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Madjid Eshaghi Gordji

Semnan University

Mohamadali Berahman

`mohamadali_berahman@semnan.ac.ir`

Semnan University

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Certifying When Temporal Order Can Be Ignored in Continuous-Time Quantum Walks

Madjid Eshaghi Gordji

Faculty of Mathematics, Statistics and Computer Science,
Semnan University, Semnan, Iran
meshaghi@semnan.ac.ir

Mohamadali Berahman*

Faculty of Mathematics, Statistics and Computer Science,
Semnan University, Semnan, Iran

Abstract

Continuous-time quantum walks on dynamic graphs are generated by an ordered sequence of Hamiltonians. Replacing that sequence by one average Hamiltonian can simplify analysis and control, but it may change the transported quantum state. We study this replacement in the single-excitation sector for sequential, undirected edge couplings. Nonzero edge Hamiltonians commute precisely when their edges coincide or are vertex-disjoint, giving a pulse-independent exact reduction for schedules supported on a matching. For general schedules, we derive a nonperturbative spectral-norm bound between the ordered unitary and the exponential of the integrated Hamiltonian. In the edge model, the bound becomes a weighted count of temporal wedges—pairs of distinct event edges meeting at one vertex—and is computable in expected linear time. The same certificate controls state-transfer amplitudes and probabilities, process fidelity, average gate fidelity, and sensitivity to event reordering. A weak-pulse expansion shows that its quadratic scaling is generally sharp. An exact three-vertex calculation and reproducible numerical tests illustrate when the certificate is informative and when cancellation makes it conservative. The result is a directly checkable sufficient condition for replacing a dynamic quantum walk by its weighted aggregate.

Keywords: continuous-time quantum walks; dynamic quantum walks; quantum state transfer; average Hamiltonian; noncommuting Hamiltonians; product-formula error

1 Introduction

Continuous-time quantum walks describe coherent transport through a network by using a graph Hamiltonian to move amplitude between localized states [1–4]. In the single-excitation sector, the same description applies to exchange-coupled qubit or spin networks, where the transition probability between two vertices is a state-transfer probability [5, 6]. Recent work has continued to examine how graph structure, local potentials, chirality, and edge weights control this transport [5–7].

The Hamiltonian need not remain fixed during a computation or transfer task. Programmable couplings lead instead to a dynamic quantum walk: an ordered sequence of graph Hamiltonians acts on the state. Such walks have been used for search, state transfer, and universal quantum computation [8, 9]. The order matters for a specifically quantum reason. Two edge Hamiltonians that share a vertex generally fail to commute, so changing the switching schedule can change amplitudes, phases, and the implemented unitary even when the same weighted edges are used.

*Corresponding author: mohamadali_berahman@semnan.ac.ir

This is also where temporal-network language is useful. A temporal network records not only which vertices interact but when each interaction is available [10, 11]. The question studied here is whether a sequential quantum walk can be replaced by the exponential of its integrated Hamiltonian without losing the effect of that chronology.

Herrman and Wong identified several exact simplifications of dynamic continuous-time quantum walks, including transformations based on commuting graph Hamiltonians [12]. We address a different point. When exact commutativity is absent, how large can the aggregation error be, and can that error be certified directly from the edge-event list rather than from a full matrix calculation?

This question is related to average-Hamiltonian methods, the Magnus expansion, and product-formula error analysis [13–17]. Those tools explain why commutators matter and provide general approximation techniques. A temporal edge network, however, has more structure than an arbitrary sum of matrices. Its commutators are determined by a simple local pattern: two distinct edges meeting at one vertex. This observation turns a matrix error estimate into a graph-readable certificate.

The answer is a temporal-wedge certificate. We first identify the edge structures that give a pulse-independent commuting schedule. We then bound the unitary aggregation error by a weighted count of pairs of distinct edges that meet at one vertex and show that this count can be evaluated in expected linear time. The bound is translated into statements about quantum state-transfer probabilities and channel fidelity, and a separate estimate measures the effect of reordering a schedule. Complete proofs remain in the main text so that the scope of each result is explicit. The Supplemental Material contains the matrix calculations and numerical protocol.

