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Convolution and Correlation Theorems Associated with the Linear Canonical Empirical Wavelet Transform and Their Applications

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Abstract

We develop a rigorous convolution–correlation calculus for a fixed-partition realization of the linear canonical empirical wavelet transform. The construction begins with a consistently normalized unitary linear canonical transform and a locally finite empirical window system in canonical frequency. We prove an exact representation theorem connecting the multiplier formulation with scale–centre canonical empirical wavelet atoms, establish frame bounds and reconstruction on canonical spectral subspaces, and derive chirp-weighted convolution and correlation factorizations. The principal results treat two independently generated empirical window systems. Their product windows form a common refinement whose overlap function is the product of the two input overlap functions. The corresponding canonical dual is proved to be the unique pointwise minimum-norm dual, while the synthesized convolution and correlation identities are invariant under the admissible choice of dual. We further establish Lipschitz stability of the coefficient map, the product refinement, and canonical duals under perturbations of empirical windows. Reproducible computations using compactly supported windows verify the cross-partition identities, dual invariance, and stability estimates, and illustrate canonical-frequency component separation.

Keywords: linear canonical transform; empirical wavelet transform; convolution theorem; correlation theorem; common refinement; frame dual; stability; canonical-frequency analysis.

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1 Introduction

Nonstationary and multicomponent signals call for representations that combine localization, spectral selectivity, and enough algebraic structure to support exact manipulation. Adaptive decompositions address the mismatch between a fixed frequency tiling and signal-dependent spectral geometry, while canonical transforms introduce a quadratic-phase geometry that is well suited to oscillatory content organized along chirped trajectories. Their combination creates a mathematical difficulty that is absent from either ingredient separately. A convolution changes the signal before empirical analysis, a correlation conjugates one factor and shifts another, and independently detected empirical partitions generally produce coefficient families with different index sets and different overlap regions. Thus a coefficient-domain convolution or correlation theorem requires more than a chirp modification of the classical Fourier identity. One must specify the canonical-frequency geometry, identify a common refinement of the input empirical systems, distinguish intrinsic statements from representations that depend on a chosen frame dual, and quantify the effect of perturbing empirical boundaries. These issues lead naturally to a fixed-partition analysis: empirical windows are detected first, then frozen, and the resulting coefficient map is linear. The adaptive stage and the operator-theoretic stage are thereby

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separated cleanly, allowing the convolution and correlation structure to be studied on a precise Hilbert-space domain.

Linear canonical transformations arise from the symplectic geometry of quadratic-phase maps, and the unitary representation of Moshinsky and Quesne provides a foundational mathematical formulation [1]. Convolution structures for the linear canonical transform (LCT) were subsequently developed to recover useful product rules in the canonical domain [4]. Wei, Ran, Li, Ma, and Tan obtained an LCT convolution with a direct transform-domain product formula and used it in multiplicative filtering [5]. A companion convolution–correlation formulation was later studied by Wei, Ran, and Li [6]. The same phenomenon occurs for the fractional Fourier transform, where a modified convolution is required away from the ordinary Fourier angle [11].

Wavelet constructions add localization to canonical geometry. Wei and Li introduced a generalized wavelet transform generated from an LCT-domain convolution and derived inversion and admissibility properties [7]. Guo and Li studied the linear canonical wavelet transform on function spaces adapted to the LCT [8]. Prasad and Ansari developed a continuous wavelet transform involving the LCT and established associated integral identities [9]. Bahri, Amir, and Ashino obtained structural properties and inequalities for a linear canonical wavelet transform through its relation with ordinary wavelet analysis [10].

Empirical wavelet analysis introduces adaptivity at the level of the spectral partition. Gilles constructed the empirical wavelet transform (EWT) by detecting spectral boundaries from the analyzed signal and building an adaptive filter bank from those boundaries [2]. A subsequent continuous empirical-wavelet framework gave reconstruction conditions for more general empirical wavelet systems and clarified the role of empirical spectral coverings [3]. These results suggest that the mathematically stable object is the window system generated after the empirical boundaries have been selected.

The unresolved issue addressed here is the interaction of convolution and correlation with *different* empirical canonical window systems. A same-partition coefficient expansion can always be obtained by inserting a frame reconstruction into the LCT product rule, so such an expansion alone carries limited structural content. The genuinely empirical problem begins when the two inputs are analyzed through distinct data-derived coverings. We show that the products of the two input window families form a canonical common refinement, derive its exact frame bounds, characterize its canonical dual by a pointwise minimum-norm principle, and prove that the synthesized convolution and correlation identities are invariant under the admissible choice of refinement dual. This separates the intrinsic common-refinement statement from the coordinate representation induced by a particular dual. We also prove quantitative stability under perturbations of the empirical windows, making the dependence on boundary detection mathematically explicit.

The remainder of the paper is organized as follows. Section 2 fixes the transform conventions, develops the empirical window framework, proves reconstruction, and connects the multiplier realization with scale–centre canonical empirical atoms. Section 3 derives canonical convolution and correlation from chirp conjugation. Section 4 establishes the common-refinement theorem, the canonical-dual extremal property, dual invariance, and the cross-partition convolution identity. Section 5 proves the corresponding correlation theorem. Section 6 develops perturbation stability and exact Fourier and fractional-canonical reductions. Section 7 gives reproducible numerical checks and a compact-window application. Section 8 summarizes the established results and remaining questions.

2 Linear Canonical Empirical Wavelet Framework

We use the unitary Fourier transform

$$(\mathcal{F}f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\omega} dt, \quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}f)(\omega) e^{it\omega} d\omega. \quad (1)$$

We use the L^2 inner product linear in its first argument. With this convention,

$$\mathcal{F}(f * g) = \sqrt{2\pi} (\mathcal{F}f)(\mathcal{F}g) \quad (2)$$

whenever the ordinary convolution is defined.

Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad AD - BC = 1,$$

and assume throughout that $B > 0$. Define

$$(\mathcal{L}_M f)(u) = \frac{1}{\sqrt{2\pi B}} \int_{\mathbb{R}} \exp \left[\frac{i}{2B} (At^2 - 2tu + Du^2) \right] f(t) dt. \quad (3)$$

For the Fourier matrix $M_F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, (3) equals (1) exactly.

Lemma 2.1 (Chirp–Fourier factorization). *For $f \in \mathcal{S}(\mathbb{R})$, define $(\mathcal{C}_M f)(t) = e^{iAt^2/(2B)} f(t)$. Then*

$$(\mathcal{L}_M f)(u) = B^{-1/2} e^{iDu^2/(2B)} (\mathcal{F}(\mathcal{C}_M f))(u/B). \quad (4)$$

Consequently $\mathcal{L}_M$ extends uniquely to a unitary operator on $L^2(\mathbb{R})$.

