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Esam H. Abdul-Hafidh

eabduhhafidh@taibahu.edu.sa

Taibah University

Mohamed H Abdelhamed

King Fahd University of Petroleum and Minerals

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Derivation of the Exact Source Functions of the Grad–Shafranov equation

Esam H. Abdul-Hafidh¹, Mohamed H Abdelhamed

¹Physics Department, Faculty of Science, Taibah University, Medina, Saudi Arabia

²EE & Physics Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

*Corresponding author: eabdulhafidh@taibahu.edu.sa

ORCID: 0000-0002-1145-0766

Abstract

Exact expressions for the source functions of the Grad–Shafranov equation (GSE), namely, the poloidal current f , plasma pressure p , and toroidal current density J_ϕ are derived through systematic mathematical manipulations. After eliminating the derivatives with respect to the poloidal flux $\left(\frac{\partial}{\partial\psi}\right)$, the GSE is transformed into a fourth-order partial differential equation that governs plasma equilibrium in tokamaks and other similar axisymmetric devices. By solving this fourth-order partial differential equation, we obtained four independent solutions. A systematic procedure was then employed to determine the flux and the exact source functions. To validate this method, a numerical computation of the poloidal flux was performed. The algorithm was subsequently applied to the International Thermonuclear Experimental Reactor (ITER) using published parameters for the baseline inductive scenario: a 15 MA-plasma current, a 32.86 T.m on-axis poloidal current, a 5.3 T on-axis toroidal magnetic field, and characteristic radii corresponding to pulsed operation at a fusion gain factor of $Q = 10$. To the best of our knowledge, this approach is novel and has not been previously reported for finding the source functions of the Grad-Shafranov equation.

Keywords: Grad-Shafranov equation; poloidal flux; plasma pressure; poloidal current; finite difference method.

1. Introduction

Nuclear fusion has long been considered a promising candidate for a sustainable, high-energy-density source of electricity. ITER is one of the most ambitious scientific projects ever undertaken and represents a major step toward demonstrating the feasibility of fusion energy as a future energy source. The successful realization of its planned plasma conditions requires accurate modeling of plasma equilibrium. The rescheduling of ITER's expected start of operations to 2039 provides an opportunity for further contributions to the understanding and modeling of the plasma physics underlying this major

project.

Over the past few decades, researchers have attempted to solve the Grad-Shafranov equation (GSE) and determine the variation of the source functions of thermonuclear plasma as a function of magnetic flux [1-5], with the aim of facilitating electricity generation in future fusion power plants. These source functions include plasma pressure p , poloidal current f and current density J_φ . Once these source functions are known, the GSE can be solved numerically to determine the spatial variation of the plasma flux in any axisymmetric magnetic confinement configuration such as tokamaks, spheromaks, or other similar systems [1].

The tokamak is currently the most promising device for achieving controlled fusion reactions. It is a combination of a toroidal chamber and a group of magnetic coils that confine the fusion plasma. The GSE, derived by combining magnetohydrodynamic (MHD) with Maxwell's equations, is used to compute the poloidal magnetic flux ψ inside the tokamak and to determine the source functions (p, f, J_φ) . GSE is a nonlinear second-order partial differential equation in space coordinates (r, z) and in the poloidal flux.

Since the 1950s, several attempts have been made to find a closed-form solution for GSE. These solutions assume an axisymmetric configuration that yields a smooth constant flux ψ independent of the azimuthal angle φ in cylindrical coordinates.

In previous studies, researchers followed a single systematic procedure to solve the GSE. They suggested expressions for p and f^2 dependence on flux ψ . This enabled them to remove the derivatives with respect to ψ from the GSE and leave only the dependence of ψ on the space coordinates. This procedure causes the flux ψ to depend exclusively on r and z . p and f^2 have been suggested by Solov'ev to be linear in ψ [2]. This leads to an expression for J_φ which depends only on r . However, Herrnegger-Maschke assumed expressions for p and f^2 that depended on ψ^2 only (parabolic source) and found J_φ depends on ψ/r as well as on $r\psi$ [3]. Additionally, McCarthy mixed the previous models and assumed

p to be a function of ψ , while f^2 depended on both ψ^2 and ψ . In this case, J_ϕ depends on r and ψ/r [4]. C. V. Atanasiu et al. assumed that both p and f^2 depended on a linear sum of ψ^2 and ψ and found J_ϕ that depends on ψ/r , $r\psi$ and r [5].

The above-mentioned solutions to GSE typically begin by assuming that the source functions (f and p) have linear or quadratic dependence on ψ [2, 3, 4]. This led to inhomogeneous solutions that depended on both the flux ψ and the squared radial distance r^2 . However, the assumption that f and p depend on ψ or ψ^2 lacks firm physical justification and is primarily introduced to render the GSE separable. This limitation motivated us to search for exact solutions that generalize the dependence of f , p and J_ϕ on ψ .

