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Research Article

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The Nim-Sum of a Random Integer Partition

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Abstract

An unordered position of ordinary Nim with N tokens is an integer partition of N. By Bouton's theorem, the position is losing if and only if the bitwise exclusive OR (XOR) of its parts is zero. Let a_N be the number of losing positions and let $p(N)$ be the partition number. This paper determines the first-order behavior of $a_N/p(N)$. Put $\tau_N = \pi/\sqrt{6N}$ and $B_N = \log(1/\tau_N)/\tau_N$. Let J_N be a power of two nearest to B_N in multiplicative distance, and set $\Lambda_N = e^{-\tau_N J_N}/\tau_N$. For even N,

$$\frac{a_N}{p(N)} = \frac{1 + e^{-2\Lambda_N} + o(1)}{J_N},$$

whereas $a_N = 0$ for odd N.

The formula has a simple explanation. Small parts randomize the lower binary digits, large parts are negligible, and the parity of one critical dyadic block determines the remaining probability. As N crosses a dyadic boundary, the normalized density follows a discontinuous sawtooth. The apparent jump is smoothed, on a shrinking scale, by the parity law of a Poisson random variable. Consequently, along even N, losing positions have density of order $1/(\sqrt{N}\log N)$, but the normalized density does not converge. The quantity $B_N a_N/p(N)$ has the entire interval $[1, 2]$ as its set of subsequential limits. A finite hierarchy of dyadic formulas refines the associated Boltzmann approximation to arbitrary algebraic order. For every fixed target nim-sum x, the same first-order profile holds along the parity class $N \equiv x \pmod 2$. We also compute the exact counts through total size $N = 20{,}000$, extending sequence A048833 in the On-Line Encyclopedia of Integer Sequences through index 10,000.

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1 Introduction

Take N counters and divide them into nonempty heaps. The resulting multiset is a position of ordinary Nim. Bouton's theorem gives a complete answer for any given position. The player who moves next loses if and only if the bitwise XOR of the heap sizes is zero (Bouton 1902). Equivalently, the nim-sum is the Sprague–Grundy value of the position.

Bouton's theorem does not say how often this happens when only the total number of counters is fixed. If heap order is ignored, the Nim positions with N counters are the integer partitions of N. Let

$$a_N = \#\left\{\lambda \vdash N \,|\, \mathrm{nim}(\lambda) = 0\right\},$$

where $\mathrm{nim}(\lambda)$ is the XOR of the parts of λ. Let $p(N)$ be the number of integer partitions of N. The ratio $a_N/p(N)$ is the fraction of unordered Nim positions with N counters that are losing.

This is not a probability generated by play. The position is selected uniformly from the partitions of N, and the players are assumed to play optimally after the position has been selected. The question is therefore an enumerative one about the distribution of losing positions in the fixed-size Nim state space.

Problem E15 in Nowakowski (2025) asks for a formula for the number of losing Nim positions with $2n$ counters. Earlier work studies positions with a fixed number of heaps or with bounds on the heap sizes. The unrestricted sequence is recorded as OEIS A048833 (Khovanova and Xiong 2014; OEIS Foundation Inc. 2026a). We determine its complete first-order asymptotic behavior.

The parity restriction is immediate. Ordinary addition and XOR have the same least significant bit, so $a_N = 0$ whenever N is odd.

Put $\tau_N = \pi/\sqrt{6N}$ and $B_N = \log(1/\tau_N)/\tau_N$. Let J_N be a power of two nearest to B_N in multiplicative distance, and set $\Lambda_N = e^{-\tau_N J_N}/\tau_N$. For even N,

$$\frac{a_N}{p(N)} = \frac{1 + e^{-2\Lambda_N} + o(1)}{J_N}. \tag{1.1}$$

Formula (1.1) is the answer at first order. Since $J_N \asymp B_N \asymp \sqrt{N}\log N$, losing positions have density

$$\frac{a_N}{p(N)} = \Theta\left(\frac{1}{\sqrt{N}\log N}\right)$$

along even N. There is no asymptotic constant. More precisely,

$$\liminf_{\substack{N\to\infty \\ 2|N}} B_N \frac{a_N}{p(N)} = 1, \qquad \limsup_{\substack{N\to\infty \\ 2|N}} B_N \frac{a_N}{p(N)} = 2.$$

The set of subsequential limits is the entire interval $[1, 2]$.

The nonconvergence has a simple structure. If $\{\log_2 B_N\} \to \theta$ for some $0 < \theta < 1$, then

$$B_N \frac{a_N}{p(N)} \longrightarrow 2^\theta.$$

Thus the normalized density follows a dyadic sawtooth. It rises from 1 to 2 during each binary phase and returns to 1 when the relevant scale crosses a power of two.

The return is not an actual discontinuity at finite N. Let K be a power of two. If $\tau_N K - \log(1/\tau_N) \to c$, then

$$K \frac{a_N}{p(N)} \longrightarrow 1 + e^{-2e^{-c}}.$$

The apparent jump is therefore resolved inside a shrinking boundary window. Its shape is the parity law of a Poisson random variable.

The proof rests on a decomposition of a partition into two parts. If part j appears c_j times, write $c_j = 2r_j + \varepsilon_j$, where $\varepsilon_j \in \{0, 1\}$. The even copies do not affect the XOR. Only the parts for which $\varepsilon_j = 1$ remain. Every partition is therefore the union of two identical copies of an ordinary partition and a partition into distinct parts. The XOR constraint is carried entirely by the distinct-part component.

Under Boltzmann sampling, the indicators of the distinct parts are independent. This turns the problem into the study of the XOR of independent Bernoulli variables. Three ranges matter.

Parts below J randomize the lower binary digits. Parts above $2J$ are rare. The block $[J, 2J)$ is the only range that matters at first order. Every integer in this block has the same leading binary digit. That digit vanishes from the total XOR exactly when the number of selected parts in the block is even. The number of selected parts is asymptotically Poisson with mean Λ. Its even probability is $(1+e^{-2\Lambda})/2$. This gives the numerator in (1.1).

The one-block formula is only the first member of a hierarchy. For every fixed r, we retain the next r upper binary digits and apply Walsh orthogonality to the corresponding dyadic blocks. This gives an explicit finite approximation with relative error $O_r(t^{2^{r-1/2}-1})$ in the Boltzmann model. By increasing r, the Boltzmann probability can be determined to any prescribed algebraic order. The transfer to partitions of exactly N is established at first order.

The result is not special to losing positions. For every fixed nonnegative integer x, let $a_N(x)$ denote the number of partitions of N with nim-sum x. Then $a_N(x) = 0$ unless $N \equiv x \pmod 2$, and on the admissible parity class,

$$\frac{a_N(x)}{p(N)} = \frac{1 + e^{-2\Lambda_N} + o(1)}{J_N}.$$

Thus every fixed target nim-sum has the same first-order dyadic profile.

The closest enumerative work studies a different sampling problem. Khovanova and Xiong (2014) consider a fixed number of heaps and obtain recursive formulas with binary and fractal structure. Hwang et al. (2024) place related bounded-heap sequences in a general class of divide-and-conquer recurrences. Their leading terms contain continuous functions of a binary phase.

The present oscillation has a different form. The number of heaps is not fixed. Only their total size is fixed. The outer profile is a discontinuous sawtooth rather than a continuous periodic function. The discontinuity is resolved only on a vanishing scale, where a Poisson parity law appears.

