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Research Article

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Posted Date: September 8th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-10763485/v1

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Additional Declarations: No competing interests reported.

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- Ethical Statement: no unethical activities were performed.
- Funding: This study received funding from the Department of Education of Guangdong Province, China, under the Young Innovative Talents Research Project (Grant No. 21400-KYC02025004), and from the Guangdong Provincial Philosophy and Social Science Planning Project, under the Youth Project (Grant No. 21400-KYB02026007).
- Conflict of Interest: None
- Informed consent: we (all the co-authors) are informed about this research work and show are consent for publication.
- Acknowledgement: we thank all co-author

Multi-period Mean-Variance Interval Portfolio Selection Model with Cardinality Constraints

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Abstract: This paper develops a multi-period mean-variance interval portfolio selection model with cardinality constraints under return uncertainty. Interval numbers are used to characterize imprecision in expected asset returns and risks, and a sentiment coefficient θ is introduced to represent investors' expectations about future market conditions. To enhance practical relevance, the model simultaneously incorporates transaction costs, borrowing constraints, lower and upper investment bounds, and cardinality constraints. Because transaction costs make successive allocation decisions path-dependent, the resulting formulation is a mixed-integer dynamic optimization problem, for which a forward dynamic programming method is developed. An empirical study based on 30 constituent stocks of the SSE 50 Index evaluates optimal allocations and the effects of portfolio cardinality and investor sentiment. In addition to the baseline comparisons, marginal-cardinality and sentiment-sensitivity analyses are performed. Terminal wealth is maximized at K=5 and displays a diversification-saturation pattern thereafter. The estimated sensitivity of wealth to θ rises from 0.01866 in stage 1 to 0.13883 in stage 5, while the wealth difference between θ=1 and θ=0 expands from 1.7395% to 9.6400%. Log-linear regressions yield R^2 values above 0.9998, quantitatively supporting the approximately exponential wealth-growth pattern observed in the baseline results. Overall, the findings suggest that a moderate cardinality level can balance diversification benefits against trading frictions and that the effect of investor sentiment accumulates over the investment horizon. The proposed framework provides a tractable approach to multi-period portfolio allocation under interval uncertainty and realistic investment constraints.
Keywords: Multi-period portfolio selection; Mean-variance interval model; Cardinality constraints; Forward dynamic programming; Investor sentiment; Sensitivity analysis

1. Introduction

Markowitz (1952) introduced the mean-variance (MV) framework, which remains a cornerstone of modern portfolio theory. In the classical MV model, expected return and variance are used to characterize portfolio return and risk, respectively. To better accommodate uncertainty in financial markets, subsequent studies have extended the framework by adopting alternative risk measures, including semivariance (Vercher et al., 2007), absolute deviation (Zhang and Zhang, 2008), VaR (Zhang, 2008), CVaR (Rockafellar and Uryasev, 2000), and entropy (Zhang et al., 2012). Other studies replace point estimates with interval numbers (Alefeld and Herzberger, 1983), fuzzy numbers (Zadeh, 1965), or admissible-error representations (Zhang and Nie, 2004, 2005) to describe uncertainty in expected asset returns.

A large body of portfolio research treats asset returns as random variables. In practice, however, information asymmetry and changing market conditions make the returns of risky assets difficult to estimate precisely. Fuzzy-set and interval-analysis approaches provide a useful way to represent such imprecision. For example, Zhang (2005) developed a multi-period interval mean-absolute-deviation portfolio model incorporating transaction costs, borrowing limits, threshold constraints, and cardinality constraints, and proposed a forward dynamic programming solution. Moghadam et al. (2020) incorporated behavioral factors, cardinality constraints, entropy, and risk-free lending into a multi-period robust portfolio model. Ma et al. (2021) considered mean, variance, skewness, kurtosis, and return entropy simultaneously in a many-objective fuzzy portfolio framework. Related studies have also examined fuzzy portfolios with transaction costs and background risk (Lu and He, 2021), and multi-period multi-objective portfolios using downside-risk measures and practical constraints (Gupta et al., 2021). More recently, Bayesian estimation and investor psychological characteristics have been introduced to improve parameter estimation and behavioral realism (Zhang and Lin, 2024; Yang et al., 2024).

Because investment decisions are revised over time, portfolio selection is naturally a multi-period problem. Existing studies have extended single-period models by considering time-consistent mean-standard-deviation criteria (Bannister et al., 2016), CVaR-based stochastic

programming (Chen and Yang, 2017), transaction costs (Mei et al., 2016), dynamic risk measures (Chen and Xu, 2014), and mean-VaR formulations with practical constraints (Babazadeh and Esfahanipour, 2019). Other work incorporates fuzzy returns, liquidity, and evolutionary optimization into multi-period allocation (Liagkouras and Metaxiotis, 2018). Recent studies further consider possibilistic mean-semivariance-entropy models (Hu et al., 2024), time-consistent strategies under realistic constraints (Zhang et al., 2024), and asymmetric return-risk measures reflecting investors' coherent perceptions (Li et al., 2025).

In practice, investors generally hold only a subset of the available assets because diversification benefits must be balanced against transaction, monitoring, and rebalancing costs. Cardinality constraints therefore provide a natural way to control portfolio size. Earlier studies addressed cardinality-constrained portfolio selection using discrete approximation (Zhang and Zhang, 2014), hybrid differential evolution (Ma et al., 2012), particle swarm optimization (Deng et al., 2012), semivariance-based heuristics (Yang et al., 2011), variable neighborhood search (Akbay et al., 2020), and classical metaheuristics such as genetic algorithms, tabu search, and simulated annealing (Chang et al., 2000). These studies establish the practical importance of explicitly controlling the number of assets held in a portfolio.

Recent research confirms a continuing shift toward multi-period portfolio models that combine uncertainty representation with realistic trading constraints and more adaptive solution methods. El Kharrim (2023) formulates a multi-period fuzzy portfolio model with CVaR, transaction costs and cardinality constraints and solves the resulting mixed-integer dynamic problem by dynamic programming. Subsequent studies extend this direction through stochastic prediction and heuristic optimization in multi-period settings (Ahmadi and Ghasemi, 2025), interval-analysis frameworks that jointly consider diversification, downside risk and liquidity (Șerban, 2025), and discrete multi-period formulations with transaction fees and minimum-investment restrictions. More recent work also studies cardinality-constrained multi-period mean-variance allocation under regime-switching market parameters (Wang et al., 2026) and interval-valued fuzzy multi-period portfolio decisions with realistic constraints (Zhang et al., 2026). Collectively, these developments reinforce the relevance of jointly modeling uncertainty, dynamic rebalancing and implementable portfolio restrictions, while motivating a closer examination of how cardinality and investor attitudes affect wealth accumulation over time.