This paper introduces a rigorous method to determine the exact source functions f , p and J_ϕ of confined plasma that obey the Grad–Shafranov equation (GSE). This equation is based on the force balance between the plasma pressure gradient $-\vec{\nabla}p$ and the magnetic force $\vec{F}_B$. This balance is guaranteed by the poloidal beta β_p [1-4] which must be less than unity for a confined plasma. Poloidal beta is defined as the ratio of the plasma pressure to the magnetic pressure. For International Thermonuclear Experimental Reactor (ITER), β_p is typically set to a value between 0.6 and 0.65[6,7], as will be clarified in Section 3. In addition, the internal inductance l_i is a parameter related to plasma confinement. It is defined as the ratio of the volume-averaged square of the poloidal magnetic field to the square of the volume-averaged poloidal magnetic field. l_i is required to lie in the range $0.7 < l_i < 1.0$ [7,8].

After determining the source functions, we solved the GSE numerically and determined the variations in the flux contour as a function of the space coordinates (r, z) . Customized code using MATLAB was developed for this purpose.

We begin by systematically differentiating the GSE with respect to r and z , thereby reducing it to a single fourth-order partial differential equation in the magnetic flux ψ and its derivatives $\frac{\partial\psi}{\partial r}$, $\frac{\partial\psi}{\partial z}$, $\frac{\partial^2\psi}{\partial r^2}$,

$\frac{\partial^2\psi}{\partial z^2}$ and $\frac{\partial^2\psi}{\partial r\partial z}$. Fortunately, this process yields exact expressions for f , p and J_ϕ , as explained in Section

2. Section 3 is devoted to testing the validity of these solutions, where the method is applied to the International Thermonuclear Experimental Reactor (ITER) using published values of its size, plasma

current, toroidal field, and other key parameters [9,10]. Section 4 provides concluding remarks.

2. Method of calculation

The GSE for stationary and conducting magnetically confined plasma is given in cylindrical coordinates by

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = -\mu_0 r^2 \frac{\partial p}{\partial \psi} - f \frac{\partial f}{\partial \psi} \quad (1)[11,12]$$

where, f , p and ψ are the poloidal current, plasma pressure, and flux, respectively. r and z are the spatial coordinates in the cylindrical coordinate system. The right-hand side of (1) is equal to $-\mu_0 r J_\varphi$, by definition [11]. Under an ideal MHD, both ψ and f are proportional to the radial coordinates r and are given by $\psi = r A_\varphi$ and $f = r B_\varphi$ [11]. A_φ and B_φ are the azimuthal components of the vector potential and the magnetic field, respectively. Using these $\psi = r A_\varphi$, $f = r B_\varphi$ and $\vec{B} = \nabla \times \vec{A}$, it is straight forward to reach to the expressions of $\vec{B}$ in terms of $\frac{\partial \psi}{\partial r}$ and $\frac{\partial \psi}{\partial z}$ as: $B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}$ and $B_z = \frac{1}{r} \frac{\partial \psi}{\partial r}$, where the derivatives of ψ , f and p with respect to the azimuthal coordinate φ vanish.

For convenience, the notation $Y_{x_i} = \frac{\partial Y}{\partial x_i}$ is used throughout this study, where $Y = f, \psi$ or p and $x_i = r$ or z . Defining $-\mu_0 r J_\varphi = G(\psi, r)$, equation (1) is written as $\psi_{rr} + \psi_{zz} - \frac{1}{r} \psi_r = G(\psi, r)$.

Multiplying equation (1) by ψ_r and by ψ_z , respectively, yields the following equations:

$$f f_r + \psi_r G = -\mu_0 r^2 p_r \quad (2)$$

$$f f_z + \psi_z G = -\mu_0 r^2 p_z \quad (3)$$

Assuming that the n^{th} derivatives ($n = 1, 2, 3, \text{ and } 4$) of G , p , f and ψ exist and are continuous, equations (2) and (3) are differentiated with respect to z and r , respectively. The resulting equations were then subtracted to obtain:

$$\psi_z G_r - \psi_r G_z = -2\mu_0 r p_z \quad (4)$$

Substituting this equation in (3) and multiplying by $\frac{1}{r^2} \frac{\psi_r}{\psi_z}$ gives:

$$\frac{1}{2r} \left(\psi_r G_r - \frac{\psi_r^2}{\psi_z} G_z \right) = \frac{1}{r^2} f f_z \frac{\psi_r}{\psi_z} + \frac{\psi_r G}{r^2} \quad (5)$$

From the relations, $p_r = p_z \frac{\psi_r}{\psi_z}$ and $f_r = f_z \frac{\psi_r}{\psi_z}$ and using equation (2), it is clear that

$$\frac{1}{2r} \left(\psi_r G_r - \frac{\psi_r^2}{\psi_z} G_z \right) = -\mu_0 p_r \quad (6)$$