2 Event-sequence model and comparison dynamics

Let $V = \{1, \dots, n\}$ and let $\mathcal{H} = \mathbb{C}^n$ be the single-excitation Hilbert space with localized basis $\{|v\rangle : v \in V\}$. An event sequence is $\mathcal{E} = (e_1, \dots, e_m)$, ordered from earliest to latest. Event r activates the unordered edge

$$e_r = \{u_r, v_r\}, \quad u_r \neq v_r,$$

for duration $\Delta_r > 0$ with real coupling J_r . Its Hamiltonian and integrated action are

$$H_r = J_r X_{e_r}, \quad X_{\{u,v\}} = |u\rangle\langle v| + |v\rangle\langle u|, \quad A_r = \Delta_r H_r = \alpha_r X_{e_r}, \quad (1)$$

where $\alpha_r = J_r \Delta_r$ is the pulse area. The sign of J_r is allowed to change because controlled couplings need not have a fixed sign. This hopping Hamiltonian is the single-excitation restriction of an exchange interaction between qubits or spins; it also describes coherent transfer between coupled modes. The basis vector $|v\rangle$ therefore represents a localized excitation, not a classical probability state. Relative phases are retained throughout the evolution and can affect later transfer events. The temporal propagator is

$$U_{\text{temp}} = e^{-iA_m} \dots e^{-iA_2} e^{-iA_1}. \quad (2)$$

The first event acts first, hence later events occur on the left.

The comparison dynamics preserves the integrated action but removes the order. Put

$$S = \sum_{r=1}^m A_r, \quad U_{\text{av}} = e^{-iS}. \quad (3)$$

If $T = \sum_r \Delta_r$, then $H_{\text{av}} = S/T$ is the weighted average Hamiltonian and $U_{\text{av}} = e^{-iH_{\text{av}}T}$. We use *weighted aggregate* or *average dynamics* rather than *static projection*, because S retains durations, coupling strengths, and repeated activations. It discards only chronological order.

The model is deliberately narrow. Events are sequential and nonoverlapping. If several couplings are applied simultaneously, their sum is one Hamiltonian layer and need not be a

single-edge event. Idle gaps generate the identity only when there is no background Hamiltonian. A nonzero drift must be included explicitly in each layer. Finally, the Hamiltonian in [equation \(1\)](#) is undirected: the ordered pairs (u, v) and (v, u) would describe the same operator. Intrinsically directed transport requires a different, generally open-system, model.

For later use, define the operator error

$$\varepsilon_{\text{op}} = \|U_{\text{temp}} - U_{\text{av}}\|_2. \quad (4)$$

For localized input s and output w , write

$$p_{\text{temp}}(s, w) = |\langle w | U_{\text{temp}} | s \rangle|^2, \quad p_{\text{av}}(s, w) = |\langle w | U_{\text{av}} | s \rangle|^2.$$

The phase-insensitive process fidelity and its corresponding average gate fidelity are [\[18\]](#)

$$F_{\text{pro}} = \frac{|\text{Tr}(U_{\text{temp}}^\dagger U_{\text{av}})|^2}{n^2}, \quad F_{\text{avg}} = \frac{nF_{\text{pro}} + 1}{n + 1}. \quad (5)$$

These two fidelities are related but are not the same quantity.

3 Graph structure behind exact aggregation

Before estimating an error, we need to know when no error can arise for structural reasons. The following theorem identifies the pulse-independent commuting case for the edge Hamiltonians in [equation \(1\)](#). It gives an exact reduction for every ordering and every choice of nonzero pulse areas. It does not rule out accidental equalities at isolated pulse values.

Theorem 1 (Commuting edge structure). *Let e and f be loop-free undirected edges and let $A_e = \alpha X_e$, $A_f = \beta X_f$, with $\alpha\beta \neq 0$. Then*

$$[A_e, A_f] = 0 \iff e = f \text{ or } e \cap f = \emptyset.$$

If the distinct edges in an event sequence are pairwise commuting, they form a matching. Repeated activations of an edge are allowed. In this case $U_{\text{temp}} = U_{\text{av}}$ for every ordering of the events.

Proof. If $e = f$, then A_e and A_f are scalar multiples of the same matrix and commute. If $e \cap f = \emptyset$, the two operators act on orthogonal two-dimensional coordinate subspaces. Both products $X_e X_f$ and $X_f X_e$ are then zero.

