

Supplementary information for “Experimental zero-added-loss multiplexing Bell-pair source for long-haul quantum networks”

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SUPPLEMENTARY NOTE 1: Simulation using Gaussian-state characteristic-function method

We present a detailed procedure for calculating the relation between the fidelity F_{exp} and the total swapping rate r_{tot} or r_{ZALM} . First, the four-fold coincidence probabilities for each measurement basis are calculated, from which r_{tot} is obtained. Next, using the probability distributions corresponding to 16 independent basis states, the density operator is reconstructed via quantum state tomography and the fidelity is evaluated. This procedure is identical to that applied to the experimentally obtained probability distributions. The coincidence probabilities are calculated based on Ref. [1–3] using the Gaussian-state characteristic-function method. Since the theoretical model and detailed procedures closely follow those in Ref. [3], we describe only the overall framework here. We define a density operator acting on the N -dimensional Hilbert space

$\mathcal{H}^{\otimes N}$ as ρ . The characteristic function of ρ is defined by

$$\chi(\boldsymbol{\xi}) = \text{Tr}[\rho \mathcal{W}(\boldsymbol{\xi})], \quad (1)$$

where

$$\mathcal{W}(\boldsymbol{\xi}) = \exp(-i\boldsymbol{\xi}^T \mathbf{R}) \quad (2)$$

is the Weyl operator. Here, $\mathbf{R} = (x_1, \dots, x_N, p_1, \dots, p_N)^T$ is the vector of quadrature operators, and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{2N})^T \in \mathbb{R}^{2N}$ is a real-valued phase-space vector. When the characteristic function of the quantum state has a Gaussian distribution as

$$\chi(\boldsymbol{\xi}) = \exp\left(-\frac{1}{4}\boldsymbol{\xi}^T \boldsymbol{\gamma} \boldsymbol{\xi} - i\mathbf{d}^T \boldsymbol{\xi}\right), \quad (3)$$

the quantum state is simply characterized by a $2N \times 2N$ matrix $\boldsymbol{\gamma}$ (the covariance matrix) and a $2N$ dimensional vector $\mathbf{d}$ (the displacement vector). As shown in Supplementary Fig. 1a, each EPS contains a pair of two-mode squeezed vacuum states (TMSVs), generating photon pairs in the H and V polarizations, respectively. For example, the characteristic function of the output state from EPS I is given by

$$\gamma_{H_1 V_1 H_3 V_3}^I(\mu_H^I, \mu_V^I) = \gamma_{H_1 V_1 H_3 V_3}^+(\mu_H^I, \mu_V^I) \oplus \gamma_{H_1 V_1 H_3 V_3}^-(\mu_H^I, \mu_V^I), \quad (4)$$

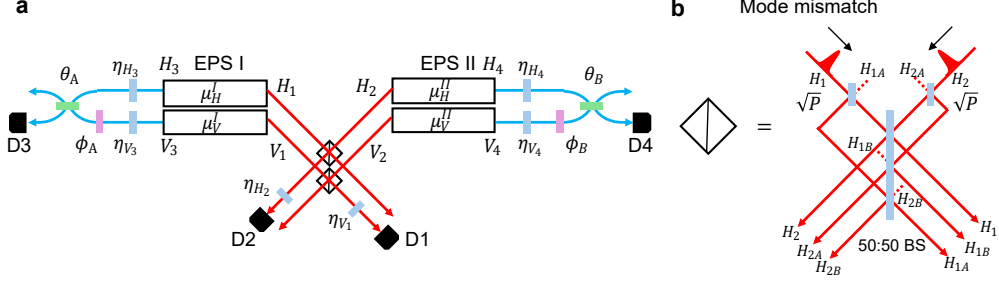
where the matrix elements are expressed as

$$\gamma_{H_1 V_1 H_3 V_3}^{\pm}(\mu_H^I, \mu_V^I) = \begin{pmatrix} 2\mu_H^I + 1 & 0 & \pm 2\sqrt{\mu_H^I(\mu_H^I + 1)} & 0 \\ 0 & 2\mu_V^I + 1 & 0 & \pm 2\sqrt{\mu_V^I(\mu_V^I + 1)} \\ \pm 2\sqrt{\mu_H^I(\mu_H^I + 1)} & 0 & 2\mu_H^I + 1 & 0 \\ 0 & \pm 2\sqrt{\mu_V^I(\mu_V^I + 1)} & 0 & 2\mu_V^I + 1 \end{pmatrix}.$$