Similar procedures with $p_z = p_r \frac{\psi_z}{\psi_r}$ and $f_z = f_r \frac{\psi_z}{\psi_r}$ and using equation (3) yields:

$$\frac{1}{2r} (\psi_z G_r - \psi_r G_z) = -\mu_0 p_z \quad (7)$$

From equations (2) and (7),

$$f f_z = \frac{r}{2} (\psi_z G_r - \psi_r G_z) - \psi_z G \quad (8)$$

Multiplying (5) by r^2 and differentiating (5) and (6) with respect to r and rearranging, we obtain the following equation:

$$\psi_z \left(G_{rr} + \frac{\psi_r^2}{\psi_z^2} G_{zz} + 2 \left(\frac{\psi_r \psi_{rz}}{\psi_z^2} G_z - \frac{\psi_r}{\psi_z} G_{zr} \right) \right) - G_z \left(\psi_{rr} + \frac{\psi_r^2}{\psi_z^2} \psi_{zz} \right) = -2\mu_0 p_z \quad (9)$$

Using (7), equation (9) can be written as:

$$\psi_z \left(G_{rr} + \frac{\psi_r^2}{\psi_z^2} G_{zz} + 2 \left(\frac{\psi_r \psi_{rz}}{\psi_z^2} G_z - \frac{\psi_r}{\psi_z} G_{zr} \right) \right) - G_z \left(\psi_{rr} + \frac{\psi_r^2}{\psi_z^2} \psi_{zz} \right) = \frac{1}{r} (\psi_z G_r - \psi_r G_z) \quad (10)$$

Rearranging, (10) becomes:

$$G_z \left(\psi_{rr} + \frac{\psi_r^2}{\psi_z^2} \psi_{zz} - \frac{1}{r} \psi_r \right) = \psi_z \left(G_{rr} + \frac{\psi_r^2}{\psi_z^2} G_{zz} - \frac{1}{r} G_r + 2 \left(\frac{\psi_r \psi_{rz}}{\psi_z^2} G_z - \frac{\psi_r}{\psi_z} G_{zr} \right) \right) \quad (11)$$

Equation (11) is a fourth-order nonlinear partial differential equation (PDE). Any solution to G must satisfy (11). Upon finding G , straightforward procedures will lead to finding expressions for the plasma pressure p , poloidal current f and toroidal current density J_ϕ in terms of ψ as described later in this section. The solution to G follows from solving the second-order differential equation.

$$G(\psi, r) = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} \quad (12)$$

Unlike previous approaches to solving the GSE, this study starts by finding an expression for G that includes four independent solutions. These solutions include logarithmic terms, where all powers of ψ are naturally imposed, because they exist in its series expansion. Upon finding G , the expressions for p and f^2 were derived. It is not difficult to show that the expression for G that satisfies (12) is

$$G = -[(ar^2 + \lambda)\psi \ln(\beta\psi) + br^2] \quad (13)$$

where a , b , λ and β are constants determined from plasma equilibrium conditions [5]. The four independent solutions are: a constant β , r^2 , $\psi \ln(\beta\psi)$ and $r^2\psi \ln(\beta\psi)$. G is valid for all $\psi \geq 0$, where $\psi \ln(\beta\psi) = \ln(\beta\psi)/(1/\psi)$. According to L'Hopital's rule, it is clear that the limit of $\ln(\beta\psi)/(1/\psi)$ approaches zero as ψ approaches zero (at the boundaries of the plasma). Substituting the expression for G in Equation (13) into Equation (7) yields the following equation: $\frac{\partial}{\partial z}(\mu_0 p) = \frac{\partial}{\partial z}\left(\frac{a}{2}\left(\psi^2 \ln(\psi) - \frac{\psi^2}{2}\right) + b\psi\right)$. Integrating both sides, the exact pressure p is

$$p = \frac{1}{\mu_0} \left[\frac{a}{2} \left(\psi^2 \ln(\beta\psi) - \frac{\psi^2}{2} \right) + b\psi \right] \quad (14)$$

Substituting (13) into (8) leads to the expression $\frac{\partial}{\partial z}\left(\frac{1}{2}f^2\right) = \frac{\partial}{\partial z}\left(\frac{\lambda}{2}\left(\psi^2 \ln(\beta\psi) - \frac{\psi^2}{2}\right) + \frac{f_0^2}{2}\right)$.

Integrating both sides, the exact poloidal current f^2 is

$$f^2 = \lambda \left(\psi^2 \ln(\beta\psi) - \frac{\psi^2}{2} \right) + f_0^2, \quad (15)$$

where f_0 is the on-axis poloidal current given by $f_0 = r_0 B_\phi = 32.86 \text{ T.m}$ (for ITER).

Based on the previous definition, $-\mu_0 r J_\phi = G(\psi, r)$, it follows that:

$$J_\phi = \frac{1}{\mu_0} [(ar + \lambda/r)\psi \ln(\beta\psi) + br] \quad (16)$$

Equations (14), (15), and (16) provide the full solution to the problem, where GSE is written in the following form:

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = -[(ar^2 + \lambda)\psi \ln(\beta\psi) + br^2] \quad (17)$$

This equation is free of derivatives with respect to ψ and can be solved numerically for ψ , given β , a , b and λ . To obtain a unique solution that fully satisfies the GSE for a specific tokamak, such as ITER, at

least three equations must be formulated based on the equilibrium and boundary conditions, with one of the four arbitrary constants held fixed. In this study, β was an adjustable parameter fixed at 2.0.

