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SOME NONEXISTENCE RESULTS FOR CERTAIN CLASSES OF WARPED PRODUCT RICCI SOLITONS

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Abstract

We establish an ambient-fiber compatibility principle for a general class of warped product Ricci solitons $M = B \times_f F$ over compact bases, bypassing the standard gradient potential ansatz. By analyzing the vertical projection of the ambient soliton equation coupled with a homothetic vector field on the Einstein fiber, we isolate a governing scalar elliptic PDE. A maximum principle argument reveals that the ambient manifold and the fiber must belong to identical geometric regimes; mixed shrinking-expanding structures are structurally obstructed. This rigidity framework reduces steady configurations to Ricci flat configurations with constant warping function. It also yields sharp nonexistence bounds dictated entirely by vertical symmetry constraints.

1. Introduction

The Ricci flow, introduced by Hamilton [1], has become one of the central tools in modern geometric analysis. Through the evolution equation $\partial g / \partial t = -2 \text{Ric}$, the flow deforms a Riemannian metric in a manner analogous to heat diffusion, revealing the underlying geometric structure of the manifold. The subject gained further prominence following the groundbreaking work of Perelman [4], whose analysis of singularities and entropy functionals led to the resolution of the Poincaré and Geometrization Conjectures.

Within this framework, Ricci solitons arise naturally as self-similar solutions to the Ricci flow and serve as canonical models for singularity formation.

A Ricci soliton is a Riemannian manifold (M, g) equipped with a vector field X satisfying the structural equation

$$\text{Ric} + (1/2)L_X g = \lambda g,$$

where L_X denotes the Lie derivative along X and $\lambda \in \mathbb{R}$ is a constant. According to the sign of λ , the soliton is classified as shrinking ($\lambda > 0$), steady ($\lambda = 0$), or expanding ($\lambda < 0$). Ricci solitons generalize Einstein metrics and occupy an important position in the study of geometric evolution equations. Their geometric and analytic properties have been extensively investigated by many authors, including Hamilton, Perelman, Cao, Petersen–Wylie, Catino, and others, leading to numerous rigidity, classification, and structural results.

Among the various geometric settings in which Ricci solitons have been studied, warped product manifolds form a particularly rich and natural class. Given a base manifold (B, g_B) , a fiber manifold (F, g_F) , and a positive smooth warping function $f \in C^\infty(B)$, the warped product

$$M = B \times_f F$$

provides a geometric structure in which the curvature of the ambient manifold is determined by a delicate interaction between the base geometry, the fiber geometry, and the warping function. Consequently, warped product Ricci solitons have attracted considerable attention and have become an important source of examples, rigidity phenomena, and classification problems in geometric analysis.

A fundamental problem in the theory of warped Ricci solitons is to understand how the Einstein geometry of the fiber constrains the global soliton structure of the ambient manifold. Most existing investigations approach this question through the full tensorial Ricci soliton equation or through equations induced on

the base manifold. Comparatively less attention has been devoted to the information encoded in the vertical component of the soliton equation. Since the fiber geometry enters the warped product structure through the vertical distribution, it is natural to ask whether the vertical component alone can impose meaningful restrictions on the ambient Ricci soliton.

Motivated by this question, we consider warped product Ricci solitons of the form $M = B \times_f F$, where (B, g_B) is compact, (F, g_F) is an Einstein manifold with Einstein constant λ_F , and the soliton vector field admits the decomposition

$$X_M = X_B + fX_F,$$

where X_B is a vector field on B and X_F is a homothetic vector field on the fiber. Rather than analyzing the full ambient soliton equation, we project the Ricci soliton equation onto the vertical distribution and derive a governing relation in the fiber direction. This vertical formulation captures the interaction between the ambient soliton constant, the Einstein constant of the fiber, and the warping function, and forms the analytical foundation of the present work.

Our principal result establishes a compatibility principle between the geometric type of the Einstein fiber and the geometric type of the ambient Ricci soliton. Under the assumption that the conformal constant of X_F vanishes, namely $c = 0$ (meaning X_F is Killing on the fiber), we derive a fundamental estimate involving the minimum and maximum values of the warping function. As a consequence, we prove that

$$\lambda_F \lambda_M > 0,$$

where λ_M denotes the Ricci soliton constant of the ambient manifold.

This theorem shows that the Einstein fiber and the ambient Ricci soliton must belong to the same geometric regime. In particular, shrinking Einstein fibers cannot occur inside expanding warped Ricci solitons, while expanding Einstein fibers cannot occur inside shrinking warped Ricci solitons. Thus, non-compatible ambient-fiber configurations are ruled out entirely. The result may therefore be interpreted as an ambient-fiber compatibility theorem for warped Ricci solitons.

The significance of this result lies in the fact that the obstruction is obtained solely from the vertical component of the Ricci soliton equation. Consequently, local Einstein data on the fiber impose global restrictions on the geometry of the ambient soliton. This reveals a compatibility phenomenon between the fiber and ambient structures and demonstrates that the vertical geometry alone is capable of determining strong existence and nonexistence properties of warped Ricci solitons.