Here, μ_p^k denotes the average photon number of the p -polarized ($p = H, V$) TMSV state generated by EPS k ($k = \text{I, II}$). Therefore, the covariance matrix of the input state is given by

$$\gamma_{\text{in}} := \gamma_{H_1 V_1 H_3 V_3}^I(\mu_H^I, \mu_V^I) \oplus \gamma_{H_2 V_2 H_4 V_4}^{\text{II}}(\mu_H^{\text{II}}, \mu_V^{\text{II}}). \quad (5)$$

Next, the photons in modes 1 and 2 used for the BSM are mixed at a BS. The BSM including mode mismatch is modeled following Ref. [1, 3], as illustrated in Supplementary Fig. 1b. We assume that both input photons possess the same Schmidt-mode distribution, such that the only source of imperfect interference is the mixedness arising from the finite spectral purity, while any relative spectral or temporal mismatch between the photons is neglected. Specifically, a single input mode is split into an interfering and a non-interfering mode using a virtual BS with transmittance of $\sqrt{P}$, and the interfering modes are subsequently mixed at a 50:50 BS, where P is the spectral purity defined in the main manuscript. Here, we introduce the ancillary vacuum



Supplementary Fig. 1 Theoretical model of entanglement swapping. **a** A pair of two-mode squeezed vacuum states is used to prepare polarization entangled photon pairs. The Bell-state measurement is realized by mixing photons in modes 1 and 2 by a 50:50 beamsplitter (BS) followed by polarization measurements. Polarization rotations on photons in modes 3 and 4 are realized by phase shifters and BSs in polarization degree of freedom. All of the photon detectors are threshold detectors. **b** Mode mismatch is modeled by virtual BSs with a transmittance of $\sqrt{P}$.

modes $H_{1A}, H_{1B}, H_{2A}, H_{2B}$ and $V_{1A}, V_{1B}, V_{2A}, V_{2B}$ for the H and V polarizations, respectively. The characteristic function after mode mixing can then be written as

$$\gamma_{\text{BSM}} = \mathbf{S}_{\text{BS}}^T \mathbf{S}_{\text{MM}}^T (\gamma_{\text{in}} \oplus \mathbf{I}_{H_{1A}, H_{1B}, H_{2A}, H_{2B}} \oplus \mathbf{I}_{V_{1A}, V_{1B}, V_{2A}, V_{2B}}) \mathbf{S}_{\text{MM}} \mathbf{S}_{\text{BS}} \quad (6)$$

where

$$\mathbf{S}_{\text{BS}} := \prod_{\nu \in \{H, V\}} \left(\mathbf{S}_{\nu_1 \nu_2}^{1/2} \mathbf{S}_{\nu_1 B \nu_2 A}^{1/2} \mathbf{S}_{\nu_1 A \nu_2 B}^{1/2} \right) \quad (7)$$

and

$$\mathbf{S}_{\text{MM}} := \prod_{\nu \in \{H, V\}} \left(\mathbf{S}_{\nu_1 \nu_1 A}^{\sqrt{P}} \mathbf{S}_{\nu_2 \nu_2 A}^{\sqrt{P}} \right) \quad (8)$$

are the symplectic matrices corresponding to the Gaussian unitary operation for mode matching and mode mixing using a 50:50 BS, respectively. Here, we define the BS transformation between modes i and j with transmittance t by $\mathbf{S}_{ij}^t$.

The loss transformation is then applied to every detected mode as

$$\gamma_{\text{L}} = \mathcal{L}(\gamma_{\text{BSM}}) \quad (9)$$

with

$$\mathcal{L} = \mathcal{L}_{V_1}^{\eta_{V_1}} \circ \mathcal{L}_{V_{1A}}^{\eta_{V_1}} \circ \mathcal{L}_{V_{1B}}^{\eta_{V_1}} \circ \mathcal{L}_{H_2}^{\eta_{H_2}} \circ \mathcal{L}_{H_{2A}}^{\eta_{H_2}} \circ \mathcal{L}_{H_{2B}}^{\eta_{H_2}} \circ \mathcal{L}_{H_3}^{\eta_{H_3}} \circ \mathcal{L}_{V_3}^{\eta_{V_3}} \circ \mathcal{L}_{H_4}^{\eta_{H_4}} \circ \mathcal{L}_{V_4}^{\eta_{V_4}}, \quad (10)$$

where $\mathcal{L}_j^\eta$ denotes the operation corresponding to the loss with transmittance η acting on the mode j .