The first of these three equations is derived from the integral of the current density, which by definition equals the global plasma current, I_{pl} .

Using equation (16), $\iint J_\phi drdz = I_{pl}$ gives:

$$a \iint r\psi \ln(\beta\psi) drdz + \lambda \iint \frac{1}{r}\psi \ln(\beta\psi) drdz + b \iint r drdz = \mu_0 I_{pl} \quad (18)$$

The second equation is based on the condition for the poloidal beta β_p , which is proportional to the ratio of the plasma pressure to the square of the poloidal magnetic field [12].

As mentioned in the Introduction, the poloidal beta describes the effectiveness of the poloidal magnetic field at plasma confinement, where the plasma pressure gradient acts outward and is balanced by the inward magnetic force [1-4].

Using this definition for the poloidal beta, $\beta_p = \frac{2\mu_0 \langle p \rangle_V}{\langle B_p \rangle^2}$ [13] along with the expression of the pressure given in (14), yields

$$a \iint r \left(\psi^2 \ln(\beta\psi) - \frac{\psi^2}{2} \right) drdz + 2b \iint r\psi drdz = \mu_0^2 I_{pl}^2 \beta_p \frac{\iint r drdz}{(\oint dl)^2} \quad (19)$$

The third equation was derived from the boundary condition $\psi = 0$ at the plasma edges. Although the derivation of this third equation is not directly available, it can be obtained using the following straightforward steps:

Multiplying equation (1) by ψ gives $\psi \left[\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} \right] = -\mu_0 r \psi J_\phi$. Using vector calculus, the left-hand side becomes: $\nabla \cdot \left(\psi \nabla \psi - \frac{\psi^2}{r} \hat{r} \right) - |\nabla \psi|^2$. That is: $\nabla \cdot \left(\psi \nabla \psi - \frac{\psi^2}{r} \hat{r} \right) / r - |\nabla \psi|^2 / r = \frac{\psi}{r} \left[\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} \right]$ and $\frac{\nabla \cdot \left(\psi \nabla \psi - \frac{\psi^2}{r} \hat{r} \right)}{r} = \nabla \cdot \left(\frac{\psi \nabla \psi}{r} - \frac{\psi^2}{r^2} \hat{r} \right) + \left(\frac{\psi \frac{\partial \psi}{\partial r}}{r^2} - \frac{\psi^2}{r^3} \right)$. Integrating both sides of this equation and applying Green's theorem (two-dimensional divergence theorem), the integral of the divergence term vanishes because of the boundary condition $\psi = 0$ and the second integral is negligible. In this case, Green's theorem reveals the following:

$\iint \nabla \cdot \left(\frac{\psi \nabla \psi}{r} - \frac{\psi^2}{r^2} \hat{r} \right) dr dz + \iint \left(\frac{\psi \frac{\partial \psi}{\partial r}}{r^2} - \frac{\psi^2}{r^3} \right) dr dz = \oint \left(\frac{\psi \nabla \psi}{r} - \frac{\psi^2}{r^2} \hat{r} \right) \cdot \hat{n} dl + \iint \left(\frac{\psi \frac{\partial \psi}{\partial r}}{r^2} - \frac{\psi^2}{r^3} \right) dr dz$. Thus,

$\iint \mu_0 J_\phi \psi dr dz = \iint \frac{|\nabla \psi|^2}{r} dr dz$. As mentioned earlier, the components of the poloidal magnetic

field are $B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}$ and $B_z = \frac{1}{r} \frac{\partial \psi}{\partial r}$. Therefore, $B_p^2 r = \frac{|\nabla \psi|^2}{r} = \frac{\left[\left(\frac{\partial \psi}{\partial r} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 \right]}{r}$. This result leads to the

identity: $\frac{1}{\mu_0} \iint B_p^2 r dr dz = \iint J_\phi \psi dr dz$. This relation enables us to determine the third constant, allowing the determination of the unknown equilibrium parameters a , b and λ , as follows:

$$a \iint r \psi^2 \ln(\beta \psi) dr dz + \lambda \iint \frac{\psi^2}{r} \ln(\beta \psi) dr dz + b \iint r \psi dr dz = \mu_0^2 I_{pl}^2 l_i \frac{\iint r dr dz}{(\oint dl)^2} \quad (20)$$

The left-hand side is related to the internal inductance (l_i) of the plasma, owing to the poloidal magnetic field. This l_i is given by:

$$l_i = \frac{\langle B_p^2 \rangle_V}{\langle B_p \rangle^2}, \text{ where } \langle B_p^2 \rangle_V = \frac{\iint B_p^2 r dr dz}{\iint r dr dz} \text{ and } \langle B_p \rangle \text{ comes from Ampere's law (for axis symmetry) as } \langle B_p \rangle = \frac{\mu_0 I_{pl}}{\oint dl} [12].$$

Therefore, $\iint B_p^2 r dr dz = \mu_0^2 I_{pl}^2 l_i \frac{\iint r dr dz}{(\oint dl)^2}$. As a result, $\iint \frac{\left[\left(\frac{\partial \psi}{\partial r} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 \right]}{r} dr dz = \mu_0^2 I_{pl}^2 l_i \frac{\iint r dr dz}{(\oint dl)^2}$. By

calculating the integrals on the right-hand side of this equation and using the prescribed value of l_i , the left-hand side can be determined. Upon finding a , b and λ , the GSE is solved and the flux ψ contours are fully determined.