It remains to consider two distinct edges sharing one vertex. Relabeling the three involved vertices gives $e = \{a, b\}$ and $f = \{b, c\}$, where a, b, c are distinct. Direct multiplication yields

$$X_{\{a,b\}} X_{\{b,c\}} = |a\rangle \langle c|, \quad X_{\{b,c\}} X_{\{a,b\}} = |c\rangle \langle a|.$$

Consequently,

$$[A_e, A_f] = \alpha\beta(|a\rangle \langle c| - |c\rangle \langle a|) \neq 0.$$

This exhausts all possible relations between two loop-free edges and proves the equivalence. Hence a pairwise commuting family has no two distinct edges with a common endpoint, which is precisely the matching condition.

For a pairwise commuting family, the matrices $-iA_r$ commute. Repeated application of $e^X e^Y = e^{X+Y}$ for commuting matrices gives

$$e^{-iA_m} \dots e^{-iA_1} = e^{-i\sum_r A_r} = U_{\text{av}}.$$

The same argument applies to every permutation of the events. $\square$

Thus the structural guarantee is quite restrictive. For the present hopping model, a connected set of several distinct active edges cannot be pairwise commuting: some two edges meet at a vertex and generate a nonzero commutator. This statement concerns a guarantee valid for all pulse areas. Special pulse values can still produce isolated coincidences between two propagators, and the theorem makes no claim about classifying them.

4 A graph-readable error certificate

Exact commutativity is too restrictive for most event sequences, so we need a quantitative statement that remains valid when the couplings are not small and does not assume convergence of a Magnus series. We use a standard type of first-order product-formula estimate [15, 17]. Its unitary form is proved here because it is the bridge to the graph certificate and makes the factor $1/2$ explicit.

Lemma 2 (Two-factor unitary bound). *For finite-dimensional Hermitian matrices A and B ,*

$$\left\| e^{-iA} e^{-iB} - e^{-i(A+B)} \right\|_2 \leq \frac{1}{2} \| [A, B] \|_2. \quad (6)$$

Proof. Set $X = -iA$ and $Y = -iB$, so X , Y , and $X + Y$ are skew-Hermitian. Let $Z(t) = e^{t(X+Y)}$ and $W(t) = e^{tX} e^{tY}$. The residual of W in the differential equation for Z is

$$W'(t) - (X + Y)W(t) = (e^{tX} Y - Y e^{tX}) e^{tY}.$$

Variation of constants, together with $Z(0) = W(0) = I$, gives

$$Z(1) - W(1) = - \int_0^1 e^{(1-t)(X+Y)} (e^{tX} Y - Y e^{tX}) e^{tY} dt.$$

The inner difference has the exact representation

$$e^{tX} Y - Y e^{tX} = \int_0^t e^{sX} [X, Y] e^{(t-s)X} ds.$$

Every exponential appearing here is unitary. Taking spectral norms and using unitary invariance therefore yields

$$\|Z(1) - W(1)\|_2 \leq \int_0^1 \int_0^t \| [X, Y] \|_2 ds dt = \frac{1}{2} \| [X, Y] \|_2.$$

Since $[X, Y] = -[A, B]$, this is [equation \(6\)](#). □

The lemma says that the cost of combining two consecutive unitary steps is controlled by their commutator, with no exponential prefactor. This clean form is possible because the generators are Hermitian and all intermediate propagators are unitary.

We now apply the lemma repeatedly. This theorem is the central error result: it converts the full time-ordering problem into a sum of pairwise local obstructions.

Theorem 3 (Nonperturbative aggregate-error bound). *For any finite sequence of Hermitian actions $A_1, \dots, A_m$,*

$$\|U_{\text{temp}} - U_{\text{av}}\|_2 \leq \frac{1}{2} \sum_{r>q} \| [A_r, A_q] \|_2. \quad (7)$$

For the edge actions in [equation \(1\)](#), define the temporal-wedge budget

$$\mathcal{W}(\mathcal{E}) = \frac{1}{2} \sum_{r>q} |\alpha_r \alpha_q| \mathbf{1}\{|e_r \cap e_q| = 1\}. \quad (8)$$

Then

$$\varepsilon_{\text{op}} \leq \min\{2, \mathcal{W}(\mathcal{E})\}. \quad (9)$$

Proof. Put $P_k = e^{-iA_k} \dots e^{-iA_1}$, $S_k = \sum_{r=1}^k A_r$, and $Q_k = e^{-iS_k}$. For $k \geq 2$, insert and subtract $e^{-iA_k} Q_{k-1}$:

$$\begin{aligned} \|P_k - Q_k\|_2 &\leq \left\| e^{-iA_k} (P_{k-1} - Q_{k-1}) \right\|_2 \\ &\quad + \left\| e^{-iA_k} e^{-iS_{k-1}} - e^{-i(A_k + S_{k-1})} \right\|_2 \\ &\leq \|P_{k-1} - Q_{k-1}\|_2 + \frac{1}{2} \| [A_k, S_{k-1}] \|_2 \\ &\leq \|P_{k-1} - Q_{k-1}\|_2 + \frac{1}{2} \sum_{q < k} \| [A_k, A_q] \|_2. \end{aligned}$$