Projective measurements onto the basis states of the photons in modes 3 and 4 are simulated by applying appropriate phase shifts between the H- and V-polarization components, followed by interference at a polarization beam splitter and detection of the H-polarized component. The covariance matrix after the polarization rotation is then given by

$$\gamma_{\text{Final}} = \mathbf{M}^T \gamma_{\text{L}} \mathbf{M}, \quad (11)$$

where

$$\mathbf{M} = \mathbf{R}_{V_3}(\phi_A) \mathbf{R}_{V_4}(\phi_B) \mathbf{S}_{H_3 V_3}^{\cos^2 \theta_B} \mathbf{S}_{H_4 V_4}^{\cos^2 \theta_A}. \quad (12)$$

Here, $\mathbf{R}_j(\phi)$ is the symplectic matrix corresponding to the phase shift ϕ on mode j . In the entanglement swapping experiment, we choose the event where H and V polarized photons are respectively detected by D1 and D2 as a successful event as show in Supplementary Fig. 1. Note that the maximum number of successful events allowed for a BSM using linear optical elements is four times the value described above.

Finally, the four-fold coincidence probability is given by

$$P_{4\text{fold}}(\phi_A, \theta_A, \phi_B, \theta_B) = \text{Tr} \left[\rho^{\gamma_{\text{Final}}} \prod_{i=1}^4 \Pi_{\text{on}}^{B_i} \right] \quad (13)$$

$$= \sum_{S \subseteq \{1,2,3,4\}} (-1)^{|S|} \frac{2^{|B_S|}}{\sqrt{\det(\gamma_{B_S} + \mathbf{I})}}, \quad (14)$$

where

$$\Pi_{\text{on}}^{B_i} = \mathbf{I} - |0\rangle\langle 0|_{B_i} \quad (15)$$

is the POVM element of the photon detection in mode B_i and

$$\begin{aligned} B_1 &= \{H_2, H_{2A}, H_{2B}\}, \\ B_2 &= \{V_1, V_{1A}, V_{1B}\}, \\ B_3 &= \{H_3\}, \\ B_4 &= \{H_4\}, \\ B_S &= \bigcup_{i \in S} B_i. \end{aligned}$$

andwhere γ_{B_S} denotes the principal submatrix of γ corresponding to the modes in B_S , with $\gamma_{B_\emptyset}$ understood as the 0×0 matrix. In Supplementary Eq. (14), we have used the identity for the overlap between an m -mode Gaussian state with covariance matrix γ and the m -mode vacuum state as

$$\text{Tr} [\rho^\gamma |0\rangle\langle 0|^{\otimes m}] = \frac{2^m}{\sqrt{\det(\gamma + \mathbf{I})}}. \quad (16)$$

The four-fold coincidence probability $P_{4\text{fold}}(\phi_A, \theta_A, \phi_B, \theta_B)$ is calculated using the experimentally characterized system parameters summarized in Supplementary Table 1, including the Klyshko efficiencies and the spectral purity P , together with the average photon number of each TMSV state. The quantum efficiencies of the detectors and the heralding efficiencies are included in the Klyshko efficiency. It is important to note that $P_{4\text{fold}}$ is not equivalent to the success probability of the local BSM. The latter includes erroneous events in which two photon pairs are generated from one EPS while no photon pair is generated from the other. Such events do not contribute to the generation of swapped Bell pairs and are therefore excluded from r_{ZALM} .

Supplementary Table 1 The experimental parameters used for the simulation

η_{V_1}	η_{H_2}	η_{H_3}	η_{V_3}	η_{H_4}	η_{V_4}	P
7.2 %	5.9 %	4.2 %	4.2 %	4.6 %	5.0 %	97 %

The entanglement fidelity F_{exp} is evaluated from $P_{\text{4fold}}(\phi_A, \theta_A, \phi_B, \theta_B)$ for the 16 polarization projection settings corresponding to the basis states shown in Fig. 4d.

The total swapping rate r_{tot} shown in Fig. 5a is evaluated by

$$r_{\text{tot}} = f_{\text{rep}} M_{\text{eff}} \sum_{\theta_A, \theta_B \in \{0, \pi/2\}} P_{\text{4fold}}(0, \theta_A, 0, \theta_B), \quad (17)$$

where the average photon numbers of all TMSV states are assumed to be identical, i.e. $\mu_H^I = \mu_V^I = \mu_H^{\text{II}} = \mu_V^{\text{II}} \equiv \mu$, and r_{tot} is evaluated as a function of the common average photon number μ . Here, $f_{\text{rep}} = 3.0$ GHz is the pump repetition rate and M_{eff} is the effective multiplexing factor. Specifically, no multiplexing correction is applied in the single-mode case, and thus $M_{\text{eff}} = 1$. For the multiplexed cases, M_{eff} is obtained by multiplying the nominal multiplexing number by a correction factor determined from the measured spectral profile shown in Fig. 1d, yielding $M_{\text{eff}} = 0.95M$ for the 800 GHz bandwidth and $M_{\text{eff}} = 0.86M$ for the full 8.7 THz SPDC bandwidth, where M denotes the nominal multiplexing number.