Proof. Separating the factor that depends only on u in (3) gives

$$\begin{aligned} (\mathcal{L}_M f)(u) &= \frac{e^{iDu^2/(2B)}}{\sqrt{2\pi B}} \int_{\mathbb{R}} e^{iAt^2/(2B)} f(t) e^{-itu/B} dt \\ &= B^{-1/2} e^{iDu^2/(2B)} \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{C}_M f)(t) e^{-it(u/B)} dt \right], \end{aligned}$$

which is (4). Multiplication by $e^{iAt^2/(2B)}$ is unitary on L^2 , the Fourier transform is unitary, and the map $H(\omega) \mapsto B^{-1/2} H(u/B)$ is unitary because $u = B\omega$. Multiplication by $e^{iDu^2/(2B)}$ is also unitary. Their composition is therefore an isometry on $\mathcal{S}(\mathbb{R})$ and extends by density to a unitary operator on $L^2(\mathbb{R})$. $\square$

Let $E \subseteq \mathbb{R}$ be measurable. Define the canonical spectral subspace

$$\mathcal{H}_E^M = \{f \in L^2(\mathbb{R}) : (\mathcal{L}_M f)(u) = 0 \text{ for a.e. } u \notin E\}. \quad (5)$$

Taking $E = \mathbb{R}$ gives the full L^2 space.

Definition 2.2 (Admissible empirical canonical window system). Let J be countable. A family $\mathfrak{M} = \{m_j\}_{j \in J}$ is an admissible empirical canonical window system on E if each $m_j \in L^\infty(\mathbb{R})$ has compact essential support contained in E , the support family is locally finite in E , and

$$S_{\mathfrak{M}}(u) = \sum_{j \in J} |m_j(u)|^2 \quad (6)$$

satisfies

$$0 < s_{\mathfrak{M}} \leq S_{\mathfrak{M}}(u) \leq S_{\mathfrak{M}}^* < \infty \quad \text{for a.e. } u \in E. \quad (7)$$

The empirical boundary detector is not part of the linear analysis operator: it produces the windows, after which the family is frozen. For $f \in \mathcal{H}_E^M$, define

$$(\mathcal{W}_j^{M, \mathfrak{M}} f)(b) = \mathcal{F}_b^{-1}[(\mathcal{L}_M f)m_j](b). \quad (8)$$

Then

$$\mathcal{F}_b(\mathcal{W}_j^{M, \mathfrak{M}} f)(u) = (\mathcal{L}_M f)(u)m_j(u) \quad (9)$$

in L^2 .

Proposition 2.3 (Analyzing atoms). *For $j \in J$ and $b \in \mathbb{R}$, define*

$$\Phi_{j,b}^{M,\mathfrak{M}} = \mathcal{L}_M^{-1} \left[\frac{1}{\sqrt{2\pi}} \overline{m_j(\cdot)} e^{-ib(\cdot)} \right]. \quad (10)$$

Then $\Phi_{j,b}^{M,\mathfrak{M}} \in L^2(\mathbb{R})$ and, for every $f \in \mathcal{H}_E^M$,

$$(\mathcal{W}_j^{M,\mathfrak{M}} f)(b) = \langle f, \Phi_{j,b}^{M,\mathfrak{M}} \rangle \quad (11)$$

for almost every b . Moreover

$$\|\Phi_{j,b}^{M,\mathfrak{M}}\|_2 = \frac{1}{\sqrt{2\pi}} \|m_j\|_2. \quad (12)$$

Proof. Compact essential support and boundedness imply $m_j \in L^2$. Unitarity of $\mathcal{L}_M$ gives (12). Furthermore,

$$\begin{aligned} \langle f, \Phi_{j,b}^{M,\mathfrak{M}} \rangle &= \left\langle \mathcal{L}_M f, \frac{1}{\sqrt{2\pi}} \overline{m_j(\cdot)} e^{-ib(\cdot)} \right\rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_E (\mathcal{L}_M f)(u) m_j(u) e^{ibu} du. \end{aligned}$$

The integrand is in L^1 because $(\mathcal{L}_M f)m_j \in L^2$ and has support of finite measure. The last integral is the inverse Fourier transform in (8), proving (11). $\square$

Theorem 2.4 (Frame bounds and reconstruction). *For every $f \in \mathcal{H}_E^M$,*

$$s_{\mathfrak{M}} \|f\|_2^2 \leq \sum_{j \in J} \|\mathcal{W}_j^{M,\mathfrak{M}} f\|_2^2 \leq S_{\mathfrak{M}}^* \|f\|_2^2. \quad (13)$$

If $J_1 \subset J_2 \subset \dots$ are finite and $\bigcup_N J_N = J$, then

$$\mathcal{L}_M f = \lim_{N \rightarrow \infty} \sum_{j \in J_N} \frac{\overline{m_j}}{S_{\mathfrak{M}}} \mathcal{F}_b(\mathcal{W}_j^{M,\mathfrak{M}} f) \quad \text{in } L^2(E), \quad (14)$$

and applying $\mathcal{L}_M^{-1}$ gives reconstruction of f in $L^2(\mathbb{R})$.

Proof. By Plancherel in b , (9), and Tonelli's theorem,

$$\begin{aligned} \sum_{j \in J} \|\mathcal{W}_j^{M,\mathfrak{M}} f\|_2^2 &= \sum_j \int_E |(\mathcal{L}_M f)(u)|^2 |m_j(u)|^2 du \\ &= \int_E |(\mathcal{L}_M f)(u)|^2 S_{\mathfrak{M}}(u) du. \end{aligned}$$

The bounds (7) and unitarity of $\mathcal{L}_M$ give (13). For reconstruction let $S_N = \sum_{j \in J_N} |m_j|^2$. The partial sum on the right of (14) equals $(\mathcal{L}_M f) S_N / S_{\mathfrak{M}}$. Local finiteness gives $S_N \rightarrow S_{\mathfrak{M}}$ almost everywhere and $0 \leq S_N / S_{\mathfrak{M}} \leq 1$. Hence

$$\left| (\mathcal{L}_M f) \left(1 - \frac{S_N}{S_{\mathfrak{M}}} \right) \right|^2 \leq |\mathcal{L}_M f|^2,$$

and dominated convergence proves (14). Unitarity of $\mathcal{L}_M^{-1}$ completes the proof. $\square$

The next proposition relates the filter-bank formulation to the scale–centre canonical empirical atom used in wavelet-style constructions.

