. On multi-order nonlinear langevin type of FDE subject to multi-point boundary conditions. *Qual. Theory Dyn. Syst.*, 24(1):58, 2025.
7. A. Boutiara, S.K. Ntouyas, and T.A. Assiri. On the cauchy problem for nonlinear fractional systems with lipschitzian matrices under the generalized metric spaces. *Qual. Theory Dyn. Syst.*, 23(Suppl 1):264, 2024.
8. A. Cabada, S. Dimitrijevic, T. Tomovic, and S. Aleksic. The existence of a positive solution for nonlinear fractional differential equations with integral boundary value conditions. *Mathematical Methods in the Applied Sciences*, 40(6):1880–1891, 2016.
9. A. Cabada and Z. Hamdi. Nonlinear fractional differential equations with integral boundary value conditions. *Applied Mathematics and Computation*, 228:251–257, 2014.
10. G. Chai. Positive solutions for boundary value problem of fractional differential equation with p -laplacian operator. *Boundary Value Problems*, 2012(1):18, 2012.
11. T. Chen and W. Liu. An anti-periodic boundary value problem for the fractional differential equation with a p -laplacian operator. *Applied Mathematics Letters*, 25(11):1671–1675, 2012.
12. K. Deimling. *Multivalued Differential Equations*. De Gruyter, Berlin, 1992.
13. B.C. Dhage. Existence results for neutral functional differential inclusions in banach algebras. *Nonlinear Analysis, Theory, Methods and Applications*, 64(6):1290–1306, 2006.
14. M. Günendi and I. Yaslan. Positive solutions of higher-order nonlinear multi-point fractional equations with integral boundary conditions. *Fractional Calculus and Applied Analysis*, 19(4):989–1009, 2016.
15. Z. Han, H. Lu, and C. Zhang. Positive solutions for eigenvalue problems of fractional differential equation with generalized p -laplacian. *Applied Mathematics and Computation*, 257:526–536, 2015.
16. T. Jankowski. Positive solutions to fractional differential equations involving stieltjes integral conditions. *Applied Mathematics and Computation*, 241:200–213, 2014.
17. H. Khan, J. Alzabut, and A. Alkhazzan. Qualitative dynamical study of hybrid system of pantograph equations with nonlinear p -laplacian operator in banach's space. *Results in Control and Optimization*, 15(1):100416, 2024.
18. A. Lasota and Z. Opial. An application of the kakutani-ky fan theorem in the theory of ordinary differential equations. *Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques*, 13:781–786, 1965.
19. Y. Li and A. Qi. Positive solutions for multi-point boundary value problems of fractional differential equations with p -laplacian. *Mathematical Methods in the Applied Sciences*, 39(6):1425–1434, 2016.
20. X. Liu, M. Jia, and W. Ge. The method of lower and upper solutions for mixed fractional four-point boundary value problem with p -laplacian operator. *Applied Mathematics Letters*, 65:56–62, 2017.
21. K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York, 1993.
22. S. Ntouyas and S. Etemad. On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions. *Applied Mathematics and Computation*, 266:235–243, 2015.
23. I. Podlubny. *Fractional Differential Equations*. Academic Press, New York, 1999.
24. B. Samet, C. Vetro, and P. Vetro. Fixed point theorems for α - ψ -contractive type mappings. *Nonlinear Analysis: Theory, Methods & Applications*, 75(4):2154–2165, 2012.
25. S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives, Theory and Applications*. Gordon and Breach Science Publishers, Amsterdam, 1993.