The first inequality uses the triangle inequality, the second uses unitary invariance and [theorem 2](#), and the last uses $[A_k, S_{k-1}] = \sum_{q < k} [A_k, A_q]$. Starting from $P_1 = Q_1$ and iterating to $k = m$ proves [equation \(7\)](#).

For edge events, [theorem 1](#) shows that the commutator vanishes for equal or disjoint edges. When two edges share exactly one vertex, the proof of that theorem also shows

$$\| [A_r, A_q] \|_2 = |\alpha_r \alpha_q|,$$

because $|a\rangle \langle c| - |c\rangle \langle a|$ has spectral norm one. Substitution into [equation \(7\)](#) gives $\varepsilon_{\text{op}} \leq \mathcal{W}(\mathcal{E})$. The difference of two unitary matrices has norm at most two, which gives the minimum in [equation \(9\)](#). $\square$

A repeated edge contributes nothing, as does a pair of disjoint edges. Only a temporal wedge—two distinct event edges meeting at one vertex—enters the certificate. Because the estimate adds commutator norms, it deliberately ignores cancellation. A small value certifies a good aggregate approximation; a large value does not by itself show that the approximation is poor.

Written as a double sum, $\mathcal{W}$ appears to require comparing all event pairs. That would be inconvenient for a long event log. The following identity rewrites the same quantity using vertex and edge totals, avoiding both commutator matrices and a quadratic pair scan.

Proposition 4 (Linear-time evaluation of the wedge budget). *Let $w_r = |\alpha_r|$. For each vertex v and each distinct event edge e , define*

$$s_v = \sum_{r: v \in e_r} w_r, \quad q_v = \sum_{r: v \in e_r} w_r^2, \quad s_e = \sum_{r: e_r = e} w_r, \quad q_e = \sum_{r: e_r = e} w_r^2.$$

Then

$$\mathcal{W}(\mathcal{E}) = \frac{1}{4} \sum_{v \in V} (s_v^2 - q_v) - \frac{1}{2} \sum_e (s_e^2 - q_e), \quad (10)$$

where the second sum is over the distinct edges appearing in the event list. The right-hand side can be evaluated in expected $O(m + n)$ time using a hash-based edge dictionary.

Proof. At a fixed vertex v ,

$$\frac{1}{2} (s_v^2 - q_v) = \sum_{\substack{r > q \\ v \in e_r \cap e_q}} w_r w_q$$

is the weighted sum of all unordered event pairs incident to v . Summing over vertices counts a pair of distinct edges sharing one vertex once, but counts two events on the same edge twice. Let

$$P = \sum_{\substack{r > q \\ |e_r \cap e_q| = 1}} w_r w_q, \quad E = \sum_{\substack{r > q \\ e_r = e_q}} w_r w_q.$$

It follows that

$$\frac{1}{2} \sum_v (s_v^2 - q_v) = P + 2E.$$

For each distinct edge, $\frac{1}{2}(s_e^2 - q_e)$ is the weighted sum over event pairs occurring on that edge, and hence

$$E = \frac{1}{2} \sum_e (s_e^2 - q_e).$$

Solving the preceding identity for P and using $\mathcal{W} = P/2$ proves [equation \(10\)](#). The four totals can be updated once per event; the final vertex and edge sums are linear in the number of stored entries. $\square$

The certificate can therefore be maintained while the event stream is read. Repeated activations are removed by the edge correction in [equation \(10\)](#); without that correction they would be counted twice, once at each endpoint.

An operator bound is most useful when it controls quantities that can be measured or used in a decision rule. The following corollary is included for that purpose.