The probabilistic tools also come from the theory of random partitions. The scale B_N is the classical extreme-part scale of Erdős and Lehner (1941). See Andrews (1998) for general background on integer partitions and Mutafchiev (2013) for an extension of largest-part limit theory to weighted partitions. The passage between fixed-size partitions and independent coordinates is related to the conditioning method of Fristedt (1993), the independent-process approach of Arratia and Tavaré (1994), and the total-variation approximations of Pittel (1997). For partitions into distinct parts, a canonical local limit theory is developed by Freiman et al. (2000). These results concern additive statistics, extreme parts, or local features of the partition. They do not determine the point mass of the nonlocal XOR statistic studied here.

We also give two exact representations of a_N. The first is a finite Walsh-character formula. The second is a convolution based on the distinct-part core. The Walsh formula leads to an exact modular algorithm. We use it to calculate the sequence through total size $N = 20{,}000$. These calculations are not used in the proof. They provide an independent check of the asymptotic formulas and reveal the finite-size corrections predicted by the dyadic hierarchy.

The rest of the paper proceeds as follows. Section 2 gives the exact formulas. Section 3 studies the Boltzmann model and derives the dyadic hierarchy. Section 4 extends the argument to a complex neighborhood of the saddle. Section 5 transfers the result to partitions of fixed total size. Section 6 states the fixed-size consequences. Section 7 presents the exact calculations and their relation to the asymptotic formulas. The appendices give a direct nonasymptotic bound and an independent real-variable proof of the order of magnitude.

2 Exact representations

This section gives two exact representations of a_N. The first separates a partition into paired parts and an odd core. It will be used in the probabilistic analysis. The second is a finite Walsh formula. It will be used in the exact computation.

For a partition $\lambda = (\lambda_1, \dots, \lambda_r)$, write $\operatorname{nim}(\lambda) = \lambda_1 \oplus \cdots \oplus \lambda_r$. Define

$$A(q) = \sum_{N \ge 0} a_N q^N, \qquad P(q) = \prod_{j \ge 1} \frac{1}{1 - q^j} = \sum_{N \ge 0} p(N) q^N.$$

The least significant binary digit gives the first restriction.

Lemma 2.1 (Parity) *For every finite multiset of nonnegative integers,*

$$\sum_j x_j \equiv \bigoplus_j x_j \pmod 2.$$

Consequently, $a_N = 0$ for odd N, and $A(-q) = A(q)$.

Proof Ordinary addition and XOR have the same least significant bit. The congruence follows by applying this observation repeatedly. If the XOR is zero, the ordinary sum is even. □

Remark 2.2 Let a_N^{mis} be the number of losing positions in misère Nim with total size N. The normal and misère criteria differ only when every heap has size one. Hence, for $N \geq 1$,

$$a_N^{\mathrm{mis}} = \begin{cases} a_N - 1, & N \text{ even}, \\ 1, & N \text{ odd}. \end{cases}$$

In particular, all first-order asymptotic conclusions along even N remain unchanged.

We now isolate the part of a partition that affects its XOR. Suppose that part j appears c_j times. Write $c_j = 2r_j + \varepsilon_j$, where $\varepsilon_j \in \{0, 1\}$. The $2r_j$ copies cancel in pairs under XOR. Only the parts for which $\varepsilon_j = 1$ remain.

Let $D(q) = \prod_{j \geq 1}(1 + q^j)$ be the generating function for partitions into distinct parts. Define

$$D_0(q) = \sum_{\substack{S \subset \mathbb{N} \text{ finite} \\ \bigoplus_{j \in S} j = 0}} q^{\sum_{j \in S} j}.$$

Proposition 2.3 (Odd core) *Define $R(q) = A(q)/P(q)$. The following identities hold as formal power series. They also hold analytically for $|q| < 1$.*

$$A(q) = P(q^2)D_0(q), \qquad P(q) = P(q^2)D(q), \qquad R(q) = \frac{D_0(q)}{D(q)}. \tag{2.1}$$

For $k \geq 0$, let

$$d_k = \#\left\{ S \subset \mathbb{N} \text{ finite} \,\middle|\, \sum_{j \in S} j = 2k \text{ and } \bigoplus_{j \in S} j = 0 \right\}.$$

Then

$$a_{2n} = \sum_{k=0}^{n} d_k p(n-k). \tag{2.2}$$

Proof Let λ have multiplicities $(c_j)_{j \geq 1}$. Write $c_j = 2r_j + \varepsilon_j$. Construct a partition μ that contains r_j copies of j, and let $S = \{j \in \mathbb{N} \mid \varepsilon_j = 1\}$. This construction is a bijection. It satisfies

$$|\lambda| = 2|\mu| + \sum_{j \in S} j, \qquad \mathrm{nim}(\lambda) = \bigoplus_{j \in S} j.$$

The partition μ contributes the factor $P(q^2)$, while the set S contributes $D_0(q)$. Thus $A(q) = P(q^2)D_0(q)$. Also, $(1-q^j)^{-1} = (1+q^j)/(1-q^{2j})$, so taking products gives $P(q) = P(q^2)D(q)$. Dividing the two identities gives the formula for $R(q)$. Extracting the coefficient of q^{2n} from the first identity gives (2.2). □

The proposition separates the two roles played by a partition. The partition μ carries the paired multiplicities and determines part of the total size. The set S carries

the entire XOR. This is the reason that the original partition problem can be reduced to a problem about independent distinct parts.

We next give a second exact representation. Choose a power of two $M = 2^m > N$ and identify $\{0, \dots, M-1\}$ with $\mathbb{F}_2^m$. For $u, j < M$, let $\langle u, j \rangle$ denote their inner product over $\mathbb{F}_2$.

Proposition 2.4 (Walsh formula) *Let $M = 2^m > N$. Then*

$$a_N = \frac{1}{M} \sum_{u=0}^{M-1} [q^N] \prod_{j=1}^{N} \frac{1}{1 - (-1)^{\langle u,j \rangle} q^j}.$$

Proof Character orthogonality on $\mathbb{F}_2^m$ gives

$$\mathbf{1}\{x = 0\} = \frac{1}{M} \sum_{u=0}^{M-1} (-1)^{\langle u,x \rangle}$$

for every $x \in \mathbb{F}_2^m$.

Let c_j be the multiplicity of part j. Its XOR is $\bigoplus_{j=1}^{N} (c_j \bmod 2) j$. Applying character orthogonality and summing over all multiplicity vectors gives

$$\begin{aligned} a_N &= \frac{1}{M} \sum_{u=0}^{M-1} \sum_{\substack{c_1, \dots, c_N \geq 0 \\ \sum_{j=1}^{N} j c_j = N}} (-1)^{\sum_{j=1}^{N} c_j \langle u,j \rangle} \\ &= \frac{1}{M} \sum_{u=0}^{M-1} [q^N] \prod_{j=1}^{N} \sum_{c \geq 0} \left((-1)^{\langle u,j \rangle} q^j \right)^c. \end{aligned}$$

Evaluating the geometric series proves the formula. □

The two representations serve different purposes. The odd core turns the ratio $A(q)/P(q)$ into a probability in the next section. The Walsh formula gives a finite expression for every a_N and leads to the exact algorithm used later in the paper.

The number of partitions of N with each possible XOR value is recorded as OEIS A050314 (OEIS Foundation Inc. 2026b). The coefficients d_k of the distinct XOR-zero core are recorded as OEIS A307506 (OEIS Foundation Inc. 2026c).

3 The Boltzmann model

We now allow the total size of the partition to be random. The gain is independence.

Put $q = e^{-t}$, where $t > 0$. For every positive integer j, let I_j be an independent Bernoulli random variable with

$$\mathbb{P}(I_j = 1) = \frac{e^{-tj}}{1 + e^{-tj}}. \tag{3.1}$$

The expected number of selected integers is finite. Hence only finitely many I_j are equal to one almost surely. We may therefore define $Y_t = \bigoplus_{j : I_j = 1} j$.

