Supplementary Information: Emergent chirality-dependent electromotive force in Weyl semimetals enabled by generalized unitary transformation

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1 S1: Detailed derivation of the generalized unitary transformation

The original Hamiltonian describing low-energy electrons around the Weyl nodes is:

$$H = sv_f(\vec{p}\cdot\vec{\sigma}) - J\vec{M}(\vec{r},t)\cdot\vec{\sigma} \tag{1}$$

1.1 S1.1: Spin-rotation transformation U_θ

Using the unitary transformation operator in the form

$$U_\theta = \exp(\frac{-i\theta(x)\sigma_x}{2}), \tag{2}$$

gives the transformed wave-function as

$$\psi' = U_\theta\psi, \tag{3}$$

then the original Schrödinger equation

$$H\psi = i\hbar\frac{\partial}{\partial t}\psi, \tag{4}$$

can be converted to

$$\begin{aligned}
&U_\theta H U_\theta^{-1}\psi' = i\hbar U_\theta \frac{\partial}{\partial t}(U_\theta^{-1}\psi') \\
&= i\hbar U_\theta (\frac{\partial U_\theta^{-1}}{\partial t})\psi' + i\hbar U_\theta U_\theta^{-1}\frac{\partial \psi'}{\partial t} \\
&= i\hbar U_\theta U_\theta^{-1}\frac{i\sigma_x}{2}\frac{\partial\theta}{\partial t}\psi' + i\hbar\frac{\partial\psi'}{\partial t},
\end{aligned} \tag{5}$$

we can write

$$\begin{aligned}
&U_\theta H U_\theta^{-1}\psi' = \\
&U_\theta\Big[sv_f(p_x\sigma_x + p_y\sigma_y + p_z\sigma_z) - JM\hat{n}\cdot\sigma\Big]U_\theta^{-1}\psi' \\
&= -\hbar\frac{\sigma_x}{2}\frac{\partial\theta}{\partial t} + i\hbar\frac{\partial\psi'}{\partial t}
\end{aligned} \tag{6}$$

$$\begin{aligned}
\Big[sv_f(e^{\frac{-i\sigma_x\theta}{2}}p_x\sigma_x e^{\frac{i\sigma_x\theta}{2}} + \quad e^{\frac{-i\sigma_x\theta}{2}}p_y\sigma_y e^{\frac{i\sigma_x\theta}{2}} + (e^{\frac{-i\sigma_x\theta}{2}}p_z\sigma_z e^{\frac{i\sigma_x\theta}{2}}) - JM\sigma_z\Big]\psi' \\
= \frac{-\hbar}{2}\sigma_x\frac{\partial\theta}{\partial t}\psi' + i\hbar\frac{\partial\psi'}{\partial t}
\end{aligned} \tag{7}$$

Transformation operator U_θ is supposed in a way that rotates spins at point x inside the wall that converts the non-collinear magnetic configuration to the collinear magnetic structure along the z axis:

$$U_\theta\sigma\cdot n(x)U_\theta^{-1} = \sigma_z \tag{8}$$

Consequently, the other terms of Hamiltonian under the transformation become:

$$\begin{aligned}
U_\theta p_x\sigma_x U_\theta^{-1} &= e^{\frac{-i\sigma_x\theta}{2}}p_x\sigma_x e^{\frac{i\sigma_x\theta}{2}} \\
&= e^{\frac{-i\sigma_x\theta}{2}}p_x e^{\frac{i\sigma_x\theta}{2}}\sigma_x \\
&= (\frac{\hbar}{2}\cdot\frac{\partial\theta}{\partial x}\sigma_x + p_x)\sigma_x
\end{aligned} \tag{9}$$

$$\begin{aligned}
U_\theta p_y\sigma_y U_\theta^{-1} &= p_y e^{\frac{-i\sigma_x\theta}{2}}\sigma_y e^{\frac{i\sigma_x\theta}{2}} \\
&= p_y(\sin\theta\sigma_z + \cos\theta\sigma_y)
\end{aligned} \tag{10}$$

$$
\begin{aligned}
U_\theta p_z \sigma_z U_\theta^{-1} &= p_z e^{\frac{-i\sigma_x\theta}{2}} \sigma_z e^{\frac{i\sigma_x\theta}{2}} \\
&= p_z(\cos\theta\sigma_z - \sin\theta\sigma_y)
\end{aligned} \tag{11}
$$