3. Results and Discussion

The International Thermonuclear Experimental Reactor (ITER)

ITER was used to test the approach described in this project. It has a tokamak donut-shaped heart with a cross-sectional schematic, as shown in Fig.1. The structure and components of ITER have been described elsewhere [7,10,14]. In this study, we focus on the prediction of the exact pressure, poloidal current, exact current density, and numerical solution of the GSE that determines the equilibrium flux at the baseline operation of the ITER. The solution to Equation (17) was found using standard numerical methods. The finite difference method was used in this project by employing customized MATLAB code. The ITER was designed to operate under many scenarios [9,14,15].

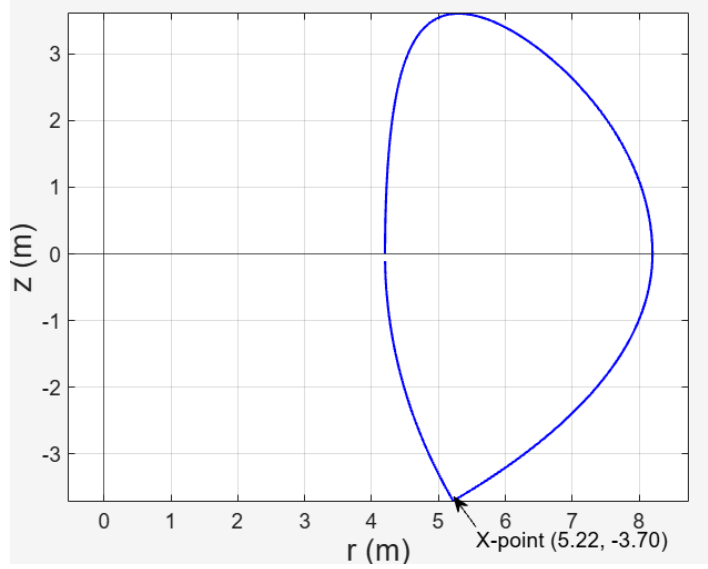


Fig.1: Schematic shape of the cross-section of the ITER reactor to the real dimensions

The current calculations were applied to the baseline inductive scenario mentioned in the introductory section of this paper with 15 MA-plasma current, 32.86 T.m on-axis poloidal current, 5.3 T on-axis toroidal magnetic field, and a major radius of 6.2 m, pulsed operation at a fusion gain factor $Q = 10$. The customized code was run on these parameters, including the elongation and triangularity of ~ 1.8 and ~ 0.45 , as given in [16]. At the X-point, however, the elongation and triangularity are 1.85 and 0.49, respectively, as given in [14,17].

Using the ITER set of the required plasma current, on-axis safety factor q_0 , edge safety factor q_{95} , toroidal magnetic field, inductance, radii, and on-axis poloidal current, the set of parameters (a , b and λ) that produce the equilibrium plasma must be unique. As mentioned in Section 2, the value of β was fixed at 2.0. The code was executed to determine the unknown equilibrium parameters, a , b and λ . The process begins by initializing random values: a in the range -0.2 to 0.2, b between 0 and 1, and λ from -3.0 to 3.0. Using these values, the code determines ψ that satisfies Equations (18), (19), and (20). The resulting error is then computed. This process was repeated iteratively until the error became negligible. The final values of a , b and λ were considered acceptable with an error of less than 1%. The values of a , b and λ determined by our customized code were verified by solving equations 18-20, simultaneously. The required integrals that enable us to verify equations 18-20 are presented in Table (1).

Table (1): The predicted ITER parameters and the required integrals to find a, b and λ from equations 18-20.

Calculated ITER parameters (this work)	Value/Unit
Plasma current, I_{p1}	15.0395 MA
Plasma major radius, r_{ax}	6.30959 m
Toroidal field, B_ϕ	5.3 T
Internal Inductance, l_i	0.828986
Poloidal beta, β_p	0.599031
$\iint r\psi \ln(\beta\psi) drdz$	1481.7 m ³ . Wb/rad
$\iint \frac{1}{r} \psi \ln(\beta\psi) drdz$	39.3404 m. Wb/rad
$\iint r drdz$	125.5597 m ³
$\left(\oint dl\right)^2$	319.5198 m ²
$\iint r\psi^2 \ln(\beta\psi) drdz$	11378 m ³ . Wb ² /rad ²
$\iint r\psi^2 drdz$	4149.3 m ³ . Wb ² /rad ²
$\iint r\psi drdz$	587.2317m ³ . Wb/rad
$\iint \frac{1}{r} \psi^2 \ln(\beta\psi) drdz$	296.9907m. Wb ² /rad ²
$\oint \frac{dl}{r^2 B_p}$	0.56661/T/m

The resulting a, b and λ along with the output operational parameters of the ITER, are listed in Table (2). As mentioned in the introductory section of this article, the poloidal beta and internal inductance were selected to satisfy the confinement requirements with β_p and l_i chosen in the ranges 0.6 – 0.65 and 0.7 – 1.0, respectively.