Novelty and Contributions of the Present Work

The study of warped product Ricci solitons has attracted considerable attention in recent years due to the rich interplay between the geometry of the base manifold, the fiber manifold, and the warping function. Existing investigations typically analyze the full Ricci soliton system or derive coupled equations on the base manifold in order to obtain classification, rigidity, and existence results. In contrast, the present work adopts a different perspective by focusing exclusively on the information encoded in the vertical component of the Ricci soliton equation.

The principal novelty of this paper lies in the observation that the vertical geometry alone is sufficient to impose strong restrictions on the global structure of a warped product Ricci soliton. By projecting the Ricci soliton equation onto the fiber directions, we derive a scalar governing relation linking the Einstein constant of the fiber, the ambient soliton constant, the warping function, and the homothetic behavior of the fiber vector field. This vertical formulation serves as the fundamental analytical tool throughout the paper.

Using this approach, we establish an ambient-fiber compatibility principle showing that, under natural geometric assumptions, the Einstein fiber and the ambient Ricci soliton must belong to the same geometric regime. In particular, shrinking fibers cannot occur inside expanding ambient solitons, and expanding fibers cannot occur inside shrinking ambient solitons. As further consequences, we obtain rigidity results for steady configurations and derive several nonexistence criteria for warped product Ricci solitons satisfying additional geometric conditions.

The main contributions of the paper may be summarized as follows:

- 1. We derive a governing equation for warped product Ricci solitons by projecting the soliton equation onto the vertical distribution.**
- 2. We obtain maximum-principle estimates relating the ambient soliton constant, the Einstein constant of the fiber, and the extrema of the warping function.**
- 3. We establish an ambient-fiber compatibility theorem, proving that the Einstein fiber and the ambient Ricci soliton must possess the same geometric type.**
- 4. We prove a steady rigidity result showing that steady ambient-fiber configurations reduce to Ricci-flat configurations with constant warping function under the hypotheses considered.**
- 5. We derive several nonexistence results and quantitative restrictions for warped product Ricci solitons under additional assumptions on the warping function and the conformal structure of the fiber.**

These results demonstrate that local Einstein data on the fiber can impose global restrictions on the ambient geometry through purely vertical considerations. The methods developed herein provide a complementary perspective to existing approaches in the theory of warped product Ricci solitons and highlight the geometric significance of the vertical component of the soliton equation.

Organization of the Paper

The remainder of this paper is organized as follows. In Section 2, we recall the necessary geometric preliminaries concerning Ricci solitons and warped product manifolds. Section 3 develops the warped product framework and derives the governing scalar equation by projecting the Ricci soliton equation onto the vertical distribution. In Section 4, we establish maximum principle estimates for the warping function and prove the main ambient-fiber compatibility theorem together with several rigidity and nonexistence results. We also present illustrative examples demonstrating the applicability of the main results. Finally, Section 5 concludes the paper with a summary of the principal contributions and discusses possible directions for future research.

2. Geometric Preliminaries and The Soliton Equation

Definition 2.1 (Ricci Curvature)

Let (M, g) be a Riemannian manifold. The Ricci curvature tensor Ric_g is the trace of the Riemann curvature tensor:

$$\text{Ric}_{ij} = R^k_{ikj}.$$

Definition 2.2 (Ricci Soliton)

Let (M, g) be a smooth Riemannian manifold. A Ricci soliton is a triple (M, g, X) satisfying

$$\text{Ric}_g + (1/2)L_X g = \lambda g,$$

where X is a smooth vector field and $\lambda \in \mathbb{R}$ is a constant.

The soliton is called:

- shrinking if $\lambda > 0$,
- steady if $\lambda = 0$,
- expanding if $\lambda < 0$.

If there exists a smooth function $\phi \in C^\infty(M)$ such that $X = \nabla\phi$, then the Ricci soliton is called a gradient Ricci soliton. In this case, using the identity $L_{\{\nabla\phi\}}g = 2\nabla^2\phi$, the soliton equation is equivalent to

$$\text{Ric}_g + \nabla^2\phi = \lambda g.$$

Throughout this paper, we shall primarily use the Lie derivative formulation

$$\text{Ric}_g + (1/2)L_X g = \lambda g.$$

Definition 2.3 (Warped Product)

Let (B, g_B) and (F, g_F) be Riemannian manifolds and $f : B \rightarrow (0, \infty)$ smooth. The warped product $M = B \times_f F$ is the manifold $B \times F$ with metric

$$g = g_B + f^2 g_F.$$

The function f is called the warping function.

Definition 2.4 (Laplacian)

The Laplace–Beltrami operator acting on smooth functions u is

$$\Delta u = \text{div}(\nabla u).$$

3. The Warped Product Ansatz and Curvature Coupling

Now we introduce the major results and give their proofs.