Proposition 2.5 (Exact scale–centre realization). *Let $a_j > 0$, $\nu_j \in \mathbb{R}$, and let $\psi \in L^2(\mathbb{R})$ satisfy*

$$h_M(z) = e^{iAz^2/(2B)}\psi(z) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Write $H_M = \mathcal{F}h_M$ and define

$$\begin{aligned} \Psi_{j,b}^M(t) &= a_j^{-1/2} \psi\left(\frac{t-b}{a_j}\right) e^{i\nu_j(t-b)} \\ &\times \exp\left[-\frac{iA}{2B} \left(t^2 - b^2 - \left(\frac{t-b}{a_j}\right)^2\right)\right]. \end{aligned} \quad (15)$$

Define

$$m_j(u) = \sqrt{\frac{2\pi a_j}{B}} e^{-iDu^2/(2B)} \overline{H_M\left(a_j\left(\frac{u}{B} - \nu_j\right)\right)}. \quad (16)$$

Then, for every $\beta \in \mathbb{R}$,

$$\Phi_{j,\beta}^{M,\mathfrak{M}} = e^{-iAB\beta^2/2} \Psi_{j,B\beta}^M. \quad (17)$$

Consequently

$$\mathcal{W}_j^{M,\mathfrak{M}} f(\beta) = e^{iAB\beta^2/2} \langle f, \Psi_{j,B\beta}^M \rangle. \quad (18)$$

If H_M is bounded and compactly supported, then the windows (16) satisfy the boundedness and compact-support requirements of [theorem 2.2](#); the remaining frame condition is precisely the overlap bound (7).

Proof. Set $b = B\beta$ and substitute $t = B\beta + a_j z$ in the Fourier factor from [theorem 2.1](#). Multiplication of (15) by the input chirp gives

$$(\mathcal{C}_M \Psi_{j,B\beta}^M)(B\beta + a_j z) = a_j^{-1/2} \psi(z) e^{i\nu_j a_j z} e^{iAB\beta^2/2} e^{iAz^2/(2B)}.$$

Therefore

$$\begin{aligned} \mathcal{F}(\mathcal{C}_M \Psi_{j,B\beta}^M)(u/B) &= \sqrt{a_j} e^{iAB\beta^2/2} e^{-i\beta u} \\ &\times H_M\left(a_j\left(\frac{u}{B} - \nu_j\right)\right). \end{aligned}$$

Using (4),

$$\begin{aligned} \mathcal{L}_M \Psi_{j,B\beta}^M(u) &= \sqrt{\frac{a_j}{B}} e^{iDu^2/(2B)} e^{iAB\beta^2/2} e^{-i\beta u} \\ &\times H_M\left(a_j\left(\frac{u}{B} - \nu_j\right)\right). \end{aligned}$$

By (16), the factor on the right that is independent of β equals $(2\pi)^{-1/2} \overline{m_j(u)}$. Hence

$$\mathcal{L}_M \Psi_{j,B\beta}^M = e^{iAB\beta^2/2} \frac{1}{\sqrt{2\pi}} \overline{m_j} e^{-i\beta(\cdot)}.$$

Applying $\mathcal{L}_M^{-1}$ proves (17). Since the L^2 inner product is linear in its first entry, (17) gives (18). $\square$

3 Generalized Linear Canonical Convolution and Correlation

The canonical signal-domain operators are derived from chirp conjugation, so their phases are fixed by the LCT convention rather than chosen independently.

Definition 3.1 (Canonical convolution). For functions for which the ordinary convolution exists, define

$$(f \star_M g)(t) = e^{-iAt^2/(2B)} [(\mathcal{C}_M f) * (\mathcal{C}_M g)](t). \quad (19)$$

Equivalently,

$$(f \star_M g)(t) = \int_{\mathbb{R}} f(\tau) g(t - \tau) \exp \left[-\frac{iA}{B} \tau(t - \tau) \right] d\tau. \quad (20)$$

Indeed, the total chirp exponent in (19) is

$$\frac{iA}{2B} \{ \tau^2 + (t - \tau)^2 - t^2 \} = -\frac{iA}{B} \tau(t - \tau).$$

Theorem 3.2 (Boundedness and LCT convolution factorization). Let $1 \leq p, q, r \leq \infty$ satisfy $1 + 1/r = 1/p + 1/q$. Then

$$\|f \star_M g\|_r \leq \|f\|_p \|g\|_q \quad (21)$$

whenever ordinary Young's inequality applies to $\mathcal{C}_M f$ and $\mathcal{C}_M g$. If $f, g \in \mathcal{S}(\mathbb{R})$, then

$$\mathcal{L}_M(f \star_M g)(u) = \sqrt{2\pi B} e^{-iDu^2/(2B)} (\mathcal{L}_M f)(u) (\mathcal{L}_M g)(u). \quad (22)$$

Proof. All chirp multipliers have modulus one. Hence

$$\|f \star_M g\|_r = \|(\mathcal{C}_M f) * (\mathcal{C}_M g)\|_r \leq \|\mathcal{C}_M f\|_p \|\mathcal{C}_M g\|_q = \|f\|_p \|g\|_q.$$

For $f, g \in \mathcal{S}$, (2) and theorem 2.1 give

$$\begin{aligned} \mathcal{L}_M(f \star_M g)(u) &= B^{-1/2} e^{iDu^2/(2B)} \mathcal{F}[(\mathcal{C}_M f) * (\mathcal{C}_M g)](u/B) \\ &= B^{-1/2} e^{iDu^2/(2B)} \sqrt{2\pi} \mathcal{F}(\mathcal{C}_M f)(u/B) \mathcal{F}(\mathcal{C}_M g)(u/B). \end{aligned}$$

Solving (4) for the two Fourier factors yields

$$\mathcal{F}(\mathcal{C}_M f)(u/B) = \sqrt{B} e^{-iDu^2/(2B)} \mathcal{L}_M f(u),$$

and the same formula for g . Substitution gives (22). $\square$

Definition 3.3 (Canonical correlation). For functions for which the integral exists, define

$$(f \diamond_M g)(t) = e^{-iAt^2/(2B)} \int_{\mathbb{R}} \overline{(\mathcal{C}_M f)(\tau)} (\mathcal{C}_M g)(\tau + t) d\tau. \quad (23)$$

Equivalently,

$$(f \diamond_M g)(t) = \int_{\mathbb{R}} \overline{f(\tau)} g(\tau + t) e^{iA\tau t/B} d\tau. \quad (24)$$

Theorem 3.4 (Correlation factorization). If $f, g \in \mathcal{S}(\mathbb{R})$, then

$$\mathcal{L}_M(f \diamond_M g)(u) = \sqrt{2\pi B} e^{iDu^2/(2B)} \overline{\mathcal{L}_M f(u)} \mathcal{L}_M g(u), \quad (25)$$

and

$$|(f \diamond_M g)(t)| \leq \|f\|_2 \|g\|_2 \quad (t \in \mathbb{R}). \quad (26)$$

Proof. Let $x = \mathcal{C}_M f$ and $y = \mathcal{C}_M g$. Their ordinary cross-correlation is

$$R_{x,y}(t) = \int_{\mathbb{R}} \overline{x(\tau)} y(\tau + t) d\tau.$$

For Schwartz functions, Fubini's theorem applies. With $s = \tau + t$,

$$\begin{aligned} \mathcal{F}R_{x,y}(\omega) &= \frac{1}{\sqrt{2\pi}} \int \int \overline{x(\tau)} y(\tau + t) e^{-i\omega t} d\tau dt \\ &= \sqrt{2\pi} \overline{\mathcal{F}x(\omega)} \mathcal{F}y(\omega). \end{aligned}$$