Despite these advances, two issues remain comparatively underexplored. First, existing multi-period studies typically emphasize fuzzy-return specification, downside-risk constraints, regime dynamics, or algorithmic performance, whereas the interaction between interval-valued return uncertainty and a continuously varying sentiment coefficient has received less attention in a mean-variance framework that simultaneously includes transaction costs, borrowing, investment bounds and cardinality restrictions. Second, numerical studies commonly report optimal portfolios or compare solution procedures, but the marginal economic effect of portfolio cardinality and the intertemporal amplification of investor sentiment are less frequently quantified. This paper addresses these two issues by embedding an explicit sentiment coefficient in an interval mean-variance model and by complementing the baseline allocation results with marginal-cardinality and dynamic sentiment-sensitivity analyses.

The contributions of this study are fourfold. First, expected returns and risks are represented by interval estimates rather than point estimates, allowing the model to incorporate imprecision in asset-return information directly. Second, a sentiment coefficient θ is embedded in the interval framework so that investors' market expectations can vary continuously from extreme pessimism to extreme optimism and affect both return and risk evaluation. Third, transaction costs, borrowing constraints, lower and upper investment bounds, and cardinality constraints are incorporated within one multi-period formulation. The resulting path-dependent mixed-integer problem is solved using a forward dynamic programming method. Fourth, the empirical analysis goes beyond reporting optimal allocations: the marginal analysis of K identifies a diversification-saturation pattern with the highest reported terminal wealth at K=5, while the sentiment-sensitivity analysis shows that the influence of θ becomes progressively stronger through multi-period wealth accumulation. These features provide both a modeling contribution and additional economic interpretation of the portfolio decisions generated by the framework.

The remainder of this paper is organized as follows. Section 2 introduces interval numbers and the sentiment coefficient. Section 3 formulates the multi-period mean-variance interval portfolio model with transaction costs, borrowing, investment-bound and cardinality constraints. Section 4

presents the forward dynamic programming solution procedure. Section 5 reports the empirical study based on SSE 50 constituents and extends the baseline results through marginal-cardinality and sentiment-sensitivity analyses. Section 6 concludes with the main findings, economic implications, limitations and directions for future research.

2 Prerequisite Knowledge

Definition 1: $\hat{a}=[a^-,a^+]=\{x\in R|a^-\le x\le a^+,a^-,a^+\in R\}$, Let is called an interval number Specifically, if $a^-=a^+$, it is a real number $\hat{a}$ then.

Theorem 1: Give the interval number: $\hat{A}=[a^-,a^+],\hat{A}=[b^-,b^+],\hat{C}=[a^-+b^-,a^++b^+]$. Then we have ① $\forall x\in\hat{A},y\in\hat{B},x+y\in\hat{C}$. ② $\forall z\in C$, There must exist $x\in\hat{A},y\in\hat{B}$, $x+y=z$ such that.

Theorem 2: Give the interval number: $\hat{A}=[a^-,a^+]$, $\hat{B}=[b^-,b^+]$.

Then $\hat{C}=[\min\{a^-b^-,a^-b^+,a^+b^-,a^+b^+\},\max\{a^-b^-,a^-b^+,a^+b^-,a^+b^+\}]$.Thus, we have ① $\forall x\in\hat{A},y\in\hat{B},xy\in\hat{C}$. ② $\forall z\in\hat{C}$, There must exist $x\in\hat{A},y\in\hat{B}$, such that $xy=z$.

Definition 2: Give the interval number $\hat{A}=[a^-,a^+]$, $\hat{B}=[b^-,b^+]$, the definitions of addition, subtraction, multiplication, and division (denominator not :equal to 0) operations are as follows

$$\hat{A}+\hat{B}=[a^-+b^-,a^++b^+] \tag{1}$$

$$\hat{A}-\hat{B}=[a^--b^-,a^+-b^+] \tag{2}$$

$$\hat{A}\times\hat{B}=[\min\{a^-b^-,a^-b^+,a^+b^-,a^+b^+\},\max\{a^-b^-,a^-b^+,a^+b^-,a^+b^+\}] \tag{3}$$

$$\hat{A}\div\hat{B}=[a^-\div b^+,a^+\div b^-],0\notin[b^-,b^+]. \tag{4}$$

Theorem 3: the sentiment $\hat{a}_i$ is θ , is an interval number Given that $\hat{a}_i=[a_i^-,a_i^+]$ coefficient, $0\le\theta\le1$, $i=1,2,...,n$, $E(\hat{a}_i(\theta))=(1-\theta)a_i^-+\theta a_i^+$, for any non-negative real numbers λ ,:there always exists

$$E(\lambda\hat{a}_i(\theta))=\lambda E(\hat{a}_i(\theta)) \tag{1}$$

proof: $E(\lambda\hat{a}_i(\theta))=(1-\theta)\lambda a_i^-+\theta\lambda a_i^+$

$$=\lambda[(1-\theta)a_i^-+\theta a_i^+]=\lambda E(\hat{a}_i(\theta))$$

Theorem 4: the sentiment $\hat{a}_i$ is θ, Given $\hat{a}_i=[a_i^-,a_i^+]$ is an interval number, EMBED Equation 3. $0\le\theta\le1$, $i=1,2,...,n$, $E(\hat{a}_i(\theta))=(1-\theta)a_i^-+\theta a_i^+$, for any λ and μ, there always exists

$$E(\lambda\hat{a}_i(\theta)+\mu)=\lambda E(\hat{a}_i(\theta))+\mu \tag{6}$$

proof: $E(\lambda\hat{a}_i(\theta)+\mu)=(1-\theta)(\lambda a_i^-+\mu)+\theta(\lambda a_i^++\mu)$

$$=\lambda[(1-\theta)(a^-+\mu)+\theta(a^++\mu)]+\mu=\lambda E(\hat{a}(\theta))+\mu$$