The random set $\{j \in \mathbb{N} \mid I_j = 1\}$ is a Boltzmann partition into distinct parts. Proposition 2.3 gives the following identity.

Lemma 3.1 *For $q = e^{-t}$,*

$$R(e^{-t}) = \frac{A(e^{-t})}{P(e^{-t})} = \mathbb{P}(Y_t = 0). \tag{3.2}$$

Proof A finite set $S \subset \mathbb{N}$ is selected with probability

$$\prod_{j \in S} \frac{q^j}{1+q^j} \prod_{j \notin S} \frac{1}{1+q^j} = \frac{q^{\sum_{j \in S} j}}{D(q)}.$$

Summing over the sets with XOR zero gives

$$\mathbb{P}(Y_t = 0) = \frac{D_0(q)}{D(q)}.$$

The result follows from (2.1). □

The problem is now to estimate the probability that the XOR of independent Bernoulli variables is zero.

3.1 The critical scale

Set

$$\ell = \log(1/t), \qquad L = L(t) = \frac{\ell}{t}. \tag{3.3}$$

This is the scale at which the expected upper tail becomes of order one, since $e^{-tL}/t = 1$.

XOR changes its dimension only at powers of two. We therefore replace L by a nearby dyadic scale. Let

$$J = J(t) = 2^{\lfloor \log_2 L + \frac{1}{2} \rfloor}. \tag{3.4}$$

Thus

$$2^{-1/2} L \le J \le 2^{1/2} L.$$

Except at an exact tie, this interval contains a single power of two. At a tie, either neighboring power may be used. This choice balances the residual nonuniformity below J against the first omitted tail above $2J$.

Define

$$\Lambda = \Lambda(t) = \frac{e^{-tJ}}{t}. \tag{3.5}$$

The value Λ is the approximate expected number of selected integers in the block $[J, 2J)$.

We divide the positive integers into the three ranges $[1, J)$, $[J, 2J)$, and $[2J, \infty)$. The first range randomizes the lower binary digits. The third range is rare. The middle range determines the first-order probability.

3.2 The lower block

Write $J = 2^m$ and define

$$X_{<J} = \bigoplus_{\substack{1 \le j < J \\ I_j = 1}} j.$$

We regard $X_{<J}$ as a random element of $\mathbb{F}_2^m$.

Let μ_t denote its distribution. For $u \in \mathbb{F}_2^m$, its Walsh coefficient is

$$\widehat{\mu}_t(u) = \mathbb{E}\left[(-1)^{\langle u, X_{<J}\rangle}\right].$$

Independence gives

$$\widehat{\mu}_t(u) = \prod_{\substack{1 \le j < J \\ \langle u, j\rangle = 1}} \tanh(tj/2).$$

For $1 \le j < J$, put $\chi_j(t) = \tanh(tj/2)$. Define

$$\beta_J(t) = \prod_{j=J/2}^{J-1} \chi_j(t) \tag{3.6}$$

and

$$\gamma_J(t) = \prod_{j=J/4}^{J/2-1} \chi_j(t) \prod_{j=3J/4}^{J-1} \chi_j(t).$$

The coefficient β_J is the largest nontrivial Walsh coefficient. All other nontrivial coefficients are much smaller.

Lemma 3.2 *Let $u_* = J/2$. Then*

$$\widehat{\mu}_t(u_*) = \beta_J(t), \qquad 0 \le \widehat{\mu}_t(u) \le \gamma_J(t) \quad \text{for } u \notin \{0, u_*\},$$

and $J\gamma_J(t) = o(\beta_J(t))$. Consequently, uniformly in $x \in \{0, \dots, J-1\}$,

$$\mathbb{P}(X_{<J} = x) = \frac{1}{J}\left[1 + (-1)^{x_{m-1}}\beta_J(t) + o(\beta_J(t))\right], \tag{3.7}$$

where x_{m-1} is the highest binary digit of x.

Proof The character $u_* = 2^{m-1}$ selects exactly the integers in $[J/2, J)$, whose highest binary digit is one. Hence

$$\widehat{\mu}_t(u_*) = \prod_{j=J/2}^{J-1} \chi_j(t) = \beta_J(t).$$

Consider a character $u \notin \{0, u_*\}$. Its restriction to the lower $m-1$ binary digits is nonzero. It therefore selects exactly $J/4$ integers from each of the half blocks $[0, J/2)$ and $[J/2, J)$. The function $j \mapsto \chi_j(t)$ is increasing. The product is thus maximized by selecting the largest $J/4$ integers from each half block. This gives

$$\widehat{\mu}_t(u) \le \gamma_J(t).$$

It remains to compare γ_J and β_J. Put $b_j = -\log \chi_j(t)$. Since $tJ/4$ tends to infinity, for all sufficiently small t,

$$2e^{-tj} \le b_j \le 4e^{-tj}$$

whenever $J/4 \le j < J$. It follows that

$$\begin{aligned}\log \frac{\gamma_J(t)}{\beta_J(t)} &= -\sum_{j=J/4}^{J/2-1} b_j + \sum_{j=J/2}^{3J/4-1} b_j \\ &\le -c_0 \frac{e^{-tJ/4}}{t}\end{aligned}$$

for some constant $c_0 > 0$.

The upper bound in (3.4) gives

$$\frac{e^{-tJ/4}}{t} \ge t^{\sqrt{2}/4-1}.$$

This grows faster than $\log J$, so $J\gamma_J(t) = o(\beta_J(t))$.

Walsh inversion now gives

$$\mathbb{P}(X_{<J} = x) = \frac{1}{J}\left[1 + (-1)^{\langle u_*, x\rangle}\beta_J(t) + O(J\gamma_J(t))\right].$$

Since $\langle u_*, x\rangle = x_{m-1}$, the result follows. □

The lower block is therefore almost uniform.

Corollary 3.3 *Uniformly in $x \in \{0, \dots, J-1\}$,*

$$\mathbb{P}(X_{<J} = x) = \frac{1 + o(1)}{J}. \tag{3.8}$$

More precisely,

$$\sup_{x<J}\left|\mathbb{P}(X_{<J} = x) - \frac{1}{J}\right| = O\left(\frac{\beta_J(t)}{J}\right). \tag{3.9}$$

Proof Both statements follow from (3.7) and $\beta_J(t) \to 0$. □

3.3 The critical block

We first remove the upper tail.

Lemma 3.4 *The probability that some selected integer is at least $2J$ satisfies*

$$\mathbb{P}(\exists j \ge 2J : I_j = 1) = O\left(\frac{e^{-2tJ}}{t}\right) = O(t^{\sqrt{2}-1}) = o(1).$$

Its contribution to $\mathbb{P}(Y_t = 0)$ is $o(1/J)$.

Proof A union bound gives

$$\mathbb{P}(\exists j \geq 2J : I_j = 1) \leq \sum_{j \geq 2J} e^{-tj} = O\left(\frac{e^{-2tJ}}{t}\right).$$

The lower bound in (3.4) gives

$$\frac{e^{-2tJ}}{t} = O(t^{\sqrt{2}-1}).$$

Condition on all selected integers at least J. Whatever their lower m binary digits are, the lower block must match one prescribed residue. Corollary 3.3 bounds this probability by

$$\frac{1 + o(1)}{J}$$

uniformly. Multiplying by the upper tail probability proves the second claim. □

Let

$$K_J = \sum_{j=J}^{2J-1} I_j, \qquad \Pi_J(t) = \prod_{j=J}^{2J-1} \frac{1 - e^{-tj}}{1 + e^{-tj}}.$$

Every integer in $[J, 2J)$ has its J-bit equal to one. The J-bit of the block XOR is therefore zero if and only if K_J is even.

