

Supplementary Information for "Reductions of QAOA Induced by Classical Symmetries: Theoretical Insights and Practical Implications"

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I. STRUCTURAL PRELIMINARIES

The foundational results and structural constructions established in the first part of this section provide the essential technical machinery required for the proofs of our main results. We then proceed to establish the rigorous verification of these properties. While these intermediate results are developed here primarily in service of our main theorems, they are formulated within a sufficiently general framework to be of independent interest.

A. Distinguished Relations and Elements in Standard DLAs

We begin by deriving several fundamental relations in the standard dynamical Lie algebras. Throughout, we adopt the following notation for iterated adjoint actions. For elements A, B of a Lie algebra $\mathfrak{g}$ and $k \in \mathbb{N}$, define

$$\text{ad}_A^k(B) := [A, \text{ad}_A^{k-1}(B)], \quad \text{ad}_A^0(B) := B. \quad (1)$$

That is, $\text{ad}_A^k(B)$ denotes the k -fold adjoint action of A on B .

Proposition I.1. The following relations hold in the algebra $\mathfrak{g}_{\Gamma, \text{std}}$:

1. $\text{ad}_Z^k(X) = -2^k i \sum_{v \in V} Y_v \left(\sum_{w \in \mathcal{N}_v} Z_w \right)^k$, for k odd,
2. $\text{ad}_Z^k(X) = -2^k i \sum_{v \in V} X_v \left(\sum_{w \in \mathcal{N}_v} Z_w \right)^k$, for k even,
3. $\text{ad}_X^3(Z) + 16 \text{ad}_X(Z) = 0$.

Proof. First, we confirm the base case for $k = 1$:

$$\text{ad}_Z(X) = [Z, X] = \left[i \sum_{(vw) \in E} Z_v Z_w, \sum_{v \in V} i X_v \right] = -2i \sum_{(vw) \in E} (Y_v Z_w + Z_v Y_w) = -2i \sum_{v \in V} Y_v \left(\sum_{w \in \mathcal{N}_v} Z_w \right) \quad (2)$$

Next, we observe that the Z -gates corresponding to any pair of vertices (including coinciding ones) commute. As a result, the adjoint operator ad_Z commutes with any polynomial $f(Z_1, \dots, Z_n)$.

Substituting the expression for $\text{ad}_Z(X)$, we proceed as follows:

$$\text{ad}_Z^2(X) = [Z, [Z, X]] = \left[Z, -2i \sum_{v \in V} Y_v \left(\sum_{w \in \mathcal{N}_v} Z_w \right) \right].$$

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Using the aforementioned commutativity, we get:

$$-2i \sum_{v \in V} \left(\sum_{w \in \mathcal{N}_v} Z_w \right) [Z, Y_v] = -2 \sum_{v \in V} \left(\sum_{w \in \mathcal{N}_v} Z_w \right) [Z, iY_v].$$

Recalling that $[iZ_w Z_v, iY_v] = 2iZ_w X_v$, we substitute:

$$\text{ad}_Z^2(X) = -2 \sum_{v \in V} \left(\sum_{w \in \mathcal{N}_v} Z_w \right)^2 (2iX_v) = 4i \sum_{v \in V} X_v \left(\sum_{w \in \mathcal{N}_v} Z_w \right)^2.$$

The assertion for higher values of k follows analogously by induction, using the same commutator properties and the fact that ad_Z commutes with polynomials in $Z_1, \dots, Z_n$.

Finally, we verify property (3). Using (2), we get that

$$\text{ad}_X(Z) = -\text{ad}_Z(X) = 2i \sum_{(vw) \in E} (Y_v Z_w + Z_v Y_w).$$

Next, we compute

$$\begin{aligned} \text{ad}_X^2(Z) &= 2 \left[\sum_{v \in V} iX_v, \sum_{(vw) \in E} i(Y_v Z_w + Z_v Y_w) \right] = 4i \sum_{(vw) \in E} (-Z_v Z_w + Y_v Y_w - Z_v Z_w + Y_v Y_w) \\ &= 8i \sum_{(vw) \in E} (Y_v Y_w - Z_v Z_w) \end{aligned}$$

and establish

$$\begin{aligned} \text{ad}_X^3(Z) &= 8 \left[i \sum_{v \in V} X_v, i \sum_{(vw) \in E} (Y_v Y_w - Z_v Z_w) \right] = -16i \sum_{(vw) \in E} (Z_v Y_w + Y_v Z_w + Y_v Z_w + Z_v Y_w) \\ &= -32i \sum_{(vw) \in E} (Z_v Y_w + Y_v Z_w) = -16 \text{ad}_X(Z). \end{aligned}$$

□

Definition I.2. Let $b \in \mathbb{B}^n$ be a binary string of length n , and let

$$\{i_1, i_2, \dots, i_m\} \subseteq \{1, 2, \dots, n\}$$

be a set of distinct indices. The *balance* of b supported on these indices, denoted by

$$\text{Bal}_{i_1 \dots i_m}(b),$$

is defined as the number of 0's minus the number of 1's appearing in b at positions $i_1, i_2, \dots, i_m$. Equivalently,

$$\text{Bal}_{i_1 \dots i_m}(b) := |\{j \in \{i_1, \dots, i_m\} : b_j = 0\}| - |\{j \in \{i_1, \dots, i_m\} : b_j = 1\}|.$$

Proposition I.3. Consider the polynomial

$$f(Z_{i_1}, Z_{i_2}, \dots, Z_{i_m}) = (Z_{i_1} + Z_{i_2} + \dots + Z_{i_m})^k.$$

Then for any standard basis vector $|b\rangle$, we have

$$f(Z_{i_1}, Z_{i_2}, \dots, Z_{i_m})(|b\rangle) = (\text{Bal}_{i_1 \dots i_m}(b))^k |b\rangle.$$

Proof. Recall that for any $j \in \{i_1, i_2, \dots, i_m\}$, the action of Z_{i_j} on the standard basis vector $|b\rangle$ is given by

$$Z_{i_j}(|b\rangle) = \begin{cases} |b\rangle, & \text{if } b_{i_j} = 0, \\ -|b\rangle, & \text{if } b_{i_j} = 1. \end{cases}$$

Therefore, the sum of operators $Z_{i_1} + Z_{i_2} + \dots + Z_{i_m}$ acting on $|b\rangle$ results in

$$(Z_{i_1} + Z_{i_2} + \dots + Z_{i_m})(|b\rangle) = \text{Bal}_{i_1 \dots i_m}(b) |b\rangle.$$

This follows because each Z_{i_j} contributes $+1$ if $b_j = 0$ and -1 if $b_j = 1$ with $j \in \{i_1, \dots, i_m\}$.

Hence, applying $f(Z_{i_1}, Z_{i_2}, \dots, Z_{i_m}) = (Z_{i_1} + Z_{i_2} + \dots + Z_{i_m})^k$ to $|b\rangle$ gives

$$f(Z_{i_1}, Z_{i_2}, \dots, Z_{i_m})(|b\rangle) = (\text{Bal}_{i_1 \dots i_m}(b))^k |b\rangle.$$

This concludes the proof. $\square$

Let $f(t) = \sum_{j=1}^n \alpha_j t^j$ be a polynomial with real coefficients, i.e., $f(t) \in \mathbb{R}[t]$. We associate the operator

$$F(\text{ad}_Z^2) := \sum_{j=1}^n \alpha_j \text{ad}_Z^{2j} \quad (3)$$

to $f(t)$ and observe that the element $F(\text{ad}_Z^2)(X)$ resides in the standard dynamical Lie algebra $\mathfrak{g}_{\Gamma, \text{std}}$.

Using part (2) of Proposition I.1, we find that

$$F(\text{ad}_Z^2)(X) = -i \sum_{j=1}^n 2^{2j} \alpha_j \sum_{v \in V} X_v \left(\sum_{w \in \mathcal{N}_v} Z_w \right)^{2j}.$$

This can be rewritten as

$$F(\text{ad}_Z^2)(X) = -i \sum_{v \in V} X_v \sum_{i=1}^n 2^{2i} \alpha_i \left(\sum_{w \in \mathcal{N}_v} Z_w \right)^{2i} = -i \sum_{v \in V} X_v \widehat{f} \left(\left(\sum_{w \in \mathcal{N}_v} Z_w \right)^2 \right), \quad (4)$$

where $\widehat{f}(t) := \sum_{j=1}^n 2^{2j} \alpha_j t^j$.

Combining this result with Proposition I.3, we have

$$F(\text{ad}_Z^2)(X)(|b\rangle) = -i \sum_{v \in V} X_v \widehat{f}(\text{Bal}_{\mathcal{N}_v}(|b\rangle)^2), \quad (5)$$

for any standard basis vector $|b\rangle$.

Let

$$\mathcal{V}_{\text{even}} := \{v \in V(\Gamma) \mid \deg(v) \equiv 0 \pmod{2}\} \text{ and } \mathcal{V}_{\text{odd}} := \{v \in V(\Gamma) \mid \deg(v) \equiv 1 \pmod{2}\} \quad (6)$$

denote, respectively, the subsets of vertices of even and odd degree in the graph Γ .

With this notation in place, we are ready to establish the following result.

Theorem I.4. Let Γ be a connected graph. The standard dynamical Lie algebra $\mathfrak{g}_{\Gamma, \text{std}}$ contains the following elements:

1. $\mathcal{X}_{\text{even}} := i \sum_{v \in \mathcal{V}_{\text{even}}} X_v,$
2. $\mathcal{X}_{\text{odd}} := i \sum_{v \in \mathcal{V}_{\text{odd}}} X_v,$

$$3. \mathcal{Z}_{\text{even/odd}} := i \sum_{\substack{(vw) \in E \\ v \in \mathcal{V}_{\text{even}} \\ w \in \mathcal{V}_{\text{odd}}}} Z_v Z_w,$$

Proof. Let m denote the largest even degree among the vertices of Γ . We introduce the polynomials

$$\hat{o}_m(t) := \sum_{j=1}^{m+1} \alpha_j t^j \in \mathbb{R}[t], \quad \hat{e}_m(t) := \sum_{j=1}^{m+1} \alpha_j t^j \in \mathbb{R}[t], \quad (7)$$

characterized by the interpolation conditions

$$\hat{o}_m(k) = \begin{cases} 0, & k = 0^2, 2^2, \dots, m^2, \\ 1, & k = 1^2, 3^2, \dots, (m+1)^2, \end{cases} \quad \hat{e}_m(k) = \begin{cases} 1, & k = 0^2, 2^2, \dots, m^2, \\ 0, & k = 1^2, 3^2, \dots, (m+1)^2. \end{cases}$$

Such polynomials can be constructed explicitly using Lagrange interpolation. Observe that the range of the function $\text{Bal}_{i_1 \dots i_k}$ on the set of standard basis vectors is $\{-k, -k+2, \dots, k-2, k\}$. Therefore, the union of the ranges of the functions $\text{Bal}_{i_1 \dots i_k}$ for $k \in \{2, 4, \dots, m\}$ (with m even) is $\{-m, -m+2, \dots, m-2, m\}$ (see Definition 1.2). Consequently, the union of the ranges of the squared functions $\text{Bal}_{i_1 \dots i_k}^2$ on the set of standard basis vectors is $\{0^2, 2^2, 4^2, \dots, m^2\}$.

Let $\mathcal{O}_m(\text{ad}_Z^2)$ and $\mathcal{E}_m(\text{ad}_Z^2)$ be the operators associated with the polynomials $o_m(t)$ and $e_m(t)$ according to the formula in (3). Using the formula from the operator expansion in (4), we get

$$F_m(\text{ad}_Z^2)(iX_v) = i \left(\sum_{j=1}^m \alpha_j \left(\sum_{w \in \mathcal{N}_v} Z_w \right)^{2j} \right) X_v.$$

operator $F_m(\text{ad}_Z^2)(X) \in \mathfrak{g}_{\Gamma, \text{std}}$. For any standard basis vector $b \in \mathcal{B}_{0,1}$, define $w_{b,v} := \text{Bal}_{\mathcal{N}_v}(b)$, where $\mathcal{N}_v$ denotes the neighborhood of v . Then, according to Proposition 1.3, we have:

$$\mathcal{O}_m(\text{ad}_Z^2)(iX_v)(b) = i \hat{o}_m(w_{b,v}^2) X_v(b).$$

If the degree of v is an even number k , then $w_{b,v}$ ranges over the even numbers between 0 and m . Therefore, $\hat{o}_m(w_{b,v}^2) = 0$, and the operator $\mathcal{O}_m(\text{ad}_Z^2)(iX_v)$ acts identically as zero on the Hilbert space W .

On the other hand, for any vertex v of odd degree and any $b \in \mathcal{B}_{0,1}$, the range of $w_{b,v}$ consists of odd values between 1 and $m+1$. In this case, $\hat{o}_m(w_{b,v}^2) = 1$.

This implies that

$$\mathcal{O}_m(\text{ad}_Z^2)(iX_v) = \begin{cases} iX_v, & \text{if } v \text{ has odd degree,} \\ 0, & \text{if } v \text{ has even degree.} \end{cases}$$

Therefore, we conclude

$$\mathcal{O}_m(\text{ad}_Z^2)(X) = i \sum_{v \in \mathcal{V}_{\text{odd}}} X_v = \mathcal{X}_{\text{odd}} \in \mathfrak{g}_{\Gamma, \text{std}}.$$

Hence, the elements $\mathcal{X}_{\text{odd}}$ and $\mathcal{X}_{\text{even}} = X - \mathcal{X}_{\text{odd}}$ both belong to $\mathfrak{g}_{\Gamma, \text{std}}$.

Next, observe that the operator $\text{ad}_{\mathcal{X}_{\text{odd}}}^2(\text{ad}_{\mathcal{X}_{\text{even}}}^2(Z))$ is proportional to $\mathcal{Z}_{\text{even/odd}}$. Thus, $\mathcal{Z}_{\text{even/odd}}$ is in $\mathfrak{g}_{\Gamma, \text{std}}$. Furthermore, since:

$$\mathcal{Z}_{\text{even/even}} + \mathcal{Z}_{\text{odd/odd}} = Z - \mathcal{Z}_{\text{even/odd}},$$

it follows that $\mathcal{Z}_{\text{even/even}} + \mathcal{Z}_{\text{odd/odd}}$ is also in $\mathfrak{g}_{\Gamma, \text{std}}$.

This completes the proof. $\square$

B. Distinguished Elements in Reduced DLAs

We begin with two structural lemmas describing elements that necessarily belong to the standard and free reduced dynamical Lie algebras. The first shows that in the standard reduced DLA one can separate the single-site Z -terms supported on neighbors of v from the Z - Z interactions internal to the reduced graph. The second lemma identifies natural Lie subalgebras contained in the free and standard reduced DLAs. In particular, the free reduced DLA contains, for each vertex adjacent to the reduction vertex v , a copy of $\mathfrak{su}(2)$ supported on the corresponding qubit Hilbert space.

Lemma I.5. The standard reduced dynamical Lie algebra $\mathfrak{g}_{\Gamma, \text{std}}^v$ contains the elements

$$\mathcal{Z}_{\hat{v},1} := i \sum_{(vw) \in E(\Gamma)} Z_w, \quad \mathcal{Z}_{\hat{v},2} := i \sum_{(ww') \in E(\Gamma_v)} Z_w Z_{w'}. \quad (8)$$

Proof. We compute:

$$\begin{aligned} [\mathcal{X}_{\hat{v}}, \mathcal{Z}_{\hat{v}}] &= -2i \sum_{(ww') \in E(\Gamma_v)} (Y_w Z_{w'} + Z_w Y_{w'}) - 2i \sum_{(vw) \in E(\Gamma)} Y_w; \\ \text{ad}_{\mathcal{X}_{\hat{v}}}^3(\mathcal{Z}_{\hat{v}}) &= 32i \sum_{(ww') \in E(\Gamma_v)} (Y_w Z_{w'} + Z_w Y_{w'}) + 8i \sum_{(vw) \in E(\Gamma)} Y_w. \end{aligned}$$

From this we isolate the two contributions. Define

$$\tilde{Y}_1 := \frac{1}{24} \cdot (\text{ad}_{\mathcal{X}_{\hat{v}}}^3(\mathcal{Z}_{\hat{v}}) + 16[\mathcal{X}_{\hat{v}}, \mathcal{Z}_{\hat{v}}]) = i \sum_{(vw) \in E(\Gamma)} Y_w,$$

and then

$$-0.5[\mathcal{X}_{\hat{v}}, \tilde{Y}_1] = i \sum_{(vw) \in E} Z_w = \mathcal{Z}_{\hat{v},1},$$

yields the desired elements $\mathcal{Z}_{\hat{v},1}$ and $\mathcal{Z}_{\hat{v},2}$ in $\mathfrak{g}_{\Gamma, \text{free}}^v$ (see 8). $\square$

Lemma I.6. Let Γ be a graph and $v \in V(\Gamma)$ a vertex. Then the free reduced dynamical Lie algebra $\mathfrak{g}_{\Gamma, \text{free}}^v$ contains the following natural subalgebras:

- (1) The free dynamical Lie algebra on the reduced graph Γ_v (see Definition I.7):

$$\mathfrak{g}_{\Gamma_v, \text{free}} = \left\langle \{iX_w \mid w \in V(\Gamma) \setminus \{v\}\}, \{iZ_w Z_{w'} \mid (ww') \in E(\Gamma), w, w' \neq v\} \right\rangle_{\text{Lie}},$$

generated by all single-qubit iX terms except the one corresponding to v , together with all $Z - Z$ interactions on edges not incident to v .