Definition 3.1 (Structure-Preserving Soliton Vector Field)

Let $M = B \times_f F$ be a warped product manifold equipped with the metric

$$g_M = g_B + f^2 g_F.$$

Let $TM = H \oplus V$ denote the canonical splitting of the tangent bundle into the horizontal distribution $H \cong TB$ and the vertical distribution $V \cong TF$.

A vector field $X \in \Gamma(TM)$ is said to preserve the warped product structure if its local flow preserves both the horizontal and vertical distributions. Equivalently, for every point $p \in M$,

$$X(p) = X_B(p) + X_V(p),$$

where $X_B \in \Gamma(H)$, $X_V \in \Gamma(V)$, and no horizontal component is mapped into the vertical distribution (or vice versa) under the infinitesimal action of X .

Under this assumption, the vertical component can be written in the scaled form

$$X_V = f^2 X_F,$$

where X_F is a vector field on the fiber F . Since the warping function f depends only on the base manifold B , one has $X_F(f) = 0$.

Proposition 3.2 (Adapted decomposition of the soliton vector field)

Let $M = B \times_f F$ be a warped product manifold equipped with the metric $g_M = g_B + f^2 g_F$, where $f : B \rightarrow (0, \infty)$ is smooth.

Suppose that the soliton vector field X_M preserves the warped product structure. Then it is natural to decompose X_M into horizontal and vertical components

$$X_M = X_B + f^2 X_F,$$

where X_B is tangent to the base B and X_F is tangent to the fiber F . Furthermore, since the warping function depends only on the base variable, one has $X_F(f) = 0$. Consequently,

$$L_{\{f^2 X_F\}}(f^2 g_F) = f^4 L_{\{X_F\}} g_F.$$

Proof

The tangent bundle of a warped product splits orthogonally as $TM = TB \oplus TF$. Hence every vector field admits a decomposition into horizontal and vertical parts, $X_M = X_B + \tilde{X}_F$. Since the fiber metric appears in the warped metric through the factor $f^2 g_F$, it is convenient to write the vertical component in the scaled form $\tilde{X}_F = f^2 X_F$. Because f depends only on the base variable, any vector field tangent to the fiber satisfies $X_F(f) = 0$. Therefore $L_{\{f^2 X_F\}}(f^2 g_F) = f^4 L_{\{X_F\}} g_F$, which shows that the chosen decomposition is naturally adapted to the warped product geometry. $\square$

3.1. The Warping-Preserving Condition on the Horizontal Soliton Component

The warped product structure contains two distinct geometric directions: the horizontal distribution TB , corresponding to the base manifold, and the vertical distribution TF , corresponding to the fiber. Since the warping function $f : B \rightarrow (0, \infty)$ depends only on the base variable, every vector field tangent to the fiber automatically annihilates f . In particular, $X_F(f) = 0$.

The corresponding statement for the horizontal component of the soliton vector field is different. In the decomposition $X_M = X_B + f^2 X_F$, the vector field X_B is tangent to B , and therefore its action on f need not vanish. We impose the additional condition

$$X_B(f) = 0.$$

Equivalently, $df(X_B) = 0$. Thus, wherever $\nabla f \neq 0$,

$$\langle X_B, \nabla f \rangle_{g_B} = 0.$$

Geometrically, this means that the horizontal component X_B is tangent to the level sets of the warping function. Hence the flow generated by X_B preserves the warping levels and does not change the value of f along its trajectories.

The purpose of this assumption is to isolate the genuinely vertical contribution of the soliton vector field in the vertical projection of the soliton equation. Indeed, the vertical part of the warped product metric is $g_M|_{TF} = f^2 g_F$. Consequently,

$$L_{\{X_B\}}(f^2 g_F) = X_B(f^2 g_F) = 2f X_B(f) g_F.$$

Under condition $X_B(f) = 0$, this term vanishes:

$$L_{\{X_B\}}(f^2 g_F) = 0.$$

On the other hand, since $X_F(f) = 0$ automatically,

$$L_{\{f^2 X_F\}}(f^2 g_F) = f^4 L_{\{X_F\}} g_F.$$

If X_F is homothetic with constant factor c ,

$$L_{\{X_F\}} g_F = c g_F,$$

and therefore

$$L_{\{f^2 X_F\}}(f^2 g_F) = cf^2 g_F.$$

Thus, under the warping-preserving condition $X_B(f) = 0$, the vertical component of the Lie derivative is determined entirely by the fiber vector field:

$$L_{\{X_M\}}g_M|_{\{TF\}} = cf^2 g_F.$$

This assumption is therefore not a restriction on the fiber component X_F , nor is it a consequence of the fact that f is defined on the base. Rather, it is a structural compatibility condition between the horizontal part of the soliton vector field and the warping function. It allows the vertical projection of the Ricci soliton equation to decouple from the variation of the warping function along the horizontal soliton flow.

Under this condition, the vertical soliton equation takes the scalar form derived in the following lemma. The subsequent maximum-principle arguments will be based entirely on this reduced equation.