For a sum of independent Bernoulli variables, $\mathbb{P}(K_J \text{ is even}) = (1 + \Pi_J(t))/2$.

The next theorem gives the first-order Boltzmann probability.

Theorem 3.5 *As t tends to zero,*

$$R(e^{-t}) = \mathbb{P}(Y_t = 0) = \frac{1 + e^{-2\Lambda(t)} + o(1)}{2J(t)}. \tag{3.10}$$

The error is uniform over the two choices allowed at an exact nearest-dyadic tie.

Proof Corollary 3.3 and Lemma 3.4 give

$$\mathbb{P}(Y_t = 0) = \frac{1 + o(1)}{J} \mathbb{P}(K_J \text{ is even}) + o(1/J). \tag{3.11}$$

Since $tJ \to \infty$, $\sum_{j=J}^{2J-1} e^{-3tj} = o(1)$. Using $\log((1-x)/(1+x)) = -2x + O(x^3)$, we obtain

$$\log \Pi_J(t) = -2 \sum_{j=J}^{2J-1} e^{-tj} + o(1).$$

Also,

$$\sum_{j=J}^{2J-1} e^{-tj} = \frac{e^{-tJ}(1 - e^{-tJ})}{1 - e^{-t}} = \Lambda(t)(1 + o(1)).$$

If $\Lambda(t)$ is bounded, then $\log \Pi_J(t) = -2\Lambda(t) + o(1)$. If $\Lambda(t) \to \infty$, both $\Pi_J(t)$ and $e^{-2\Lambda(t)}$ tend to zero. Thus $\Pi_J(t) = e^{-2\Lambda(t)} + o(1)$ in all cases. Substitution into (3.11) proves the result. □

The exact block product also gives a useful error bound.

Proposition 3.6 *Uniformly as t tends to zero,*

$$R(e^{-t}) = \frac{1 + \Pi_J(t)}{2J} + O\left(\frac{\beta_J(t)}{J} + \frac{e^{-2tJ}}{tJ}\right). \tag{3.12}$$

The relative error is

$$O\left(\beta_J(t) + \frac{e^{-2tJ}}{t}\right). \tag{3.13}$$

The first term is smaller than every power of t. The second term is at worst $O(t^{\sqrt{2}-1})$. If

$$\frac{tJ}{\log(1/t)} \longrightarrow \sqrt{2},$$

it is $O(t^{2\sqrt{2}-1})$.

Proof Let T be the event that no selected integer is at least $2J$. On T, zero XOR requires K_J to be even. Conditional on the middle block, the lower block must match one prescribed residue. Equation (3.9) gives

$$\frac{1}{J} + O\left(\frac{\beta_J(t)}{J}\right)$$

uniformly.

The complement of T has probability

$$O\left(\frac{e^{-2tJ}}{t}\right).$$

The lower block still has to match one prescribed residue. Its maximal point probability is

$$\frac{1}{J} + O\left(\frac{\beta_J(t)}{J}\right).$$

This proves (3.12).

The main term lies between $1/(2J)$ and $1/J$. Division gives (3.13). Finally,

$$\beta_J(t) \le \exp\left(-2\sum_{j=J/2}^{J-1} e^{-tj}\right) \le \exp\left(-c\frac{e^{-tJ/2}}{t}\right)$$

for some $c > 0$. The stated upper tail bounds follow from (3.4). □

3.4 The dyadic hierarchy

The first-order formula retains one upper binary digit. We now retain r such digits.

Fix $r \ge 1$. Identify $0, \dots, 2^r - 1$ with $\mathbb{F}_2^r$. For $1 \le a < 2^r$, define

$$\Pi_{a,J}(t) = \prod_{j=aJ}^{(a+1)J-1} \frac{1 - e^{-tj}}{1 + e^{-tj}}. \tag{3.14}$$

Set

$$\mathcal{H}_r(t, J) = 2^{-r} \sum_{u \in \mathbb{F}_2^r} \prod_{\substack{1 \le a < 2^r \\ \langle u, a \rangle = 1}} \Pi_{a,J}(t). \tag{3.15}$$

An empty product is equal to one.

Theorem 3.7 *For every fixed $r \geq 1$,*

$$R(e^{-t}) = \frac{\mathcal{H}_r(t,J)}{J} + O_r\left(\frac{\beta_J(t)}{J} + \frac{e^{-2^r tJ}}{tJ}\right) \tag{3.16}$$

uniformly as t tends to zero.

Moreover,

$$\mathcal{H}_r(t,J) \geq 2^{-r},$$

and

$$\frac{R(e^{-t})}{\mathcal{H}_r(t,J)/J} = 1 + O_r\left(\beta_J(t) + \frac{e^{-2^r tJ}}{t}\right) = 1 + O_r\left(t^{2^{r-1/2}-1}\right). \tag{3.17}$$

Proof Let E_r be the event that no selected integer is at least $2^r J$. For $1 \leq a < 2^r$, let $C_a = \sum_{j=aJ}^{(a+1)J-1} I_j \pmod 2$. On E_r, the r binary digits above the lowest m digits have XOR $U_r = \bigoplus_{a:C_a=1} a$.

Character orthogonality and independence give

$$\begin{aligned}
\mathbb{P}(U_r = 0) &= 2^{-r} \sum_{u \in \mathbb{F}_2^r} \mathbb{E}\left[(-1)^{\langle u, U_r\rangle}\right] \\
&= 2^{-r} \sum_{u \in \mathbb{F}_2^r} \prod_{\substack{1 \leq a < 2^r \\ \langle u,a\rangle = 1}} \mathbb{E}\left[(-1)^{C_a}\right] \\
&= \mathcal{H}_r(t,J).
\end{aligned}$$

Here $\mathbb{E}[(-1)^{C_a}] = \Pi_{a,J}(t)$.

Conditional on the selected integers in $[J, 2^r J)$, the lower block can cancel their lowest m digits only when $U_r = 0$. When this condition holds, (3.9) gives the required lower block probability as

$$\frac{1}{J} + O\left(\frac{\beta_J(t)}{J}\right)$$

uniformly in the residue.

Also, $\mathbb{P}(E_r^c) = O(e^{-2^r tJ}/t)$. On E_r^c, the lower block must again match one prescribed residue. These observations give (3.16).

For real t, every factor in (3.15) is nonnegative. The trivial character contributes one. Hence

$$\mathcal{H}_r(t,J) \geq 2^{-r}.$$

Dividing (3.16) by $\mathcal{H}_r(t,J)/J$ gives the first estimate in (3.17).

The term $\beta_J(t)$ is smaller than every power of t. The bound

$$tJ \geq 2^{-1/2} \log(1/t)$$

gives

$$\frac{e^{-2^r tJ}}{t} = O\left(t^{2^{r-1/2}-1}\right).$$

This proves the final estimate. □

The first member is the one-block formula $\mathcal{H}_1(t,J) = (1 + \Pi_J(t))/2$. The second member has a simple explicit form.