- (2) The direct sum of single-qubit $\mathfrak{su}(2)$ algebras acting on the neighbors of v ,

$$\mathfrak{su}(2)^{\oplus |\mathcal{N}_v|} = \left\langle \{iX_j \mid w \in \mathcal{N}_v\}, \{iZ_j \mid w \in \mathcal{N}_v\} \right\rangle_{\text{Lie}},$$

corresponding to the independent local actions on each vertex adjacent to v .

Similarly, the standard reduced dynamical Lie algebra $\mathfrak{g}_{\Gamma, \text{std}}^v$ contains the following subalgebras:

- (1) The standard dynamical Lie algebra on the reduced graph Γ_v :

$$\mathfrak{g}_{\Gamma_v, \text{std}} = \left\langle \mathcal{X}_{\hat{v}}, \mathcal{Z}_{\hat{v},2} \right\rangle_{\text{Lie}}.$$

- (2) The $\mathfrak{su}(2)$ subalgebra, arising as

$$\mathfrak{su}(2) = \left\langle \mathcal{X}_{\hat{v}}, \mathcal{Z}_{\hat{v},1} \right\rangle_{\text{Lie}}.$$

Proof. By Lemma I.5, the specified generators indeed belong to the corresponding dynamical Lie algebras. It then follows directly that their Lie brackets generate the subalgebras listed above. $\square$

C. More Distinguished Elements in Reduced DLAs

Recall that $\mathcal{N}_{v,j}$ denotes the set of vertices in Γ_v at distance j from the distinguished vertex v . Below, we establish Theorem IV.2, which asserts that if every vertex $w \in \mathcal{N}_{v,j}$ has an odd number of neighbors in the preceding layer $\mathcal{N}_{v,j-1}$ —that is,

$$|\{u \in \mathcal{N}_{v,j-1} \mid (u,w) \in E\}| \equiv 1 \pmod{2}, \quad (9)$$

then the standard reduced DLA $\mathfrak{g}_{\Gamma,\text{std}}^v$ contains uniform iX -type operators supported on individual distance layers relative to v . In the subsequent section, we refine this construction to isolate finer vertex subsets characterized by parity data along distance-increasing paths.

Proof of Theorem IV.2. We proceed by induction on j . For the base case $j = 1$, we first observe that the degree of any vertex $w \in \mathcal{N}_{v,1}$ is one greater than its degree in the reduced graph Γ_v . A direct computation shows that

$$\begin{aligned} \mathcal{X}_{\widehat{v},1} &= 0.25 \text{ad}_{\mathcal{Z}_{\widehat{v},1}}^2(\mathcal{X}_{\widehat{v}}), \\ \mathcal{X}_{v,1,\text{even}} &= \mathcal{O}_m(\text{ad}_{\mathcal{Z}_{\widehat{v},2}}^2)(\mathcal{X}_{\widehat{v},1}), \\ \mathcal{X}_{v,1,\text{odd}} &= \mathcal{X}_{\widehat{v},1} - \mathcal{X}_{v,1,\text{even}}, \end{aligned} \quad (10)$$

where $\mathcal{X}_{\widehat{v}}$, $\mathcal{Z}_{\widehat{v},1}$, and $\mathcal{Z}_{\widehat{v},2}$ are defined as

$$\mathfrak{g}_{\Gamma,\text{std}}^v = \left\langle \mathcal{X}_{\widehat{v}}, \mathcal{Z}_{\widehat{v}} \right\rangle_{\text{Lie}}, \quad (11)$$

with generators

$$\mathcal{X}_{\widehat{v}} := iH_M^v \text{ and } \mathcal{Z}_{\widehat{v}} := iH_P^v. \quad (12)$$

From (10), it follows that $\mathcal{X}_{\widehat{v},1}$, $\mathcal{X}_{v,1,\text{even}}$, and $\mathcal{X}_{v,1,\text{odd}}$ are elements of the reduced dynamical Lie algebra $\mathfrak{g}_{\Gamma,\text{std}}^v$, thereby establishing the base case.

Now, assume that $\mathcal{X}_{\widehat{v},j}$, $\mathcal{X}_{v,j,\text{even}}$, $\mathcal{X}_{v,j,\text{odd}} \in \mathfrak{g}_{\Gamma,\text{std}}^v$ for all $j \leq n$. To prove the statement for $j = n+1$, we construct an auxiliary operator consisting of two-qubit Pauli- Z terms supported on the vertices in $\mathcal{N}_{v,n} \cup \mathcal{N}_{v,n+1}$:

$$\mathcal{Z}_n := -0.25 \text{ad}_{\mathcal{X}_{\widehat{v},n}}^2(\mathcal{Z}_{\widehat{v},2}) = i \sum_{\substack{(ww') \in E \\ w \in \mathcal{N}_{v,n} \\ w' \notin \mathcal{N}_{v,n}}} Z_w Z_{w'} - 2i \sum_{\substack{(ww') \in E \\ w, w' \in \mathcal{N}_{v,n}}} (Y_w Y_{w'} - Z_w Z_{w'}) \quad (13)$$

Utilizing this element and the assumption in (9), we obtain $\mathcal{X}_{\widehat{v},n+1}$ via the relation:

$$\mathcal{X}_{\widehat{v},n+1} = \mathcal{O}_m(\text{ad}_{\mathcal{Z}_n}^2)(\mathcal{X}_{\widehat{v}} - \mathcal{X}_{\widehat{v},n}). \quad (14)$$

Finally, the parity-specific components are recovered following the same logic as the base case:

$$\begin{aligned} \mathcal{X}_{v,n+1,\text{odd}} &= \mathcal{O}_m(\text{ad}_{\mathcal{Z}_{\widehat{v},2}}^2)(\mathcal{X}_{\widehat{v},n+1}), \\ \mathcal{X}_{v,n+1,\text{even}} &= \mathcal{X}_{\widehat{v},n+1} - \mathcal{X}_{v,n+1,\text{odd}}. \end{aligned} \quad (15)$$

This completes the inductive step and hence the proof. $\square$

We proceed with presenting the construction of the extension graph and verifying the properties listed in Theorem II.5.

We now introduce a graph-theoretic reduction that plays a central role in our analysis of dynamical Lie algebras.

Definition I.7. Let $\Gamma = (V, E)$ be a graph and let $v \in V(\Gamma)$ be a vertex. The *reduced graph at v* , denoted by Γ_v , is the graph obtained from Γ by removing the vertex v together with all edges incident to v . Equivalently, Γ_v is the graph with vertex and edge sets

$$V(\Gamma_v) = V(\Gamma) \setminus \{v\}, \quad E(\Gamma_v) = E(\Gamma) \setminus \{(v, j) \mid j \in \mathcal{N}_{v,1}\}. \quad (16)$$

D. Construction of the Extension Graph and Verification of the Required Properties

We now describe the construction of the extended graph $\widehat{\Gamma}_v$, which proceeds in three steps (labeled 0–2).

Step 0 is a preprocessing stage ensuring that all vertices satisfy the parity condition in (9). Steps 1 and 2 then implement the extension procedure, which guarantees that the standard reducedDLA coincides with the free one.

Finally, we verify that $\widehat{\Gamma}_v$ satisfies the properties required by Theorem II.5.

Step 0. Correcting parity. For each $j \geq 2$ and every vertex $w \in \mathcal{N}_{v,j}$ that violates the parity assumption (9), we select a vertex $w_{j-2} \in \mathcal{N}_{v,j-2}$ such that $\text{dist}(w_{j-2}, w) = 2$. We then extend the graph by attaching the triangular segment shown in Figure 1.

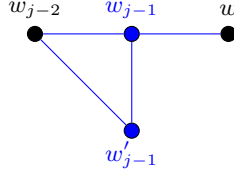


FIG. 1. Local modification for parity correction. Vertices and edges belonging to the original graph Γ are shown in black, while the auxiliary elements introduced during Step 0 are highlighted in blue.

Step 1. Equalizing distances. For each vertex $w \in V(\Gamma_v)$, attach a path of length $j(v) - \text{dist}_\Gamma(v, w)$ to w . Vertices already at maximal distance $j(v)$ from v remain unchanged. Denote the resulting graph by Γ' .

By construction, every original vertex $w \neq v$ now determines a unique vertex in Γ' at distance $j(v)$ from v , obtained by following the newly attached path from w to its endpoint. Hence there is a natural injective map

$$\psi : V(\Gamma_v) \hookrightarrow V(\Gamma') \cap \mathcal{N}'_{v,j(v)},$$

where $\mathcal{N}'_{v,j(v)}$ denotes the set of vertices in Γ' at distance $j(v)$ from v .

Step 2. Layered attachment. Fix an indexing of the vertices in $\mathcal{N}'_{v,j(v)}$ by

$$\{0, 1, \dots, |\mathcal{N}'_{v,j(v)}| - 1\}.$$

For each vertex with index j , attach an additional path of length j . Denote the resulting graph by $\widehat{\Gamma}_v$.

An illustrative example of this procedure is shown in Figures 2, 3, and 4, which display the original graph, the intermediate extension after Step 1, and the final graph $\widehat{\Gamma}_v$, respectively.

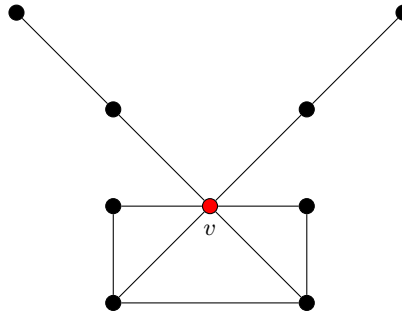


FIG. 2. A sample graph Γ with a distinguished vertex v .

Remark I.8. MaxCut solutions on Γ are recovered from solution strings on $\widehat{\Gamma}_v$ by deleting the bits corresponding to the auxiliary vertices introduced during the construction.

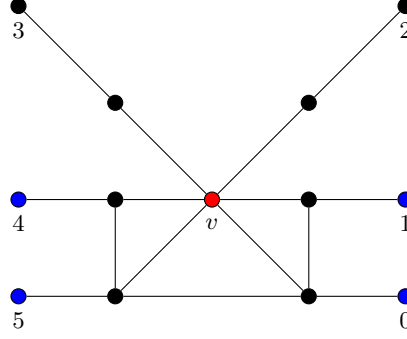


FIG. 3. The extended graph Γ' obtained from Γ after Step 1. Vertices added in this step are highlighted in blue. All vertices at maximal graph distance $j(v) = 2$ from the distinguished vertex v are labeled, preparing the indexing used in Step 2.

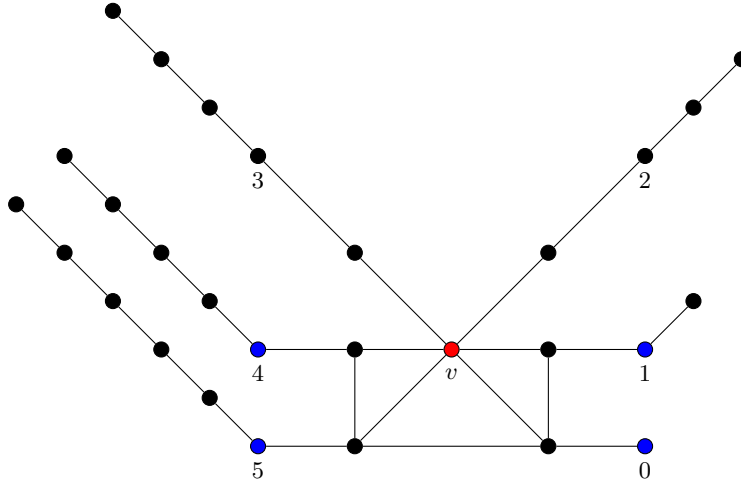


FIG. 4. The completed graph $\hat{\Gamma}_v$ obtained after Step 2. To each vertex at maximal distance $j(v) = 2$ from the distinguished vertex v , an additional path is attached whose length equals its assigned label.

Next, we prove the properties of extension graphs stated in Theorem II.5.

Proof of Theorem II.5. By monitoring the change in the number of vertices and edges at each modification step, we verify that $\hat{\Gamma}_v$ satisfies the theorem's structural requirements.

Step 0. The initial reduced graph Γ_v contains $n - 1$ vertices. Since each vertex violating the parity assumption (9) triggers an extension via the gadget shown in Figure 1, adding two vertices and four edges per instance, the total number of vertices in the resulting graph is bounded by $1 + 3(n - 1) = 3n - 2$. Similarly, the number of edges is at most $|E(\Gamma)| + 4(n - 1)$.

Step 1. Let $v \in V(\Gamma)$ be the distinguished vertex and let $j(v)$ denote the maximal distance from v to any vertex of Γ . Notice that it is not altered on the previous step. For each vertex $w \in V(\Gamma_v)$, we attach to w a path graph of length

$$j(v) - \text{dist}(v, w).$$

If $\text{dist}(v, w) = j(v)$, no modification is performed.

Denote by Γ' the resulting graph. By construction, every vertex $w \in V(\Gamma_v)$ now either lies at distance $j(v)$ from v in Γ' , or is the ancestor of a unique leaf at that distance. Hence there is a bijection

$$V(\Gamma_v) \longrightarrow \mathcal{N}'_{v, j(v)},$$

obtained by following the attached path from w to its endpoint.

Since each added vertex contributes exactly one new edge, the total number of added vertices and edges in this step is bounded above by

$$(j(v) - 1)(3n - 3).$$

Step 2. Label the vertices of $\mathcal{N}'_{v,j(v)}$ from 0 to $|\mathcal{N}'_{v,j(v)}| - 1$. To each vertex with index j , attach a path of length j . Let $\widehat{\Gamma}_v$ denote the resulting graph.

By construction, the newly added vertices in this step have pairwise distinct distances to v . Moreover, each such vertex is the unique vertex in $\widehat{\Gamma}_v$ at its respective distance from v .

Vertex and edge bounds.

We verify that $\widehat{\Gamma}_v$ possesses the properties stated in the theorem. Specifically, the total vertex count is bounded by:

$$|V(\widehat{\Gamma}_v)| < 1 + 3(j(v) - 1)(n - 1) + \sum_{j=0}^{|\mathcal{N}'_{v,j(v)}|-1} j < 1 + 3(j(v) - 1)(n - 1) + \frac{(3n - 2)(3n - 3)}{2},$$

where the leading 1 represents the vertex v and the final inequality follows from the bound $|\mathcal{N}'_{v,j(v)}| < n$.

Similarly, the total number of edges in $\widehat{\Gamma}_v$ satisfies

$$|E(\widehat{\Gamma}_v)| < |E(\Gamma)| + 3(j(v) - 1)(n - 1) + \sum_{j=0}^{|\mathcal{N}'_{v,j(v)}|-1} j < |E(\Gamma)| + 3(j(v) - 1)(n - 1) + \frac{(3n - 2)(3n - 3)}{2}.$$

The vertices added in the second step have pairwise distinct distances to v by construction. Moreover, each such vertex is the unique vertex in $\widehat{\Gamma}_v$ at its respective distance from v . It therefore follows from Theorem IV.2 that

$$iX_w \in \mathfrak{g}_{\widehat{\Gamma}_v, \text{std}}^v$$

for each of these vertices.

Applying Remark II.7 together with Theorem IV.2 to the vertices in $\mathcal{N}'_{v,j(v)}$, we conclude that

$$iX_w \in \mathfrak{g}_{\widehat{\Gamma}_v, \text{std}}^v$$

for all remaining vertices as well, allowing to establish the required containment of DLAs.

Finally, we establish a canonical extension-reduction principle between the original and extended graphs. Specifically, every MaxCut configuration on Γ admits a unique extension to a configuration on $\widehat{\Gamma}_v$. Conversely, any solution on $\widehat{\Gamma}_v$ restricts uniquely to a solution on Γ by discarding the values assigned to the auxiliary vertices.

Each triangular subgraph introduced in Step 0 (cf. Figure 1) possesses a maximum cut value of 3. A crucial property of this construction is that this maximum is attained regardless of whether the states assigned to w_{j-2} and w_j are identical or opposite, thereby preserving the original cut constraints.

This local property, combined with the path attachments in the subsequent steps, allows the MaxCut objective function on $\widehat{\Gamma}_v$ to be decomposed into two independent contributions: the original objective function and a constant term arising from the optimal cuts of the auxiliary edges in $\widehat{\Gamma}_v \setminus \Gamma$. Because the auxiliary components do not impose additional constraints on the states of the original vertices, every solution on Γ extends to a unique solution on $\widehat{\Gamma}_v$. Conversely, any MaxCut solution on $\widehat{\Gamma}_v$ reduces to a solution on Γ by restriction. This ensures a natural one-to-one correspondence between the set of MaxCut solutions on Γ and those on $\widehat{\Gamma}_v$, which completes the proof. $\square$

We turn now to the formal verification of Theorem, which addresses pure-state controllability on the reduced Hilbert space for the extension graphs introduced above.

Theorem I.9. If the reduction of the extension graph $\widehat{\Gamma}_v$ at vertex v is connected and is not a path, a star, or an even cycle, then the reduced Quantum Approximate Optimization Algorithm, QAOA $_v$, for the MaxCut problem on the extended graph $\widehat{\Gamma}_v$ exhibits pure-state controllability.