Theorem 5: Given $\hat{a}_i=[a_i^-,a_i^+]$ and $\hat{b}=[b_i^-,b_i^+]$ is interval number，θ is a_i^+ and b_i^- the sentiment coefficient, $0\le\theta\le1$, $E(\hat{a}_i(\theta))=(1-\theta)a_i^-+\theta a_i^+$, $E(\hat{b}_i(\theta))=(1-\theta)b_i^-+\theta b_i^+$, for any λ and μ , :there always exists

$$E(\lambda\hat{a}_i(\theta)+\mu\hat{b}_i(\theta))=\lambda E(\hat{a}_i(\theta))+\lambda E(\hat{b}_i(\theta)) \tag{7}$$

proof: $E(\lambda\hat{a}_i(\theta)+\mu\hat{b}_i(\theta))=(1-\theta)(\lambda a_i^-+\mu b_i^-)+\theta(\lambda a_i^++\mu b_i^+)$

$$= \lambda[(1-\theta)a_i^- + \theta a_i^+] + \mu[(1-\theta)b_i^- + \mu b_i^+) = \lambda E(\hat{a}_i(\theta)) + \mu E(\hat{b}_i(\theta))$$

Theorem 3: Given $\hat{a}_i = [a_i^-, a_i^+]$ is a number of intervals, θ is the sentiment $\hat{a}_i$ coefficient, $0 \le \theta \le 1$, $i = 1, 2, ..., n$, the variance of the number of intervals can be defined as:

$$\sigma^2(\hat{a}_i(\theta)) = \int_0^1 r(a_i^+ - (\theta a_i^+ + (1-\theta)a_i^-))^2 dr + \int_0^1 r((\theta a_i^+ + (1-\theta)a_i^-) - a_i^-)^2 dr \quad (8)$$

Theorem 6: Given $\hat{a}_i = [a_i^-, a_i^+]$ is a number of intervals, θ is the sentiment $\hat{a}_i$ coefficient, $0 \le \theta \le 1$, $i = 1, 2, ..., n$, $E(\hat{a}_i) = (1-\theta)a_i^- + \theta a_i^+$, then the variance of the number of intervals is:

$$\sigma^2(\hat{a}_i) = (\theta^2 + (1-\theta)^2)(a_i^+ - a_i^-)^2 / 2 \quad (9)$$

Proof:

$$\begin{aligned}\sigma^2(\hat{a}_i(\theta)) &= \int_0^1 r(a_i^+ - (\theta a_i^+ + (1-\theta)a_i^-))^2 dr + \int_0^1 r((\theta a_i^+ + (1-\theta)a_i^-) - a_i^-)^2 dr \\ &= (\theta^2 (a_i^+ - a_i^-)^2 + (1-\theta)^2 (a_i^+ - a_i^-)^2) / 2 \\ &= (\theta^2 + (1-\theta)^2)(a_i^+ - a_i^-)^2 / 2\end{aligned}$$

Theorem 7: Given $\hat{a}_i = [a_i^-, a_i^+]$ and $\hat{a}_j = [a_j^-, a_j^+]$ is interval number, θ is the sentiment $\hat{a}_i$ and $\hat{a}_j$, $0 \le \theta \le 1$, $i = 1, 2, ..., n$, $E(\hat{a}_i(\theta)) = (1-\theta)a_i^- + \theta a_i^+$, $E(\hat{a}_j(\theta)) = (1-\theta)a_j^- + \theta a_j^+$, then the covariance *Cov* of the number of intervals $\hat{a}_i$ and $\hat{a}_j$ is:

$$Cov(\hat{a}_i(\theta), \hat{a}_j(\theta)) = (\theta^2 + (1-\theta)^2)(a_i^+ - a_i^-)(a_j^+ - a_j^-) / 2 \quad (10)$$

Proof:

$$\begin{aligned}Cov(\hat{a}_i(\theta), \hat{a}_j(\theta)) &= \int_0^1 r(a_i^+ - (\theta a_i^+ + (1-\theta)a_i^-))(a_j^+ - (\theta a_j^+ + (1-\theta)a_j^-))dr \\ &+ \int_0^1 r((\theta a_i^+ + (1-\theta)a_i^-) - a_i^-)(\theta a_j^+ + (1-\theta)a_j^-) - a_j^-)dr \\ &= (\theta^2 (a_i^+ - a_i^-)(a_j^+ - a_j^-) + (1-\theta)^2 (a_i^+ - a_i^-)(a_j^+ - a_j^-)) / 2 \\ &= (\theta^2 + (1-\theta)^2)(a_i^+ - a_i^-)(a_j^+ - a_j^-) / 2\end{aligned}$$

Theorem 8: Given $\hat{a}_i = [a_i^-, a_i^+]$ is interval number, θ is the sentiment $\hat{a}_i$ coefficient, $0 \le \theta \le 1$, $i = 1, 2, ..., n$, $E(\hat{a}_i) = (1-\theta)a_i^- + \theta a_i^+$, for any non-negative real numbers λ ,:there always exists

$$\sigma^2(\lambda \hat{a}_i) = \lambda^2 \sigma^2(\hat{a}_i) \quad (11)$$

proof:

$$\begin{aligned}\sigma^2(\lambda \hat{a}_i(\theta)) &= \int_0^1 r(\lambda a_i^+ - (\theta \lambda a_i^+ + (1-\theta)\lambda a_i^-))^2 dr \\ &+ \int_0^1 r((\theta \lambda a_i^+ + (1-\theta)\lambda a_i^-) - \lambda a_i^-)^2 dr \\ &= (\lambda^2 \theta^2 (a_i^+ - a_i^-)^2 + \lambda^2 (1-\theta)^2 (a_i^+ - a_i^-)^2) / 2 \\ &= \lambda^2 (\theta^2 + (1-\theta)^2)(a_i^+ - a_i^-)^2 / 2 \\ &= \lambda^2 \sigma^2(\hat{a}_i(\theta))\end{aligned}$$