Corollary 3.8 *For $a = 1, 2, 3$, write $\Pi_a = \Pi_{a,J}(t)$. Then*

$$R(e^{-t}) = \frac{1 + \Pi_1\Pi_2 + \Pi_1\Pi_3 + \Pi_2\Pi_3}{4J} + O\left(\frac{\beta_J(t)}{J} + \frac{e^{-4tJ}}{tJ}\right). \tag{3.18}$$

Proof The three nonzero block labels in $\mathbb{F}_2^2$ are

$$01, \qquad 10, \qquad 11.$$

The four characters contribute

$$1, \qquad \Pi_1\Pi_3, \qquad \Pi_2\Pi_3, \qquad \Pi_1\Pi_2$$

to (3.15). The result follows from Theorem 3.7. □

Corollary 3.9 *For every $A > 0$, there is a fixed integer $r = r(A)$ such that*

$$R(e^{-t}) = \frac{\mathcal{H}_r(t, J)}{J}\left(1 + O_A(t^A)\right). \tag{3.19}$$

Proof Choose r so that $2^{r-1/2} - 1 > A$, and apply (3.17). □

3.5 The sawtooth

The transition factor in Theorem 3.5 has a direct probabilistic meaning. Suppose that $\Lambda(t)$ has a finite limit. The inclusion probabilities in $[J, 2J)$ vanish and sum to $\Lambda(t)(1+o(1))$. Hence K_J is asymptotically Poisson, whose even probability is $(1 + e^{-2\lambda})/2$. This is the source of the factor in (3.10).

The next result describes the outer phase and the transition between two neighboring phases.

Corollary 3.10 *Let $\theta \in (0, 1)$ and suppose that $\{\log_2 L(t)\} \to \theta$. Then*

$$L(t)R(e^{-t}) \longrightarrow 2^{\theta-1}.$$

Consequently,

$$\liminf_{t\downarrow 0} L(t)R(e^{-t}) = \frac{1}{2}, \qquad \limsup_{t\downarrow 0} L(t)R(e^{-t}) = 1.$$

Let K be a power of two. If $tK - \log(1/t) \to c \in \mathbb{R}$, then

$$KR(e^{-t}) \longrightarrow \frac{1 + e^{-2e^{-c}}}{2}.$$

Proof Write $2^m < L < 2^{m+1}$. Away from a dyadic boundary, the nearest power is either 2^m or 2^{m+1}.

If $J = 2^m$, then $\Lambda \to \infty$ and $R(e^{-t}) \sim 2^{-m-1}$. If $J = 2^{m+1}$, then $\Lambda \to 0$ and the same asymptotic relation holds. Since $L/2^m \to 2^\theta$, we obtain

$$L(t)R(e^{-t}) \longrightarrow 2^{\theta-1}.$$

In the boundary window, $J = K$ and $\Lambda(t) = e^{-tK}/t \to e^{-c}$. The transition formula follows from Theorem 3.5. □

A simpler model gives the same sawtooth without the Poisson smoothing.

Proposition 3.11 *Let Z_n be the XOR of a uniformly selected subset of $\{1, \dots, n\}$. Then*

$$\mathbb{P}(Z_n = 0) = 2^{-\lfloor \log_2 n \rfloor - 1}.$$

Hence

$$n\mathbb{P}(Z_n = 0) = 2^{\{\log_2 n\} - 1} \in [1/2, 1).$$

Proof Let $s = \lfloor \log_2 n \rfloor$. Consider the linear map

$$\mathbb{F}_2^n \longrightarrow \mathbb{F}_2^{s+1}$$

defined by

$$(\varepsilon_1, \dots, \varepsilon_n) \longmapsto \bigoplus_{j:\varepsilon_j = 1} j.$$

Its image contains

$$1, 2, 4, \dots, 2^s.$$

The map is therefore surjective. All fibers have the same size, so the probability of every XOR value is $2^{-(s+1)}$. □

4 The saddle arc

To recover coefficients, we must evaluate $R(q)$ near the positive real axis. The argument of the preceding section continues to work. The three blocks remain the same. Only their weights become complex.

Let $z = t - i\vartheta$ and suppose that $|\vartheta| \le t^{11/8}$. The quantities $J = J(t)$ and $\Lambda = \Lambda(t)$ retain their real-axis definitions.

Remark 4.1 The exponent $11/8$ is convenient but not essential. With the cutoff $|\vartheta| \le t^\alpha$, the cubic remainder, Gaussian tail, and variation bounds require, respectively, $3\alpha - 4 > 0$, $2\alpha - 3 < 0$, and $2(\alpha - 1) > 1 - 1/\sqrt{2}$. Thus every $4/3 < \alpha < 3/2$ works. We use $11/8$, which lies strictly inside this interval.

4.1 Complex weights

For each positive integer j, define

$$w_{j,0}(z) = \frac{1}{1 + e^{-zj}}, \qquad w_{j,1}(z) = \frac{e^{-zj}}{1 + e^{-zj}}.$$

The two weights sum to one. They need not be real or positive.

For any finite set of indices, these weights define a complex measure on its subsets. The XOR of a subset is defined as before.

4.2 The lower block

Let $\mu_{z,J}(x)$ be the normalized complex weight of subsets of $\{1, \ldots, J-1\}$ whose XOR is x.

Lemma 4.2 *Uniformly for $|\vartheta| \le t^{11/8}$ and $x < J$,*

$$\mu_{z,J}(x) = \frac{1 + o(1)}{J}.$$

Proof Walsh inversion remains valid over the complex numbers. For $u \neq 0$, the corresponding transform is

$$\widehat{\mu}_z(u) = \prod_{\substack{1 \le j < J \\ \langle u,j \rangle = 1}} \tanh(zj/2).$$

We first show that $x \mapsto |\tanh(zx/2)|$ is increasing on $[0, J]$ when t is sufficiently small. We have

$$\frac{d}{dx} \log |\tanh(zx/2)| = \Re\left(\frac{z}{\sinh(zx)}\right).$$

Put $a = tx$ and $b = \vartheta x$. After multiplication by a positive denominator, the numerator of this real part is

$$t \sinh(a)\cos(b) + \vartheta \cosh(a)\sin(b).$$

This expression is positive because $|b| \le |\vartheta| J = o(1)$ and $\vartheta \sin(\vartheta x) \ge 0$.

Every nonzero character selects $J/2$ indices. The modulus of its transform is therefore maximized by selecting the largest $J/2$ indices. Hence

$$|\widehat{\mu}_z(u)| \le \prod_{j=J/2}^{J-1} |\tanh(zj/2)|.$$

Put $\rho = e^{-tj}$ and $\phi = \vartheta j$. Then

$$\left|\frac{1 - \rho e^{i\phi}}{1 + \rho e^{i\phi}}\right|^2 = \frac{1 - 2\rho\cos(\phi) + \rho^2}{1 + 2\rho\cos(\phi) + \rho^2}.$$

Uniformly on the saddle arc, $\cos(\phi) \ge 1/2$, so $\log|\tanh(zj/2)| \le -ce^{-tj}$ for some $c > 0$. Therefore

$$|\widehat{\mu}_z(u)| \le \exp\left(-c \sum_{j=J/2}^{J-1} e^{-tj}\right) \le \exp\left(-c' \frac{e^{-tJ/2}}{t}\right)$$

for some $c' > 0$.

The final expression is smaller than every power of t. This bound is uniform over all nonzero characters. Summing the $J-1$ nontrivial terms in the Walsh inversion formula proves the result. □

4.3 The other blocks

The complex weights need not have total variation one. We now show that their excess variation is negligible on the two remaining blocks.

Lemma 4.3 *Uniformly for $|\vartheta| \le t^{11/8}$, the total variation norm of the normalized complex product measure on $[J, 2J)$ is $1+o(1)$. The total variation mass of the configurations that use at least one integer $j \ge 2J$ is $o(1)$.*

Proof Put $r = e^{-t}$ and $\phi_j = \vartheta j$. For one coordinate, the total variation factor is

$$\frac{1 + r^j}{|1 + r^j e^{i\phi_j}|}.$$

On $[J, 2J)$, we have $r^j = o(1)$ and $|\phi_j| \le 2|\vartheta|J = o(1)$. It follows that

$$\log \frac{1 + r^j}{|1 + r^j e^{i\phi_j}|} \ll r^j \phi_j^2.$$

The logarithm of the total variation norm of the block is therefore at most

$$C(\vartheta J)^2 \sum_{j=J}^{2J-1} e^{-tj}.$$

Using $J \asymp \ell/t$ and $|\vartheta| \le t^{11/8}$, this expression is bounded by

$$t^{3/4} \ell^2 t^{-(1-1/\sqrt{2})},$$

which tends to zero.