Substituting Eqs. (9)–(11) into Eq. (7), the transformed Schrödinger equation becomes:

$$
\begin{aligned}
&\Big[sv_f\big((\frac{\hbar}{2}\cdot\frac{\partial\theta}{\partial x}\sigma_x + p_x)\sigma_x + p_y(\sin\theta\sigma_z + \cos\theta\sigma_y) + p_z(\cos\theta\sigma_z - \sin\theta\sigma_y)\big) - JM\sigma_z\Big]\psi' \\
&= -\hbar\frac{\sigma_x}{2}\frac{\partial\theta}{\partial t} + i\hbar\frac{\partial\psi'}{\partial t}
\end{aligned} \tag{12}
$$

1.2 S1.2: Dimensionality-reduction transformation U_λ

In contrast to the exchange interaction, which has been transformed into a collinear effective Hamiltonian, the nature of spatial dependence of θ means that the U_θ-transformation cannot change the spin-momentum coupling term into a collinear diagonalizable operator. Nonetheless, we can address the transverse direction in momentum space by applying a new transformation, U_λ, which removes the p_y and p_z dependent component of the Hamiltonian. This transformation is defined as:

$$
U_\lambda = \exp(i\lambda(p_y y + y p_y + p_z z + z p_z)) \tag{13}
$$

$$
\psi'' = \exp(i\lambda(p_y y + y p_y + p_z z + z p_z))\psi' \tag{14}
$$

As a result

$$
\psi' = \exp(-i\lambda(p_y y + y p_y + p_z z + z p_z))\psi'' \tag{15}
$$

It can be shown that:

$$
\begin{aligned}
U_\lambda p_i U_\lambda^{-1} &= e^{i\lambda(p_i r_i + r_i p_i)} p_i e^{-i\lambda(p_i r_i + r_i p_i)} \\
&= p_i + i\lambda(2i\hbar p_i) + \frac{(i\lambda)^2}{2!}(2i\hbar)^2 p_i + \frac{(i\lambda)^3}{3!}(2i\hbar)^3 p_i + ... \\
&= \left(1 - 2\hbar\lambda + \frac{(2\hbar\lambda)^2}{2!} - \frac{(2\hbar\lambda)^3}{3!} + ...\right) p_i \\
&= e^{(-2\hbar\lambda)} p_i,
\end{aligned} \tag{16}
$$

where $i = y, z$ and $r_i = y, z$.

Owing to this fact, one can realize at the large positive λ that $U_\lambda p_y U_\lambda^{-1} \simeq 0$ and $U_\lambda p_z U_\lambda^{-1} \simeq 0$. Accordingly, it can be realized that:

$$
\big(sv_f(\frac{\hbar}{2}\cdot\frac{\partial\theta}{\partial x}\sigma_x + p_x)\sigma_x + JM\sigma_z\big)\psi'' = i\hbar\frac{\partial\psi''}{\partial t} \tag{17}
$$

Accordingly, the effective Hamiltonian can be given by:

$$H' = sv_f p_x \sigma_x + (\frac{\hbar}{2} \cdot \frac{\partial \theta}{\partial x} \sigma_x)\sigma_x - JM\sigma_z \tag{18}$$

2 S2: Berry phase calculation

Applying the reversed transformations defined in Section S1, eigenfunctions of the original Hamiltonian are given as:

$$\psi_{k_x^\pm} = e^{ik_x x} e^{\frac{i\sigma_x \theta}{2}} e^{-i\lambda(p_y y + y p_y + p_z z + z p_z)} \psi''_{k_x^\pm} \tag{19}$$