Theorem 9: Given $\hat{a}_i = [a_i^-, a_i^+]$ is interval number, θ is the sentiment $\hat{a}_i$ coefficient, $0 \le \theta \le 1$, $i = 1, 2, ..., n$, $E(\hat{a}_i) = (1-\theta)a_i^- + \theta a_i^+$, for any non-negative real numbers λ and μ, :there always exists

$$\sigma^2(\lambda \hat{a}_i + \mu) = \lambda^2 \sigma^2(\hat{a}_i) \quad (12)$$

proof:

$$\sigma^2(\lambda\hat{a}_i(\theta)+\mu)=\int_0^1 r(\lambda a_i^+ +\mu)-(\theta(\lambda a_i^+ +\mu)+(1-\theta)(\lambda a_i^- +\mu)))^2 dr$$
$$+\int_0^1 r((\theta(\lambda a_i^+ +\mu)+(1-\theta)(\lambda a_i^- +\mu))-(\lambda a_i^- +\mu))^2 dr$$
$$=(\lambda^2\theta^2(a_i^+ -a_i^-)^2+\lambda^2(1-\theta)^2(a_i^+ -a_i^-)^2)/2$$
$$=\lambda^2(\theta^2+(1-\theta)^2)(a_i^+ -a_i^-)^2/2$$
$$=\lambda^2\sigma^2(\hat{a}_i(\theta))$$

Theorem 10: Given $\hat{a}_i=[a_i^-,a_i^+]$ and $\hat{a}_j=[a_j^-,a_j^+]$ is interval number，$\theta$is the sentiment $\hat{a}_i$ and $\hat{a}_j$, $0\le\theta\le 1$, $i=1,2,...,n$, $E(\hat{a}_i(\theta))=(1-\theta)a_i^- +\theta a_i^+$, $E(\hat{a}_j(\theta))=(1-\theta)a_j^- +\theta a_j^+$, for any non-negative real numbers λ and μ, there are always covariances:

$$Cov(\lambda\hat{a}_i(\theta),\mu\hat{a}_j(\theta))=\lambda\mu(\theta^2+(1-\theta)^2)(a_i^+ -a_i^-)(a_j^+ -a_j^-)/2 \quad (13)$$

proof:

$$Cov(\lambda\hat{a}_i(\theta),\mu\hat{a}_j(\theta))=\int_0^1 r(\lambda a_i^+ -(\lambda\theta a_i^+ +\lambda(1-\theta)a_i^-))(\mu a_j^+ -(\mu\theta a_j^+ +\mu(1-\theta)a_j^-))dr$$
$$+\int_0^1 r((\lambda\theta a_i^+ +\lambda(1-\theta)a_i^-)-\lambda a_i^-)(\mu\theta a_j^+ +\mu(1-\theta)a_j^-)-\mu a_j^-)dr$$
$$=\lambda\mu(\theta^2(a_i^+ -a_i^-)(a_j^+ -a_j^-)+(1-\theta)^2(a_i^+ -a_i^-)(a_j^+ -a_j^-))/2$$
$$=\lambda\mu(\theta^2+(1-\theta)^2)(a_i^+ -a_i^-)(a_j^+ -a_j^-)/2$$
$$=\lambda\mu Cov(\hat{a}_i(\theta),\hat{a}_j(\theta))$$

3Multi-period Mean-variance Portfolio Model

3.1 Problem description and notation

Consider a financial market with n risky assets and one risk-free asset. The investor allocates initial wealth at the beginning of the first period and may rebalance the portfolio over the subsequent T periods. Thus, at each stage, wealth can be redistributed among the n risky assets and the risk-free asset. The notation used throughout the model is defined as follows:

x_{it} represents the investment proportion of risky asset i in stage t;
x_{i0} is the initial proportion of investment on risky asset i in stage 1;
x_t is the stage t portfolio, where $x_t=(x_{1t}, x_{2t}, \ldots, x_{nt})$;
x_{ft} is the proportion of investment in the risk-free asset in stage t, with $x_{ft}=1-(x_{1t}+x_{2t}+\ldots+x_{nt})$;
x_{ft}^b is the lower bound of the risk-free asset in stage t, where $x_{ft}\ge x_{ft}^b$;
R_{it} is the rate of return on the risky asset i in stage t;
r_{pt} is the expected rate of return on the portfolio x_t in stage t;
r_N is the net rate of return on the portfolio x_t in stage t;
u_{it} is the upper bound constraint on asset x_{it};
l_{it} is the lower bound constraint on asset x_{it};
Wt is the initial wealth in period t;
r_{bt} is the borrowing rate on the risk-free asset;
r_{lt} is the lending rate on the risk-free asset;
c_{it} is the unit transaction cost of the risky asset i in period t;
K denotes the maximum number of risky assets held in the portfolio at each period.

3.2 Returns, risks, and constraints

Assuming the uncertain return rate of risky asset i is denoted by $\hat{r}_{it}=[r_{it}^-,r_{it}^+]$, r_{it}^- and r_{it}^+ represents the sentiment coefficient of θ, which is an interval number , According to Definition 2, the weighted investment portfolio. $0\le\theta\le 1$ where : interval mean is obtained as

$$E(\hat{r}_{it}) = (1-\theta)r_{it}^{-} + \theta r_{it}^{+} \tag{14}$$

Indicates the optimism of investors towards θ. Where the sentiment coefficient higher value suggests a more θ the expected returns in the capital market. A it indicates an extremely pessimistic expectation of θ=0 optimistic outlook. If it signifies an extremely optimistic θ=1 returns in the capital market. Conversely, if lies between 0 and 1, it indicates an expectation, 0< θ<1 expectation. When somewhere between extreme optimism and extreme pessimism.

Let the borrowing rate of the risk-free asset be the same, and if the risk-free asset is allowed to borrow, the interval expected return of the portfolio is, according to **Theorem 3:**

$$\hat{r}_{pt} = \sum_{i=1}^{n} E(\hat{r}_{it}) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) = \sum_{i=1}^{n} ((1-\theta)r_{it}^{-} + \theta r_{it}^{+}) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) \tag{15}$$

Where, $r_{ft} = \begin{cases} r_{bt}, 1 - \sum_{i=1}^{n} x_{it} \le 0 \\ r_{lt}, 1 - \sum_{i=1}^{n} x_{it} \ge 0 \end{cases}, r_{bt} \ge r_{lt}, 1 - \sum_{i=1}^{n} x_{it} \ge x_{ft}^{b}$.