For the upper block, phase control is not needed. Its nonempty total variation mass is bounded by

$$O\left(\sum_{j \ge 2J} e^{-tj} \right) = o(1).$$

□

4.4 Stability

We now combine the three blocks. The result is the complex counterpart of Theorem 3.5.

Theorem 4.4 *Uniformly for $|\vartheta| \le t^{11/8}$,*

$$R(e^{-t+i\vartheta}) = \frac{1 + e^{-2\Lambda(t)} + o(1)}{2J(t)} = R(e^{-t})(1 + o(1)). \tag{4.1}$$

Proof Let $\mu_<, \mu_0, \mu_>$ be the normalized complex XOR measures generated by $[1, J)$, $[J, 2J)$, and $[2J, \infty)$. Write $\|\cdot\|_{\mathrm{TV}}$ for total variation.

Lemma 4.2 gives

$$\mu_<(x) = \frac{1}{J} + \varepsilon_x, \qquad \sup_{x<J} |\varepsilon_x| = o(J^{-1}), \qquad \sup_x |\mu_<(x)| = O(J^{-1}).$$

Lemma 4.3 gives $\|\mu_0\|_{\mathrm{TV}} = 1 + o(1)$.

Separate the empty upper configuration by writing $\mu_> = w_\varnothing(z)\delta_0 + \nu_>$, where $w_\varnothing(z) = \prod_{j \ge 2J}(1 + e^{-zj})^{-1}$ and $\nu_>$ is the contribution of all nonempty configurations.

Lemma 4.3 gives $\|\nu_>\|_{\mathrm{TV}} = o(1)$. The total mass of $\mu_>$ is one. Hence

$$|w_\varnothing(z) - 1| = \left|\sum_x \nu_>(x)\right| \le \|\nu_>\|_{\mathrm{TV}} = o(1).$$

Define $\eta = (w_\varnothing(z) - 1)\delta_0 + \nu_>$. Then $\mu_> = \delta_0 + \eta$ and $\|\eta\|_{\mathrm{TV}} = o(1)$.

The part of the full XOR convolution that contains η is bounded by

$$\sup_x |\mu_<(x)| \|\mu_0\|_{\mathrm{TV}} \|\eta\|_{\mathrm{TV}} = o(J^{-1}).$$

We may therefore retain only the δ_0 component of the upper block.

Every integer in $[J, 2J)$ has its J-bit equal to one. The J-bit of the middle XOR is zero if and only if the number of selected integers in that block is even. When this condition holds, the lower block must match the remaining m binary digits.

It follows that

$$\begin{aligned} R(e^{-z}) &= \sum_{\substack{y<2J \\ J\text{-bit of } y \text{ is zero}}} \mu_0(y)\mu_<(y) + o(J^{-1}) \\ &= \frac{1}{J} \sum_{\substack{y<2J \\ J\text{-bit of } y \text{ is zero}}} \mu_0(y) + O\left(\sup_x |\varepsilon_x| \|\mu_0\|_{\mathrm{TV}}\right) + o(J^{-1}). \end{aligned}$$

The normalized complex weight of the even middle configurations is $(1+\Pi_J(z))/2$, where $\Pi_J(z) = \prod_{j=J}^{2J-1}(1 - e^{-zj})/(1 + e^{-zj})$. Thus

$$R(e^{-z}) = \frac{1 + \Pi_J(z)}{2J} + o(J^{-1}).$$

It remains to estimate $\Pi_J(z)$. Uniformly on the saddle arc,

$$\log \Pi_J(z) = -2 \sum_{j=J}^{2J-1} e^{-zj} + o(1).$$

Also,

$$\sum_{j=J}^{2J-1} e^{-zj} = \frac{e^{-zJ}(1 - e^{-zJ})}{1 - e^{-z}}.$$

Suppose first that $\Lambda(t)$ is bounded. Since $|\vartheta|J = o(1)$, the last expression is $\Lambda(t)(1+o(1))$, and hence $\Pi_J(z) = e^{-2\Lambda(t)} + o(1)$.

Suppose next that $\Lambda(t) \to \infty$. The real part of the sum is $\Lambda(t)(1+o(1))$, so $\Pi_J(z) = o(1)$. Since also $e^{-2\Lambda(t)} = o(1)$, the same formula holds in this case.

Therefore

$$R(e^{-z}) = \frac{1 + e^{-2\Lambda(t)} + o(1)}{2J(t)}.$$

The real-axis formula in Theorem 3.5 gives the second equality in (4.1). □

5 Fixed total size

The Boltzmann model allows the total size to vary. We now condition on the total size being N.

The denominator has the usual partition saddle. The numerator has two equal saddles. One is near $q = e^{-t_N}$. The other is near $q = -e^{-t_N}$. The second saddle appears because $A(q)$ is an even function.

5.1 The partition saddle

For $t > 0$, define

$$\mu(t) = \sum_{j \geq 1} \frac{j}{e^{tj} - 1}, \qquad \sigma^2(t) = \sum_{j \geq 1} \frac{j^2 e^{tj}}{(e^{tj} - 1)^2}.$$

The first quantity is the expected total size of a Boltzmann partition. The second is its variance.

For every $N \geq 1$, let t_N be the unique solution of $\mu(t_N) = N$. This is the standard saddle for unrestricted partitions (Fristedt 1993; Hardy and Ramanujan 1918).

For $\psi \in \mathbb{R}$, put $d(\psi) = \min_{k \in \mathbb{Z}} |\psi - 2\pi k|$.

Lemma 5.1 *There are constants $s_0 > 0$ and $c_0 > 0$ such that, for every $0 < s \leq s_0$ and every $\psi \in \mathbb{R}$,*

$$\frac{|P(e^{-s+i\psi})|}{P(e^{-s})} \leq \exp\left[-c_0 \min\left\{\frac{d(\psi)^2}{s^3}, \frac{1}{s}\right\}\right]. \tag{5.1}$$

Proof The product expansion gives

$$\log \frac{P(e^{-s})}{|P(e^{-s+i\psi})|} = \sum_{j,k \geq 1} \frac{e^{-sjk}}{k} \left(1 - \cos(jk\psi)\right).$$

By periodicity, we may assume that $|\psi| = d(\psi) \leq \pi$. The result is immediate when $\psi = 0$.

Put $n_s = \lfloor 1/s \rfloor$ and retain only the terms with $k = 1$ and $1 \leq j \leq n_s$.