The condition $X_B(f) = 0$ may consequently be interpreted as a warping-preserving symmetry assumption: the soliton flow is allowed to act nontrivially on the base, but its horizontal component preserves the warping function.

Lemma 3.3

Let (M, g_M) be a warped product manifold $M = B \times_f F$ with metric $g_M = g_B + f^2 g_F$, where the fiber (F, g_F) is an Einstein manifold satisfying $\text{Ric}_F = \lambda_F g_F$. Suppose (M, g_M) is a Ricci soliton with vector field X_M satisfying

$$\text{Ric}_M + (1/2)L_{\{X_M\}}g_M = \lambda_M g_M.$$

If the soliton vector field splits as $X_M = X_B + f^2 X_F$, where X_F is a homothetic vector field on F with constant homothetic factor c , i.e.

$$L_{\{X_F\}}g_F = cg_F,$$

where $c \in \mathbb{R}$ is constant, then the warping function f satisfies

$$\lambda_F/f^2 - \Delta f/f - (m-1)|\nabla f|^2/f^2 + (1/2)cf^2 = \lambda_M,$$

where $m = \dim F$, and Δf and $|\nabla f|^2$ are evaluated with respect to the base metric g_B .

Proof

Assume that $M = B \times_f F$ is a warped product Ricci soliton satisfying the soliton equation, $\lambda_M > 0$. Multiplying by 2, we obtain

$$2\text{Ric}_M + L_{\{X_M\}}g_M = 2\lambda_M g_M.$$

Since F is Einstein, $\text{Ric}_F = \lambda_F g_F$. By the classical O'Neill formula for the Ricci curvature of warped products, the vertical component is

$$\begin{aligned} \text{Ric}_M|_{\{TF\}} &= \\ (\lambda_F/f^2 - \Delta f/f - (m-1)|\nabla f|^2/f^2) g_M|_{\{TF\}}. \end{aligned}$$

Since $g_M|_{\{TF\}} = f^2 g_F$, this becomes

$$\begin{aligned} \text{Ric}_M|_{\{TF\}} &= \\ (\lambda_F/f^2 - \Delta f/f - (m-1)|\nabla f|^2/f^2) g_M|_{\{TF\}}. \end{aligned}$$

Assume that the soliton vector field splits as $X_M = X_B + f^2 X_F$, where $L_{\{X_F\}}g_F = cg_F$. Since f depends only on the base and X_F is tangent to the fiber, $X_F(f) = 0$. Therefore the fiber contribution of the Lie derivative is

$$L_{\{X_M\}}g_M|_F = cf^2 g_F.$$

Substituting the fiber components into the soliton equation and dividing by $f^2 g_F$ gives

$$\lambda_F/f^2 - \Delta f/f - (m-1)|\nabla f|^2/f^2 + (1/2)cf^2 = \lambda_M,$$

which completes the proof. $\square$

4. Rigidity under Maximum Principle

Throughout this section, let $M = B \times_f F$ be a warped product Ricci soliton satisfying the assumptions of Lemma 3.3:

- B is compact without boundary.
- F is Einstein with $\text{Ric}_F = \lambda_F g_F$.
- $L_{\{X_F\}}g_F = cg_F$.
- M is a warped product soliton with constant λ_M .
- $m = \dim(F) \geq 2$.

Then the warping function satisfies

$$\lambda_F/f^2 - \Delta f/f - (m-1)|\nabla f|^2/f^2 + (1/2)cf^2 = \lambda_M.$$

Multiplying by f^2 yields

$$\lambda_F - f\Delta f - (m-1)|\nabla f|^2 + (1/2)cf^2 = \lambda_M f^2.$$

Why the assumption $\dim(F) \geq 2$?

The restriction $\dim(F) = m \geq 2$ is imposed to ensure that the fiber dimension contributes nontrivially to the vertical soliton equation. In particular, the term $(m-1)|\nabla f|^2/f^2$ appears with a definite sign and plays a crucial role in the maximum principle arguments used later. When $m = 1$, this term vanishes identically and the governing equation reduces to

$$\lambda_F/f^2 - \Delta f/f + (1/2)cf^2 = \lambda_M.$$

As a consequence, the maximum principle estimates become substantially simpler and the compatibility inequalities obtained in Theorem 4.4 follow in a comparatively trivial manner. Since the geometric interaction between the warping function and the fiber dimension disappears in the one-dimensional case, we restrict our attention to Einstein fibers of dimension at least two; but we also demonstrate that, when the $m = 1$ case is combined with the harmonicity of f and some other constraints, the warped product cannot exist. Therefore we provide a separate subsection (subsection 4.3, Theorem 4.12) for the analysis of the one-dimensional fiber case.

4.1. Rigidity Under Killing Fiber Flow

Theorem 4.1 (Maximum Principle Bounds for the Warping Function)

Assume that $M = B \times_f F$ is a warped product Ricci soliton satisfying the assumptions of Lemma 3.3. Assume additionally that $L_{\{X_F\}}g_F = 0$ ($c = 0$), and that B is compact without boundary. Then the warping function f satisfies

$$f_{\min}^2 \leq \lambda_F/\lambda_M \leq f_{\max}^2,$$

where $f_{\min} := \min_B f$ and $f_{\max} := \max_B f$.