Proof. This assertion is a direct consequence of the equality between the standard and free reduced DLAs associated with the extension graph $\widehat{\Gamma}_v$ and vertex v , which was established in Theorem II.5. Specifically, when the graph obtained by reducing the extension graph at v does not belong to the list of exceptions appearing in Theorem II.4, it implies that the dynamical Lie group associated with the restricted DLA $\mathfrak{g}_{\widehat{\Gamma}_v, \text{std}}^v \cong \mathfrak{su}(\widehat{W}_v)$ is the full special unitary group acting on the reduced Hilbert space, i.e.,

$$G := \exp(\mathfrak{g}_{\widehat{\Gamma}_v, \text{std}}^v) \cong SU(\widehat{W}_v). \quad (17)$$

Because the unitary group $SU(\widehat{W}_v)$ acts transitively on the projectivization of $\widehat{W}_v$, there must exist a group element $g \in G$ that maps the initial state $|h\rangle$ to the target state $|t\rangle$ up to a global phase.

Furthermore, since G is a compact and connected Lie group, standard results in quantum control theory [4, Corollary 3.2.6] guarantee that any such element $g \in G$ can be realized via a finite-depth alternating product of the exponentials of the DLA's generators. This yields the desired parameterized quantum circuit $U(\beta, \gamma)$ satisfying

$$U(\beta, \gamma) |h\rangle = e^{i\phi} |t\rangle. \quad (18)$$

□

II. PARITY-DEGREE SEPARATION SEQUENCES

We introduce a refinement of distance layers based on parity patterns of vertex degrees along shortest paths to the fixed node. These subsets will allow us to identify structured elements in the reduced dynamical Lie algebras.

Definition II.1. Let v be a vertex in a graph Γ and let $\mathbf{a} := (a_1, a_2, \dots, a_j) \in \mathbb{Z}_2^j$ be a binary sequence. We denote by $\mathcal{C}_{j, \mathbf{a}}^v \subseteq \mathcal{N}_{v, j}$ the subset of vertices such that for each vertex $w \in \mathcal{C}_{j, \mathbf{a}}^v$, there exists a path

$$w_0 = v - w_1 - \dots - w_j = w$$

such that $\deg(w_s) \equiv a_s \pmod{2}$ for each $1 \leq s \leq j$.

Theorem II.2. Let Γ be a connected graph and let $v \in V(\Gamma)$ be a vertex. Assume the following:

- (i) For every $j \geq 1$ and every $w \in \mathcal{N}_{v, j}$,

$$|\{u \in \mathcal{N}_{v, j-1} : (u, w) \in E\}| \equiv 1 \pmod{2}.$$

- (ii) For every $j \geq 1$, every $\mathbf{a} \in \mathbb{Z}_2^j$, and every $w \in \mathcal{C}_{j, \mathbf{a}}^v$,

$$|\{u \in \mathcal{C}_{j-1, \mathbf{a}'}^v : (u, w) \in E\}| \equiv 1 \pmod{2},$$

where $\mathbf{a}' = (a_1, \dots, a_{j-1})$.

Then the standard reduced dynamical Lie algebra $\mathfrak{g}_{\Gamma, \text{std}}^v$ contains the elements

$$\mathcal{X}_{\mathcal{C}_{j, \mathbf{a}}^v}^v := i \sum_{w \in \mathcal{C}_{j, \mathbf{a}}^v} X_w \quad (19)$$

for every subset $\mathcal{C}_{j, \mathbf{a}}^v$.

Proof. We argue by induction on j , beginning with the base case $j = 1$.

Recall that the containment $\mathcal{X}_{\hat{v},1} \in \mathfrak{g}_{\Gamma,\text{std}}^v$ was established in Theorem IV.2. Next, proceeding as in the proof of Theorem I.4, we obtain

$$\begin{aligned}\mathcal{X}_{\mathcal{C}_{1,0}^v} &= \mathcal{E}_m(\text{ad}_{\mathcal{Z}_{\hat{v},2}}^2)(\mathcal{X}_{\hat{v},1}) = 4i \sum_{\substack{w \in \mathcal{N}_{v,1} \\ \deg(w) \equiv 0 \pmod{2}}} X_w, \\ \mathcal{X}_{\mathcal{C}_{1,1}^v} &= \mathcal{X}_{\hat{v}} - \mathcal{X}_{\mathcal{C}_{1,0}^v},\end{aligned}\tag{20}$$

both of which lie in $\mathfrak{g}_{\Gamma,\text{std}}^v$. This establishes the claim for $j = 1$.

Now assume that for some $n \geq 1$ and for every sequence $\mathbf{a} := (a_1, a_2, \dots, a_j) \in \mathbb{Z}_2^j$ with $j \leq n$, the elements

$$\mathcal{X}_{\mathcal{C}_{n,\mathbf{a}}^v} \in \mathfrak{g}_{\Gamma,\text{std}}^v$$

are contained in the reduced dynamical Lie algebra. We show that the same holds for $j = n + 1$.

Let $\mathbf{a} := (a_1, a_2, \dots, a_{n+1}) \in \mathbb{Z}_2^{n+1}$ and denote by $\mathbf{a}' := (a_1, a_2, \dots, a_n) \in \mathbb{Z}_2^n$ its truncation. By the induction hypothesis, the element $\mathcal{X}_{\mathcal{C}_{n,\mathbf{a}'}^v}$ belongs to $\mathfrak{g}_{\Gamma,\text{std}}^v$. We define

$$\mathcal{X}_{\hat{v},>n} := \mathcal{X}_{\hat{v}} - \sum_{k=1}^n \mathcal{X}_{\hat{v},k},$$

which belongs to $\mathfrak{g}_{\Gamma,\text{std}}^v$ by Theorem IV.2. Using this operator, we introduce the following auxiliary element:

$$\begin{aligned}\mathcal{Z}_{\mathbf{a}'} &:= \frac{1}{16} \cdot \text{ad}_{\mathcal{X}_{\hat{v},>n}}^2 \left(\text{ad}_{\mathcal{X}_{\mathcal{C}_{n,\mathbf{a}'}^v}}^2 (\mathcal{Z}_{\hat{v},2}) \right) \\ &= \frac{1}{16} \cdot \text{ad}_{\mathcal{X}_{\hat{v},>n}}^2 \left(-4i \sum_{\substack{(ww') \in E \\ w \in \mathcal{C}_{n,\mathbf{a}'}^v \\ w' \notin \mathcal{C}_{n,\mathbf{a}'}^v}} Z_w Z_{w'} + 8i \sum_{\substack{(ww') \in E \\ w, w' \in \mathcal{C}_{n,\mathbf{a}'}^v}} (Y_w Y_{w'} - Z_w Z_{w'}) \right) \\ &= i \sum_{\substack{(ww') \in E \\ w \in \mathcal{C}_{n,\mathbf{a}'}^v \\ w' \in \mathcal{X}_{\hat{v},>n}}} Z_w Z_{w'}.\end{aligned}\tag{21}$$

Finally, we obtain

$$\mathcal{X}_{\mathcal{C}_{n+1,\mathbf{a}',1}^v} = \text{ad}_{\mathcal{Z}_{\mathbf{a}'}}^2 (\mathcal{X}_{\hat{v},>n,\text{odd}}) \text{ and } \mathcal{X}_{\mathcal{C}_{n+1,\mathbf{a}',0}^v} = \text{ad}_{\mathcal{Z}_{\mathbf{a}'}}^2 (\mathcal{X}_{\hat{v},>n,\text{even}})\tag{22}$$

This establishes the inductive step and completes the proof. $\square$

Example II.3. Let $m \geq 3$ be an integer. Define the acyclic graph $\mathcal{T}_m$ as follows. Start with the path graph P_m on vertices $\{1, 2, \dots, m\}$. Attach a copy of the path graph P_2 to each vertex $j = 1, \dots, m-3$, and attach a single additional edge to the vertex $m-2$.

Next, let $v - \mathcal{T}_m$ denote the graph obtained from $\mathcal{T}_m$ by adjoining an auxiliary vertex v and connecting it to vertex 1 of $\mathcal{T}_m$. We view $v - \mathcal{T}_m$ as a rooted tree with root v . By construction, v has degree one, and removing the root v (together with its incident edge) recovers the original tree $\mathcal{T}_m$.

The neighborhood layers $\mathcal{N}_{v,j}$ in Γ have the following degree parities:

- $\mathcal{N}_{v,1}$ consists of a single vertex of degree 1;
- $\mathcal{N}_{v,2}$ consists of two vertices of degrees 1 and 0;

- for $3 \leq j \leq m-2$, the set $\mathcal{N}_{v,j}$ consists of three vertices of degrees 1, 0, and 1;
- $\mathcal{N}_{v,m-1}$ consists of three vertices of degrees 0, 1, and 1;
- $\mathcal{N}_{v,m}$ consists of a single vertex of degree 1.

See Figure 5 for the case $\mathcal{T}_5$, where all vertex degrees are indicated.

It follows that no two vertices lie in the same equivalence class, and hence the induced partition of the vertex set $V(\Gamma)$ consists entirely of singleton sets. By Theorem II.5, the standard reduced dynamical Lie algebra $\mathfrak{g}_{v-\mathcal{T}_m, \text{std}}^v$ contains the Pauli operator iX_w for every vertex w of the reduced graph $\mathcal{T}_m$. Consequently, we obtain the inclusion

$$\mathfrak{g}_{\mathcal{T}_m, \text{free}} \subseteq \mathfrak{g}_{v-\mathcal{T}_m, \text{std}}^v. \quad (23)$$

It is straightforward to verify that when $m = 2k$ is even, the tree $\mathcal{T}_m$ is an even-odd bipartite graph. Hence, by Theorem 1 of [5], the free DLA associated with $\mathcal{T}_m$ for even m has exponential dimension in n , the total number of vertices of $\mathcal{T}_m$. A direct count gives

$$n = 3(m-3) + 2 + 1 + 1 = 3m - 5. \quad (24)$$

Therefore, the family of graphs $\{v - \mathcal{T}_m\}_{m \geq 3, m \text{ even}}$ provides an explicit class of acyclic graphs for which the dimension of the standard reduced DLA grows exponentially in the number of vertices.

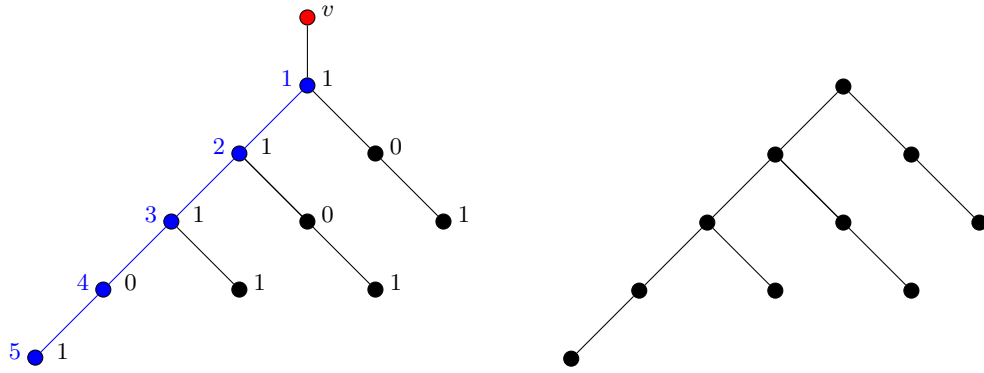


FIG. 5. The graph $v - \mathcal{T}_5$, constructed from the path graph P_5 (shown in blue). Each vertex is labeled by its index (left) and the parity of its degree (right). Also shown is the reduced graph $\mathcal{T}_5$, obtained by removing the vertex v together with its incident edge.

Definition II.4. Let Γ be a connected graph and $v \in V(\Gamma)$ be a vertex. For integers $j \geq 0$ and $\ell \geq 1$, and for a binary sequence $\mathbf{a} = (a_1, \dots, a_\ell) \in \mathbb{Z}_2^\ell$, define $\mathcal{C}_{j,\ell,\mathbf{a}}^v \subseteq \mathcal{N}_{v,j}$ as follows:

A vertex $w \in \mathcal{N}_{v,j}$ belongs to $\mathcal{C}_{j,\ell,\mathbf{a}}^v$ if and only if there exists a vertex $w' \in \mathcal{N}_{v,j+\ell}$ and a path

$$w = w_0 - w_1 - \dots - w_\ell = w'$$

such that

$$\deg(w_s) \equiv a_s \pmod{2} \quad \text{for all } s = 1, \dots, \ell.$$

Theorem II.5. Let Γ be a connected graph and let $v \in V(\Gamma)$ be a vertex. Assume the following:

- (i) For every $j \geq 1$ and every $w \in \mathcal{N}_{v,j}$,

$$|\{u \in \mathcal{N}_{v,j-1} : (u, w) \in E\}| \equiv 1 \pmod{2}.$$

- (ii) For every $j \geq 1$, every $\mathbf{a} \in \mathbb{Z}_2^j$, and every $w \in \mathcal{C}_{j,\mathbf{a}}^v$,

$$|\{u \in \mathcal{C}_{j-1,\mathbf{a}'}^v : (u, w) \in E\}| \equiv 1 \pmod{2},$$

where $\mathbf{a}' = (a_1, \dots, a_{j-1})$.

- (iii) For every $j \geq 1$ and for every pair of distinct vertices $w \neq w' \in \mathcal{N}_{v,j}$ there exists a binary sequence $\mathbf{a} \in \mathbb{Z}_2^j$ such that

$$w \in \mathcal{C}_{j,\mathbf{a}}^v \quad \text{and} \quad w' \notin \mathcal{C}_{j,\mathbf{a}}^v.$$

Equivalently, no two vertices in $\mathcal{N}_{v,j}$ share identical parity profiles along all shortest paths from v .

Then for every vertex $w \in \mathcal{N}_{v,j}$, the corresponding single-site Pauli generator lies in the standard reduced dynamical Lie algebra:

$$iX_w \in \mathfrak{g}_{\Gamma,\text{std}}^v, \quad \forall w \in \mathcal{N}_{v,j}. \quad (25)$$

If the above condition holds for all $j \geq 1$, then the standard and free reduced dynamical Lie algebras coincide, namely

$$\mathfrak{g}_{\Gamma,\text{std}}^v = \mathfrak{g}_{\Gamma,\text{free}}^v. \quad (26)$$

In order to establish the statement in Theorem II.5, we begin with a technical lemma that allows us to isolate elements supported on intersections of parity-defined subsets.

Lemma II.6. Let $j \geq 1$ and let $\mathbf{a} \neq \mathbf{b}$ be binary sequences in $\mathbb{Z}_2^j$. Let

$$\mathcal{S}_{\mathbf{a}} \subseteq \mathcal{C}, \quad \mathcal{S}_{\mathbf{b}} \subseteq \mathcal{C}', \quad \mathcal{S}_{\mathbf{a}} \cap \mathcal{S}_{\mathbf{b}} \neq \emptyset,$$

where $(\mathcal{C}, \mathcal{C}')$ denotes $(\mathcal{C}_{j,\mathbf{a}}^v, \mathcal{C}_{j,\mathbf{b}}^v)$. Assume that

$$i \sum_{w \in \mathcal{S}_{\mathbf{a}}} X_w, \quad i \sum_{w \in \mathcal{S}_{\mathbf{b}}} X_w \in \mathfrak{g}_{\Gamma,\text{std}}^v.$$

Then

$$i \sum_{w \in \mathcal{S}_{\mathbf{a}} \cap \mathcal{S}_{\mathbf{b}}} X_w \in \mathfrak{g}_{\Gamma,\text{std}}^v. \quad (27)$$

Proof. We prove the statement in the case $\mathcal{C} = \mathcal{C}_{j,\mathbf{a}}^v$ and $\mathcal{C}' = \mathcal{C}_{j,\mathbf{b}}^v$; the argument for $\mathcal{C}_{j,\ell,\mathbf{a}}^v$ and $\mathcal{C}_{j,\ell,\mathbf{b}}^v$ is analogous. For $j = 1$, the sequences $\mathbf{a}$ and $\mathbf{b}$ are single binary digits. Since $\mathbf{a} \neq \mathbf{b}$, the sets $\mathcal{C}_{1,\mathbf{a}}^v$ and $\mathcal{C}_{1,\mathbf{b}}^v$ are disjoint, hence

$$\mathcal{S}_{\mathbf{a}} \cap \mathcal{S}_{\mathbf{b}} = \emptyset,$$

and the claim is vacuous.

Assume now that $j \geq 2$. We distinguish two cases.

Case 1. The truncated sequences

$$\mathbf{a}' = (a_1, \dots, a_{j-1}) \quad \text{and} \quad \mathbf{b}' = (b_1, \dots, b_{j-1})$$

are distinct. Define

$$\mathcal{Z}_{\mathbf{a}'} := -0.25 \operatorname{ad}_{\mathcal{X}_{\mathcal{C}_{j,\mathbf{a}'}^v}^2}^2(\mathcal{Z}_{\widehat{v},2}).$$

Then the element $\sum_{w \in \mathcal{S}_{\mathbf{a}} \cap \mathcal{S}_{\mathbf{b}}} X_w$ can be obtained via the identity

$$\sum_{w \in \mathcal{S}_{\mathbf{a}} \cap \mathcal{S}_{\mathbf{b}}} X_w = \mathcal{O}_m(\operatorname{ad}_{\mathcal{Z}_{\mathbf{a}'}}^2) \left(\sum_{w \in \mathcal{S}_{\mathbf{b}}} X_w \right), \quad (28)$$

and, therefore, lies in $\mathfrak{g}_{\Gamma,\text{std}}^v$.