Since $\mathcal{C}_M(f \diamond_M g) = R_{x,y}$, [theorem 2.1](#) gives

$$\mathcal{L}_M(f \diamond_M g)(u) = B^{-1/2} e^{iDu^2/(2B)} \sqrt{2\pi \mathcal{F}x(u/B)} \mathcal{F}y(u/B).$$

Using [\(4\)](#) for f and g , the opposite input phases cancel and [\(25\)](#) follows. Finally, [\(24\)](#) and Cauchy–Schwarz give

$$|(f \diamond_M g)(t)| \leq \int |f(\tau)| |g(\tau + t)| d\tau \leq \|f\|_2 \|g\|_2.$$

□

4 Cross-Partition Refinement and the LCEWT Convolution Theorem

Let $\mathfrak{M} = \{m_j\}_{j \in J}$ and $\mathfrak{N} = \{n_k\}_{k \in K}$ be two admissible empirical canonical window systems on the same spectral set E , with overlap functions $S_{\mathfrak{M}}$ and $S_{\mathfrak{N}}$. They may arise from different empirical partitions. Let $\mathfrak{P} = \{p_\ell\}_{\ell \in L}$ be an output window system on E .

Theorem 4.1 (Product-refinement frame). *Define*

$$r_{jk}(u) = m_j(u) n_k(u), \quad (j, k) \in J \times K. \quad (27)$$

Then $\mathfrak{R} = \{r_{jk}\}_{J \times K}$ is locally finite, each r_{jk} has compact essential support, and its overlap function is

$$S_{\mathfrak{R}}(u) = \sum_{j,k} |r_{jk}(u)|^2 = S_{\mathfrak{M}}(u) S_{\mathfrak{N}}(u). \quad (28)$$

Consequently

$$s_{\mathfrak{M}} s_{\mathfrak{N}} \leq S_{\mathfrak{R}}(u) \leq S_{\mathfrak{M}}^* S_{\mathfrak{N}}^* \quad (29)$$

for almost every $u \in E$. Thus the product family is an admissible empirical refinement of the two input systems.

Proof. The support relation

$$\text{ess supp } r_{jk} \subseteq \text{ess supp } m_j \cap \text{ess supp } n_k$$

shows compactness. Let $K_0 \subset E$ be compact. Local finiteness of the two input systems implies that only finitely many supports of m_j and finitely many supports of n_k meet K_0 . Hence only finitely many intersections can meet K_0 , proving local finiteness of $\mathfrak{R}$. At a fixed u , all sums are finite, so

$$\begin{aligned} S_{\mathfrak{R}}(u) &= \sum_j \sum_k |m_j(u)|^2 |n_k(u)|^2 \\ &= \left(\sum_j |m_j(u)|^2 \right) \left(\sum_k |n_k(u)|^2 \right) = S_{\mathfrak{M}}(u) S_{\mathfrak{N}}(u). \end{aligned}$$

The bounds [\(29\)](#) follow immediately. □

A measurable family $d = \{d_{jk}\}$ is called a dual multiplier family for $\mathfrak{R}$ if

$$\sum_{j,k} d_{jk}(u) r_{jk}(u) = 1 \quad \text{for a.e. } u \in E. \quad (30)$$

The canonical dual is

$$q_{jk}^-(u) = \frac{\overline{m_j(u)n_k(u)}}{S_{\mathfrak{M}}(u)S_{\mathfrak{N}}(u)}. \quad (31)$$

Theorem 4.2 (Pointwise minimum-norm characterization of the canonical dual). *Fix $u \in E$ outside a null set and regard $r(u) = \{r_{jk}(u)\}$ as a finite nonzero vector. Among all vectors $d(u)$ satisfying (30), the canonical dual (31) is the unique minimizer of the pointwise $\ell^2(J \times K)$ norm. Its norm is*

$$\|q^-(u)\|_{\ell^2} = \frac{1}{\sqrt{S_{\mathfrak{R}}(u)}}. \quad (32)$$

Proof. For any dual $d(u)$, Cauchy–Schwarz gives

$$1 = \left| \sum_{j,k} d_{jk}(u) r_{jk}(u) \right| \leq \|d(u)\|_{\ell^2} \|r(u)\|_{\ell^2}.$$

Since $\|r(u)\|_{\ell^2}^2 = S_{\mathfrak{R}}(u)$,

$$\|d(u)\|_{\ell^2} \geq S_{\mathfrak{R}}(u)^{-1/2}.$$

For (31),

$$\|q^-(u)\|_{\ell^2}^2 = \frac{\sum_{j,k} |r_{jk}(u)|^2}{S_{\mathfrak{R}}(u)^2} = \frac{1}{S_{\mathfrak{R}}(u)},$$

so equality is attained. Equality in Cauchy–Schwarz requires $d(u)$ to be proportional to $\overline{r(u)}$. The constraint (30) fixes the proportionality constant to $S_{\mathfrak{R}}(u)^{-1}$, proving uniqueness. $\square$

Theorem 4.3 (Cross-partition LCEWT convolution theorem). *Let $f, g \in \mathcal{S}(\mathbb{R}) \cap \mathcal{H}_E^M$. For every output index $\ell \in L$,*

$$\begin{aligned} \mathcal{F}_b[\mathcal{W}_\ell^{M,\mathfrak{P}}(f \star_M g)] &= \sqrt{2\pi B} e^{-iDu^2/(2B)} p_\ell(u) \\ &\times \sum_{j \in J} \sum_{k \in K} q_{jk}^-(u) \mathcal{F}_b(\mathcal{W}_j^{M,\mathfrak{M}} f)(u) \mathcal{F}_b(\mathcal{W}_k^{M,\mathfrak{N}} g)(u). \end{aligned} \quad (33)$$

For fixed ℓ , only pairs satisfying

$$\text{ess sup } p_\ell \cap \text{ess sup } m_j \cap \text{ess sup } n_k \quad (34)$$

of positive measure can contribute, and there are only finitely many such pairs.

Proof. Write $F_M = \mathcal{L}_M f$ and $G_M = \mathcal{L}_M g$. By (9) and theorem 3.2,

$$\mathcal{F}_b[\mathcal{W}_\ell^{M,\mathfrak{P}}(f \star_M g)] = \sqrt{2\pi B} e^{-iDu^2/(2B)} p_\ell F_M G_M. \quad (35)$$

The input coefficient spectra are

$$C_j = \mathcal{F}_b(\mathcal{W}_j^{M,\mathfrak{M}} f) = F_M m_j, \quad D_k = \mathcal{F}_b(\mathcal{W}_k^{M,\mathfrak{N}} g) = G_M n_k.$$

Hence, using (31) and (28),

$$\begin{aligned} \sum_{j,k} q_{jk}^- C_j D_k &= F_M G_M \sum_{j,k} \frac{\overline{m_j n_k}}{S_{\mathfrak{M}} S_{\mathfrak{N}}} m_j n_k \\ &= F_M G_M \frac{S_{\mathfrak{R}}}{S_{\mathfrak{M}} S_{\mathfrak{N}}} = F_M G_M \end{aligned}$$

almost everywhere on E . Substitution in (35) proves (33).