Proof

Under the assumption $c = 0$, Lemma 3.3 reduces to

$$\lambda_F/f^2 - \Delta f/f - (m-1)|\nabla f|^2/f^2 = \lambda_M.$$

Multiplying by f^2 gives

$$\lambda_F - f\Delta f - (m-1)|\nabla f|^2 = \lambda_M f^2.$$

Rearranging,

$$f\Delta f = \lambda_F - \lambda_M f^2 - (m-1)|\nabla f|^2.$$

Since B is compact and f is smooth, f attains a global maximum at some point $p \in B$ and a global minimum at some point $q \in B$.

At the maximum point p , $\nabla f(p) = 0$ and $\Delta f(p) \leq 0$. Evaluating at p gives

$$f(p)\Delta f(p) = \lambda_F - \lambda_M f(p)^2.$$

Since $f(p) > 0$ and $\Delta f(p) \leq 0$,

$$\lambda_F - \lambda_M f(p)^2 \leq 0.$$

Hence $f_{\max}^2 \geq \lambda_F/\lambda_M$.

Similarly, at the minimum point q , $\nabla f(q) = 0$ and $\Delta f(q) \geq 0$. Evaluating at q gives

$$f(q)\Delta f(q) = \lambda_F - \lambda_M f(q)^2.$$

Since $f(q) > 0$ and $\Delta f(q) \geq 0$,

$$\lambda_F - \lambda_M f(q)^2 \geq 0.$$

Therefore $f_{\min}^2 \leq \lambda_F/\lambda_M$.

Combining these yields

$$f_{\min}^2 \leq \lambda_F/\lambda_M \leq f_{\max}^2.$$

This completes the proof. $\square$

Corollary 4.2 (Characterization of Constant Warping Functions)

Under the assumptions of Theorem 4.1, if the warping function f is constant, then

$$f^2 = \lambda_F/\lambda_M.$$

Conversely, any constant warping function must satisfy this identity.

Proof

If $f \equiv f_0 > 0$ is constant, then $\nabla f = 0$ and $\Delta f = 0$. Substituting into the governing equation gives

$$\lambda_F/f_0^2 = \lambda_M.$$

Therefore $f_0^2 = \lambda_F/\lambda_M$. Conversely, if $f \equiv f_0$ is constant, then the soliton equation reduces precisely to the above algebraic identity. Hence every constant warped product solution necessarily satisfies $f^2 = \lambda_F/\lambda_M$. $\square$

Remark 4.3 (On the Sharpness of Rigidity Conditions)

The rigidity result established under the assumption $c = 0$ is sharp within the framework of the maximum principle on compact manifolds. Indeed, the proof relies crucially on the positivity of the curvature terms and the sign structure of the governing equation at extremal points, and these conditions cannot be relaxed without altering the argument fundamentally.

However, these assumptions should be understood as sufficient rather than universally necessary. If $\lambda_F < 0$, the curvature contribution of the fiber changes sign, and nontrivial warped configurations may arise. Similarly, when $c > 0$, the homothetic forcing term introduces a competing effect that can balance the shrinking condition and permit non-constant solutions.

Moreover, in noncompact settings, where the maximum principle may fail or require additional control at infinity, the rigidity phenomenon is expected to weaken. These observations suggest that the rigidity conditions identified here delineate a sharp boundary within the compact, unforced regime, while more general geometric configurations may admit richer solution structures.

Now We are ready to give the proof of the Main Result of this Paper.

Theorem 4.4 (Ambient–Fiber Compatibility)

Let (M, g_M, X_M, λ_M) be a warped product Ricci soliton of the form $M = B \times_f F$, where (F, g_F) is an Einstein manifold satisfying $\text{Ric}_F = \lambda_F g_F$. Assume that the warping function f attains both its maximum and minimum values on B and that the fiber vector field satisfies $L_{X_F} g_F = c g_F$, $c = 0$.

Then

$$f_{\min}^2 \leq \lambda_F / \lambda_M \leq f_{\max}^2.$$

Consequently,

$$\lambda_F \lambda_M > 0.$$

In particular, the Einstein fiber and the ambient Ricci soliton must have the same geometric type. More precisely:

- 1. If $\lambda_M > 0$, then $\lambda_F > 0$.**
- 2. If $\lambda_M < 0$, then $\lambda_F < 0$.**
- 3. There does not exist a warped product Ricci soliton satisfying the above hypotheses for which λ_M and λ_F have opposite signs.**

Proof

By Theorem 4.1,

$$f_{\min}^2 \leq \lambda_F / \lambda_M \leq f_{\max}^2.$$

Since the warping function is strictly positive on B , $f_{\min} > 0$, and hence $\lambda_F / \lambda_M \geq f_{\min}^2 > 0$. Therefore $\lambda_F / \lambda_M > 0$, which implies $\lambda_F \lambda_M > 0$. The remaining conclusions follow immediately. $\square$

Corollary 4.5

There exists no warped product Ricci soliton satisfying the hypotheses of Theorem 4.4 for which the ambient soliton is shrinking and the Einstein fiber is expanding.