The ZALM Bell-pair rate r_{ZALM} shown in Fig. 5b is evaluated by removing the detection losses in modes 3 and 4 while retaining the intrinsic heralding efficiency associated with spectral filtering. Accordingly, the system transmittances are set to $\eta_{H_3} = \eta_{V_3} = \eta_{H_4} = \eta_{V_4} = 57\%$, where 57 % is the intrinsic heralding efficiency due to spectral filtering (see Methods for details). When assuming an ideal linear-optical BSM with a success probability of $P_{\text{BSM}} = 1/2$, the four-fold coincidence probability is multiplied by a factor of four. Furthermore, for the idealized scenarios represented by the cyan and purple curves, perfect transmission is assumed for all modes, i.e., $\eta_{V_1} = \eta_{H_2} = \eta_{H_3} = \eta_{V_3} = \eta_{H_4} = \eta_{V_4} = 1$.

Supplementary Note 2: Experimental validation of the simulated spectral purity

To validate the numerical model used to estimate the spectral purity in Methods, we compare the simulated spectral purity with that experimentally inferred from measurements of the intensity correlation function of the idler photons without detecting the signal photons. Since the source is pulsed, we denote by $g_n^{(2)}$ the intensity correlation between n th neighbor pulses, where $g_0^{(2)}$ corresponds to the autocorrelation function. The relationship between $g_0^{(2)}$ and P can be derived as follows. We consider a statistical mixture of light in K independent Schmidt modes. Let λ_i denote the normalized Schmidt coefficient of the i th Schmidt mode, satisfying $\sum_{i=1}^K \lambda_i^2 = 1$. In the low-gain regime, the average photon number in the i th Schmidt mode is proportional

to λ_i^2 , and we define the normalized modal weight as

$$p_i = \frac{\langle N_i \rangle}{\sum_j \langle N_j \rangle} \simeq \lambda_i^2, \quad (18)$$

where N_i is the photon-number operator for the i th Schmidt mode.

The intensity correlation within a single pulse is then given by

$$\begin{aligned} g_0^{(2)} &= \frac{\left\langle : \left(\sum_{i=1}^K N_i \right)^2 : \right\rangle}{\left\langle \sum_{i=1}^K N_i \right\rangle^2} \\ &= 2 \sum_{i=1}^K p_i^2 + 2 \sum_{i>j} p_i p_j, \end{aligned} \quad (19)$$

where $\langle : \dots : \rangle$ denotes the normally ordered expectation value. The second equality follows from the statistical independence of the Schmidt modes and the fact that each reduced Schmidt mode of the SPDC state exhibits an ideal thermal photon-number distribution. Using

$$2 \sum_{i>j} p_i p_j = 1 - \sum_i p_i^2, \quad (20)$$

we obtain

$$g_0^{(2)} = 1 + \sum_i p_i^2 \simeq 1 + \sum_i \lambda_i^4 = 1 + P, \quad (21)$$

where the last equality follows from the definition of the spectral purity,

$$P = \sum_i \lambda_i^4. \quad (22)$$

This result is consistent with Ref. [4].

In the experiment of $g_0^{(2)}$ measurement, the idler photons are first passed through a DWDM filter and then split by a 50:50 FBS. The intensity correlation function $g_n^{(2)}$ is estimated from the measured coincidence counts as

$$g_n^{(2)} = \frac{C_{12}^{(n)} f_{\text{rep}}}{S_1 S_2}, \quad (23)$$

where $C_{12}^{(n)}$ is the coincidence count between a photon detected at D1 and a photon detected at D2 separated by n pulse intervals, and S_1 and S_2 are the single-count rates at D1 and D2, respectively. Assuming $g_\infty^{(2)} = 1$, we obtain $g_0^{(2)} = \frac{C_{12}^{(0)}}{C_{12}^{(\infty)}}$. Since the pulse duration is much shorter than the pulse interval, we estimated the uncorrelated

coincidence count $C_{12}^{(\infty)}$ by averaging the coincidence counts from neighboring pulses,

$$C_{12}^{(\infty)} = \frac{1}{4} \sum_{n=-2,-1,1,2} C_{12}^{(n)}. \quad (24)$$

Using this method, we obtain $g_0^{(2)} = 1.84 \pm 0.03$, corresponding to a spectral purity of $P = 84 \pm 3\%$. This value agrees well with the simulated value of 86 %, calculated from the JSA $F_i(\omega')\Phi(\omega, \omega')$, thereby validating the numerical model used to estimate the spectral purity.

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