Corollary 5 (Operational consequences). *Let $\epsilon = \min\{2, \mathcal{W}(\mathcal{E})\}$ and $\rho = \min\{1, 2\epsilon\}$. Then, for all localized states s, w ,*

$$|\langle w | (U_{\text{temp}} - U_{\text{av}}) | s \rangle| \leq \epsilon, \quad (11)$$

$$|p_{\text{temp}}(s, w) - p_{\text{av}}(s, w)| \leq \rho. \quad (12)$$

For a threshold $\eta \in (0, 1)$, the two classifications $\mathbf{1}\{p_{\text{temp}}(s, w) \geq \eta\}$ and $\mathbf{1}\{p_{\text{av}}(s, w) \geq \eta\}$ agree whenever

$$|p_{\text{av}}(s, w) - \eta| > \rho. \quad (13)$$

Moreover,

$$F_{\text{pro}} \geq \left[\max \left\{ 0, 1 - \frac{\epsilon^2}{2} \right\} \right]^2. \quad (14)$$

Consequently,

$$F_{\text{avg}} \geq \frac{n [\max \{0, 1 - \epsilon^2/2\}]^2 + 1}{n + 1}. \quad (15)$$

Proof. The amplitude bound follows from

$$|\langle w | (U_{\text{temp}} - U_{\text{av}}) | s \rangle| \leq \|U_{\text{temp}} - U_{\text{av}}\|_2 \leq \epsilon.$$

If $a = \langle w | U_{\text{temp}} | s \rangle$ and $b = \langle w | U_{\text{av}} | s \rangle$, then $|a|, |b| \leq 1$, and

$$||a|^2 - |b|^2| \leq (|a| + |b|)|a - b| \leq 2\epsilon.$$

Probabilities differ by at most one, proving [equation \(12\)](#). If the aggregate probability lies farther than ρ from the threshold, the probability bound prevents the temporal probability from crossing that threshold, which proves [equation \(13\)](#).

Finally,

$$\begin{aligned} \|U_{\text{temp}} - U_{\text{av}}\|_{\text{F}}^2 &= 2n - 2 \operatorname{Re} \operatorname{Tr}(U_{\text{temp}}^\dagger U_{\text{av}}), \\ \|U_{\text{temp}} - U_{\text{av}}\|_{\text{F}} &\leq \sqrt{n} \|U_{\text{temp}} - U_{\text{av}}\|_2 \leq \sqrt{n} \epsilon. \end{aligned}$$

Hence

$$\frac{\operatorname{Re} \operatorname{Tr}(U_{\text{temp}}^\dagger U_{\text{av}})}{n} \geq 1 - \frac{\epsilon^2}{2}.$$

Taking the nonnegative part, using $|z| \geq \operatorname{Re} z$, and squaring proves [equation \(14\)](#). Substitution into the exact relation between F_{pro} and F_{avg} in [equation \(5\)](#) gives [equation \(15\)](#). $\square$

The threshold statement is most useful when the aggregate probability has a clear margin. If it lies almost exactly on the chosen threshold, a worst-case estimate cannot guarantee that the decision is unchanged.

The bound in [theorem 3](#) is quadratic in the pulse areas. To see whether this is merely an artifact of the estimate, we examine the weak-pulse limit. The result identifies the leading mechanism while leaving the nonperturbative bound in place for finite pulses.

Proposition 6 (Weak-pulse leading term). *Scale all actions by a common $\lambda \in \mathbb{R}$, and define*

$$U_{\text{temp}}(\lambda) = e^{-i\lambda A_m} \dots e^{-i\lambda A_1}, \quad U_{\text{av}}(\lambda) = e^{-i\lambda \sum_r A_r}.$$

Then, in operator norm as $\lambda \rightarrow 0$,

$$U_{\text{temp}}(\lambda) - U_{\text{av}}(\lambda) = -\frac{\lambda^2}{2} \sum_{r>q} [A_r, A_q] + O(\lambda^3). \quad (16)$$

If $C = \sum_{r>q} [A_r, A_q] \neq 0$, then

$$\lim_{\lambda \rightarrow 0} \frac{\|U_{\text{temp}}(\lambda) - U_{\text{av}}(\lambda)\|_2}{\lambda^2} = \frac{1}{2} \|C\|_2.$$

Proof. Expanding each exponential to second order and multiplying in chronological order gives

$$U_{\text{temp}}(\lambda) = I - i\lambda \sum_r A_r - \lambda^2 \left(\frac{1}{2} \sum_r A_r^2 + \sum_{r>q} A_r A_q \right) + O(\lambda^3).$$

Expanding the aggregate exponential gives

$$U_{\text{av}}(\lambda) = I - i\lambda \sum_r A_r - \frac{\lambda^2}{2} \left(\sum_r A_r^2 + \sum_{r>q} (A_r A_q + A_q A_r) \right) + O(\lambda^3).$$

Subtracting the second expression from the first yields [equation \(16\)](#). Dividing by λ^2 , taking spectral norms, and using continuity of the norm proves the limit when $C \neq 0$. $\square$

Thus a nonzero signed sum of pairwise commutators produces a genuinely quadratic discrepancy. The distinction between the signed sum C and the unsigned budget $\mathcal{W}$ is important: C can cancel even when several temporal wedges are present, whereas $\mathcal{W}$ is designed as a safe certificate.