Corollary 4.6

There exists no warped product Ricci soliton satisfying the hypotheses of Theorem 4.4 for which the ambient soliton is expanding and the Einstein fiber is shrinking.

4.2. The Steady Case

The steady case requires separate consideration, since the maximum-principle estimate obtained in the non-steady regime involves the quotient λ_F/λ_M and therefore does not apply when $\lambda_M = 0$. Importantly, the governing vertical equation itself remains well-defined in this case.

Assume that $\lambda_M = 0$ and $c = 0$. Then the scalar equation reduces to

$$\lambda_F - f\Delta f - (m-1)|\nabla f|^2 = 0.$$

Since B is compact, the positive smooth function f attains both its maximum and minimum. Let $p_{\max}, p_{\min} \in B$ be points such that $f(p_{\max}) = f_{\max}$ and $f(p_{\min}) = f_{\min}$.

At $p_{\max}$, $\nabla f(p_{\max}) = 0$ and $\Delta f(p_{\max}) \leq 0$. Evaluating gives

$$\lambda_F = f(p_{\max})\Delta f(p_{\max}) \leq 0.$$

Similarly, at $p_{\min}$, $\nabla f(p_{\min}) = 0$ and $\Delta f(p_{\min}) \geq 0$, and hence

$$\lambda_F = f(p_{\min})\Delta f(p_{\min}) \geq 0.$$

Consequently, $\lambda_F = 0$.

Substituting $\lambda_F = 0$ gives

$$f\Delta f + (m-1)|\nabla f|^2 = 0.$$

For the positive function f ,

$$\Delta(f^m) = mf^{m-1}\Delta f + m(m-1)f^{m-2}|\nabla f|^2.$$

Therefore $\Delta(f^m) = 0$. Since B is compact without boundary, every harmonic function on B is constant. Hence $f \equiv \text{constant}$.

Theorem 4.7 (Steady Rigidity)

Let $(M, g_M) = (B \times_f F, g_B + f^2 g_F)$ be a warped product Ricci soliton satisfying the hypothesis of Lemma 3.3. Assume that B is compact without boundary, that the fiber component is Killing ($c = 0$), and that the ambient soliton is steady ($\lambda_M = 0$). Then $\lambda_F = 0$ and the warping function is constant:

$$f \equiv f_0 > 0.$$

In particular, the steady configuration reduces to a constant-warping product with Ricci-flat fiber.

Thus, the steady case is not excluded by the vertical soliton equation. Rather, under the compactness and warping-preserving hypotheses, it is rigid: the Einstein fiber must be Ricci-flat and the warping function must be constant.

4.3. Examples and Geometric Applications

The following examples illustrate the applicability of Theorem 4.4 and demonstrate how the ambient-fiber compatibility principle manifests itself in several classical geometric settings.

Example 4.8 (Compact Kähler–Einstein Product)

Let $B = \mathbb{CP}^m$ be the complex projective space equipped with the Fubini–Study metric, normalized so that $\text{Ric}_B = \lambda g_B$, $\lambda > 0$. Let $F = \mathbb{CP}^n$ be another complex projective space with the same Einstein constant, $\text{Ric}_F = \lambda g_F$. Choose the constant warping function $f \equiv 1$ and let $X_M = 0$.

Then $M = B \times F$ is an Einstein manifold satisfying $\text{Ric}_M = \lambda g_M$, and hence is a shrinking Ricci soliton. Since $\lambda_F = \lambda_M = \lambda > 0$, Theorem 4.4 applies and yields $\lambda_F \lambda_M > 0$. Thus the ambient Ricci soliton

and its Einstein fiber belong to the same shrinking regime, illustrating the ambient-fiber compatibility principle.

Example 4.9 (Compact Hyperbolic Einstein Product)

Let $B = \Gamma_1 \backslash \mathbb{H}^m$ and $F = \Gamma_2 \backslash \mathbb{H}^n$, where Γ_1 and Γ_2 are cocompact discrete groups of isometries of hyperbolic space. Equip both manifolds with their standard hyperbolic metrics normalized so that $\text{Ric}_B = \text{Ric}_F = \lambda g$, $\lambda < 0$. Take $f \equiv 1$ and $X_M = 0$.

Then $M = B \times F$ is Einstein with $\text{Ric}_M = \lambda g_M$, and therefore is an expanding Ricci soliton. Since $\lambda_F = \lambda_M = \lambda < 0$, Theorem 4.4 gives $\lambda_F \lambda_M > 0$, confirming that expanding Einstein fibers occur only inside expanding ambient Ricci solitons under the hypotheses of the theorem.