When indicating the borrowing of the risk-free-assets. When $1 - \sum_{i=1}^{n} x_{it} \ge 0$, represents risk-free-asset deposits, when $1 - \sum_{i=1}^{n} x_{it} \le 0$, represents a risk-asset for borrowing.

Assuming the transaction cost for the t period investment portfolio, where $x_t = (x_{1t}, x_{2t}, \ldots, x_{nt})$ and the t (1 period investment portfolio $x_{1t\text{-}1} = (x_{1t\text{-}1}, x_{2t\text{-}1}, \ldots, x_{nt\text{-}1})$) is a V-shaped function. That is, the transaction cost for the i risky asset in the t period is $c_{it}\left|x_{it} - x_{it-1}\right|$. Therefore, the total transaction cost for the investment portfolio $x_t = (x_{1t}, x_{2t}, \ldots, x_{nt})$

$$C_t = \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{it-1}\right|, t = 1, 2, \ldots, T \tag{16}$$

Therefore, the interval net income of the portfolio x is:

$$\hat{r}_{Nt} = \sum_{i=1}^{n} ((1-\theta)r_{it}^{-} + \theta r_{it}^{+}) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{it-1}\right| \tag{17}$$

Wealth changes in periods t and t+1 are

$$W_{t+1} = W_t(1 + \hat{r}_{Nt}) = W_t(1 + \sum_{i=1}^{n} ((1-\theta)r_{it}^{-} + \theta r_{it}^{+}) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it}\left|x_{it} - x_{it-1}\right|) \tag{18}$$

Assuming the random return R_{it} and expected return r_{it} of risky asset i are interval numbers, then we have $E(\hat{R}_{it}) = (1-\theta)R_{it}^{-} + \theta R_{it}^{+}$, $E(\hat{r}_{it}) = (1-\theta)r_{it}^{-} + \theta r_{it}^{+}$, where θ represents the sentiment coefficient, where $0 \le \theta \le 1$. According to **Theorem 6** and **7**, the variance of the portfolio x_t is

$$\sigma^2(x) = (\theta^2 + (1-\theta)^2) x_t' H_{t^*} x_t / 2 \tag{19}$$

where, $H_{t^*} = (Cov_{t^*}(r_{it}, r_{jt}))_{n \times n}$

To calculate the cardinality constraint of the investment portfolio model, we introduce 0-1 binary decision variables z_{it}={0,1}, then the cardinality constraint for portfolio x_t is

$$\sum_{i=1}^{n} z_{it} \le K \tag{20}$$

Where the upper and lower bounds of the portfolio x_t are expressed as

$$l_{it} z_{it} \le x_{it} \le u_{it} z_{it}, \quad i = 1, 2, \ldots, n, \quad t = 1, 2, \ldots, T \tag{21}$$

3.3 Multi-stage interval portfolio model

Taking transaction costs, investment bounds, borrowing constraints, and cardinality constraints into account, the multi-period mean-variance interval portfolio model is formulated as follows:

$$\max \prod_{t=1}^{T} W_1(1+\sum_{i=1}^{n}((1-\theta)r_{it}^{-}+\theta r_{it}^{+})x_{it}+r_{ft}(1-\sum_{i=1}^{n}x_{it})-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{it-1}\right|)$$

$$s.t\begin{cases} W_{t+1}=(1+(\sum_{i=1}^{n}((1-\theta)r_{it}^{-}+\theta r_{it}^{+})x_{it}+r_{ft}(1-\sum_{i=1}^{n}x_{it})-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{it-1}\right|))W_t \quad (b_1) \\ (\theta^2+(1-\theta)^2)x_t{}'H_{t*}x_t/2\le \upsilon_t \qquad (b_2) \\ 1-\sum_{i=1}^{n}x_{it}\ge x_{ft}^{b} \qquad (b_3) \\ \sum_{i=1}^{n}z_{it}\le K,\ \ z_{it}=\{0,1\} \qquad (b_4) \\ l_{it}z_{it}\le x_{it}\le u_{it}z_{it},\ \ i=1,2,...,n,\ \ t=1,2,...,T \qquad (b_5) \end{cases} \tag{22}$$

In model (22), the investor's terminal wealth goal maximization in T stages is considered. Among them, represents the sentiment coefficient of investor's θ, considered among them, expected returns from the future capital market (b_1) is the cumulative wealth; constraint (b_2) is the risk of the investment portfolio x_t at each stage that cannot exceed the given υ_t; (b_3) is the risk-free asset in the t period the investment ratio lower bound; (b_4) is the number of assets invested does not exceed the F value; (b_5) is the upper and lower bound constraints of x_{it} the number of assets invested does not exceed the K value.

Definition: when $\theta = 1$ it means that investors are absolutely optimistic about the expected returns of the asset market, when $\theta = 0$ it means that investors are absolutely, pessimistic about the expected returns on the future capital market are between absolute optimism and absolute pessimism; when $\theta = 0.5$ it means that investors are neither pessimistic nor optimistic, and model (22) is transformed into a classic portfolio model.

4 Forward Dynamic Planning Method

The subproblem of the model (22) at stage t can be transformed into

$$\max \sum_{i=1}^{n}((1-\theta)r_{it}^{-}+\theta r_{it}^{+})x_{it}+r_{ft}(1-\sum_{i=1}^{n}x_{it})-\sum_{i=1}^{n}c_{it}\left|x_{it}-x_{it-1}\right|$$

$$s.t\begin{cases} (\theta^2+(1-\theta)^2)x_t{}'H_{t*}x_t/2\le \upsilon_t \\ 1-\sum_{i=1}^{n}x_{it}\le x_{ft}^{b} \\ \sum_{i=1}^{n}z_{it}\le K,\ \ z_{it}=\{0,1\} \\ l_{it}z_{it}\le x_{it}\le u_{it}z_{it},\ \ i=1,2,...,n\ ,\ \ t=1,2,...,T \end{cases} \tag{23}$$

Let $y_{it}=\left|x_{it}-x_{it-1}\right|$, model (23) can be transformed into

$$\max \sum_{i=1}^{n}((1-\theta)r_{it}^{-}+\theta r_{it}^{+})x_{it}+r_{ft}(1-\sum_{i=1}^{n}x_{it})-\sum_{i=1}^{n}c_{it}y_{it}$$

$$s.t\begin{cases}(\theta^2+(1-\theta)^2)x_t' H_{t*} x_t / 2 \le \upsilon_t \\ 1-\sum_{i=1}^{n} x_{it} \le x_{ft}^b \\ \sum_{i=1}^{n} z_{it} \le K, \ \ z_{it} = \{0,1\} \\ l_{it} z_{it} \le x_{it} \le u_{it} z_{it}, \\ y_{it} \ge x_{it} - x_{it-1}, \\ y_{it} \ge -(x_{it} - x_{it-1}), \ \ i = 1,2,...,n, \ \ t = 1,2,...,T \end{cases} \tag{24}$$

The resulting formulation is a path-dependent mixed-integer dynamic optimization problem. Following Zhang et al. (2024), we solve it using a forward dynamic programming procedure.