It remains to justify the active-set statement. The support of $C_j D_k$ is contained in $\text{ess supp } m_j \cap \text{ess supp } n_k$. Multiplication by p_ℓ removes every pair for which (34) has measure zero. Since $\text{ess supp } p_\ell$ is compact and each input support family is locally finite, it meets only finitely many supports from each input system, leaving finitely many pairs. Each product $C_j D_k$ lies in L^1 by Cauchy–Schwarz, so the finite sum is well defined in L^1 . $\square$

The individual coefficients in (33) depend on the chosen dual of the product refinement. The synthesized result does not.

Theorem 4.4 (Dual invariance of the synthesized convolution identity). *Let $d = \{d_{jk}\}$ satisfy (30). Assume that, for a fixed ℓ , every product $p_\ell d_{jk}$ associated with an active pair is essentially bounded. Then (33) remains valid with q_{jk}^- replaced by d_{jk} .*

Proof. With $C_j = F_M m_j$ and $D_k = G_M n_k$,

$$\sum_{j,k} d_{jk} C_j D_k = F_M G_M \sum_{j,k} d_{jk} m_j n_k = F_M G_M$$

almost everywhere on E by (30). The active sum is finite on $\text{ess supp } p_\ell$, so no unconditional-convergence issue occurs there. Substituting this identity into (35) proves the claim. $\square$

Remark 4.5 (Intrinsic and coordinate-dependent content). The common refinement $\{m_j n_k\}$, its overlap $S_{\mathfrak{M}} S_{\mathfrak{N}}$, and the active triple intersections (34) are determined by the two empirical window systems. The individual coupling weights depend on a dual choice. The canonical dual is distinguished by theorem 4.2; the synthesized identity is independent of that choice by theorem 4.4.

5 Cross-Partition LCEWT Correlation Theorem

For correlation the first spectral factor is conjugated, so the appropriate common refinement is

$$\rho_{jk}(u) = \overline{m_j(u)} n_k(u). \quad (36)$$

Its squared overlap is again

$$\sum_{j,k} |\rho_{jk}(u)|^2 = S_{\mathfrak{M}}(u) S_{\mathfrak{N}}(u). \quad (37)$$

Hence its canonical dual is

$$q_{jk}^+(u) = \frac{m_j(u) \overline{n_k(u)}}{S_{\mathfrak{M}}(u) S_{\mathfrak{N}}(u)}. \quad (38)$$

The same Cauchy–Schwarz proof as in theorem 4.2 shows that (38) is the unique pointwise minimum- ℓ^2 dual of (36).

Theorem 5.1 (Cross-partition LCEWT correlation theorem). *Let $f, g \in \mathcal{S}(\mathbb{R}) \cap \mathcal{H}_E^M$. For every $\ell \in L$,*

$$\begin{aligned} \mathcal{F}_b \left[\mathcal{W}_\ell^{M, \mathfrak{P}}(f \diamond_M g) \right] &= \sqrt{2\pi B} e^{iDu^2/(2B)} p_\ell(u) \\ &\times \sum_{j,k} q_{jk}^+(u) \overline{\mathcal{F}_b(\mathcal{W}_j^{M, \mathfrak{M}} f)(u)} \mathcal{F}_b(\mathcal{W}_k^{M, \mathfrak{N}} g)(u). \end{aligned} \quad (39)$$

Only triples with

$$\text{ess supp } p_\ell \cap \text{ess supp } m_j \cap \text{ess supp } n_k$$

of positive measure contribute. The synthesized identity remains unchanged if q_{jk}^+ is replaced by any dual e_{jk} satisfying

$$\sum_{j,k} e_{jk}(u) \overline{m_j(u)} n_k(u) = 1 \quad (40)$$

with the same local boundedness condition as in [theorem 4.4](#).

Proof. Let $F_M = \mathcal{L}_M f$ and $G_M = \mathcal{L}_M g$. From [theorem 3.4](#) and (9),

$$\mathcal{F}_b[\mathcal{W}_\ell^{M,\mathfrak{P}}(f \diamond_M g)] = \sqrt{2\pi B} e^{iDu^2/(2B)} p_\ell \overline{F_M} G_M. \quad (41)$$

Set

$$C_j = F_M m_j, \quad D_k = G_M n_k.$$

Then

$$\overline{C_j} D_k = \overline{F_M} G_M \overline{m_j} n_k.$$

Using (38),

$$\begin{aligned} \sum_{j,k} q_{jk}^+ \overline{C_j} D_k &= \overline{F_M} G_M \sum_{j,k} \frac{m_j \overline{n_k}}{S_{\mathfrak{M}} S_{\mathfrak{N}}} \overline{m_j} n_k \\ &= \overline{F_M} G_M. \end{aligned}$$

Substitution into (41) proves (39). The active-set and finiteness arguments are identical to those in the proof of [theorem 4.3](#). Finally, if e_{jk} satisfies (40), then

$$\sum_{j,k} e_{jk} \overline{C_j} D_k = \overline{F_M} G_M \sum_{j,k} e_{jk} \overline{m_j} n_k = \overline{F_M} G_M,$$

which proves dual invariance. $\square$

Corollary 5.2 (Autocorrelation). *For $f \in \mathcal{S}(\mathbb{R}) \cap \mathcal{H}_E^M$, [theorem 5.1](#) with $g = f$ gives an exact cross-partition coefficient representation of $f \diamond_M f$. If the same empirical system is used for both inputs, the correlation refinement is $\{\overline{m_j} m_k\}_{j,k}$.*

Proof. Set $g = f$ in (39). All hypotheses remain unchanged, so the stated specialization follows directly. $\square$

6 Stability and Reduction Principles

The empirical content of the construction is meaningful only if small perturbations of the detected windows produce controlled changes in the coefficient geometry. The next results give such estimates directly in the natural mixed norm.