Berry phase is defined as $\gamma^\pm = i \oint < \psi_\pm | \frac{\partial}{\partial r} | \psi_\pm > dr$, where $i < \psi_\pm | \frac{\partial}{\partial r} | \psi_\pm >$ is the Berry connection $A^\pm(\overrightarrow{r})$ and $\overrightarrow{r}$ is the atomic position. For each of the eigenstates, considering that integration is computable on a closed route including the magnetic wall along the system and a semicircular contour outside the system, and since only the routes with spatial variations of magnetization can contribute in the integration, the semicircular path does not contribute. Then we can write:

$$\gamma^\pm = i \int_c (\alpha^*_\pm, \beta^*_\pm)(ik_x + \frac{i\sigma_x}{2} \cdot \frac{\partial \theta}{\partial r}) \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} dr + i \int_0^L (\alpha^*_\pm, \beta^*_\pm)(ik_x + \frac{i\sigma_x}{2} \cdot \frac{\partial \theta}{\partial x}) \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} dx \tag{20}$$

Contribution of the first term in the integral is equal to zero and finally the Berry phase reads:

$$\gamma^\pm = -k_x L - (\alpha_\pm, \beta_\pm)[\theta(L,t) - \theta(0,t)] \tag{21}$$

In which L is the dimension of system along the x direction.

3 S3: Detailed SMF integration over occupied states

The spin-dependent motive force is:

$$\varepsilon^\pm = -\frac{h}{e} \cdot \frac{d\gamma^\pm}{dt} \tag{22}$$

Supposing that the magnetic wall of substrate moves with velocity v_D and dynamics of the wall is considered by $x_0(t) = v_D t$, the SMF for a single state reads:

$$\varepsilon^+_{k_x,s} = \frac{\hbar^2 s v_f k_x}{e} \cdot \frac{JM + E_{k_x}}{s^2 v_f^2 \hbar^2 k_x^2 + (JM + E_{k_x})^2} \cdot \frac{v_D}{d} \cdot \left(sech(\frac{L - v_D t}{d}) - sech(\frac{-v_D t}{d}) \right) \tag{23}$$

In large scale of system, $L \to \infty$, expression $sech(\frac{L - v_D t}{d}) \sim 0$, then the result can be written as:

$$\varepsilon^+_{k_x,s} = \frac{\hbar^2 s v_f k_x}{e} \cdot \frac{JM + E_{k_x}}{s^2 v_f^2 \hbar^2 k_x^2 + (JM + E_{k_x})^2} \cdot \frac{v_D}{d} \cdot [-sech(\frac{-v_D t}{d})] \tag{24}$$

The total spin-motive force is obtained by taking into account the occupied states of the band energy:

$$\begin{aligned}\varepsilon^{+}_{tot,s} &= \sum_{k_x} \varepsilon^{+}_{k_x} \\ &= \int \frac{L}{2\pi} dk_x \frac{\hbar^2 s v_f k_x}{e} \cdot \frac{JM + E^{+}}{(s v_f \hbar k_x)^2 + (JM + E^{+})^2} \frac{v_D}{d} \cdot sech(\frac{v_D t}{d})\end{aligned} \tag{25}$$

Considering $s^2 = 1$ and using the fact that $E^2_{k_x} = \hbar^2 k_x^2 s^2 v_f^2 + J^2 M^2$ it can be understood that:

$$\begin{aligned}&\varepsilon^{+}_{tot,s} = \int_0^{E_f} \frac{L dk_x}{2\pi} \cdot \frac{\hbar^2 s v_f k_x}{e} \frac{v_D}{d} \frac{JM + E^{+}}{E_{k^2_{x,s}} - J^2 M^2 + (JM + E_{k_{x,s}})^2} \cdot (-sech(\frac{-v_D t}{d})) \\ &= \frac{Ls}{2\pi e d} \cdot \frac{v_D}{v_f} \cdot sech(\frac{-v_D t}{d}) \int_0^{E_f} \frac{dE_{k_x}}{2}\end{aligned} \tag{26}$$

At the limit of steady state, time scale $t \sim \frac{L}{v_D}$, the final expression of spin-motive force of system is:

$$\varepsilon^{+}_{tot,s} = \frac{s E_f}{2\pi e d} \frac{v_D}{v_f} e^{-\frac{L}{d}} \tag{27}$$