The weighted aggregate is only one reference. Sometimes the direct question is how much the output changes when the same events are rearranged. The final theorem answers that question by charging only event pairs whose relative order is reversed.

Theorem 7 (Sensitivity to event reordering). *Let π and σ be two permutations of the same actions, and let $\mathcal{I}(\pi, \sigma)$ be the set of unordered pairs $\{r, q\}$ that appear in opposite relative order in the two permutations. If U_π and U_σ are the corresponding propagators, then*

$$\|U_\pi - U_\sigma\|_2 \leq \sum_{\{r, q\} \in \mathcal{I}(\pi, \sigma)} \|[A_r, A_q]\|_2. \quad (17)$$

For edge events, the summand is zero unless the two distinct edges share exactly one vertex, in which case it is $|\alpha_r \alpha_q|$.

Proof. First consider two adjacent factors. By the triangle inequality and [theorem 2](#),

$$\begin{aligned} \|e^{-iA} e^{-iB} - e^{-iB} e^{-iA}\|_2 &\leq \|e^{-iA} e^{-iB} - e^{-i(A+B)}\|_2 \\ &\quad + \|e^{-i(A+B)} - e^{-iB} e^{-iA}\|_2 \\ &\leq \|[A, B]\|_2. \end{aligned}$$

Multiplying the two products on the left or right by the same unitary matrices does not change their distance. Any permutation can be transformed into another by adjacent swaps, and a standard inversion-sorting construction swaps each pair in $\mathcal{I}(\pi, \sigma)$ exactly once. Applying the adjacent-swap bound at each step and summing proves [equation \(17\)](#). The edge specialization follows from [theorem 1](#). $\square$

Exchanging two disjoint events is free under this bound, while reversing two events that meet at one vertex has a cost proportional to the product of their pulse areas. The estimate also distinguishes a local schedule change from a complete reversal.

5 Exact benchmark and numerical stress test

The smallest sequence containing a temporal wedge has three vertices a, b, c . Apply edge ab for pulse area θ , followed by edge bc with the same pulse area:

$$A_1 = \theta X_{\{a,b\}}, \quad A_2 = \theta X_{\{b,c\}}.$$

Direct exponentiation gives

$$\langle c | U_{\text{temp}} | a \rangle = -\sin^2 \theta, \quad p_{\text{temp}}(a, c) = \sin^4 \theta, \quad (18)$$

$$\langle c | U_{\text{av}} | a \rangle = -\frac{1}{2} [1 - \cos(\sqrt{2}\theta)], \quad p_{\text{av}}(a, c) = \frac{1}{4} [1 - \cos(\sqrt{2}\theta)]^2. \quad (19)$$

If the event order is reversed, the bc interaction acts before any amplitude reaches b , and

$$\langle c | U_{\text{rev}} | a \rangle = 0.$$

The signs in [equations \(18\) and \(19\)](#) are both negative for the Hamiltonian convention in [equation \(1\)](#); they matter for subsequent interference even though they disappear from the displayed probabilities. Here $\mathcal{W} = \theta^2/2$, and the weak-pulse probabilities satisfy

$$p_{\text{temp}}(a, c) = \theta^4 + O(\theta^6), \quad p_{\text{av}}(a, c) = \frac{\theta^4}{4} + O(\theta^6).$$

This is a comparison of fixed-time transfer probabilities, not a claim about classical topological reachability.

We also tested the certificate on random event sequences. For each $n \in \{8, 16, 32\}$, we set $m = 2n$, drew every unordered edge uniformly with replacement, and sampled each positive pulse area uniformly from $[a/2, a]$. Each row in [table 1](#) reports the median over 200 independent sequences generated with seed 20260821. The aggregate exponential was evaluated as a dense matrix exponential, while each event unitary was applied through its exact two-dimensional block. Numerical details are given in the Supplemental Material.