Theorem 6.1 (Lipschitz stability of the LCEWT analysis map). *Let $\mathfrak{M} = \{m_j\}_{j \in J}$ and $\widetilde{\mathfrak{M}} = \{\widetilde{m}_j\}_{j \in J}$ be two admissible window systems on the same E . Define*

$$\delta_{\mathfrak{M}} = \operatorname{ess\,sup}_{u \in E} \left(\sum_{j \in J} |m_j(u) - \widetilde{m}_j(u)|^2 \right)^{1/2}. \quad (42)$$

Then for every $f \in \mathcal{H}_E^M$,

$$\left(\sum_j \|\mathcal{W}_j^{M,\mathfrak{M}} f - \mathcal{W}_j^{M,\widetilde{\mathfrak{M}}} f\|_2^2 \right)^{1/2} \leq \delta_{\mathfrak{M}} \|f\|_2. \quad (43)$$

Proof. By Plancherel and Tonelli,

$$\begin{aligned} \sum_j \|\mathcal{W}_j^{M, \mathfrak{M}} f - \mathcal{W}_j^{M, \widetilde{\mathfrak{M}}} f\|_2^2 &= \int_E |\mathcal{L}_M f(u)|^2 \sum_j |m_j(u) - \widetilde{m}_j(u)|^2 du \\ &\leq \delta_{\mathfrak{M}}^2 \|\mathcal{L}_M f\|_2^2 = \delta_{\mathfrak{M}}^2 \|f\|_2^2. \end{aligned}$$

Taking square roots proves (43). $\square$

Theorem 6.2 (Stability of the common refinement). *Let $\mathfrak{M}, \widetilde{\mathfrak{M}}$ and $\mathfrak{N}, \widetilde{\mathfrak{N}}$ be pairs of admissible systems on E . Let*

$$r_{jk} = m_j n_k, \quad \widetilde{r}_{jk} = \widetilde{m}_j \widetilde{n}_k,$$

and define $\delta_{\mathfrak{M}}$ and $\delta_{\mathfrak{N}}$ as in (42). If

$$S_{\mathfrak{N}}(u) \leq S_{\mathfrak{N}}^*, \quad S_{\widetilde{\mathfrak{M}}}(u) \leq S_{\widetilde{\mathfrak{M}}}^*,$$

then

$$\begin{aligned} \operatorname{ess\,sup}_{u \in E} \left(\sum_{j,k} |r_{jk}(u) - \widetilde{r}_{jk}(u)|^2 \right)^{1/2} \\ \leq \delta_{\mathfrak{M}} \sqrt{S_{\mathfrak{N}}^*} + \sqrt{S_{\widetilde{\mathfrak{M}}}^*} \delta_{\mathfrak{N}}. \end{aligned} \quad (44)$$

Proof. At a fixed u , regard $m(u)$ and $\widetilde{m}(u)$ as vectors in $\ell^2(J)$ and $n(u)$ and $\widetilde{n}(u)$ as vectors in $\ell^2(K)$. The refinement vector is the tensor product $r(u) = m(u) \otimes n(u)$. Thus

$$r - \widetilde{r} = (m - \widetilde{m}) \otimes n + \widetilde{m} \otimes (n - \widetilde{n}).$$

The Hilbert tensor norm satisfies $\|a \otimes b\|_2 = \|a\|_2 \|b\|_2$, so

$$\begin{aligned} \|r(u) - \widetilde{r}(u)\|_2 &\leq \|m(u) - \widetilde{m}(u)\|_2 \|n(u)\|_2 \\ &\quad + \|\widetilde{m}(u)\|_2 \|n(u) - \widetilde{n}(u)\|_2. \end{aligned}$$

Now $\|n(u)\|_2^2 = S_{\mathfrak{N}}(u)$ and $\|\widetilde{m}(u)\|_2^2 = S_{\widetilde{\mathfrak{M}}}(u)$. Taking essential suprema yields (44). $\square$

Theorem 6.3 (Stability of canonical dual windows). *Let $\mathfrak{M} = \{m_j\}$ and $\widetilde{\mathfrak{M}} = \{\widetilde{m}_j\}$ be admissible systems on E with lower bounds $s_{\mathfrak{M}}, s_{\widetilde{\mathfrak{M}}} > 0$ and upper bounds $S_{\mathfrak{M}}^*, S_{\widetilde{\mathfrak{M}}}^*$. Define their canonical dual multipliers*

$$d_j = \frac{\overline{m_j}}{S_{\mathfrak{M}}}, \quad \widetilde{d}_j = \frac{\overline{\widetilde{m}_j}}{S_{\widetilde{\mathfrak{M}}}^*}.$$

Then

$$\operatorname{ess\,sup}_{u \in E} \left(\sum_j |d_j(u) - \widetilde{d}_j(u)|^2 \right)^{1/2} \leq C_{\mathfrak{M}, \widetilde{\mathfrak{M}}} \delta_{\mathfrak{M}}, \quad (45)$$

where

$$C_{\mathfrak{M}, \widetilde{\mathfrak{M}}} = \frac{1}{s_{\mathfrak{M}}} + \frac{\sqrt{S_{\widetilde{\mathfrak{M}}}^*} (\sqrt{S_{\mathfrak{M}}^*} + \sqrt{S_{\widetilde{\mathfrak{M}}}^*})}{s_{\mathfrak{M}} s_{\widetilde{\mathfrak{M}}}}. \quad (46)$$

Proof. At a fixed u , write $S = S_{\mathfrak{M}}(u)$ and $\widetilde{S} = S_{\widetilde{\mathfrak{M}}}(u)$. Then

$$d - \widetilde{d} = \frac{\overline{m} - \widetilde{m}}{S} + \widetilde{m} \left(\frac{1}{S} - \frac{1}{\widetilde{S}} \right).$$

Hence

$$\|d - \tilde{d}\|_2 \leq \frac{\|m - \tilde{m}\|_2}{S} + \|\tilde{m}\|_2 \frac{|S - \tilde{S}|}{S\tilde{S}}. \quad (47)$$

Moreover,

$$\begin{aligned} |S - \tilde{S}| &= |\|m\|_2^2 - \|\tilde{m}\|_2^2| \\ &\leq (\|m\|_2 + \|\tilde{m}\|_2)\|m - \tilde{m}\|_2. \end{aligned}$$

Using $S \geq s_{\mathfrak{M}}$, $\tilde{S} \geq s_{\tilde{\mathfrak{M}}}$, $\|m\|_2 \leq \sqrt{S_{\mathfrak{M}}^*}$, and $\|\tilde{m}\|_2 \leq \sqrt{S_{\tilde{\mathfrak{M}}}^*}$ in (47) gives the pointwise bound with the constant (46). Taking the essential supremum proves (45). $\square$

Proposition 6.4 (Fourier reduction). *For $M_F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,*

$$\mathcal{L}_{M_F} = \mathcal{F}, \quad f \star_{M_F} g = f * g, \quad f \diamond_{M_F} g = R_{f,g},$$

where $R_{f,g}(t) = \int \overline{f(\tau)}g(\tau + t) d\tau$. The factorization formulas become

$$\mathcal{F}(f * g) = \sqrt{2\pi} \mathcal{F}f \mathcal{F}g, \quad \mathcal{F}R_{f,g} = \sqrt{2\pi} \overline{\mathcal{F}f} \mathcal{F}g.$$

The cross-partition refinement theorems reduce to the corresponding empirical Fourier-filter-bank identities.