Table (2): Configurational output ITER parameters.

Output equilibrium parameters	Value/Unit
λ	0.253960 T.rad/Wb
f_0	33.4408 T.m
b	0.0728978 T/m ²
β	2.0 rad/Wb
a	-0.000165128 rad/m ⁴

The code was run for values of β_p less than 0.6 and greater than 0.65 and the results are shown in Fig.2. A radially oscillating pressure that indicates non-steady MHD operation appears for values of 0.35 and 0.4. For $\beta_p = 0.8$, the maximum pressure shifts from the center of the plasma toward the left edge of the reactor. This indicates that the choice of β_p within the above-mentioned range satisfies the required plasma stability.

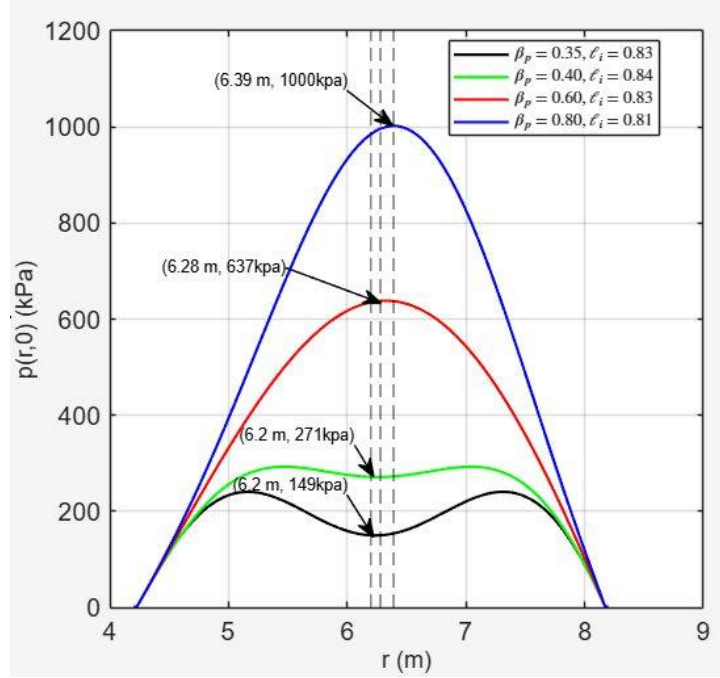


Fig.2: The projection of the pressure profile at $z = 0$ for values of β_p 0.35, 0.4, 0.6 and 0.8 and its correlated l_i^s

The values of l_i are correlated with β_p as shown in Fig.3, where a minute decrease (within the prescribed values) in the internal inductance is shown as β_p is changed. Therefore, a well-behaved pressure and steady flux are shown at $\beta_p \approx 0.6$ and $l_i = 0.83$.

The solutions to equations (18-20) to find β , a , b and λ guarantee a unique solution to the GSE for ITER.