Case 2. The truncated sequences coincide:

$$(a_1, \dots, a_{j-1}) = (b_1, \dots, b_{j-1}),$$

but $a_j \not\equiv b_j \pmod{2}$. Without loss of generality, assume $a_j \equiv 1 \pmod{2}$. Then we compute

$$\begin{aligned} & \mathcal{O}_m(\text{ad}_{\mathcal{Z}_{\hat{v},2}}^2) \left(\sum_{w \in \mathcal{S}_{\mathbf{a}}} X_w + \sum_{w \in \mathcal{S}_{\mathbf{b}}} X_w \right) - \sum_{w \in \mathcal{S}_{\mathbf{b}}} X_w \\ &= \sum_{w \in \mathcal{S}_{\mathbf{b}}} X_w + 2 \sum_{w \in \mathcal{S}_{\mathbf{a}} \cap \mathcal{S}_{\mathbf{b}}} X_w - \sum_{w \in \mathcal{S}_{\mathbf{b}}} X_w \\ &= \sum_{w \in \mathcal{S}_{\mathbf{a}} \cap \mathcal{S}_{\mathbf{b}}} X_w, \end{aligned} \tag{29}$$

which again belongs to $\mathfrak{g}_{\Gamma, \text{std}}^v$. $\square$

Proof of Theorem II.5. Consider the distance shell $\mathcal{N}_{v,j}$. By Theorem II.2, for every binary sequence $\mathbf{a} \in \mathbb{Z}_2^j$, the Lie algebra $\mathfrak{g}_{\Gamma, \text{std}}^v$ contains the element

$$\mathcal{X}_{\mathcal{C}_{j,\mathbf{a}}^v}^v = i \sum_{w \in \mathcal{C}_{j,\mathbf{a}}^v} X_w.$$

Let $\ell := |\mathcal{N}_{v,j}|$, and enumerate the vertices of $\mathcal{N}_{v,j}$ as $w_1, \dots, w_\ell$. If $\mathbf{a} \in \mathbb{Z}_2^j$ is such that $w' \in \mathcal{C}_{j,\mathbf{a}}^v$ while $w'' \notin \mathcal{C}_{j,\mathbf{a}}^v$, then the element

$$X_{\hat{v},j} - \mathcal{X}_{\mathcal{C}_{j,\mathbf{a}}^v}^v$$

belongs to $\mathfrak{g}_{\Gamma, \text{std}}^v$ (by Theorems IV.2 and II.2) and contains $X_{w''}$ but not $X_{w'}$. Thus, for each parity class $\mathbf{a}$, we have access to uniform X -operators supported both on $\mathcal{C}_{j,\mathbf{a}}^v$ and on its complement within $\mathcal{N}_{v,j}$.

For $t \in \{2, \dots, \ell\}$, let $\mathcal{C}_{j,t}^v \subseteq \mathcal{N}_{v,j}$ be the subset such that $\mathcal{C}_{j,t}^v$ contains w_1 but not w_t . Such subsets can be chosen either directly from the parity classes appearing in the statement of the theorem or from their complements in $\mathcal{N}_{v,j}$.

Applying Lemma II.6 iteratively to the intersections

$$\mathcal{C}_{j,2}^v \cap \mathcal{C}_{j,3}^v, \quad \mathcal{C}_{j,2}^v \cap \mathcal{C}_{j,3}^v \cap \mathcal{C}_{j,4}^v, \quad \dots,$$

and observing that

$$\mathcal{C}_{j,2}^v \cap \dots \cap \mathcal{C}_{j,\ell}^v = \{w_1\},$$

we conclude that the element iX_{w_1} belongs to $\mathfrak{g}_{\Gamma, \text{std}}^v$. The same argument applies to each $w_t \in \mathcal{N}_{v,j}$, yielding the containment in (25). $\square$

Remark II.7. The proofs of Theorems IV.2, II.2, and II.5 are local in nature, relying only on the presence of a single-site Pauli- X generator at the distinguished vertex. Consequently, all containment results remain valid with v replaced by any vertex $w \neq v$ such that $iX_w \in \mathfrak{g}_{\Gamma, \text{std}}^v$ and the parity condition (9), centered at w , is satisfied.

Observation 1. The constructions in Theorem II.2 and Remark II.7 can be combined to systematically isolate individual vertices within a fixed distance layer and thereby establish the containment of all single-site Pauli- X operators in the standard reduced dynamical Lie algebra. This is illustrated in Example II.8

Example II.8. In order to show the practical importance of the proposed theorems and how they give the distinguished Pauli- X gates in the reduced DLAs, we give a full analysis of the graph Γ' depicted in Figure 6. First, we give the full collection of subsets

$$\mathcal{N}_{v,j} := \{w \in V(\Gamma) : \text{dist}(v, w) = j\}. \tag{30}$$

:

$$\begin{aligned}
\mathcal{N}_{v,1} &= \{v_{1,1}\} \\
\mathcal{N}_{v,2} &= \{v_{2,1}, v_{2,2}\} \\
\mathcal{N}_{v,3} &= \{v_{3,1}, v_{3,2}, v_{3,3}, v_{3,4}\}; \\
\mathcal{N}_{v,4} &= \{v_{4,1}, v_{4,2}, v_{4,3}, v_{4,4}\}; \\
\mathcal{N}_{v,5} &= \{v_{5,1}\}.
\end{aligned}$$

Next, we list out the subsets of vertices $\mathcal{C}_{j,\mathbf{a}}^v$ according to Definition II.1 for the graph under consideration:

$$\begin{aligned}
\mathcal{C}_{1,\mathbf{a}=1}^v &= \{v_{1,1}\}; \\
\mathcal{C}_{2,\mathbf{a}=11}^v &= \{v_{2,1}, v_{2,2}\}; \\
\mathcal{C}_{3,\mathbf{a}=110}^v &= \{v_{3,2}, v_{3,4}\}; \\
\mathcal{C}_{3,\mathbf{a}=111}^v &= \{v_{3,1}, v_{3,3}\}; \\
\mathcal{C}_{4,\mathbf{a}=1101}^v &= \{v_{4,3}, v_{4,4}\}; \\
\mathcal{C}_{4,\mathbf{a}=1110}^v &= \{v_{4,1}\}; \\
\mathcal{C}_{4,\mathbf{a}=1111}^v &= \{v_{4,2}\}; \\
\mathcal{C}_{5,\mathbf{a}=11101}^v &= \{v_{5,1}\}.
\end{aligned}$$

For the four remaining subsets $\mathcal{C}_{2,\mathbf{a}=11}^v, \mathcal{C}_{3,\mathbf{a}=110}^v, \mathcal{C}_{3,\mathbf{a}=111}^v$ and $\mathcal{C}_{4,\mathbf{a}=1101}^v$ of cardinality 2, we can use the observation from Remark II.7 by choosing node $v_{5,1}$ as the marked node for the reduced settings, to partition the pair of vertices according to the respective distances from $v_{5,1}$.

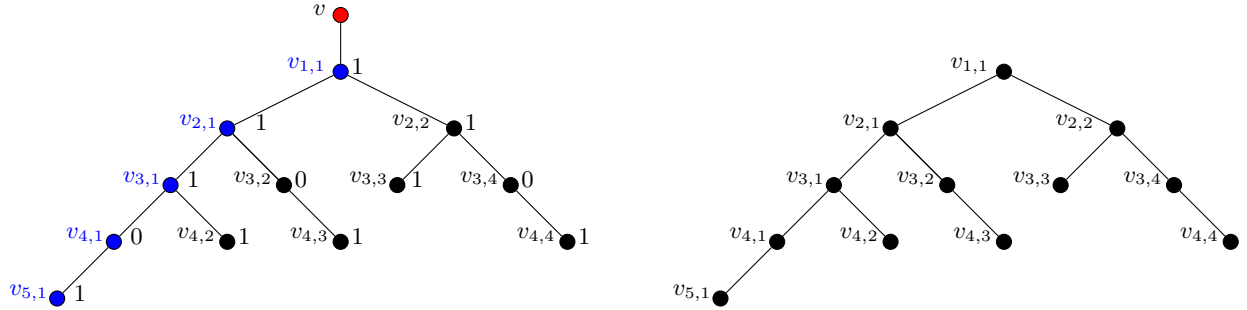


FIG. 6. The graph Γ' , where each vertex is labeled by its index (left) and by the parity of its degree (right), and the reduced graph Γ_v , obtained by removing the vertex v together with all edges incident to it.

We conclude this section by establishing Theorem II.4, which provides sufficient conditions for the free reduced dynamical Lie algebra to coincide with the full special unitary algebra on the reduced Hilbert space.

Proof of Theorem II.4. We first consider the case where the reduced graph Γ_v is connected, non-bipartite, and not an odd cycle (the latter case will be addressed separately). Then Γ_v satisfies the hypotheses of Theorem 1 of [5], which implies that the associated free DLA, $\mathfrak{g}_{\Gamma_v, \text{free}}$, is isomorphic to

$$\mathfrak{su}(2^{n-2}) \oplus \mathfrak{su}(2^{n-2}),$$

acting irreducibly on the two parity sectors $W_{\text{even}}, W_{\text{odd}} \subset W_v$, the subspaces spanned by the Hadamard basis states $\{|\pm\rangle^{\otimes n-1}\}$ containing an even (respectively, odd) number of $|-\rangle$ tensor factors.

Number the qubits corresponding to the vertices of Γ so that the distinguished vertex v corresponds to the n^{th} (last) qubit. With this convention, the above Lie algebra can be characterized as the centralizer in $\mathfrak{su}(W_v) = \mathfrak{su}(2^{n-1})$ of the global parity operator

$$\tau := X_1 X_2 \cdots X_{n-1}.$$

A convenient basis description of $\mathfrak{g}_{\Gamma_v, \text{free}}$ is obtained in terms of Pauli strings. For a Pauli string $\mathcal{P} = \sigma_1 \otimes \cdots \otimes \sigma_{n-1}$ with $\sigma_j \in \{I, X, Y, Z\}$, one verifies directly that

$$\tau \mathcal{P} \tau = (-1)^{k(\mathcal{P})} \mathcal{P},$$

where $k(\mathcal{P})$ is the total number of factors equal to Y or Z . Hence $\mathcal{P}$ commutes with τ if and only if $k(\mathcal{P})$ is even. Therefore the reduced algebra is spanned by

$$\{i\mathcal{P} \mid \mathcal{P} \neq I, \tau; k(\mathcal{P}) \equiv 0 \pmod{2}\}, \quad (31)$$

that is, by all traceless Pauli strings containing an even total number of Y and Z factors, except for identity and τ itself (see Table V in [5]).

By definition of $\mathfrak{g}_{\Gamma, \text{free}}^v$, it contains iZ_w for all $w \in \mathcal{N}_{v,1}$. Let $w' \in V(\Gamma_v) \setminus \mathcal{N}_{v,1}$ be any vertex not adjacent to v in Γ , and choose $w \in \mathcal{N}_{v,1}$.

Observe that the Pauli strings $X_w X_{w'}$ and $Y_w Y_{w'}$ satisfy (31). Hence

$$iX_w X_{w'}, \quad iY_w Y_{w'} \in \mathfrak{g}_{\Gamma, \text{free}}^v.$$

Consequently, the element

$$[iX_w X_{w'}, [iZ_w, iY_w Y_{w'}]]$$

is proportional to $iZ_{w'}$. Thus we confirm that $iZ_{w'} \in \mathfrak{g}_{\Gamma, \text{free}}^v$ for every $w' \in V(\Gamma_v)$. Since, by definition, all $iX_{w'}$ also belong to $\mathfrak{g}_{\Gamma, \text{free}}^v$, we obtain the full single-site algebra

$$\text{span}\{iX_{w'}, iY_{w'}, iZ_{w'}\} \cong \mathfrak{su}(2)$$

at each vertex $w' \in V(\Gamma_v)$.

We now prove that the enlarged algebra equals $\mathfrak{su}(2^{n-1})$. Let $\mathcal{Q} \notin \{I, \tau\}$ be an arbitrary Pauli string.

Case 1: $k(\mathcal{Q})$ is even.

Then $i\mathcal{Q}$ already belongs to the free DLA of the reduced subgraph by (31).

Case 2: $k(\mathcal{Q})$ is odd.

Then there exists a site w' such that $\mathcal{Q}_{w'} \in \{Y, Z\}$. Define a Pauli string $\mathcal{Q}'$ by

$$\mathcal{Q}'_w = \begin{cases} X, & w = w', \\ \mathcal{Q}_w, & \text{otherwise,} \end{cases}$$

that is, $\mathcal{Q}'$ coincides with $\mathcal{Q}$ at every site except w' , where the factor is replaced by X . Then $k(\mathcal{Q}')$ is even, and therefore

$$i\mathcal{Q}' \in \mathfrak{g}_{\Gamma, \text{free}}^v.$$

Since $\mathcal{Q}$ and $\mathcal{Q}'$ differ only at the site w' , and since we have the full local algebra $\mathfrak{su}(2)$ available at w' , suitable commutators with

$$iX_{w'}, \quad iY_{w'}, \quad iZ_{w'}$$

transform $i\mathcal{Q}'$ into $i\mathcal{Q}$ (up to a nonzero scalar multiple). Hence $i\mathcal{Q}$ also lies in $\mathfrak{g}_{\Gamma, \text{free}}^v$.

Thus every traceless Pauli string belongs to the free reduced DLA. Since the traceless Pauli strings form a basis of $\mathfrak{su}(2^{n-1})$, we conclude that $\mathfrak{g}_{\Gamma, \text{free}}^v = \mathfrak{su}(2^{n-1})$.

It remains to address the case where Γ_v is a cycle graph with $n = 2k + 1$ vertices ($k \in \mathbb{Z}_{\geq 1}$). We proceed by induction on k .

Base case ($k = 1$). Here Γ_v is a triangle with vertices $\{w_1, w_2, w_3\}$. As at least one vertex (say w_1) is adjacent to v , we have $iZ_{w_1} \in \mathfrak{g}_{\Gamma, \text{free}}^v$. Utilizing the relations:

$$\begin{aligned} iX_{w_1} X_{w_2} Z_{w_3} &\propto [iX_{w_1} Z_{w_2}, iY_{w_2} Z_{w_3}], \\ iZ_{w_1} X_{w_2} &\propto [iX_{w_1} X_{w_2} Z_{w_3}, iY_{w_1} Z_{w_3}], \\ iZ_{w_2} &\propto [iZ_{w_1} X_{w_2}, iZ_{w_1} Y_{w_2}], \end{aligned} \quad (32)$$

we obtain $iZ_{w_2} \in \mathfrak{g}_{\Gamma, \text{free}}^v$, and by symmetry, $iZ_{w_3} \in \mathfrak{g}_{\Gamma, \text{free}}^v$. Generation of all single-site algebras implies

$$\mathfrak{g}_{\Gamma, \text{free}}^v \simeq \mathfrak{su}(8).$$

Induction step. Assume the claim holds for odd cycles of length not exceeding $2k - 1$. Let Γ_v be a cycle of length $2k + 1$. Given $iZ_w \in \mathfrak{g}_{\Gamma, \text{free}}^v$, we compute:

$$\begin{aligned} \mathcal{A}_1 &:= iZ_w X_{w_1} Z_{w_2} \propto [iZ_w Z_{w_1}, iY_{w_1} Z_{w_2}], \\ \mathcal{A}_2 &:= iZ_{w-1} X_w X_{w_1} Z_{w_2} \propto [\mathcal{A}_1, iY_w Z_{w-1}], \\ \mathcal{A}_3 &:= iZ_{w-2} Z_{w-1} X_w X_{w_1} Z_{w_2} \propto [\mathcal{A}_2, iY_{w-1} Z_{w-2}], \\ \mathcal{A}_4 &:= iZ_{w-2} Y_w X_{w_1} Z_{w_2} \propto [\mathcal{A}_3, iZ_{w-1} Z_w], \\ \mathcal{A}_5 &:= iZ_{w-2} Y_w Y_{w_1} \propto [\mathcal{A}_4, iZ_{w_1} Z_{w_2}], \\ &\quad iZ_{w-2} Z_w \propto [\mathcal{A}_5, iX_w Y_{w_1}], \end{aligned}$$

where $\{w_{-2}, w_{-1}, w, w_1, w_2\}$ forms a path in Γ_v . An analogous calculation yields $iZ_w Z_{w_2}$.

By applying the inductive hypothesis to the effective graph obtained by bypassing w_1 and w_{-1} (specifically, connecting w directly to w_2 and w_{-2}), we find that all single-site algebras, except possibly at w_{-1} and w_1 , are contained in $\mathfrak{g}_{\Gamma, \text{free}}^v$. These remaining sites are then recovered via commutators analogous to (32) applied to the local triangles (w, w_1, w_2) and (w, w_{-1}, w_{-2}) . This completes both the induction step and the proof. $\square$

The next result is a direct consequence of Theorem II.4.

Corollary II.9. Let $v \in V(\Gamma)$ be such that the reduced graph Γ_v is connected and is neither bipartite nor a cycle. Assume moreover that the containment of algebras in

$$\mathfrak{g}_{\Gamma, \text{std}}^v \supseteq \mathfrak{g}_{\Gamma_v, \text{free}}, \quad (33)$$

holds. Then the standard reduced dynamical Lie algebra $\mathfrak{g}_{\Gamma, \text{std}}^v$ is equal to the free reduced dynamical Lie algebra $\mathfrak{g}_{\Gamma, \text{free}}^v$ and, hence, is isomorphic to the special unitary Lie algebra on the reduced Hilbert space W_v :

$$\mathfrak{g}_{\Gamma, \text{std}}^v = \mathfrak{g}_{\Gamma, \text{free}}^v. \quad (34)$$

III. PROOFS OF THEOREMS II.10 AND II.13

The proofs of Theorems II.10 and II.13 follow the same general strategy. They rely on the fact that, for a connected graph Γ , the image of the Lie algebra $\mathfrak{g}_{\Gamma, \text{free}}$ inside $\mathfrak{su}(W)$ generates, in its associative envelope, all two-body Pauli operators $Z_w Z_{w'}$ for arbitrary pairs of vertices $w, w' \in V$. In the reduced setting, an analogous statement holds for $\mathfrak{g}_{\Gamma, \text{free}}^v$, whose image generates all single-site operators iZ_w in the associative envelope inside $\mathfrak{su}(W_v)$.