Proof. For M_F , $A = D = 0$ and $B = 1$. Substitution in (3), (20), (24), (22), and (25) gives the stated formulas. The refinement windows and their duals contain no additional LCT phase, so theorems 4.3 and 5.1 reduce directly to the empirical Fourier-filter-bank case. $\square$

Proposition 6.5 (Fractional-canonical specialization). *Let*

$$M_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad 0 < \theta < \pi.$$

Then

$$(f \star_{M_\theta} g)(t) = \int f(\tau)g(t - \tau)e^{-i \cot \theta \tau(t - \tau)} d\tau, \quad (48)$$

and

$$\mathcal{L}_{M_\theta}(f \star_{M_\theta} g)(u) = \sqrt{2\pi \sin \theta} e^{-i(\cot \theta)u^2/2} \mathcal{L}_{M_\theta}f(u) \mathcal{L}_{M_\theta}g(u). \quad (49)$$

At $\theta = \pi/2$ the Fourier formulas of theorem 6.4 are recovered.

Proof. For M_θ , $A/B = D/B = \cot \theta$ and $B = \sin \theta$. Substitution into (20) and (22) gives (48) and (49). The value $\theta = \pi/2$ gives $A = D = 0$ and $B = 1$. $\square$

Remark 6.6 (Same-band formulas). If both inputs use the same disjoint indicator partition $m_j = \mathbf{1}_{\Omega_j}$, then the product refinement contains only diagonal active pairs on each output band. The spectral coefficient identity therefore reduces to a single same-band product. For $D \neq 0$ the LCT output chirp $e^{-iDu^2/(2B)}$ remains, so an ordinary convolution in the translation variable requires an additional phase-compensating formulation. This identifies the precise obstruction without assuming same-bank closure.

7 Numerical Validation and a Compact-Window Application

All calculations in this section are generated by the accompanying script `validate.py`; the script also writes `verification_metrics.csv` and the three figures used below. The numerical tests instantiate the theory on the finite canonical-frequency interval $E = [-10, 10]$ using compact cosine bumps. Two distinct input systems are normalized pointwise on their common coverage, so their overlap functions equal one there. The first system has centers $-7, -3.5, 0, 3.5, 7$, while the second has centers $-6.2, -2, 2.2, 6.4$. The output system is different again. This deliberately tests the cross-partition rather than same-partition formulas.

For the convolution and correlation checks we prescribe smooth compact canonical spectra F_M and G_M inside the common coverage and evaluate the two sides of [theorems 4.3](#) and [5.1](#) directly on a 4097-point grid. We use $B = 1.3$ and $D = 1.1$; the value of A does not enter the coefficient-domain product once the LCT spectra have been specified. The canonical-refinement dual is also perturbed by a pointwise null-space component to produce a different admissible dual. The residual in its reconstruction constraint and the resulting convolution residual test [theorem 4.4](#) independently of the minimum-norm choice.

Table 1: Reproducible checks of the cross-partition identities and dual invariance.

quantity	value
cross-partition convolution residual	1.97×10^{-16}
cross-partition correlation residual	1.98×10^{-16}
alternative-dual constraint residual	5.57×10^{-16}
alternative-dual convolution residual	1.99×10^{-16}
minimum sampled excess norm of alternative dual	1.61×10^{-1}

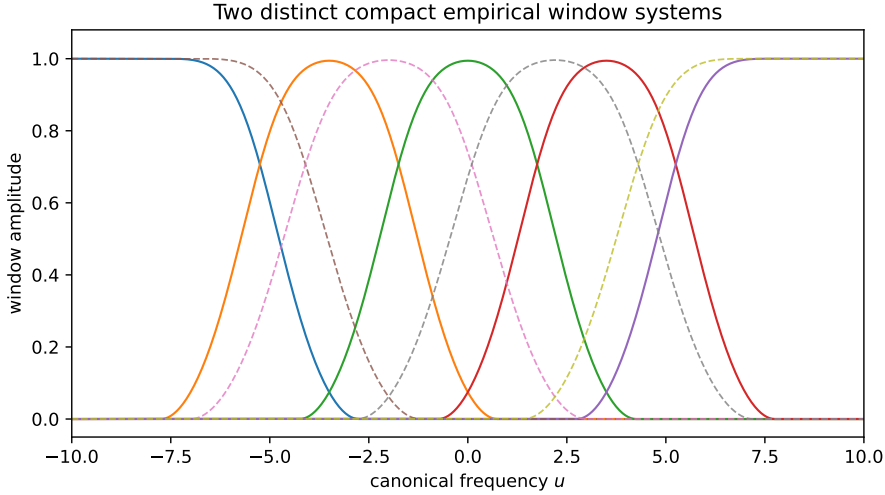


Figure 1: Two distinct compact empirical canonical window systems used in the cross-partition validation. Solid and dashed curves belong to different input coverings.

We next shift every center of the first empirical system by 0.15 while retaining the same compact window profile. The computed window distance [\(42\)](#) is

$$\delta_{\mathfrak{M}} = 8.36 \times 10^{-2}.$$

For the test spectrum used by the script, the observed mixed-norm coefficient change divided by the signal norm is

$$6.50 \times 10^{-2},$$

which is below the theorem bound. Perturbing both input systems produces a product-refinement distance 1.19×10^{-1} , while the right side of (44) equals 1.81×10^{-1} .

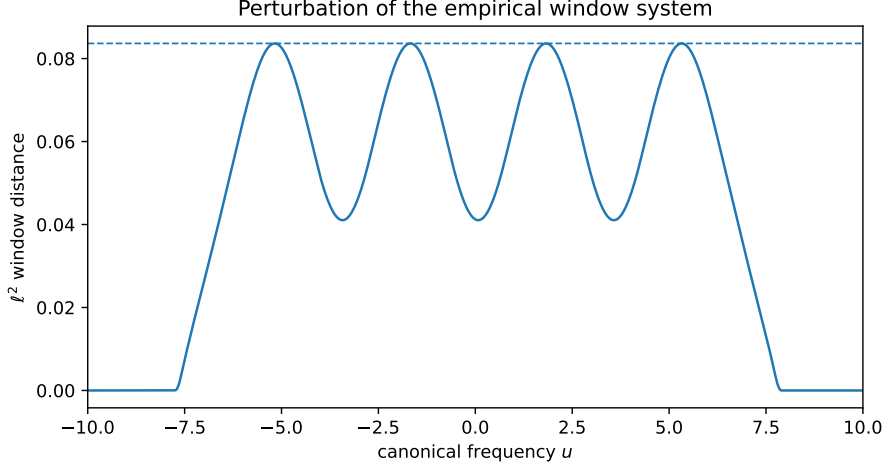


Figure 2: Pointwise ℓ^2 distance between an empirical window system and the system obtained by a small center perturbation. The horizontal dashed line is the essential-supremum value used in [theorem 6.1](#).

For a compact-window application, two canonical spectra are generated with peaks near -3.30 and 3.80 . The midpoint $u_* = 0.249$ is obtained from the two detected maxima and is used as an empirical boundary. A compact two-window partition of unity is constructed on $[-8, 8]$, with a smooth interior transition across the detected boundary. Define $X_j = (\mathcal{L}_M x)m_j$ and $x_j = \mathcal{L}_M^{-1} X_j$. By unitarity, spectral and time-domain relative L^2 errors coincide. The two reconstructed components have relative errors

$$4.02 \times 10^{-3}, \quad 5.11 \times 10^{-3}, \quad (50)$$

while the relative total reconstruction error is 1.53×10^{-18} . The component errors quantify leakage through the deliberately smooth transition region; the total error is at roundoff because the two compact windows form a partition of unity on the support of the synthetic spectrum.