Suppose first that $d(\psi) \leq 4s$. Then $|j\psi| \leq 4$ for $j \leq n_s$. Since $e^{-sj} \geq e^{-1}$ and $1 - \cos x \geq cx^2$ on $[-4, 4]$, the retained sum is at least

$$c' d(\psi)^2 \sum_{j=1}^{n_s} j^2 \geq c'' \frac{d(\psi)^2}{s^3}.$$

Suppose next that $4s \leq d(\psi) \leq \pi$. Then

$$\begin{aligned} \sum_{j=1}^{n_s} (1 - \cos(j\psi)) &\geq n_s - \left|\sum_{j=1}^{n_s} e^{ij\psi}\right| \\ &\geq n_s - \frac{1}{|\sin(\psi/2)|} \\ &\geq \frac{c'''}{s} \end{aligned}$$

for all sufficiently small s. Since $e^{-sj} \geq e^{-1}$, the same lower bound holds, up to a constant, for the retained weighted sum. This proves (5.1). □

Lemma 5.2 *As N tends to infinity,*

$$t_N = \frac{\pi}{\sqrt{6N}} + O(N^{-1}) \tag{5.2}$$

and

$$\sigma^2(t_N) \sim \frac{\pi^2}{3t_N^3}.$$

Put $t = t_N$, $r = e^{-t}$, and $\delta = t^{11/8}$. Uniformly for $|\theta| \le \delta$,

$$\frac{P(re^{i\theta})}{P(r)} e^{-iN\theta} = \exp\left(-\frac{\sigma^2(t)\theta^2}{2}\right)(1 + o(1)). \tag{5.3}$$

Moreover,

$$\int_{\delta \le |\theta| \le \pi} \frac{|P(re^{i\theta})|}{P(r)}\, d\theta = o(t^{3/2}). \tag{5.4}$$

Consequently,

$$p(N) = \frac{e^{Nt} P(e^{-t})}{\sqrt{2\pi}\,\sigma(t)}(1 + o(1)). \tag{5.5}$$

Proof Euler–Maclaurin summation applied to $f(x) = x/(e^x - 1)$ gives

$$\mu(t) = \frac{\pi^2}{6t^2} - \frac{1}{2t} + O(1).$$

Since $\mu(t_N) = N$, this proves (5.2).

Differentiating gives $-\mu'(t) = \sigma^2(t)$. A second Euler–Maclaurin estimate gives

$$\sigma^2(t) = \frac{\pi^2}{3t^3} + O(t^{-2}).$$

We next study the central arc. The product formula for P gives

$$\log P(e^{-t+i\theta}) = \sum_{j,k \ge 1} \frac{e^{-tjk} e^{i\theta jk}}{k}.$$

Expanding at $\theta = 0$, the linear and quadratic terms are $i\mu(t)\theta$ and $-\sigma^2(t)\theta^2/2$. The remainder is bounded by

$$C|\theta|^3 \sum_{j,k \ge 1} j^3 k^2 e^{-tjk}.$$

The sum is $O(t^{-4})$. Hence the remainder is $O(|\theta|^3 t^{-4}) = O(t^{1/8})$ uniformly for $|\theta| \le t^{11/8}$. Since $\mu(t) = N$, this proves (5.3).

For the remaining arc, Lemma 5.1 gives

$$\frac{|P(e^{-t+i\theta})|}{P(e^{-t})} \le \exp\left[-c_0 \min\left\{\frac{\theta^2}{t^3}, \frac{1}{t}\right\}\right].$$

Therefore

$$\int_{\delta \le |\theta| \le \pi} \frac{|P(e^{-t+i\theta})|}{P(e^{-t})}\, d\theta \le 2\int_{\delta}^{4t} \exp\left(-c_0 \frac{\theta^2}{t^3}\right) d\theta + 2\pi e^{-c_0/t}.$$

Since

$$\frac{\delta^2}{t^3} = t^{-1/4},$$

the right side is $o(t^{3/2})$. This proves (5.4). Cauchy's formula and the Gaussian integral on the central arc give (5.5). □

5.2 Two equal saddles

We now apply the complex estimate from the preceding section.

Proposition 5.3 *For even N,*

$$\frac{a_N}{p(N)} = 2R(e^{-t_N})(1 + o(1)). \tag{5.6}$$

For odd N,

$$a_N = 0.$$

Proof Put $t = t_N$, $r = e^{-t}$, and $\delta = t^{11/8}$. Cauchy's formula gives

$$a_N = \frac{e^{Nt}}{2\pi} \int_{-\pi}^{\pi} A(re^{i\theta}) e^{-iN\theta}\, d\theta.$$

Consider first the arc $|\theta| \le \delta$. Theorem 4.4 gives

$$A(re^{i\theta}) = R(r)P(re^{i\theta})(1 + o(1))$$

uniformly on this arc. Lemma 5.2 then shows that the arc contributes

$$R(r)p(N)(1 + o(1)).$$

There is an equal contribution near $\theta = \pi$. By Lemma 2.1, $A(-q) = A(q)$. When N is even, the substitution $\theta = \pi + \phi$ identifies the arcs $|\theta - \pi| \le \delta$ and $|\phi| \le \delta$.

It remains to control the complement. Proposition 2.3 gives $A(q) = P(q^2)D_0(q)$. The coefficients of D_0 are nonnegative. Hence

$$|A(re^{i\theta})| \le |P(r^2 e^{2i\theta})| D_0(r).$$

Since $A(r) = P(r^2)D_0(r)$, Lemma 5.1, applied with $s = 2t$ and $\psi = 2\theta$, gives

$$\frac{|A(re^{i\theta})|}{A(r)} \le \exp\left[-c_0 \min\left\{\frac{d(2\theta)^2}{(2t)^3}, \frac{1}{2t}\right\}\right].$$

Outside the two deleted arcs, $d(2\theta) \ge 2\delta$. Hence

$$\frac{d(2\theta)^2}{(2t)^3} \ge \frac{(2\delta)^2}{(2t)^3} = \frac{1}{2} t^{-1/4}.$$

It follows that

$$\frac{|A(re^{i\theta})|}{A(r)} \le \exp(-ct^{-1/4})$$

throughout the complement.

The complement contributes $o(A(r)t^{3/2})$. It is negligible relative to either major arc. Adding the two equal major arcs gives

$$a_N = 2R(r)p(N)(1 + o(1)).$$

This proves (5.6).

The assertion for odd N follows from Lemma 2.1. □

6 The asymptotic law

We can now state the answer for partitions of a fixed integer. Section 3 uses t, L, J, Λ in the Boltzmann model. In this section, the starred quantities $L_N^*, J_N^*, \Lambda_N^*$ are evaluated at the exact partition saddle t_N. The unstarred quantities B_N, J_N, Λ_N use the explicit approximation $\tau_N = \pi/\sqrt{6N}$. We first state the result at the exact saddle and then replace it with the explicit quantities.

6.1 The exact saddle

Define

$$L_N^* = \frac{\log(1/t_N)}{t_N}, \qquad J_N^* = 2^{\lfloor \log_2 L_N^* + 1/2 \rfloor}, \qquad \Lambda_N^* = \frac{e^{-t_N J_N^*}}{t_N}. \tag{6.1}$$

At an exact multiplicative tie, either neighboring power of two may be used.

Theorem 6.1 *For even N,*

$$\frac{a_N}{p(N)} = \frac{1 + e^{-2\Lambda_N^*} + o(1)}{J_N^*}. \tag{6.2}$$

For odd N,

$$a_N = 0.$$

Proof Theorem 3.5 gives the Boltzmann probability at $t = t_N$. Proposition 5.3 multiplies this probability by two. This gives (6.2). The assertion for odd N follows from Lemma 2.1. □

Formula (6.2) is exact at first order. Its only implicit quantity is the partition saddle t_N.

6.2 An explicit scale

Put

$$\tau_N = \frac{\pi}{\sqrt{6N}}, \qquad B_N = \frac{\log(1/\tau_N)}{\tau_N} = \frac{\sqrt{6N}}{\pi} \log\left(\frac{\sqrt{6N}}{\pi}\right). \tag{6.3}$$

Let J_N be a power of two nearest to B_N in multiplicative distance, and put $\Lambda_N = e^{-\tau_N J_N}/\tau_N$.

Corollary 6.2 *For even N,*

$$\frac{a_N}{p(N)} = \frac{1 + e^{-2\Lambda_N} + o(1)}{J_N}. \tag{6.4}$$

At an exact multiplicative tie, either neighboring power of two gives the same first-order value.