These observations allow us to establish irreducibility of the relevant Hilbert spaces. We present the detailed arguments below.

Proof of Theorem II.10. First, note that it follows from the actions of single-gate Pauli operators in the Hadamard basis: $X(|+\rangle) = |+\rangle$, $X(|-\rangle) = -|-\rangle$, $Z(|+\rangle) = |-\rangle$, and $Z(|-\rangle) = |+\rangle$ that the generators of $\mathfrak{g}_{\Gamma, \text{free}}$ preserve the parity of the number of $|-\rangle$ in the Hadamard basis for a standard basis vector. Consequently, the Lie algebra $\mathfrak{g}_{\Gamma, \text{free}}$, generated by $\{X_i \mid i \in V\}$ and $\{Z_i Z_j \mid (ij) \in E\}$, preserves the subspaces W_{even} and W_{odd} .

Next, we show that both W_{even} and W_{odd} are irreducible representations of $\mathfrak{g}_{\Gamma, \text{free}}$. Let $v \in W_{\text{even}}$ (or $v \in W_{\text{odd}}$) be a nonzero vector. Write

$$v = \sum \alpha_h |h\rangle, \quad (35)$$

where each $|h\rangle$ is a Hadamard basis vector with an even (or odd) number of $|-\rangle$. We claim that $\text{span}(X_1, \dots, X_n)(v) \subseteq \text{span}(\mathfrak{g}_{\Gamma, \text{free}})(v)$ contains a pure state vector in the Hadamard basis.

To prove this, we argue by induction on n , the number of vertices in Γ . For the base case $n = 1$, the statement is trivial since both $W_{\text{even}} = \mathbb{C}|+\rangle$ and $W_{\text{odd}} = \mathbb{C}|-\rangle$ are one-dimensional. For the inductive step, suppose the claim holds for $n - 1$. Consider a vector $v \in W_{\text{even}}$ (or W_{odd}). If v can be decomposed as $v = |-\rangle \otimes v'$ or $v = |+\rangle \otimes v'$, the inductive hypothesis applied to v' ensures the existence of a standard Hadamard basis vector $\hat{v}' \in \text{span}(\{X_2, \dots, X_n\})(v')$, leading to a vector $\tilde{v} = |-\rangle \otimes \hat{v}'$ or $\tilde{v} = |+\rangle \otimes \hat{v}'$ in $\text{span}(\{X_1, \dots, X_n\})(v)$.

In the second case, where v cannot be decomposed as $v = |-\rangle \otimes v'$ or $v = |+\rangle \otimes v'$, there exists a basis vector $|h\rangle$ in the decomposition (35) with $h_1 = |+\rangle$. In this scenario, the vector $v + X_1(v)$ is nonzero and decomposable as $|+\rangle \otimes v'$, reducing the problem to the first case. This establishes that $\text{span}(\{X_1, \dots, X_n\})(v)$ contains a pure Hadamard basis vector.

Finally, to prove irreducibility, it suffices to show that $\mathfrak{g}_{\Gamma, \text{free}}(h) \simeq W_{\text{even}}$ (or $\mathfrak{g}_{\Gamma, \text{free}}(h) \simeq W_{\text{odd}}$) for a Hadamard basis vector h .

Since Γ is connected, for any pair of vertices w, w' there exists a path

$$w = i_0, -i_1 - \dots - i_k = w'$$

connecting w to w' . As $Z_{i_\ell} Z_{i_{\ell+1}}$ lies in the image of $\mathfrak{g}_{\Gamma, \text{free}}$ for each edge $(i_\ell, i_{\ell+1}) \in E$, their product belongs to the associative algebra generated by this image and satisfies

$$Z_{i_0} Z_{i_1} Z_{i_1} Z_{i_2} \dots Z_{i_{k-1}} Z_{i_k} = Z_w Z_{w'}. \quad (36)$$

Finally, the set of operators

$$\{Z_w Z_{w'} \mid (i, j) \in V \times V\}$$

acts transitively on the Hadamard basis of W_{even} (or equivalently, W_{odd}). $\square$

Proof of Theorem II.13. Let $w = \sum \alpha_h |h\rangle$ be a nonzero vector in W_v , expressed in the Hadamard basis. By the same argument as in the proof of Theorem II.10, the subspace

$$\text{span}\{X_i \mid i \neq v\}(w) \subseteq \text{span}(\tilde{\mathfrak{g}}_{\Gamma, \text{free}})(w)$$

contains a pure Hadamard basis vector. It remains to show that the cyclic subspace generated by any single Hadamard basis vector under the action of $\mathfrak{g}_{\Gamma, \text{free}}^v$ contains all the other basis vectors. Equivalently, starting from one such basis vector, we must be able to produce every other basis vector using operators in the associative algebra generated by the elements of $\mathfrak{g}_{\Gamma, \text{free}}^v$.

To achieve this, it suffices to realize operators that swap $|+\rangle$ and $|-\rangle$ at any chosen qubit, while leaving the other qubits unchanged. We will show that such operators are contained in the associative algebra generated by the elements of $\mathfrak{g}_{\Gamma, \text{free}}^v$.

Since Γ is connected, for any vertex w of Γ_v there exists a path

$$v = i_0, i_1, \dots, i_k = w$$

connecting v to w . This allows to obtain the element

$$Z_{i_1} Z_{i_2} \dots Z_{i_{k-1}} Z_{i_k} = Z_j, \quad (37)$$

which provides the necessary single-qubit Z operator at position w inside the associative algebra generated by the image of $\mathfrak{g}_{\Gamma, \text{free}}^v$. $\square$

IV. ORIGINAL AND REDUCED DLAS FOR MAXCUT QAOAS ON CERTAIN GRAPHS

In this section, we compare the standard and reduced dynamical Lie algebras associated with the Quantum Approximate Optimization Algorithm for MaxCut problems on various types of graphs. Our comparison is based on the dimensions of these Lie algebras.

A. Exponential vs. Quadratic Growth: The k -Armed Spider Graphs

We start by presenting a family of graphs for which the dimension of the standard dynamical Lie algebra grows exponentially in the number of vertices, while there exists a distinguished vertex for which the dimension of the corresponding standard *reduced* DLA is only quadratic in the number of vertices. This provides a concrete and explicit separation between the full and reduced settings.

The family under consideration consists of the so-called k -armed spider graphs, denoted $\mathcal{O}_{m_1, \dots, m_k}$, which were studied in [7]. Such a graph is obtained by taking a central vertex v and attaching to it k path graphs $P_{m_1}, \dots, P_{m_k}$, each joined to v at one of its endpoints. In other words, $\mathcal{O}_{m_1, \dots, m_k}$ consists of k disjoint paths whose initial vertices are identified with the central vertex v (see Figure 7).

The reduction of $\mathcal{O}_{m_1, \dots, m_k}$ at the central vertex v removes v and disconnects the graph into a disjoint union of the path graphs $P_{m_1} \sqcup \dots \sqcup P_{m_k}$:

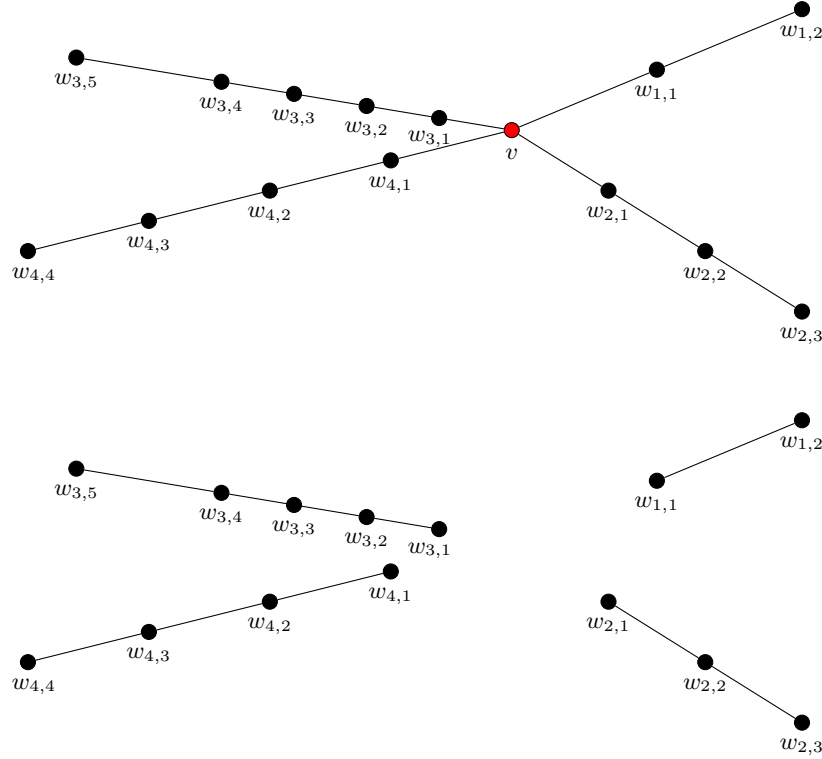


FIG. 7. Top: the 4-armed spider graph $\mathcal{O}_{2,3,4,5}$ with central vertex v and 4 path graphs P_2, P_3, P_4 and P_5 attached. Bottom: the reduced graph $\mathcal{O}_{2,3,4,5}^v$ obtained by removing the central vertex v , resulting in a disjoint union of the 4 path graphs.

Assume now that the lengths $m_1, \dots, m_k$ are pairwise distinct. It was established in [7, Corollary 26] that in this case the standard and free DLAs of the full spider graph coincide, namely

$$\mathfrak{g}_{\mathcal{O}_{m_1, \dots, m_k}, \text{std}} = \mathfrak{g}_{\mathcal{O}_{m_1, \dots, m_k}, \text{free}}.$$

Consequently, Theorem 1 of [5] allows to conclude that $\dim(\mathfrak{g}_{\mathcal{O}_{m_1, \dots, m_k}, \text{std}})$ is exponential in n .

Combining Theorem IV.2 with Remark II.7, one obtains directly the analogous statement for the reduced DLAs at the central vertex v :

$$\mathfrak{g}_{\mathcal{O}_{m_1, \dots, m_k}, \text{std}}^v = \mathfrak{g}_{\mathcal{O}_{m_1, \dots, m_k}, \text{free}}^v = \mathfrak{g}_{P_{m_1}, \text{free}} \oplus \dots \oplus \mathfrak{g}_{P_{m_k}, \text{free}}. \quad (38)$$

Thus, the reduced algebra decomposes as a direct sum of the free DLAs associated with the individual arms.

We next compute its dimension. It is known that the free DLA of the path graph P_n , with an additional single-site Z generator at one of its vertices, is isomorphic to the Lie algebra $\mathfrak{so}(2n+1)$, which has dimension $2n^2 + n$ (see Proposition B3 in [5]). Applying this to each arm yields

$$\mathfrak{g}_{\mathcal{O}_{m_1, \dots, m_k}, \text{std}}^v = \mathfrak{so}(2m_1 + 1) \oplus \dots \oplus \mathfrak{so}(2m_k + 1). \quad (39)$$

Consequently,

$$\dim(\mathfrak{g}_{\mathcal{O}_{m_1, \dots, m_k}, \text{std}}^v) = \sum_{j=1}^k (2m_j^2 + m_j) = 2 \sum_{j=1}^k m_j^2 + \sum_{j=1}^k m_j.$$

Let $n = \sum_{j=1}^k m_j$ denote the total number of non-central vertices. Then

$$\dim(\mathfrak{g}_{\mathcal{O}_{m_1, \dots, m_k}, \text{std}}^v) = 2 \sum_{j=1}^k m_j^2 + n \leq 2 \left(\sum_{j=1}^k m_j \right)^2 + n = 2n^2 + n. \quad (40)$$

This establishes a quadratic upper bound in the system size.

In contrast, for the full (non-reduced) spider graph the standard DLA grows exponentially with the number of vertices, as shown in [7]. The k -armed spider graphs therefore provide an explicit family in which symmetry reduction at a carefully chosen vertex dramatically alters the algebraic complexity: from exponential growth in the full system to quadratic growth in the reduced one.

B. Acyclic Graphs

Let Γ be a connected acyclic graph, and fix a vertex $v \in V(\Gamma)$ at which we perform the reduction. Since Γ contains no cycles, it can be naturally regarded as a rooted tree with root v . In particular, for every vertex $w \in V(\Gamma)$ there exists a unique simple path

$$v = w_0 - w_1 - \dots - w_j = w$$

connecting v to w , where $j = \text{dist}(v, w)$. As a consequence, each vertex $w \in \mathcal{N}_{v,j}$ of the reduced graph Γ_v is associated with a *unique* parity-degree profile

$$\mathbf{a} = (a_1, \dots, a_j) \in \mathbb{Z}_2^j, \quad a_s \equiv \deg(w_s) \pmod{2},$$

formed by recording the parities of the degrees of the intermediate vertices along this path. Equivalently, every vertex $w \in \mathcal{N}_{v,j}$ belongs to exactly one of the subsets $\mathcal{C}_{j,\mathbf{a}}^v$ introduced in Definition II.1. Thus, for acyclic graphs, the collections $\{\mathcal{C}_{j,\mathbf{a}}^v\}_{\mathbf{a} \in \mathbb{Z}_2^j}$ form a partition of each distance layer $\mathcal{N}_{v,j}$.

This observation allows us to formulate a simple and easily verifiable structural condition that is sufficient for the containment of dynamical Lie algebras in (33) to hold in the acyclic setting. The condition requires that leaf vertices at the same distance from v be distinguishable by their parity-degree profiles along the unique path to the root.

Theorem IV.1. Let Γ be a connected acyclic graph and let $v \in V(\Gamma)$. Suppose that for every $j \in \mathbb{Z}_{>0}$, the parity-degree sequences formed by the parities of the degrees of the vertices along the unique paths from v to the *leaf vertices* in $\mathcal{N}_{v,j}$ are pairwise distinct. Then the containment of algebras in (33) holds.

Proof. Let

$$M := \max\{j \geq 0 \mid \mathcal{N}_{v,j} \neq \emptyset\}$$

be the maximal distance from the distinguished vertex v . Since Γ is acyclic, every vertex in $\mathcal{N}_{v,M}$ is a leaf.

By assumption, the parity-degree sequences associated to the unique paths connecting v to distinct leaves in $\mathcal{N}_{v,M}$ are pairwise distinct. Therefore, by Theorem II.2 and Theorem II.5, we obtain

$$iX_w \in \mathfrak{g}_{\Gamma, \text{std}}^v \quad \text{for all } w \in \mathcal{N}_{v,M}.$$

Each vertex $u \in \mathcal{N}_{v,M-1}$ is adjacent to at least one leaf $w \in \mathcal{N}_{v,M}$. Applying Remark II.7, we conclude that the presence of iX_w in the reduced dynamical Lie algebra implies

$$iX_u \in \mathfrak{g}_{\Gamma,\text{std}}^v \quad \text{for all } u \in \mathcal{N}_{v,M-1}.$$

Iterating this argument along the unique paths connecting the leaves in $\mathcal{N}_{v,M}$ to the root v , we deduce that

$$iX_u \in \mathfrak{g}_{\Gamma,\text{std}}^v$$

for every vertex u lying on any such path.

Next, consider the set of leaves at the next maximal distance

$$M_2 := \max\{j < M \mid \mathcal{N}_{v,j} \text{ contains a leaf}\}.$$

For each leaf $w \in \mathcal{N}_{v,M_2}$, the hypothesis of the theorem again guarantees that its parity-degree profile along the path to v is distinct from those of other leaves in the same layer. Thus, Theorem II.5 implies

$$iX_w \in \mathfrak{g}_{\Gamma,\text{std}}^v \quad \text{for all leaf vertices } w \in \mathcal{N}_{v,M_2}.$$

As before, Remark II.7 allows us to propagate this containment upward along the unique paths from these leaves to v .

Proceeding inductively over decreasing distances from v , and exhausting all leaf layers of Γ , we conclude that

$$iX_u \in \mathfrak{g}_{\Gamma,\text{std}}^v \quad \text{for every vertex } u \in V(\Gamma).$$

This establishes the containment of algebras in (33) and completes the proof. $\square$

Remark IV.2. The parity-degree sequences appearing in Theorem VI.1 can be computed and compared efficiently using a breadth-first search (BFS) rooted at the distinguished vertex v . Since Γ is acyclic, BFS determines the distance layers $\mathcal{N}_{v,j}$ and uniquely fixes, for each vertex, the simple path from v to that vertex.

For each leaf $w \in \mathcal{N}_{v,j}$, the associated parity-degree sequence is obtained by traversing this path and recording the parity of the degrees of the vertices along it. To verify that these sequences are pairwise distinct within each layer $\mathcal{N}_{v,j}$, one may encode each sequence as a binary string and store it in a hash table while traversing the leaves at that distance. A collision in the hash table indicates a violation of the condition in Theorem VI.1.

Because each vertex and edge is visited a constant number of times during the BFS and each parity-degree sequence is processed once, the total time required to construct all sequences and check their pairwise distinctness over all distance layers is $O(|V(\Gamma)|)$ for acyclic graphs.