The forward dynamic programming procedure is summarized as follows:

Step 1. Set t=1. Given the initial wealth W1 and the initial portfolio $x_0=(x_{10}, x_{20}, \ldots, x_{n0})$, solve model (24) using CPLEX to obtain the optimal portfolio for the first period.

$$x_1^{\max*} = (x_{11}^{\max*}, x_{21}^{\max*}, ..., x_{n1}^{\max*})'$$

The corresponding wealth W2 at the beginning of the second period is then obtained from the wealth-transition equation.

$$W_2 = (1+(\sum_{i=1}^{n}((1-\theta)r_{i1}^{-} + \theta r_{i1}^{+})x_{i1}^{*} + r_{f1}(1-\sum_{i=1}^{n} x_{i1}^{*}) - \sum_{i=1}^{n} c_{i1}\left|x_{i1}^{*} - x_{i1-1}^{*}\right|))W_1$$

Step 2. For t=m, given the current wealth and portfolio inherited from the previous stage, solve model (24) using CPLEX to obtain the optimal portfolio for period m+1.

$$x_{m+1}^{\max*} = (x_{m+1}^{\max*}, x_{m+2}^{\max*}, ..., x_{m+n}^{\max*})'$$

The resulting portfolio determines the wealth available at the beginning of period m+2.

$$W_{m+2}^{\max*} = (1+(\sum_{i=1}^{n}((1-\theta)r_{im}^{-} + \theta r_{im}^{+})x_{im+1}^{*} + r_{ft}(1-\sum_{i=1}^{n} x_{im+1}^{*}) - \sum_{i=1}^{n} c_{it}\left|x_{im+1}^{*} - x_{im}^{*}\right|))W_{m+1}^{\max*}$$

Step 3. If t=T, terminate the procedure and report the terminal wealth and portfolio strategy; otherwise, set t=m+1 and return to Step 2.

5 Empirical Research

To evaluate the proposed model and solution procedure, we conduct an empirical analysis using 30 stocks selected from the constituents of the SSE 50 Index (n=30). The selected stocks are denoted by S1–S30, with their corresponding stock codes reported below.

Assuming that the yield of 30 stocks is expressed by the number of intervals, processing the historical data using the method proposed in literature Vercher et al., (2007) this paper collects the yield from April 2010 to the end of March 2020 as sample data and processes it in one cycle every 3 months; the transaction cost for each stage is constant c_{it}=0.003; the upper bound of x_{it} is l_{it}=0 and u_{it}=0.3(i=1,…,30； t =1,…,5); investment proportion of risk-free assets x_{ft}^b=-0.5; loan of risk-free assets r_{bt}=0.02, loan rate is r_{lt}=0.009 (t =1,…,5); portfolio x_t cannot exceed the given =0.026 in all stages; the number of assets in the portfolio is K =2,…,8; and the value of interval number is: $r_i^{-} = 0.9\hat{r}_i, r_i^{+} = 1.1\hat{r}_i$， $y_j^{-} = 0.9\hat{y}_{ij}$， $y_j^{+} = 1.1\hat{y}_{ij}$。

All numerical experiments are implemented on a laptop equipped with an Intel Core i7-10510U processor (1.80 GHz, up to 2.30 GHz) and 8 GB of memory. The empirical results are reported as follows:

(1) For K=6 and θ=0.5, we first determine the optimal portfolio allocations across the five investment periods.

Table 1. Optimal portfolio allocations across five periods for K=6 and θ=0.5.

Asset i / t	Optimal investment ratio(%)

1	Asset 2 0.299	Asset 11 0.001	Asset 12 0.3	Asset 14 0.3	Asset 17 0.3	Asset 30 0.3
2	Asset 2 0.299	Asset 12 0.3	Asset 13 0.001	Asset 14 0.3	Asset 17 0.3	Asse t30 0.3
3	Asset 2 0.3	Asset 5 0.299	Asset 12 0.3	Asset 14 0.3	Asset 17 0.3	Asset 30 0.001
4	Asset 2 0.3	Asset 5 0.299	Asset 12 0.3	Asset 14 0.3	Asset 17 0.3	Asset 30 0.001
5	Asset 2 0.3	Asset 10 0.001	Asset 12 0.3	Asset 14 0.3	Asset 17 0.3	Asset 30 0.299

Table 1 reports the optimal allocation for K=6 and θ=0.5. In the first period, positive weights are assigned to assets 2, 11, 12, 14, 17, and 30, with weights of 0.299, 0.001, 0.300, 0.300, 0.300, and 0.300, respectively; all remaining risky-asset weights are zero. Under this strategy, terminal wealth reaches 1.5088. The optimal allocations for periods 2–5 are also reported in Table 1.

(2) We next consider K=8 and θ=0.5 and compute the corresponding optimal portfolio allocations.

Table 2. Optimal portfolio allocations across five periods for K=8 and θ=0.5.