4.4. Nonexistence Under Positive Curvature and Expanding Fiber

The following theorem demonstrates more specifically why working in the case $m = 1$ can result in unwanted ambiguities, specifically nonexistence. We analyse this behavior by using a contradiction argument after imposing some other necessary conditions on the warped product structure.

Theorem 4.10 (Nonexistence for One-Dimensional Fibers)

Let $M = B \times_f F$ be a warped product shrinking Ricci soliton satisfying the assumptions of Lemma 3.3. Suppose that:

1. $\dim(F) = 1$,
2. f is harmonic,
3. $L_{\{X_F\}} g_F = c g_F$ with $c < 0$.

Then no such shrinking Ricci soliton exists.

Proof

Since $\dim(F) = 1$, every one-dimensional Riemannian manifold is Ricci-flat, and hence $\lambda_F = 0$. Since f is harmonic and $m = 1$, Lemma 3.3 reduces to

$$\lambda_F / f^2 + (1/2) c f^2 = \lambda_M.$$

Rearranging,

$$\lambda_F / f^2 = \lambda_M - (1/2) c f^2.$$

Because $\lambda_M > 0$, $c < 0$, and $f > 0$, we have $\lambda_M - (1/2) c f^2 > 0$. Therefore $\lambda_F / f^2 > 0$, which implies $\lambda_F > 0$. This contradicts the fact that $\lambda_F = 0$ for a one-dimensional fiber. Hence no such shrinking Ricci soliton exists. $\square$

Proposition 4.11 (Quadratic Bound for Harmonic Warping Functions)

Let $M = B \times_f F$ be a warped product Ricci soliton satisfying the assumptions of Lemma 3.3. Assume that $\Delta f = 0$, $\lambda_F > 0$, $c > 0$, $\lambda_M > 0$. Then the maximum value of the warping function satisfies

$$(1/2) c f_{\max}^4 - \lambda_M f_{\max}^2 + \lambda_F \geq 0,$$

where $f_{\max} := \max_B f$.

Proof

Since f is harmonic, Lemma 3.3 reduces to

$$\lambda_F / f^2 - (m - 1) |\nabla f|^2 / f^2 + (1/2) c f^2 = \lambda_M.$$

Since $-(m-1)|\nabla f|^2/f^2 \leq 0$, it follows that

$$\lambda_F/f^2 + (1/2)cf^2 \geq \lambda_M.$$

Let $p \in B$ be a point at which f attains its maximum value $f_{\max}$. Evaluating gives

$$\lambda_F/f_{\max}^2 + (1/2)cf_{\max}^2 \geq \lambda_M.$$

Multiplying by $f_{\max}^2 > 0$ gives

$$\lambda_F + (1/2)cf_{\max}^4 \geq \lambda_M f_{\max}^2,$$

which is the desired inequality. $\square$

Theorem 4.12 (Discriminant Criterion for Harmonic Warping Functions)

Let $M = B \times_f F$ be a warped product Ricci soliton satisfying the assumptions of Lemma 3.3. Assume that $\Delta f = 0$, $\lambda_F > 0$, $c > 0$, $\lambda_M > 0$. Define

$$\Delta := \lambda_M^2 - 2c\lambda_F.$$

Then:

1. If $\Delta < 0$, then

$$(1/2)cf_{\max}^4 - \lambda_M f_{\max}^2 + \lambda_F > 0$$

for every $f_{\max} > 0$.

2. If $\Delta \geq 0$, then

$$f_{\max}^2 \leq [\lambda_M - \sqrt{(\lambda_M^2 - 2c\lambda_F)}]/c$$

or

$$f_{\max}^2 \geq [\lambda_M + \sqrt{(\lambda_M^2 - 2c\lambda_F)}]/c.$$

Proof

By Proposition 4.13, the inequality

$$(1/2)cf_{\max}^4 - \lambda_M f_{\max}^2 + \lambda_F \geq 0$$

holds. Let $y = f_{\max}^2$. Then

$$(1/2)cy^2 - \lambda_M y + \lambda_F \geq 0.$$

Consider the quadratic polynomial

$$P(y) = (1/2)cy^2 - \lambda_M y + \lambda_F.$$

Since $c > 0$, the parabola opens upward. Its discriminant is

$$\Delta = \lambda_M^2 - 2c\lambda_F.$$

If $\Delta < 0$, then $P(y) > 0$ for all $y \in \mathbb{R}$, proving (1).

If $\Delta \geq 0$, the roots are

$$y_{\pm} = [\lambda_M \pm \sqrt{(\lambda_M^2 - 2c\lambda_F)}]/c.$$

Since the parabola opens upward, $P(y) \geq 0$ holds precisely when $y \leq y_-$ or $y \geq y_+$. Substituting $y = f_{\max}^2$ yields the desired inequalities. $\square$

Remark 4.13

It should be noted that the inequality

$$(1/2)cf^4 - \lambda_- Mf^2 + \lambda_- F \geq 0$$

is in fact a pointwise consequence of Lemma 3.3 under the assumptions $\Delta f = 0$, $c > 0$, $\lambda_- F > 0$. Indeed, from

$$\lambda_- F/f^2 - (m-1)|\nabla f|^2/f^2 + (1/2)cf^2 = \lambda_- M$$

and the nonnegativity of $(m-1)|\nabla f|^2/f^2$, one obtains

$$\lambda_- F/f^2 + (1/2)cf^2 \geq \lambda_- M$$

at every point of the base manifold B . Multiplying by $f^2 > 0$ yields

$$(1/2)cf^4 - \lambda_- Mf^2 + \lambda_- F \geq 0$$

pointwise on B . Consequently, the quadratic restriction is independent of the maximum principle and holds throughout the manifold.