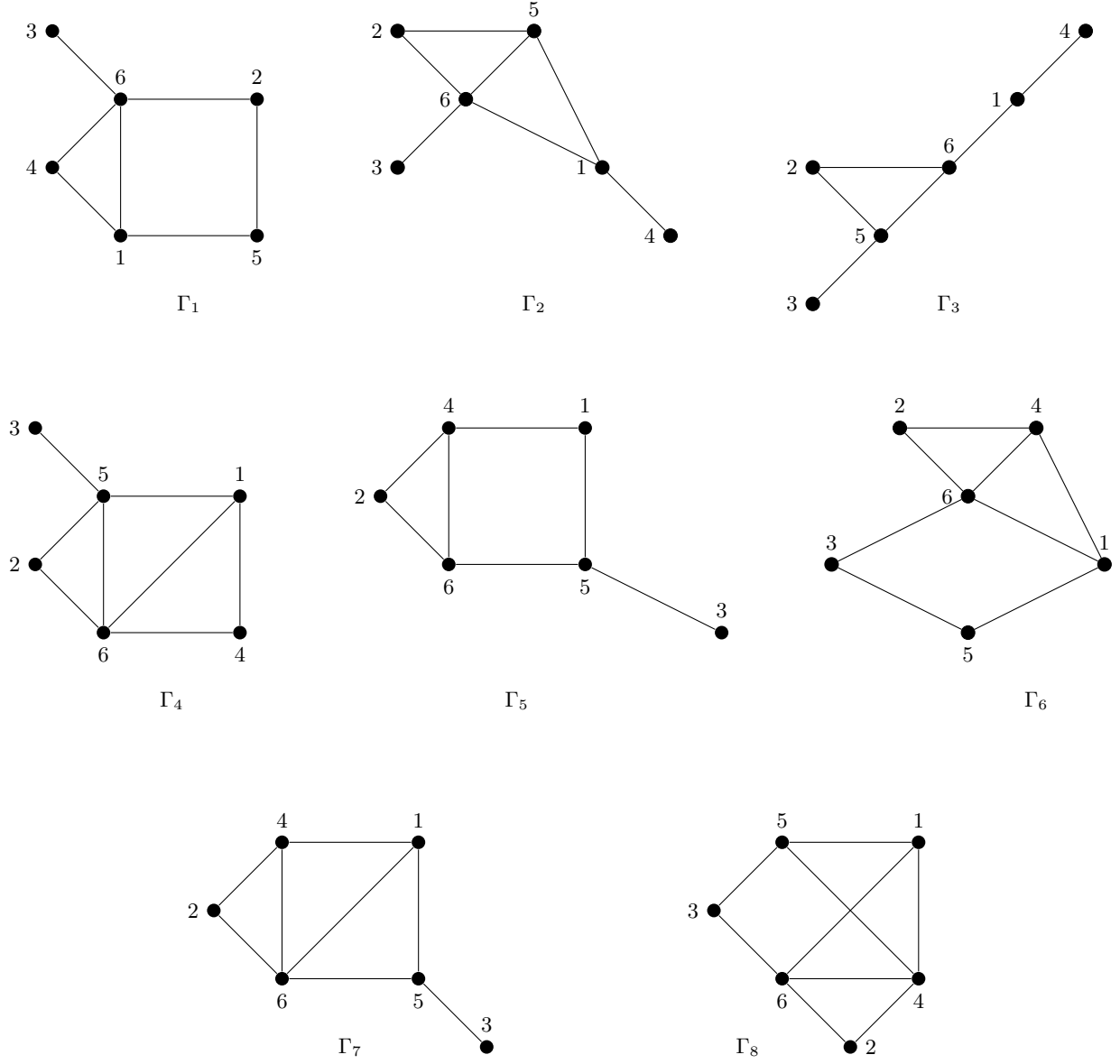
C. Asymmetric Graphs

Next, we shift our focus to asymmetric graphs, i.e., graphs whose automorphism group is trivial. The smallest such graphs have 6 vertices. Up to isomorphism, there are 8 distinct asymmetric graphs on 6 nodes, depicted in Figure 8.

Using the computational framework implemented in [3], we computed the dimensions of the standard DLAs and the standard reduced DLAs corresponding to each vertex of each asymmetric graph on 6 nodes. The results are summarized in Figure IV C.

Remark IV.3. By definition, for an n -node graph Γ and a vertex $v \in V(\Gamma)$, the standard reduced DLA $\mathfrak{g}_{\Gamma,\text{std}}^v$ is a Lie subalgebra of $\mathfrak{su}(2^{n-1}) = \mathfrak{su}(W_v)$, where W_v is a Hilbert space of dimension 2^{n-1} . In particular, for $n = 6$ we have $\dim(\mathfrak{su}(2^5)) = 4^5 - 1 = 1023$. Therefore, any algebra whose dimension entry equals 1023 in Figure IV C necessarily coincides with the full algebra $\mathfrak{su}(32)$ (up to Lie algebra isomorphism).

Similarly, we computed the dimensions of the standard and reduced dynamical Lie algebras for all 144 asymmetric graphs (up to isomorphism) on 7 nodes. The results are presented in Table I. The patterns observed in the 6- and 7-vertex cases suggest a consistent trend in how the dimensions of reduced DLAs behave relative to the full standard DLA, and motivate the following conjecture.



i	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_1})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_2})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_3})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_4})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_5})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_6})$
1	1984	956	1020	1023	1021	991	258
2	1970	258	995	1014	1023	957	258
3	1462	258	862	907	915	258	73
4	1984	1011	1022	1023	1008	258	1023
5	1913	971	979	1023	909	258	890
6	1984	1013	1023	1023	1023	1023	870
7	1984	987	964	1023	1014	258	995
8	1984	1023	1023	1023	1023	1023	1023

FIG. 8. Dimensions of the standard and reduced dynamical Lie algebras for asymmetric graphs on 6 nodes.

Conjecture IV.4. Let Γ be an asymmetric graph. Then there exists a vertex $v \in V(\Gamma)$ such that the dimension of the corresponding standard reduced dynamical Lie algebra is strictly smaller than that of the full standard DLA:

$$\dim \mathfrak{g}_{\Gamma, \text{std}}^v < \dim \mathfrak{g}_{\Gamma, \text{std}}.$$

This conjecture has been verified for all asymmetric graphs on 6 and 7 vertices (see Figure IV C and Table I).

Remark IV.5. Conjecture IV.4 may fail for certain graphs with nontrivial groups of automorphisms. For example, it does not hold for cyclic graphs with 5 or more vertices (see (46)).

i	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_1})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_2})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_3})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_4})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_5})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_6})$	$\dim(\mathfrak{g}_{\Gamma_i, \text{std}}^{v_7})$
1	1808	477	131	1122	1072	1183	521	33
2	4400	1025	1979	1816	2099	2303	843	75
3	> 3260	2045	2694	2775	3203	2844	993	911
4	> 3890	3135	3079	3668	3438	2976	1026	1026
5	> 2953	265	1929	2642	2277	2047	1026	833
6	> 4396	2141	3627	2606	3265	1024	2284	963
7	> 3145	2131	2503	2047	2708	1584	2517	675
8	> 4272	2538	3053	3584	3227	1916	1023	906
9	> 4244	2007	2738	3450	3475	3525	3169	872
10	> 4532	2884	2834	3297	3785	3466	2653	1024
11	2891	1615	1610	265	1855	697	919	84
12	> 4212	2623	3727	3609	3364	2818	2861	1026
13	> 4742	3746	3592	3432	4038	3547	3269	1027
14	> 4422	1819	2506	3032	3698	2562	1027	1891
15	> 4434	1974	2968	2499	3985	2503	1027	1788
16	> 4796	3730	3149	3304	3970	3044	1027	2508
17	4320	2609	2576	3615	3436	2306	1012	977
18	> 4846	2682	3712	2909	3474	1025	2580	1026
19	> 4208	3621	2931	3404	4051	1849	3417	1027
20	> 4716	3391	3408	3855	3397	3421	3123	1027
21	3733	3042	2821	3676	3799	2847	1026	935
22	> 3903	3939	3415	3300	3972	2763	1027	2988
23	> 4735	3612	3831	3044	3840	2738	1027	2336
24	2594	511	532	1366	1417	1500	1469	131
25	> 3766	1923	1027	2091	2237	3098	1756	266
26	> 3017	826	1019	1936	2191	2920	2071	266
27	> 4358	2547	1027	3194	3062	3785	2117	1767
28	> 4222	2095	3467	3715	3843	3167	2277	1809
29	> 4803	2560	1026	3587	2705	3330	1002	2434
30	> 3858	2141	1027	1954	1816	2832	266	2148

TABLE I. Dimensions of the standard and reduced dynamical Lie algebras for asymmetric graphs on 7 nodes. The minimal dimension values are highlighted in red.

1. Variance-Based Proxies for DLA Dimensions of Standard and Reduced QAOA

As it appears infeasible to establish the precise dimensions of standard and reduced dynamical Lie algebras corresponding to QAOAs for MaxCut problems on arbitrary asymmetric graphs with a larger number of

nodes, we instead exploit the connection between the variance of the QAOA loss function and the dimension of the corresponding DLA. Broadly speaking, the variance is inversely proportional to the DLA dimension; thus, larger variance signals a smaller-dimensional DLA.

These results are also of independent interest, since analyzing the variance is essential for understanding the emergence of barren plateaus in the optimization landscape [6, 9]. The phenomenon of *barren plateaus* refers to the exponential suppression of gradients in variational quantum algorithms, which renders classical training ineffective [6, 8]. A key diagnostic of this behavior is the variance of the loss function over parameter space: when this variance is exponentially small in the number of qubits n , the corresponding gradients also vanish exponentially [2].

We now recall this connection more explicitly. The *loss function* in QAOA is defined as the expectation value of the problem Hamiltonian H_P with respect to the parametrized QAOA state:

$$\begin{aligned}\ell_{\beta,\gamma}(\rho, H_P) &:= \langle \psi(\beta, \gamma) | H_P | \psi(\beta, \gamma) \rangle \\ &= \text{Tr}[U(\beta, \gamma) \rho U^\dagger(\beta, \gamma) H_P],\end{aligned}\tag{41}$$

where $\rho = |\xi\rangle\langle\xi|$ is the pure initial state and $U(\beta, \gamma)$ is the parameterized QAOA unitary

$$U(\beta, \gamma) := e^{-i\beta_p H_M} e^{-i\gamma_p H_P} \dots e^{-i\beta_1 H_M} e^{-i\gamma_1 H_P}.\tag{42}$$

Due to Theorems 2 and 3 of [9], for sufficiently deep QAOA circuits (i.e., for large enough value of p), the variance of the loss function over parameters (β, γ) can be approximated by the variance taken over the dynamical Lie group $G = e^{\mathfrak{g}}$. The latter was computed in Theorem 1 of [9], which we now recall.

Let the DLA decompose as

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k \oplus \mathfrak{z},$$

into simple compact Lie algebras $\mathfrak{g}_j$ together with a (possibly trivial) center $\mathfrak{z}$ (this decomposition exists by Proposition A.1 in [10]).

Then the variance of the loss function over the group G is given by

$$\text{Var}_G[\ell_{\beta,\gamma}(\rho, H_P)] = \sum_{j=1}^k \frac{\mathcal{P}_{\mathfrak{g}_j}(\rho) \mathcal{P}_{\mathfrak{g}_j}(H_P)}{\dim(\mathfrak{g}_j)}.\tag{43}$$

In the above formula, $\mathcal{P}_{\mathfrak{g}_j}(H) := \text{Tr}[H_{\mathfrak{g}_j}^2]$ and $H_{\mathfrak{g}_j}$ denotes the orthogonal projection of an operator H onto the subalgebra $\mathfrak{g}_j \subseteq \text{End}(W)$.

Remark IV.6. Two remarks are in order. First, formula (43) depends on the dimensions of the *simple components* $\mathfrak{g}_j$ rather than on the total dimension of the DLA itself. Thus, the relation between the variance and the full DLA structure can be subtle: the variance captures a weighted contribution from each simple summand rather than a single global quantity.

Second, the variance of the loss function provides an indicator of whether a given QAOA instance is susceptible to barren plateaus. Therefore, it can be used as an operational diagnostic of the comparative performance of the standard and reduced QAOA circuits for the same optimization problem.

Below, we present numerical computations of the variance for both standard and reduced QAOA circuits for the MaxCut problem on a variety of connected asymmetric graphs with 7 and 14 vertices. The variance of the gradient is computed as the mean over 400 initializations of the parametrized circuit corresponding to a MaxCut instance, with each initialization assigning randomized variational parameters between 0 and π , with a total of $2p$ parameters, where p is the circuit depth. All gradients were computed for a range of circuit depths using PennyLane's QNode with the *autograd* interface and a *backprop* differentiation method, which resulted in the fastest computation of the gradient.

Figure 9 shows the variance of the gradient for a select number of 7-node asymmetric graphs. Upon comparison to Table I, we see that there is good correspondence between the DLA dimensions and the magnitude of the variance in terms of their inverse relationship. In many cases, there is a specific vertex reduction (shown in red) that simplifies the problem significantly compared with the full (standard) problem DLA dimensions. Furthermore, for any vertex reduction (shown in gray) there is a corresponding increase in the variance of the gradient compared to the original problem, for large enough depths. For a circuit depth

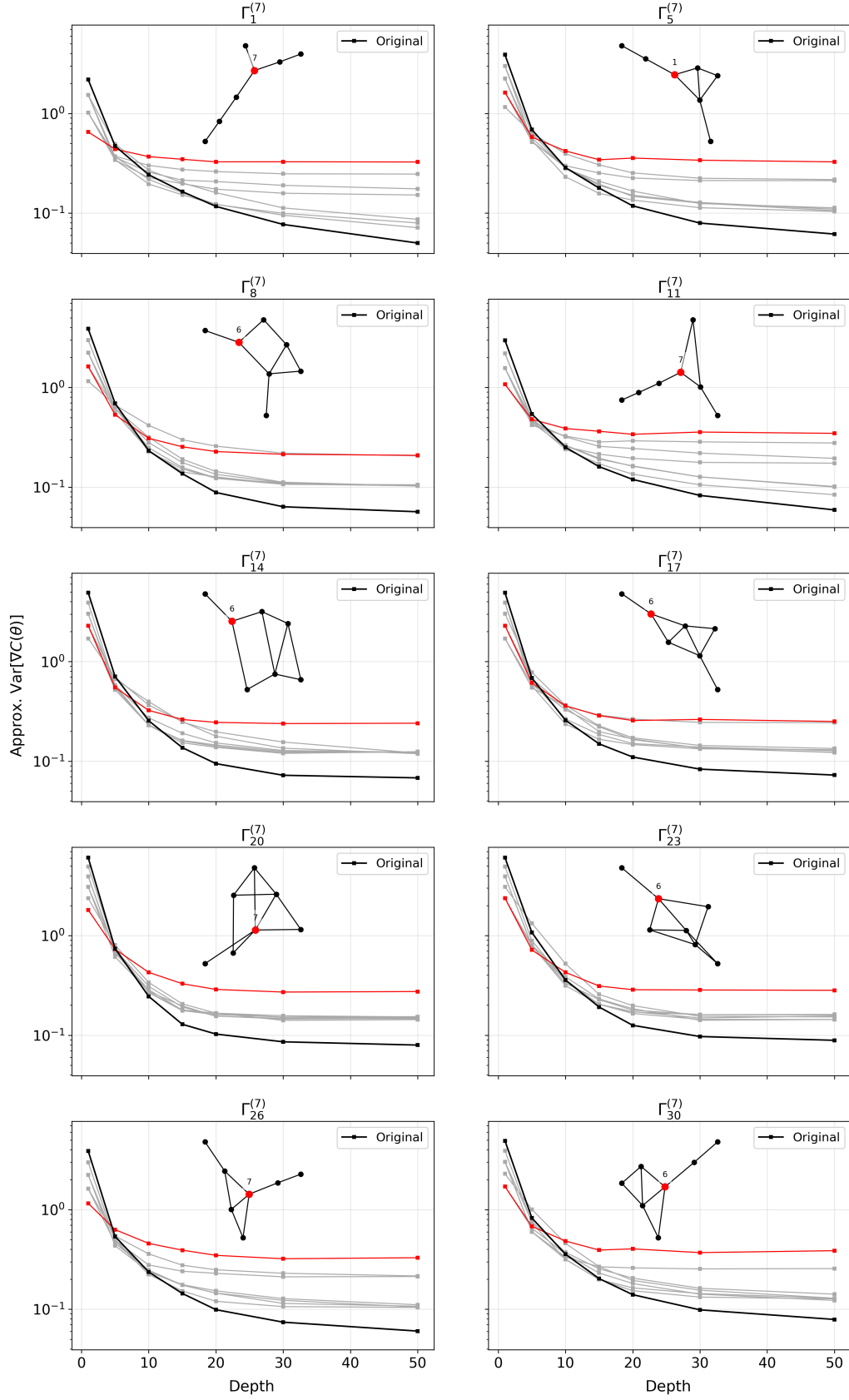


FIG. 9. Approximate variance for selected 7-node asymmetric graphs ($\Gamma^{(7)}$). The red data shows the reduced QAOA instance that results in the largest variance of the gradient at depth 50, with the corresponding vertex highlighted in red and labeled on the graph inset.

of 1, in all cases, we see a larger value for the variance in the standard problem. We also note that around a circuit depth of 5, the variance of the gradient for the original problem and that for the reduced version cross paths. This behavior, while noteworthy, is beyond the scope of this article.

In practice, computing the variance of the gradient allows us to determine a good candidate for vertex reduction, especially for cases where computing the DLA dimensions becomes intractable. To demonstrate this practical application we compute the variance for ten 14-node asymmetric graphs.