资产 *i* / *t*	Optimal investment ratio of (%)							
1	Asset 2 0.297	Asset 7 0.001	Asset 10 0.001	Asset 11 0.001	Asset 12 0.3	Asset 14 0.3	Asset 17 0.3	Asset 30 0.3
2	Asset 2 0.297	Asset 5 0.001	Asset 11 0.001	Asset 12 0.3	Asset 13 0.001	Asset 14 0.3	Asset 17 0.3	Asset 30 0.3
3	Asset 2 0.3	Asset 5 0.297	Asset 10 0.001	Asset 12 0.3	Asset 13 0.001	Asset 14 0.3	Asset 17 0.3	Asset 30 0.001
4	Asset 2 0.3	Asset 5 0.297	Asset 7 0.001	Asset 10 0.001	Asset 12 0.3	Asset 14 0.3	Asset 17 0.3	Asset 30 0.001
5	Asset 2 0.3	Asset 5 0.001	Asset 10 0.001	Asset 10 0.001	Asset 12 0.3	Asset 14 0.3	Asset 17 0.3	Asset 30 0.297

As shown in Table 2, when K=8 and θ=0.5, the first-period portfolio assigns positive weights to assets 2, 7, 10, 11, 12, 14, 17, and 30. The corresponding weights are 0.297, 0.001, 0.001, 0.001, 0.300, 0.300, 0.300, and 0.300, respectively, while the remaining risky-asset weights are zero. The resulting terminal wealth is 1.5086. The allocations for the remaining periods are reported in Table 2.

(3) Fixing θ=0.5, we examine how terminal wealth W(t+1) changes as the cardinality limit varies over K∈[2,8].

Table 3. Terminal wealth W(t+1) for different cardinality limits K, with θ=0.5.

K	2	3	4	5	6	7	8
W(*t*+1)	1.3425	1.4114	1.4663	1.5089	1.5088	1.5086	1.5086

Table 3 shows a non-monotonic relation between portfolio cardinality and terminal wealth. Terminal wealth increases as K rises from 2 to 5, reaches its maximum at K=5, and then decreases slightly for K=6 and K=7 before remaining unchanged at K=8. This pattern suggests that the diversification benefit becomes limited once the portfolio contains a moderate number of risky assets.

(4) Fixing K=6, we examine wealth accumulation across investment periods for θ∈[0,1].

Table 4. Wealth W(t+1) across five periods for different sentiment coefficients θ, with K=6.

θ / *t*	1	2	3	4	5

$\theta = 0$	1.0693	1.1557	1.2421	1.3355	1.4388
$\theta = 0.2$	1.0742	1.1655	1.2573	1.3569	1.4674
$\theta = 0.4$	1.078	1.1742	1.2713	1.3771	1.4949
$\theta = 0.6$	1.0817	1.1828	1.2854	1.3975	1.5227
$\theta = 0.8$	1.0855	1.1916	1.2996	1.4181	1.5509
$\theta = 1$	1.0879	1.1988	1.3122	1.437	1.5775

The wealth paths implied by Table 4 are plotted in Figure 1.

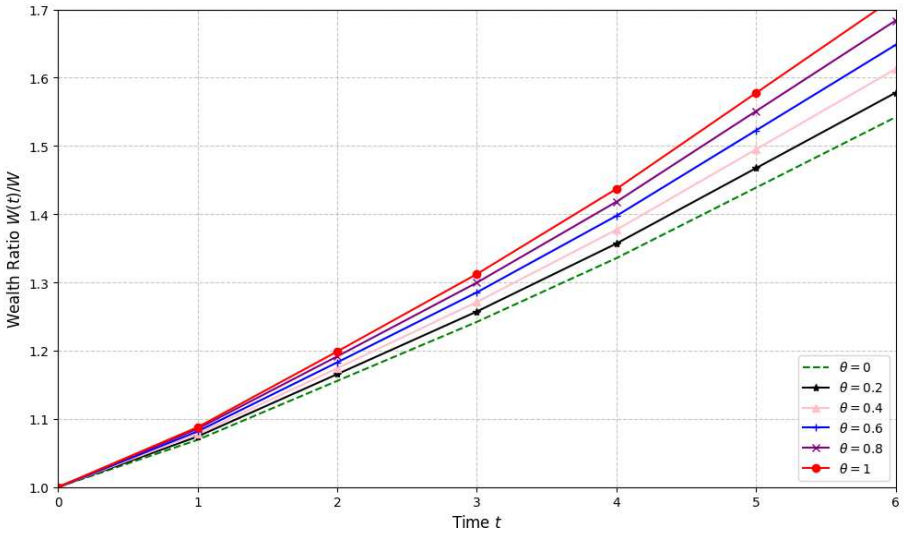


Fig. 1. Wealth evolution across investment periods for different sentiment coefficients θ, with K=6.

Table 4 and Figure 1 show that, for K=6, wealth W(t+1) increases over the investment horizon for all reported values of θ. At each stage, a larger sentiment coefficient is associated with higher wealth. The gap between optimistic and pessimistic scenarios also widens over time, indicating that the effect of the sentiment parameter accumulates through the multi-period wealth dynamics. The wealth paths appear approximately exponential; this pattern is examined quantitatively in Section 5.2.

5.1 Marginal effects of the cardinality constraint

[ADDED—2026 REVISION] To further examine the economic meaning of the cardinality constraint, we conduct a marginal sensitivity analysis based on the terminal wealth values in Table 3. This extension follows the recent literature that treats portfolio sparsity/cardinality as a practical regularization and computational issue (Li et al., 2024; Anis and Kwon, 2025; Jiang et al., 2025; Regaigui et al., 2025; Ebrahimi et al., 2026). Let ΔW(K)=W(K)-W(K-1). The results are reported in Table 5.

Table 5. Marginal effect of the cardinality constraint K on terminal wealth.

K	2	3	4	5	6	7	8
Terminal wealth W	1.3425	1.4114	1.4663	1.5089	1.5088	1.5086	1.5086
Marginal ΔW	—	0.0689	0.0549	0.0426	-0.0001	-0.0002	0.0000
Marginal change (%)	—	5.1322	3.8898	2.9053	-0.0066	-0.0133	0.0000

Table 5 provides three additional findings. First, increasing K from 2 to 5 continuously improves terminal wealth, but the marginal gain declines from 0.0689 to 0.0426, corresponding to a decrease in the marginal percentage improvement from 5.1322% to 2.9053%. Second, the maximum terminal wealth is obtained at K=5 (W=1.5089). When K rises above 5, the change in wealth is economically negligible and slightly negative: ΔW=-0.0001 for K=6 and -0.0002 for K=7, while K=8 produces no further change. Third, the result indicates a saturation effect of diversification under the model constraints. A moderate cardinality level is sufficient to capture most diversification benefits, whereas adding more assets can introduce transaction and rebalancing costs without generating additional terminal wealth. This finding is consistent with recent cardinality-constrained portfolio studies emphasizing sparse holdings and realistic trading frictions.