5. Conclusion

In this paper, we investigated warped product Ricci solitons by analyzing the vertical component of the Ricci soliton equation. Assuming that the fiber is Einstein and that the soliton vector field admits a structure-preserving decomposition, we derived a scalar equation governing the warping function. Using the maximum principle in the case of a Killing fiber vector field, we established quantitative bounds relating the extrema of the warping function to the Einstein constant of the fiber and the ambient soliton constant. As a consequence, we proved an ambient-fiber compatibility principle showing that the Einstein fiber and the ambient Ricci soliton must belong to the same geometric regime. In particular, shrinking fibers cannot occur inside expanding ambient solitons, expanding fibers cannot occur inside shrinking ambient solitons, and steady configurations are excluded under the hypotheses considered. Several additional rigidity and nonexistence results were also derived from the governing scalar equation.

The methods developed here rely solely on the vertical projection of the Ricci soliton equation, demonstrating that the geometry of the fiber alone can impose strong global restrictions on the ambient warped product structure. It would be of interest to investigate whether analogous compatibility phenomena persist for noncompact base manifolds, more general homothetic fiber vector fields, or multiply warped product Ricci solitons.

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References

- [1] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom. 17 (1982), 255–306.
- [2] A. L. Besse, Einstein Manifolds, Springer, Berlin, 1987.

- [3] B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, Math. Surveys Monogr., Vol. 110, AMS, 2004.
- [4] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159 (2002).
- [5] H.-D. Cao, Recent progress on Ricci solitons, *Adv. Lect. Math.* 11 (2009), 1–38.
- [6] P. Petersen and W. Wylie, On gradient Ricci solitons with symmetry, *Proc. Amer. Math. Soc.* 137 (2009), 2085–2092.
- [7] H.-D. Cao and D. Zhou, On complete gradient shrinking Ricci solitons, *J. Differential Geom.* 85 (2010), 175–186.
- [8] C. He, P. Petersen and W. Wylie, Warped product Einstein metrics on homogeneous spaces, *J. Reine Angew. Math.* 707 (2015), 217–245.
- [9] M. L. de Sousa and R. Pina, Gradient Ricci solitons with warped product structure, *Differential Geom. Appl.* 45 (2016), 1–12.
- [10] K. Tenenblat and V. Borges, Ricci almost solitons on semi-Riemannian warped products, *Results Math.* 72 (2017), 1265–1282.
- [11] S. D. Lee, B. H. Kim and J. H. Choi, Classification of warped product spaces with gradient Ricci solitons, *Korean J. Math.* 24 (2016), 487–499.
- [12] M. Rimoldi and G. Veronelli, Topology of gradient Ricci solitons via f -harmonic maps, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 11 (2012), 431–472.
- [13] O. Munteanu and N. Sesum, On gradient Ricci solitons, *J. Geom. Anal.* 23 (2013), 539–561.
- [14] S. Pigola, M. Rigoli and A. G. Setti, Maximum principles on Riemannian manifolds, *Mem. Amer. Math. Soc.* 174 (2005).
- [15] D. Açıkgöz Kaya and L. Onat, Ricci solitons on multiply warped product manifolds, *Int. Electron. J. Geom.* 13 (2020), 159–170.
- [16] M. Aquib, V. Sah, S. K. Yadav and J. Upreti, Ricci solitons on doubly warped products with QSMC, *Axioms* 14 (2025), 548.
- [17] S. Shenawy, N. Bin Turki, N. Syied and C. A. Mantica, Almost Ricci–Bourguignon solitons on doubly warped manifolds, *Universe* 9 (2023), 396.
- [18] J. P. Singh and R. Sumalatha, Conformal Ricci–Yamabe solitons on warped product manifolds, *Filomat* 38 (2024), 2005–2015.
- [19] M. Guediri, Rigidity of non-steady gradient Ricci solitons, *Axioms* 14 (2025), 842.
- [20] L. C. Evans, *Partial Differential Equations*, 2nd ed., AMS, 2010.
- [21] E. Hebey, *Sobolev Spaces on Riemannian Manifolds*, Springer, 1996.
- [22] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer, 1998.
- [23] N. S. Trudinger, On embeddings into Orlicz spaces and some applications, *J. Math. Mech.* 17 (1967), 473–483.
- [24] B. O’Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Pure and Applied Mathematics, Vol. 103, Academic Press, New York, 1983.

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