Figure 12 shows the mean of the variance of the gradient for increasing circuit depth on ten 14-node graphs. The vertex reduction that results in the largest variance is highlighted in red. Here we notice that as the graphs become denser¹, most vertex reductions result in a small difference in the variance of the gradient with respect to the original problem. Nevertheless, in all cases, any vertex reduction results in a larger variance of the gradient compared with the standard problem without any loss of quality in the solution.

Observation 2. In general, we see that the vertex reduction that results in the largest magnitude (at depth 50) of the variance of the gradient corresponds to a vertex that contains a leaf. This can be evidenced in Figure 10: when a leaf is added artificially to a vertex in the graph, there is an increase in the variance of the gradient compared to the original problem instance (no leaves).

Observation 3. Remarkably, as the variance of the gradient increases with the graph size (black curve in Figure 10) for the original problem instances, there is an improvement in the variance with the artificial addition of a leaf for the reduced case. Furthermore, since a leaf is added to the fixed vertex, the quality of the solution is not jeopardized. Similarly, we see that vertices with similar vertex connectivity result in similar magnitudes of the variance of the gradient upon reduction.

Refer to Appendix D for additional numerical results for graphs ranging from 11–15 nodes. Code availability for all numerical results can be found here: https://github.com/joseluisfalla/QAOA_Reductions_by_Classical_Symmetries.git.

D. Star Graphs

Consider the star graph $K_{1,n}$ consisting of a central vertex, denoted by 0, and n leaf vertices labeled $1, \dots, n$. Each leaf vertex is connected to the central vertex by a single edge, and there are no edges among the leaves. See Figure 11 for an illustration in the case $n = 5$.

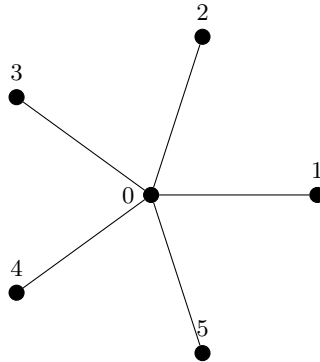


FIG. 11. The star graph $K_{1,5}$ with center vertex 0 and outer vertices $1, \dots, 5$.

The standard dynamical Lie algebra associated to $K_{1,n}$, denoted $\mathfrak{g}_{K_{1,n},\text{std}}$, is generated by the two operators

$$X = iX_0 + iX_1 + \dots + iX_n, \quad Z = iZ_0Z_1 + \dots + iZ_0Z_n.$$

¹ These 14-node graphs were generated using a range of edge-creation probabilities, resulting in asymmetric graphs of increasing edge density.

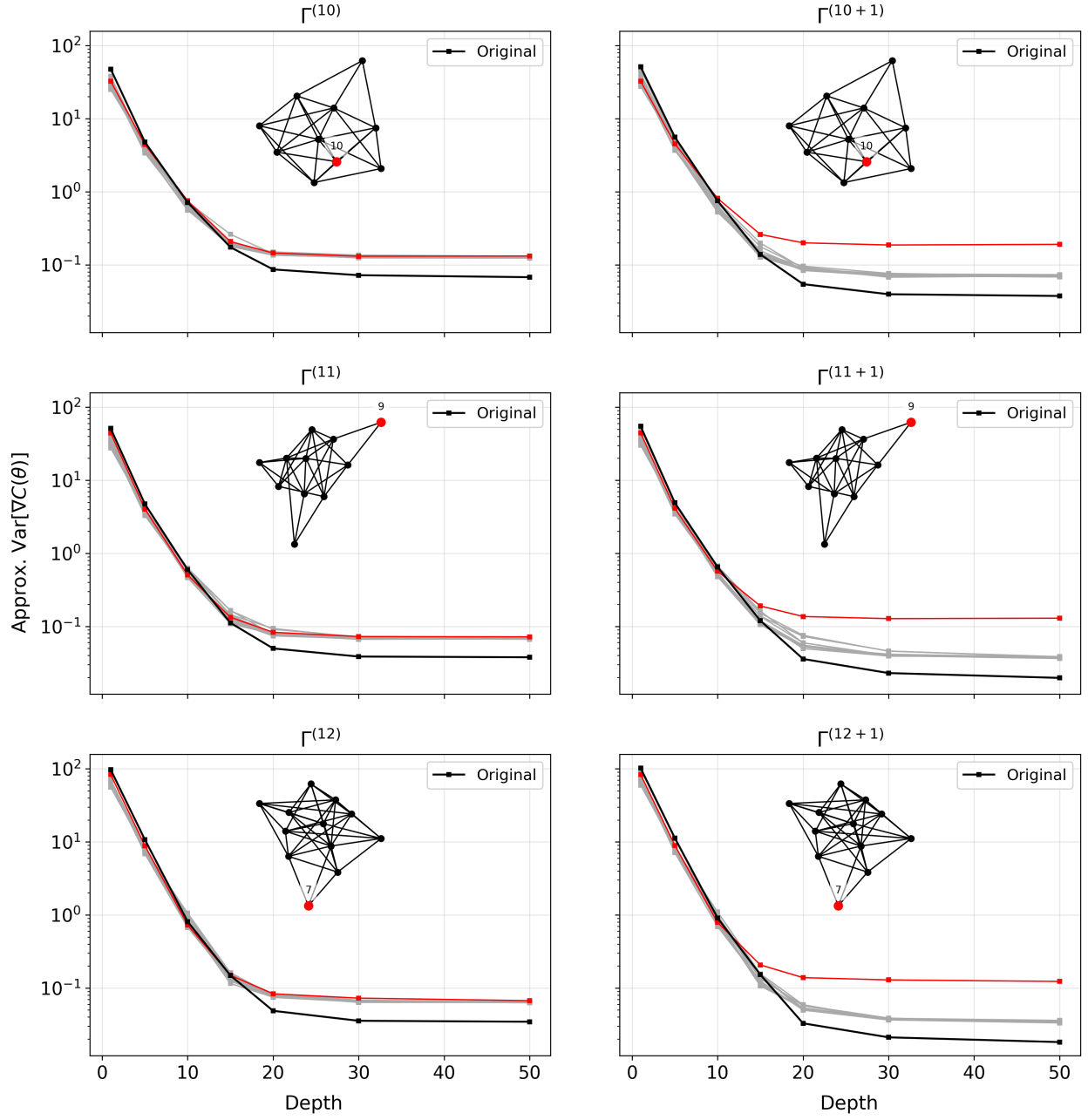


FIG. 10. Effect of artificially introducing a leaf. The graphs on the left column ($\Gamma^{(n)}$) contain no leaves and any vertex reduction results in similar magnitudes in the approximate variance of the gradient. The graphs on the right column ($\Gamma^{(n+1)}$) show an increase in the variance of the gradient when a leaf is artificially introduced to a vertex.

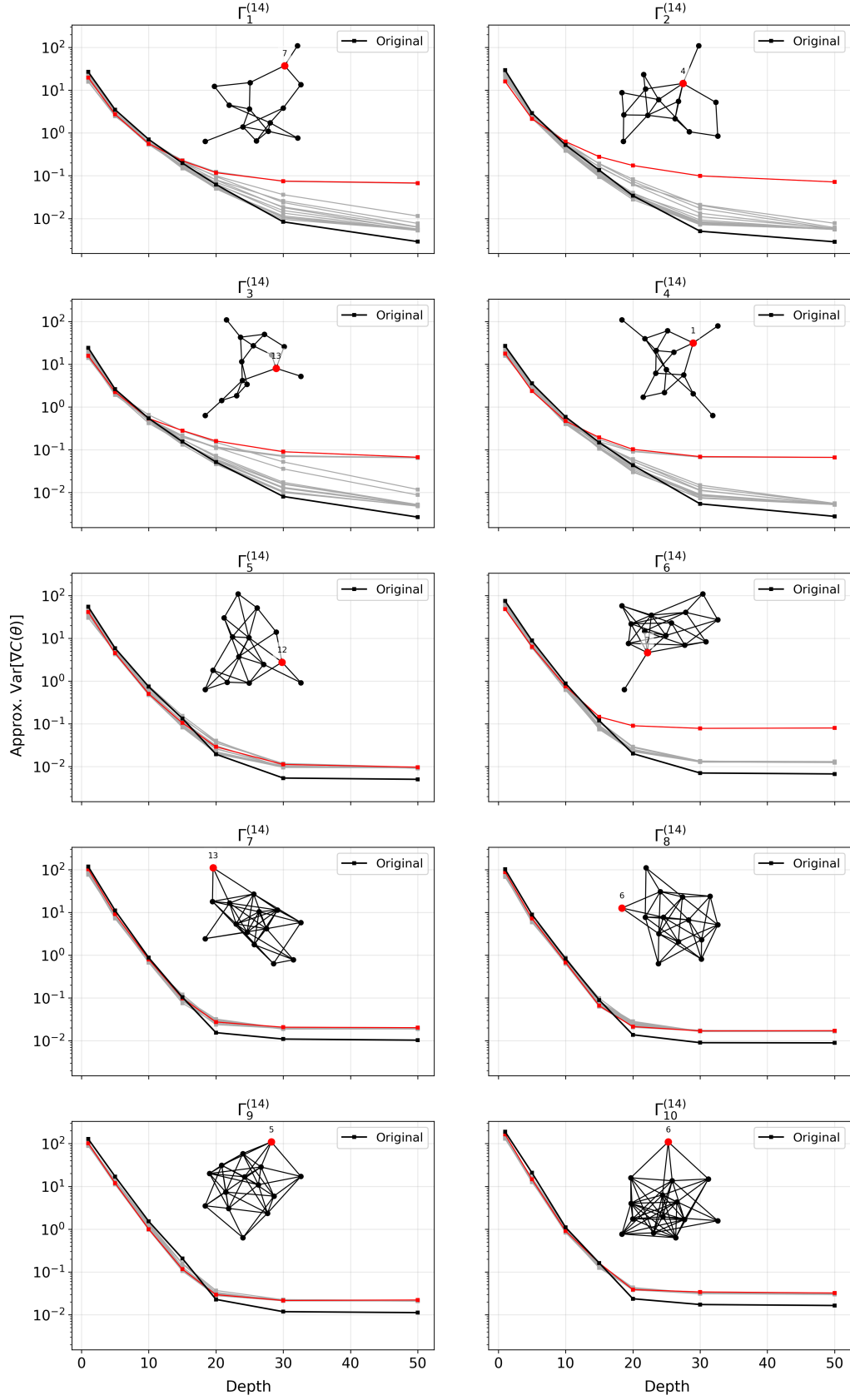


FIG. 12. Approximate variance for ten 14-node asymmetric graphs ($\Gamma^{(14)}$). The red data shows the reduced QAOA instance that results in the largest variance of the gradient at depth 50, with the corresponding vertex highlighted in red and labeled on the graph inset.

Although we did not obtain a closed-form expression for $\dim(\mathfrak{g}_{K_{1,n},\text{std}})$ as a function of n , we computed the dimensions numerically for several values of n . These values are presented in Table II.

n	2	3	4	5	6	7	8	9	10	11
$\dim(\mathfrak{g}_{K_{1,n},\text{std}})$	9	24	33	66	81	138	161	247	281	400

TABLE II. Numerically computed dimensions of the standard dynamical Lie algebra $\mathfrak{g}_{K_{1,n},\text{std}}$ for several values of n .

Remark IV.7. The *orbit Lie algebra* $\mathfrak{g}_{\Gamma,\text{orb}}$ of a graph Γ is defined as the Lie algebra generated by the $\text{Aut}(\Gamma)$ -orbit sums of the single-qubit and two-qubit Pauli operators, namely

$$\left\{ i \sum_{g \in \text{Aut}(\Gamma)} X_{g(w)} \left| w \in V(\Gamma) \right. \right\} \quad \text{and} \quad \left\{ i \sum_{g \in \text{Aut}(\Gamma)} Z_{g(w)} Z_{g(w')} \left| (ww') \in E(\Gamma) \right. \right\}$$

(see Section VI in [5]).

The automorphism group of the star graph $K_{1,n}$ is the group S_n , which acts by permuting the n leaf vertices, while fixing the central vertex. Thus there are two vertex-orbits: the singleton orbit of the center, and the orbit formed by the leaves. Consequently, the corresponding orbit-sum generators are

$$iX_0, \quad i \sum_{j=1}^n X_j,$$

and for the edges (all of which are of the form $(0, i)$), the unique orbit-sum operator

$$i \sum_{j=1}^n Z_0 Z_j.$$

When n is even, the degrees of the vertices modulo 2 satisfy:

$$\deg(0) = n \equiv 0 \pmod{2}, \quad \deg(j) = 1 \equiv 1 \pmod{2} \quad \text{for all } j \geq 1.$$

Thus the center and the leaves lie in *distinct degree-parity classes*. By Theorem I.4, this implies that the standard dynamical Lie algebra $\mathfrak{g}_{K_{1,n},\text{std}}$ contains all orbit generators of $\mathfrak{g}_{K_{1,n},\text{orb}}$. Hence, for even n , we obtain

$$\mathfrak{g}_{K_{1,n},\text{std}} = \mathfrak{g}_{K_{1,n},\text{orb}}.$$

For odd values of n , this inclusion may fail. A concrete example is $K_{1,3}$, where

$$\dim(\mathfrak{g}_{K_{1,3},\text{std}}) = 24, \quad \dim(\mathfrak{g}_{K_{1,3},\text{orb}}) = 26.$$

The standard reduced dynamical Lie algebra at the central vertex $v = 0$, denoted $\mathfrak{g}_{K_{1,n},\text{std}}^0$, is generated by the elements

$$\mathcal{X}_{\hat{0}} := iX_1 + \cdots + iX_n, \quad \mathcal{Z}_{\hat{0}} := iZ_1 + \cdots + iZ_n, \quad (44)$$

and is isomorphic to the three-dimensional simple Lie algebra $\mathfrak{su}(2)$.

Remark IV.8. The standard reduced DLA $\mathfrak{g}_{K_{1,n},\text{std}}^0$ is thus independent of n and has dimension 3, in stark contrast to the full standard DLA $\mathfrak{g}_{K_{1,n},\text{std}}$, whose dimension grows with n (see Table II).

While dimensions of $\mathfrak{g}_{K_{1,n},\text{std}}$ can be computed explicitly for small n , a closed-form expression for its dimension, as well as a uniform description of its decomposition into simple Lie algebra components and its center, remains nontrivial. This highlights the simplifying effect of the vertex-reduction procedure on the DLA structure.

FIG. 13. The path graph P_8 with 8 vertices.

E. Path Graphs

Consider the path graph P_n , illustrated in Figure 13.

It was shown in [5] that the free dynamical Lie algebra associated with P_n satisfies

$$\mathfrak{g}_{P_n, \text{free}} \cong \mathfrak{so}(2n),$$

and therefore

$$\dim(\mathfrak{g}_{P_n, \text{free}}) = 2n^2 - n.$$

Moreover, the standard and orbit dynamical Lie algebras coincide and are isomorphic to the unitary Lie algebra,

$$\mathfrak{g}_{P_n, \text{std}} = \mathfrak{g}_{P_n, \text{orb}} \cong \mathfrak{u}(n),$$

as established in Theorem 1 and Lemma A.6 of [5]. In particular,

$$\dim(\mathfrak{g}_{P_n, \text{std}}) = \dim(\mathfrak{g}_{P_n, \text{orb}}) = n^2.$$

Let $v = k \in V(P_n)$ be a fixed vertex. Then Remark II.7 together with Theorem IV.2 imply the following containments:

$$\mathfrak{g}_{P_n, \text{std}}^v \supseteq \begin{cases} \mathfrak{g}_{P_{n-1}, \text{free}}, & k = 1 \text{ or } k = n, \\ \mathfrak{g}_{P_{(n-1)/2}, \text{free}}, & k = \frac{n+1}{2}, n \equiv 1 \pmod{2}, \\ \mathfrak{g}_{P_{k-1}, \text{free}} \oplus \mathfrak{g}_{P_{n-k}, \text{free}}, & \text{otherwise.} \end{cases} \quad (45)$$

In particular, if v is a leaf vertex (i.e., $k = 1$ or $k = n$), then

$$\mathfrak{g}_{P_n, \text{std}}^v \supseteq \mathfrak{g}_{P_{n-1}, \text{free}},$$

and therefore

$$\dim(\mathfrak{g}_{P_n, \text{std}}^v) \geq \dim(\mathfrak{g}_{P_{n-1}, \text{free}}) = 2(n-1)^2 - (n-1).$$

For all $n \geq 5$, one verifies that

$$2(n-1)^2 - (n-1) > n^2,$$

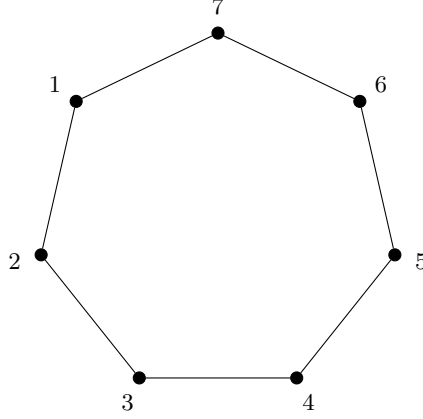
which shows that the dimension of the standard reduced dynamical Lie algebra at a leaf vertex strictly exceeds the dimension of the standard dynamical Lie algebra of the original path graph.

F. Cycle Graphs

We now consider the cycle graphs C_n , illustrated in Figure 14.

It follows from Theorem 1 in [5] that the free standard dynamical Lie algebra associated to a cycle graph satisfies

$$\mathfrak{g}_{C_n, \text{free}} \cong \mathfrak{so}(2n) \oplus \mathfrak{so}(2n).$$

FIG. 14. The cycle graph C_7 with 7 vertices.