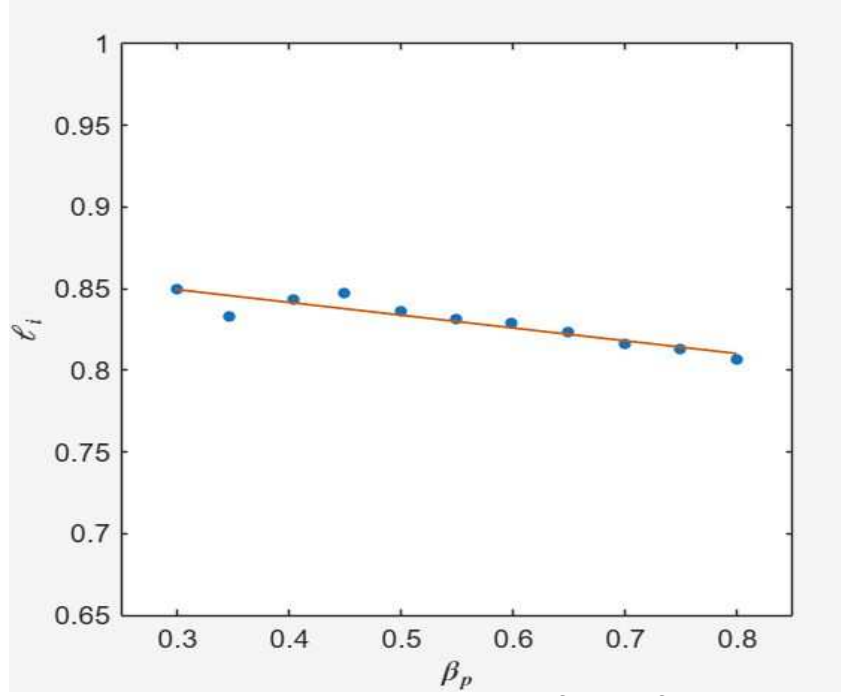


Fig.3: The correlation of β_p^s and l_i^s

Using the selected parameters, the poloidal flux, plasma pressure, poloidal current, and toroidal current density contours were determined, as shown in Fig.4. These figures illustrate the plasma behavior, with Fig.5 presenting the cross-sectional profiles of the contours in Fig.4 at $z = 0$. The flux, poloidal current, and current density were calculated from their minimum values at the boundaries to their maxima at the core of the plasma. The projections in Fig.5 present the results of this study, which are consistent with the expected operational output of the ITER. The poloidal magnetic flux was zero at the boundary and reached its maximum value of 11.39 Wb/rad at the axis of the plasma. This corresponded to a maximum pressure of 638 kPa along the axis. The poloidal current and current density were found to be on the 34.71 T.m and 1.48 MA/m², respectively, in agreement with the expected values.

All the results were obtained under the desired safety factors q_0 (on the axis) and q_{95} (edge). The safety factors q_0 , at the position where ψ reaches its maximum, and q_{95} , at the 95% poloidal-flux surface (near the edge of the plasma) are set to be approximately 1.0 and 3.0, respectively [15]. These q_0 and q_{95} are key parameters for plasma stability and confinement.

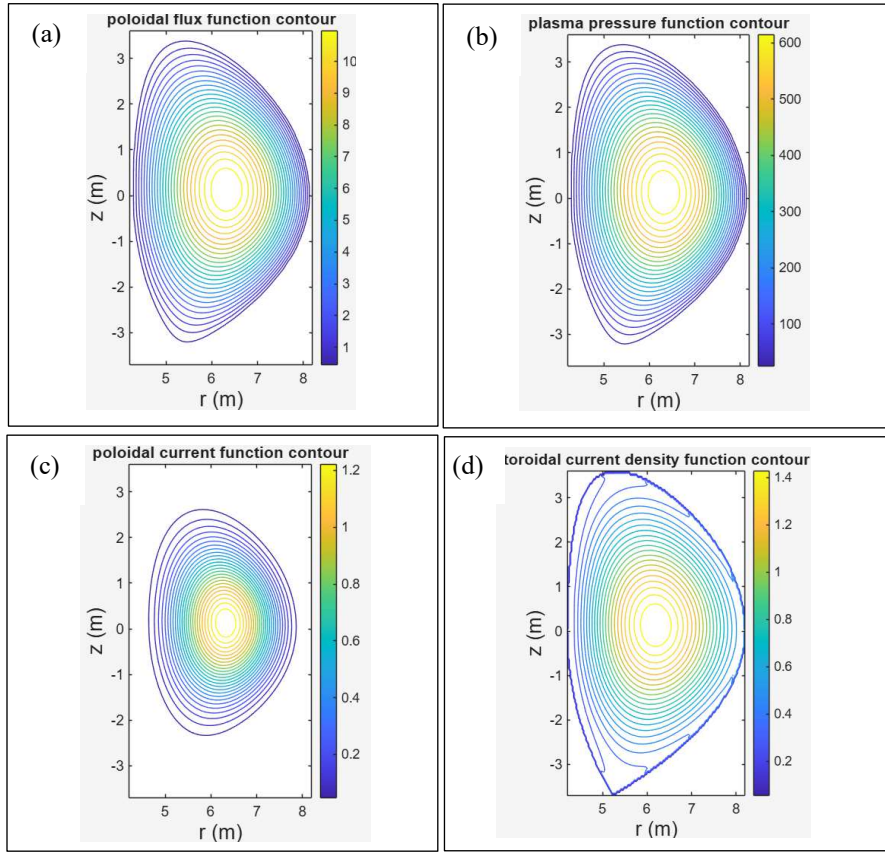


Fig.4: (a) The poloidal flux contour, (b) The plasma pressure contour and (c) The poloidal current contour (d) The toroidal current density contour, for $q_{ax} = 0.992627$, $q_{95} = 3.01552$, $a = -0.000165128 \text{ rad/m}^4$, $\lambda = 0.253960 \text{ T.rad/Wb}$, $b = 0.0728978 \text{ T/m}^2$ and $\beta = 2.0$