The numerical section is intentionally limited to consistency checks and one controlled application. It does not use external denoising benchmarks and therefore makes no comparative performance claim.

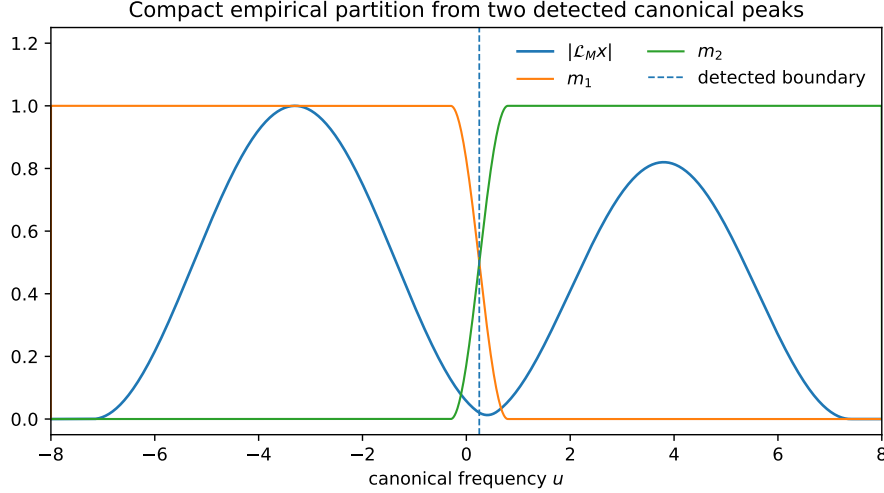


Figure 3: Compact empirical partition generated from two detected canonical-frequency peaks. The windows have compact support on the computational spectral interval and a smooth interior transition, so the application satisfies the support assumptions of the analytical framework.

8 Conclusion and Perspectives

We have developed a fixed-partition LCEWT framework in which empirical canonical windows are selected first and then treated as a locally finite frame of spectral multipliers. The construction is compatible with a unitary LCT normalization and admits genuine L^2 analyzing atoms. An exact bridge theorem shows how the multiplier atoms are related, after a rescaling of translation and a known quadratic phase, to scale-centre canonical empirical wavelet atoms. Frame bounds and reconstruction follow from the window-overlap function on the relevant canonical spectral subspace.

The convolution and correlation operators were derived by chirp conjugation of their ordinary counterparts, which fixes all phase factors and yields exact LCT product identities. The principal structural contribution concerns different empirical partitions. Products of the two input window systems form a locally finite common refinement whose overlap is the product of the input overlaps. The canonical dual of this refinement is the unique pointwise minimum-norm dual. Cross-partition convolution and correlation formulas then follow from this refinement, while a separate dual-invariance theorem shows that the synthesized result is independent of the chosen admissible dual. This distinction identifies which parts of the coefficient representation are intrinsic and which are coordinate dependent.

The empirical dependence is further quantified through Lipschitz stability of the analysis map, the product refinement, and the canonical dual under perturbations of the windows. These estimates address boundary sensitivity directly and are absent from a purely algebraic same-partition expansion. The numerical calculations use compactly supported windows throughout and verify the cross-partition identities, dual invariance, and stability bounds with reproducible data.

Several questions remain. Sharper perturbation bounds in terms of boundary locations rather than window distances would connect the theory more directly to a specific empirical boundary detector. A second problem is unconditional cross-partition synthesis when an output window can interact with infinitely many input windows. A third is to characterize regularity classes for which the refinement multipliers admit integrable translation-domain kernels. These questions concern the interaction of empirical geometry, frame duality, and canonical convolution independently of application-specific benchmarking.

Declarations

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Data availability. Data sharing is not applicable to this article, as no datasets were generated or analyzed during the present study.

References

- [1] M. Moshinsky and C. Quesne, “Linear canonical transformations and their unitary representations,” *Journal of Mathematical Physics*, vol. 12, no. 8, pp. 1772–1780, 1971. doi:10.1063/1.1665805.
- [2] J. Gilles, “Empirical Wavelet Transform,” *IEEE Transactions on Signal Processing*, vol. 61, no. 16, pp. 3999–4010, 2013. doi:10.1109/TSP.2013.2265222.
- [3] J. Gilles, “Continuous empirical wavelets systems,” *Advances in Data Science and Adaptive Analysis*, vol. 12, no. 03n04, Art. 2050006, 2020. doi:10.1142/S2424922X20500060.
- [4] B. Deng, R. Tao, and Y. Wang, “Convolution theorems for the linear canonical transform and their applications,” *Science in China Series F: Information Sciences*, vol. 49, no. 5, pp. 592–603, 2006. doi:10.1007/s11432-006-2016-4.
- [5] D. Wei, Q. Ran, Y.-M. Li, J. Ma, and L.-Y. Tan, “A Convolution and Product Theorem for the Linear Canonical Transform,” *IEEE Signal Processing Letters*, vol. 16, no. 10, pp. 853–856, 2009. doi:10.1109/LSP.2009.2026107.
- [6] D. Wei, Q. Ran, and Y.-M. Li, “A Convolution and Correlation Theorem for the Linear Canonical Transform and Its Application,” *Circuits, Systems, and Signal Processing*, vol. 31, no. 1, pp. 301–312, 2012. doi:10.1007/s00034-011-9319-4.
- [7] D. Wei and Y.-M. Li, “Generalized wavelet transform based on the convolution operator in the linear canonical transform domain,” *Optik*, vol. 125, no. 16, pp. 4491–4496, 2014. doi:10.1016/j.ijleo.2014.02.021.
- [8] Y. Guo and B.-Z. Li, “The linear canonical wavelet transform on some function spaces,” *International Journal of Wavelets, Multiresolution and Information Processing*, vol. 16, no. 1, Art. 1850010, 2018. doi:10.1142/S0219691318500108.
- [9] A. Prasad and Z. A. Ansari, “Continuous Wavelet Transform Involving Linear Canonical Transform,” *National Academy Science Letters*, vol. 42, no. 4, pp. 337–344, 2019. doi:10.1007/s40009-018-0743-x.
- [10] M. Bahri, A. K. Amir, and R. Ashino, “Linear canonical wavelet transform: Properties and inequalities,” *International Journal of Wavelets, Multiresolution and Information Processing*, vol. 19, no. 6, Art. 2150027, 2021. doi:10.1142/S0219691321500272.
- [11] A. I. Zayed, “A convolution and product theorem for the fractional Fourier transform,” *IEEE Signal Processing Letters*, vol. 5, no. 4, pp. 101–103, 1998. doi:10.1109/97.664179.