All samples satisfied $\varepsilon_{\text{op}} \leq \mathcal{W}$, as required by [theorem 3](#). The gap between the two columns also shows that the certificate is deliberately conservative. Complete reversal inverts every event pair, so [theorem 7](#) gives $\varepsilon_{\text{rev}} \leq \min\{2, 2\mathcal{W}\}$; this inequality also held in every sample. At $n = 16$, a log-log fit over the three values of a gives slopes 1.99 for ε_{op} and 3.99 for Δ_{pro} . These values agree with the quadratic unitary error in [theorem 6](#) and the corresponding fourth-order process infidelity. The numerical study is a stress test of the derived inequalities, not an experimental validation or a claim about a particular hardware platform.

6 Discussion

The results clarify what a weighted aggregate can and cannot preserve in a dynamic continuous-time quantum walk. Exact agreement for every ordering is guaranteed for repeated edges and

Table 1: Median errors for random event sequences. Here ε_{rev} compares the sampled order with its complete reversal, and $\Delta_{\text{pro}} = 1 - F_{\text{pro}}$.

n	m	a	$\mathcal{W}$	ε_{op}	ε_{rev}	Δ_{pro}
8	16	.02	5.675×10^{-3}	7.706×10^{-4}	1.537×10^{-3}	2.114×10^{-7}
8	16	.05	3.615×10^{-2}	4.961×10^{-3}	9.863×10^{-3}	8.548×10^{-6}
8	16	.08	9.144×10^{-2}	1.212×10^{-2}	2.404×10^{-2}	5.217×10^{-5}
16	32	.02	1.292×10^{-2}	9.240×10^{-4}	1.846×10^{-3}	2.179×10^{-7}
16	32	.05	8.081×10^{-2}	5.937×10^{-3}	1.182×10^{-2}	8.724×10^{-6}
16	32	.08	2.018×10^{-1}	1.451×10^{-2}	2.879×10^{-2}	5.451×10^{-5}
32	64	.02	2.689×10^{-2}	1.014×10^{-3}	2.026×10^{-3}	2.176×10^{-7}
32	64	.05	1.707×10^{-1}	6.523×10^{-3}	1.299×10^{-2}	8.485×10^{-6}
32	64	.08	4.393×10^{-1}	1.646×10^{-2}	3.263×10^{-2}	5.567×10^{-5}

disjoint edge pairs. Outside that structural case, the temporal-wedge budget gives a computable sufficient condition for preserving the implemented unitary, and therefore the state-transfer amplitudes and fidelities derived from it. The certificate can be evaluated from the event list without forming an $n \times n$ commutator for every pair. The reordering bound adds another useful distinction: two schedules are compared through the noncommuting pairs whose relative order actually changes, rather than through the total number of events.

The analysis also has clear limits. It concerns a closed, single-excitation model with ideal sequential pulses. Decoherence, measurements, and irreversible directionality are not included. Simultaneous or smoothly overlapping couplings can be represented by more general Hamiltonian layers, for which the general bound [equation \(7\)](#) remains valid but the simple wedge formula need not. A background drift can be included in the same way, although it introduces additional commutators. The certificate is a worst-case bound and can be loose when signed commutators cancel. These limitations are not small technical details; they mark the boundary of what the present theorems establish.

The most direct extension is therefore not to make broader hardware claims, but to sharpen the certificate for structured schedules. Possible routes include cancellation-aware bounds, optimized event orderings, and channel norm estimates for open-system layers. Those questions require separate analysis.

7 Conclusion

For sequential edge Hamiltonians, a pulse-independent exact reduction valid for every ordering is obtained when repeated activations are allowed but the distinct active edges form a matching. For a general schedule, the difference between the ordered quantum evolution and the weighted aggregate is bounded by a temporal-wedge budget built from pairs of edges sharing one vertex. This budget is computable in expected linear time from vertex and edge totals. It controls quantum state-transfer probabilities and channel fidelity, while an inversion bound controls direct changes of schedule. The result is a sufficient certificate, not a prediction of the exact error: cancellation or special pulse values may make the actual discrepancy smaller.

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Data and code availability. No external data were used. The numerical values in [table 1](#) were generated from the equations in this paper. The exact event exponential, random-number generator, fixed seed, algorithmic steps, and precision checks needed to reproduce them are provided in the Supplemental Material. No standalone software package was developed.

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