In addition, Theorem 33 in [1] gives an explicit decomposition:

$$\mathfrak{g}_{C_n, \text{std}} \cong \underbrace{\mathfrak{su}(2) \oplus \cdots \oplus \mathfrak{su}(2)}_{n-1 \text{ copies}} \oplus \mathfrak{u}(1)^{\oplus 2}.$$

In particular,

$$\dim(\mathfrak{g}_{C_n, \text{std}}) = 3(n-1) + 2 = 3n - 1.$$

According to Lemma 8 in [5], the natural Lie algebra $\mathfrak{g}_{C_n, \text{nat}}$ associated to the cycle graph also satisfies

$$\dim(\mathfrak{g}_{C_n, \text{nat}}) = 3n - 1.$$

We always have the containments

$$\mathfrak{g}_{C_n, \text{std}} \subseteq \mathfrak{g}_{C_n, \text{orb}} \subseteq \mathfrak{g}_{C_n, \text{nat}}.$$

Since all three Lie algebras have the same dimension, they must coincide. Therefore,

$$\mathfrak{g}_{C_n, \text{std}} = \mathfrak{g}_{C_n, \text{orb}} = \mathfrak{g}_{C_n, \text{nat}} \cong \underbrace{\mathfrak{su}(2) \oplus \cdots \oplus \mathfrak{su}(2)}_{n-1 \text{ copies}} \oplus \mathfrak{u}(1)^{\oplus 2}.$$

We turn our attention to the reduced DLAs. Let $v \in C_n$ be any vertex of the cycle graph C_n . Consider the corresponding standard reduced dynamical Lie algebra $\mathfrak{g}_{C_n, \text{std}}^v$. By Lemma 1.6, this algebra contains the standard DLA of the path graph obtained by removing v and its two incident edges from C_n , that is,

$$\mathfrak{g}_{P_{n-1}, \text{std}} \subseteq \mathfrak{g}_{C_n, \text{std}}^v.$$

As a consequence, we have the dimension bound

$$\dim(\mathfrak{g}_{C_n, \text{std}}^v) \geq \dim(\mathfrak{g}_{P_{n-1}, \text{std}}) = (n-1)^2,$$

which exceeds the dimension of the standard DLA of the cycle graph:

$$\dim(\mathfrak{g}_{C_n, \text{std}}) = 3n - 1,$$

for all $n \geq 5$, i.e.

$$\dim(\mathfrak{g}_{C_n, \text{std}}^v) > \dim(\mathfrak{g}_{C_n, \text{std}}) \text{ for } n > 5 \tag{46}$$

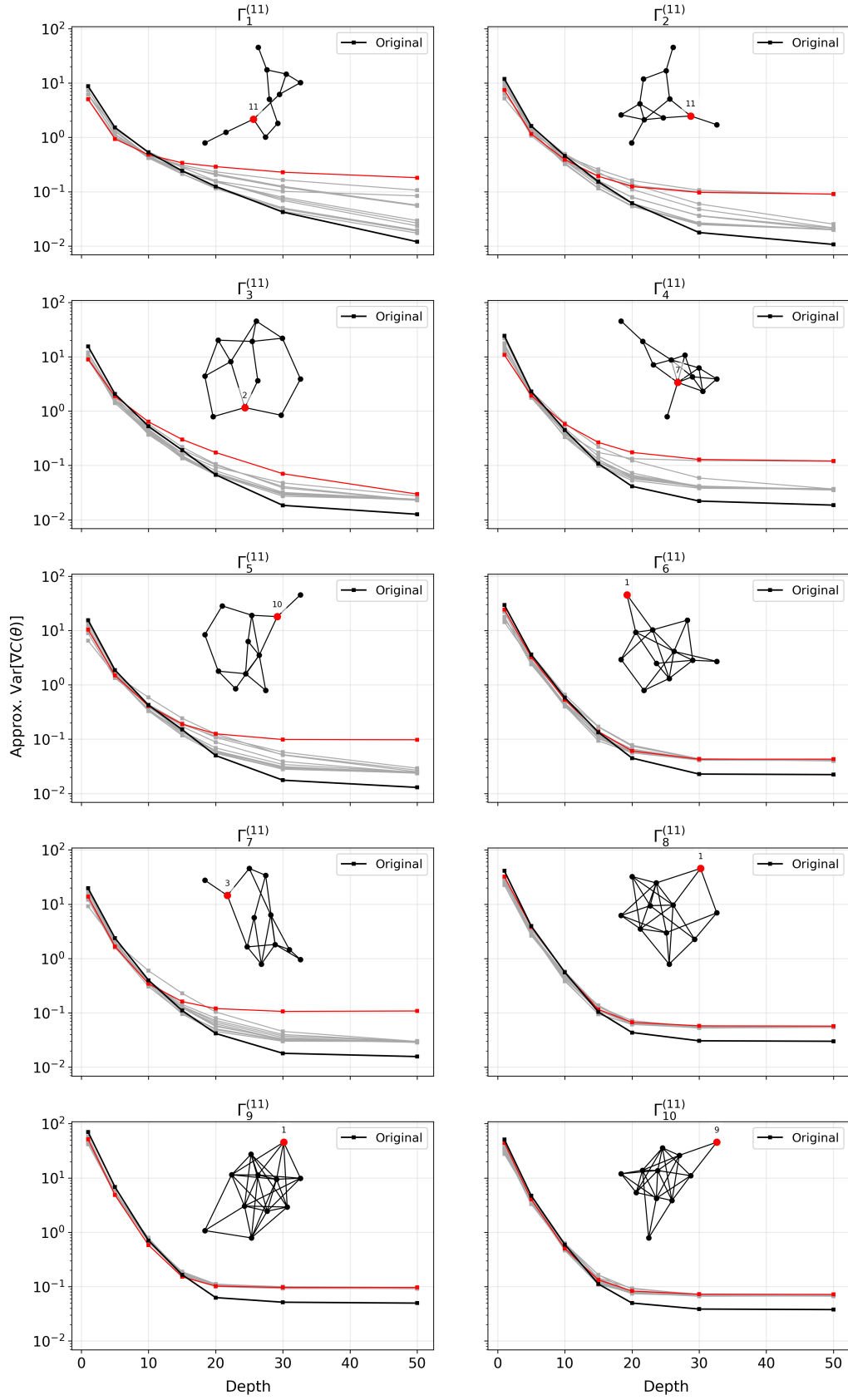


FIG. 15. Approximate variance for ten 11-node asymmetric graphs ($\Gamma^{(11)}$). The red data shows the reduced QAOA instance that results in the largest variance of the gradient at depth 50, with the corresponding vertex highlighted in red and labeled on the graph inset.

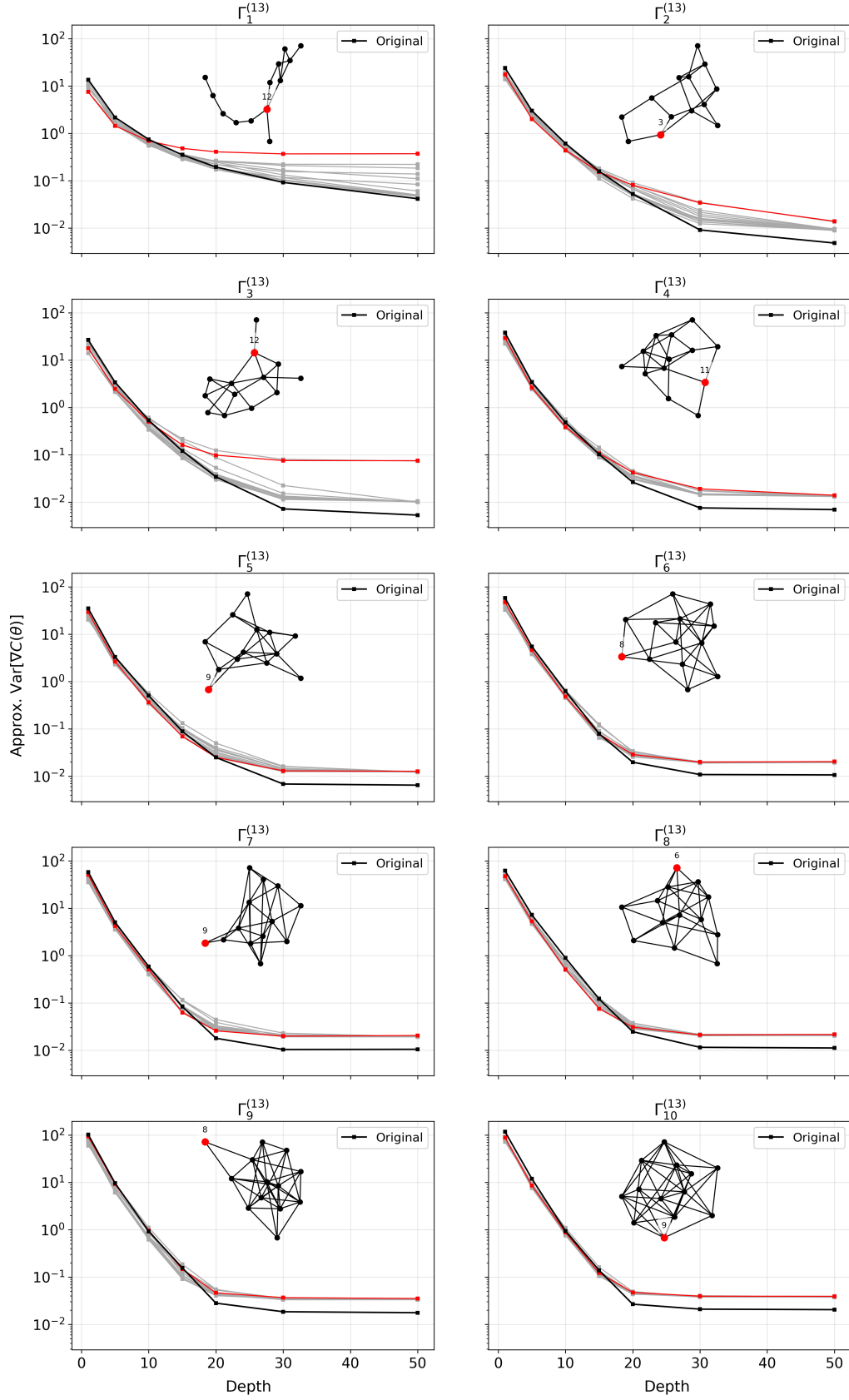


FIG. 16. Approximate variance for ten 13-node asymmetric graphs ($\Gamma^{(13)}$). The red data shows the reduced QAOA instance that results in the largest variance of the gradient at depth 50, with the corresponding vertex highlighted in red and labeled on the graph inset.

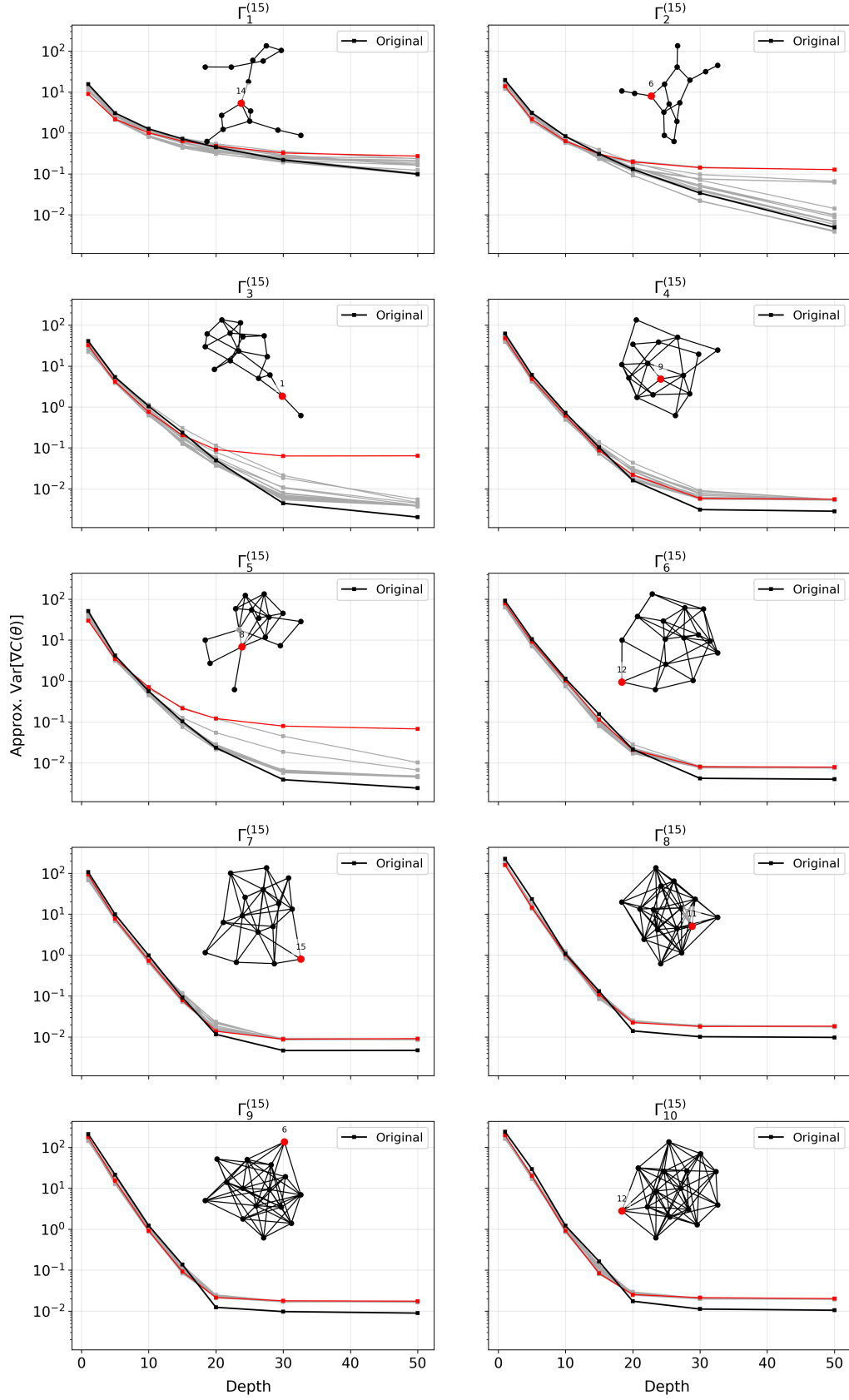


FIG. 17. Approximate variance for ten 15-node asymmetric graphs ($\Gamma^{(15)}$). The red curve data the reduced QAOA instance that results in the largest variance of the gradient at depth 50, with the corresponding vertex highlighted in red and labeled on the graph inset.

V. ADDITIONAL NUMERICAL RESULTS

In this appendix, we include additional numerical results for the variance in the gradient for 11-, 13-, and 15-node asymmetric graphs. Some of these graphs (see e.g., $\Gamma_5^{(11)}$, $\Gamma_7^{(11)}$, $\Gamma_3^{(13)}$, $\Gamma_5^{(15)}$) corroborate Observation 2.

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- [1] Jonathan Allcock, Miklos Santha, Pei Yuan, and Shengyu Zhang. On the dynamical Lie algebras of quantum approximate optimization algorithms. *Quantum*, 10:2119, May 2026. [31](#)
 - [2] Andrew Arrasmith, Zoë Holmes, M Cerezo, and Patrick J Coles. Equivalence of quantum barren plateaus to cost concentration and narrow gorges. *Quantum Science and Technology*, 7(4):045015, 2022. [24](#)
 - [3] V. Bergholm and et al. PennyLane: Automatic differentiation of hybrid quantum classical computations. *arXiv:1811.04968*, 2022. [21](#)
 - [4] D. D’Alessandro. *Introduction to Quantum Control and Dynamics*. Chapman and Hall/CRC, New York, 2nd edition, 2021. [10](#)
 - [5] S. Kazi, M. Larocca, M. Farinati, P. Coles, M. Cerezo, and R. Zeier. Analyzing the quantum approximate optimization algorithm: ansätze, symmetries, and Lie algebras. *PRX Quantum*, 6, 2025. [12](#), [15](#), [16](#), [19](#), [20](#), [29](#), [30](#), [31](#)
 - [6] M. Larocca, S. Thanasilp, S. Wang, K. Sharma, J. Biamonte, P. J. Coles, L. Cincio, J. R. McClean, and Z. Holmes. Barren plateaus in variational quantum computing. *Nature Reviews Physics*, pages 1–16, 2025. [24](#)
 - [7] R. Mao, P. Yuan, J. Allcock, and S. Zhang. Qaoa-maxcut has barren plateaus for almost all graphs. *arXiv:2512.24577*, 2026. [19](#), [20](#)
 - [8] J. R. McClean, S. Boixo, V. N. Smelyanskiy, R. Babbush, and H. Neven. Barren plateaus in quantum neural network training landscapes. *Nature Communications*, 9(1):1–6, 2018. [24](#)
 - [9] M. Ragone, B. N. Bakalov, F. Sauvage, A. F. Kemper, C. Marrero, M. Larocca, and M. Cerezo. A Lie algebraic theory of barren plateaus for deep parameterized quantum circuits. *Nat. Commun.*, 15, 2024. [24](#)
 - [10] R. Wiersema, E. Kökcü, A. F. Kemper, and B. N. Bakalov. Classification of dynamical Lie algebras of 2-local spin systems on linear, circular and fully connected topologies. *npj Quantum Information*, 10, 2024. [24](#)