5.2 Sentiment sensitivity, time amplification and exponential-growth test

[ADDED—2026 REVISION] The original analysis shows that terminal wealth increases with the sentiment coefficient θ and with investment stage t. To quantify this relationship, we conduct two supplementary tests using the data in Table 4. First, for each stage, we estimate the linear sensitivity equation $W(t+1)=a_t+b_t\theta$. Second, for each sentiment level, we fit $\ln W(t+1)=\alpha+\beta t$; $\exp(\beta)-1$ is interpreted as the fitted stage growth rate. This type of dynamic sensitivity analysis is consistent with recent fuzzy portfolio research that explicitly models investor attitudes and dynamic risk-return preferences (Zeng et al., 2024; Zhou et al., 2026).

Table 6. Stage-specific sensitivity of wealth to the sentiment coefficient θ.

Stage t	b_t (θ sensitivity)	R^2	W(θ=1)-W(θ=0)	Relative difference (%)
1	0.01866	0.99149	0.0186	1.7395
2	0.04320	0.99832	0.0431	3.7293
3	0.07021	0.99939	0.0701	5.6437
4	0.10164	0.99975	0.1015	7.6001
5	0.13883	0.99992	0.1387	9.6400

Table 7. Exponential-growth test for wealth under different sentiment coefficients.

θ	β in $\ln(W)=\alpha+\beta t$	Fitted stage growth exp(β)-1	R^2
0.0	0.07382	7.6614%	0.99984
0.2	0.07759	8.0678%	0.99985
0.4	0.08133	8.4728%	0.99985
0.6	0.08507	8.8794%	0.99986
0.8	0.08876	9.2819%	0.99985
1.0	0.09244	9.6850%	0.99986

The results in Table 6 show that the sensitivity coefficient b_t rises monotonically from 0.01866 in stage 1 to 0.13883 in stage 5, and the relative wealth gap between θ=1 and θ=0 expands from 1.7395% to 9.6400%. Thus, the effect of investor sentiment is not constant over time; it is amplified through the multi-period wealth accumulation mechanism. Table 7 further confirms the exponential pattern described in Figure 1. For all six sentiment levels, the log-linear fit has R^2 greater than 0.9998. The fitted stage growth rate increases from 7.6614% under θ=0 to 9.6850% under θ=1. Therefore, optimism in this model affects not only the level of terminal wealth but also the speed at which wealth accumulates over successive stages. Combined with the cardinality robustness results, these tests suggest that the proposed model has two distinct channels: K mainly governs diversification breadth and displays a saturation point, whereas θ governs the dynamic return path and produces an accumulating effect over time.

6 Conclusion

This study investigates a multi-period mean-variance interval portfolio selection problem with cardinality constraints under return uncertainty. Interval estimates are used in place of point estimates to represent expected returns and risks, while a sentiment coefficient θ captures investors' expectations about future market conditions. The model incorporates transaction costs, borrowing constraints, lower and upper investment bounds, and cardinality restrictions. Because portfolio rebalancing and transaction costs create path dependence, the formulation is a mixed-integer dynamic optimization problem and is solved by a forward dynamic programming method. The empirical analysis based on SSE 50 constituents evaluates both optimal allocation strategies and the economic effects of cardinality and sentiment.

(1) For $K\in[2,5]$, terminal wealth W(t+1) increases as K rises and reaches its highest reported value at K=5. For K>5, the additional wealth gain disappears and terminal wealth decreases slightly or remains essentially unchanged. The marginal analysis therefore indicates a diversification-saturation effect: under the specified transaction costs and investment constraints, expanding the number of holdings beyond a moderate level does not necessarily improve terminal wealth.

(2) For K=6 and $\theta\in[0,1]$, wealth increases over the investment horizon and is higher for larger values of θ at each reported stage. The supplementary sensitivity analysis shows that the estimated effect of θ on wealth rises from 0.01866 in stage 1 to 0.13883 in stage 5, while the relative wealth difference between θ=1 and θ=0 expands from 1.7395% to 9.6400%. Thus, within the model, the effect associated with investor sentiment is amplified through multi-period wealth accumulation.

(3) The log-linear fits produce R^2 values above 0.9998 for the reported θ scenarios, providing

quantitative support for the approximately exponential wealth-growth pattern observed in Figure 1. This result should be interpreted as an in-sample characterization of the simulated wealth paths rather than as evidence of a universal exponential law for asset markets.

Several limitations should be acknowledged. First, the empirical sample covers SSE 50 data from April 2010 to March 2020, so the analysis does not capture more recent market regimes. Second, the interval construction, transaction-cost parameters and sentiment coefficient are specified within the current modeling framework rather than jointly estimated from real-time market information. Third, the additional sensitivity tests are based on the reported in-sample numerical results and therefore do not constitute an out-of-sample performance test. Future research could update the data using rolling or out-of-sample designs, introduce regime-dependent returns and transaction costs, estimate sentiment from news or other textual data, and compare the proposed method with recent regime-switching, sparse and prediction-integrated multi-period portfolio approaches.

The supplementary analyses strengthen the internal consistency of the reported numerical results. Marginal changes in terminal wealth reproduce the non-monotonic pattern associated with K, and the log-linear fits quantify the growth pattern previously identified graphically. These exercises should be viewed as internal sensitivity checks rather than external robustness tests, because they use the same empirical design and reported wealth paths as the baseline analysis.

The findings have two practical implications within the model. For portfolio managers, the results suggest that diversification should be balanced against transaction and rebalancing frictions: once the main diversification benefit has been captured, additional holdings may not generate higher terminal wealth. For investors with heterogeneous expectations, the increasing sensitivity of wealth to θ indicates that the consequences of the assumed market outlook become more pronounced as the investment horizon lengthens. Careful calibration of the sentiment parameter is therefore important when the framework is applied to multi-period allocation decisions.

Taken together, the results show that portfolio sparsity and investor sentiment affect not only the composition of the optimal portfolio but also the path of wealth accumulation. By combining interval uncertainty, realistic constraints and a tractable dynamic solution procedure, the proposed model offers a useful extension of multi-period mean-variance portfolio selection. Its practical predictive value, however, should be assessed further using updated data and genuine out-of-sample comparisons.

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