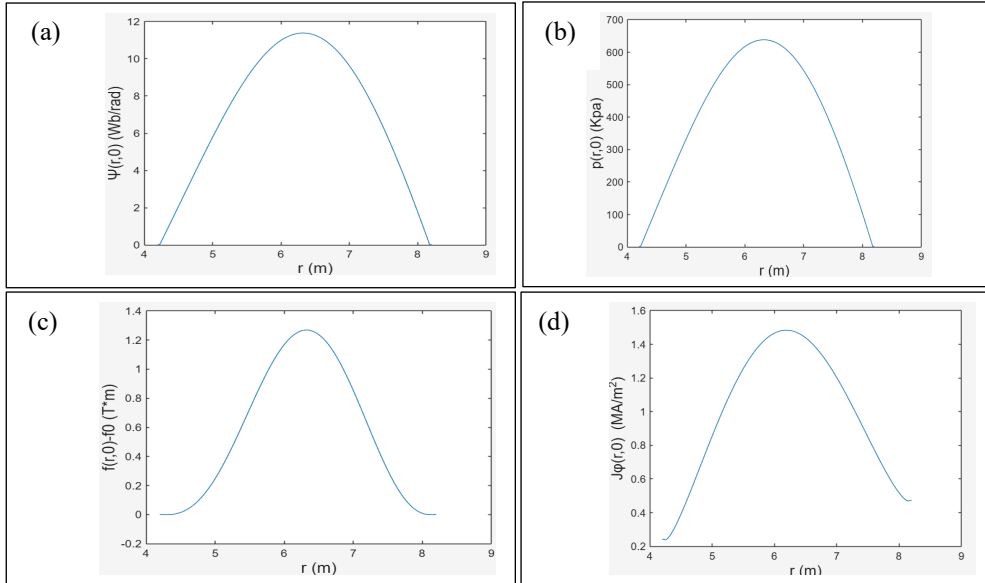


Fig.5: (a) The projection of the poloidal flux on the $r - \psi$ plane ($z = 0$); $\psi(r, 0)$, (b) The projection of the plasma pressure on the $r - p$ plane ($z = 0$); $p(r, 0)$, (c), The projection of the poloidal current on the $r - f$ plane ($z = 0$); $f(r, 0)$, (d) The projection of the toroidal current density on the $r - J_\phi$ plane ($z = 0$); $J_\phi(r, 0)$, for $q_0 = 0.992627$, $q_{95} = 3.01552$, $a = -0.000165128 \text{ rad/m}^4$, $\lambda = 0.253960 \text{ T.rad/Wb}$, $b = 0.0728978 \text{ T/m}^2$ and $\beta = 2.0$

The expressions for q_0 and q_{95} are obtained using equation (15) derived in this work for f , together with the equations derived by Oleg Bogoyavlenskij [18] and L. Guazzotto and J. P. Freidberg [13], respectively. The equations for q_0 and q_{95} are:

$$q_0 = \frac{1}{r_{ax}} \sqrt{\frac{\lambda \left(\psi_{axis}^2 \ln(\beta \psi_{axis}) - \frac{\psi_{axis}^2}{2} \right) + f_0^2}{\frac{\partial^2 \psi}{\partial r^2} \frac{\partial^2 \psi}{\partial z^2} - \left(\frac{\partial^2 \psi}{\partial r \partial z} \right)^2}} \quad (21)$$

where r_{ax} is the position at which the flux is maximum and

$$q_{95} = \frac{\sqrt{\lambda \left(\psi_{95}^2 \ln(\beta \psi_{95}) - \frac{\psi_{95}^2}{2} \right) + f_0^2}}{2\pi} \oint \frac{dl}{r^2 B_p} \quad (22)$$

q_0 is evaluated using the predicted values obtained in this work, where $\psi_{axis} = 11.39 \text{ Wb/rad}$, $f_0 = 33.4408 \text{ T.m}$, $\frac{\partial^2 \psi}{\partial r^2} \Big|_{r=r_{ax}} = -7.7005 \text{ Wb/rad/m}^2$, $\frac{\partial^2 \psi}{\partial z^2} \Big|_{r=r_{ax}} = -3.9917 \text{ Wb/rad/m}^2$ and $\left(\frac{\partial^2 \psi}{\partial r \partial z} \Big|_{r=r_{ax}} \right)^2 = 0.0241 \text{ Wb}^2/\text{rad}^2/\text{m}^4$. The resulting value is $q_0 = 0.992627$. Similarly, q_{95} is evaluated using $\psi_{95} = 0.5913 \text{ Wb/rad}$ and the other parameters given in Table (1). The resulting value is $q_{95} = 3.01552$.

The poloidal beta β_p and internal inductance l_i are related to magnetic confinement and are given by

$\frac{2\mu_0 \langle p \rangle_V}{\langle B_p \rangle^2}$ and $\frac{\langle B_p^2 \rangle}{\langle B_p \rangle^2}$, respectively. We checked the relationship between them for stable plasma flux,

pressure, poloidal current, and toroidal current density and found that they are not strongly correlated and l_i is about ~ 0.8 at different β_p , as shown in Fig.3.

4. Conclusion

In this study, exact analytical expressions for the source functions of the Grad–Shafranov equation are derived. These results make it possible to solve the Grad–Shafranov equation self-consistently. The equation was solved numerically using a finite-difference scheme implemented in a customized MATLAB code tailored to ITER parameters. The resulting equilibria show excellent agreement with the expected ITER baseline-operating scenario. The methodology presented here provides a robust and general framework for generating physically consistent equilibrium solutions, and can be readily applied to ITER and other axisymmetric tokamak fusion reactors.

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Credit author statement

Esam H. Al-Abdul-Hafidh: Formal Analysis, writing the original draft, and designing the study. supervision, reviewing and editing.

Mohamed H Abdelhamed: Formal Analysis. Acquisition of data, editing.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Conflicts of interest

There are no conflicts to declare.

ORCID iDs

Esam H. Al-Abdul-Hafidh: ORCID: 0000-0002-1145-0766

Mohamed H Abdelhamed: -

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