Proof Lemma 5.2 gives $t_N = \tau_N + O(\tau_N^2)$. Also, $J_N \asymp B_N \asymp \tau_N^{-1} \log(1/\tau_N)$. Hence $(t_N - \tau_N)J_N = o(1)$ and $\log(t_N/\tau_N) = o(1)$. Replacing t_N by τ_N therefore changes the exponent defining Λ_N by $o(1)$.

The nearest dyadic power can change only near a multiplicative tie. Let the neighboring powers be J and $2J$. For the lower choice, $\Lambda_N \to \infty$ and the numerator tends to one. For the upper choice, $\Lambda_N \to 0$ and the numerator tends to two. Both choices therefore give $1/J$ at first order. □

6.3 The dyadic phase

The explicit formula shows that the density has no single asymptotic constant.

Corollary 6.3 *Along even N,*

$$\frac{a_N}{p(N)} = \Theta\left(\frac{1}{B_N}\right) = \Theta\left(\frac{1}{\sqrt{N}\log N}\right).$$

Moreover,

$$\liminf_{\substack{N\to\infty \\ 2|N}} B_N \frac{a_N}{p(N)} = 1, \qquad \limsup_{\substack{N\to\infty \\ 2|N}} B_N \frac{a_N}{p(N)} = 2. \tag{6.5}$$

If $\{\log_2 B_N\} \to \theta$ for some $\theta \in (0,1)$, then

$$B_N \frac{a_N}{p(N)} \longrightarrow 2^\theta. \tag{6.6}$$

Let K be a power of two. If

$$\tau_N K - \log(1/\tau_N) \longrightarrow c \in \mathbb{R}, \tag{6.7}$$

then

$$K \frac{a_N}{p(N)} \longrightarrow 1 + e^{-2e^{-c}}. \tag{6.8}$$

The set of subsequential limits of $B_N a_N/p(N)$ is the full interval $[1,2]$.

Proof Write $2^m < B_N < 2^{m+1}$. Away from a dyadic boundary, the nearest power is either 2^m or 2^{m+1}.

If $J_N = 2^m$, then $\Lambda_N \to \infty$ and $a_N/p(N) \sim 2^{-m}$. If $J_N = 2^{m+1}$, then $\Lambda_N \to 0$ and the numerator in (6.4) tends to two, so again $a_N/p(N) \sim 2^{-m}$.

If $\{\log_2 B_N\} \to \theta$, then $B_N/2^m \to 2^\theta$. This proves (6.6).

Under (6.7), $e^{-\tau_N K}/\tau_N \to e^{-c}$ and $K/B_N \to 1$, so K is the nearest dyadic scale for all sufficiently large N. Substitution into (6.4) gives (6.8).

Finally, B_N is increasing and its relative increments tend to zero. Subsequences can therefore approach every binary phase and every finite value of c. Formula (6.6) gives the interior of $[1,2]$. The two endpoints follow from (6.8) by letting c tend to minus infinity or plus infinity. □

Figures 1 and 2 illustrate the outer dyadic phase and its Poisson boundary transition.

6.4 Other nim-sums

The result is not special to XOR zero.

Fix $x \in \mathbb{Z}_{\geq 0}$, and let $a_N(x)$ be the number of partitions of N with nim-sum x.

Corollary 6.4 *We have $a_N(x) = 0$ unless $N \equiv x \pmod 2$. Along the admissible parity class,*

$$\frac{a_N(x)}{p(N)} = \frac{1 + e^{-2\Lambda_N} + o(1)}{J_N}. \tag{6.9}$$

The phase limit, the boundary profile, and the interval of subsequential limits are the same for every fixed x.

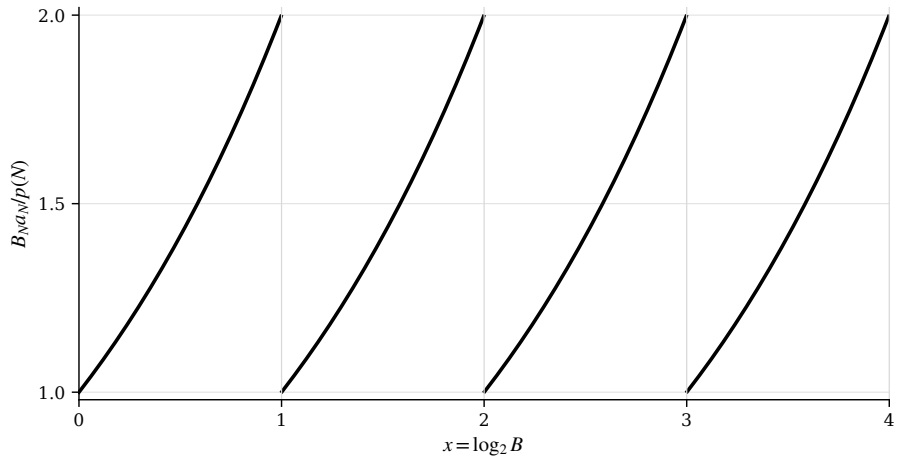


Fig. 1 The dyadic sawtooth. Away from shrinking neighborhoods of integer values of $\log_2 B_N$, the normalized density converges to $2^{\{\log_2 B_N\}}$. It rises from 1 to 2 during each dyadic epoch and returns to 1 at the next epoch.

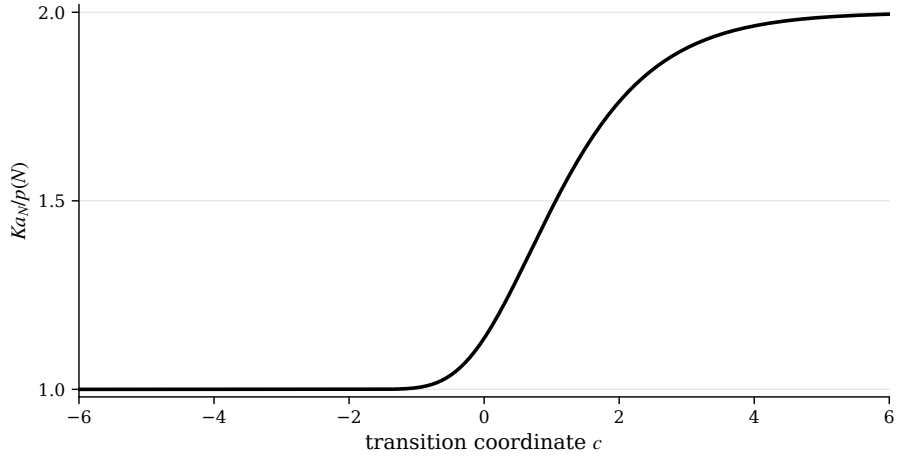


Fig. 2 The Poisson transition. At a dyadic boundary, the apparent jump from 2 to 1 is resolved by the limiting profile $1+e^{-2e^{-c}}$. As N increases, c decreases, so the curve is traversed from right to left.

Proof Define

$$D_x(q) = \sum_{\substack{S\subset\mathbb{N} \text{ finite} \\ \bigoplus_{j\in S} j = x}} q^{\sum_{j\in S} j}, \qquad A_x(q) = P(q^2)D_x(q), \qquad R_x(q) = \frac{A_x(q)}{P(q)}.$$

The odd core argument gives

$$R_x(e^{-t}) = \mathbb{P}(Y_t = x).$$

Since x is fixed and $J(t)$ tends to infinity, we have

$$x < J(t)$$

for all sufficiently small t. Conditional on the middle and upper blocks, the lower block must match a translated residue. Corollary 3.3 is uniform over the residue. Since the J-bit of x is zero, the parity condition on the critical block is unchanged. The proof of Theorem 3.5 therefore gives

$$R_x(e^{-t}) = \frac{1 + e^{-2\Lambda(t)} + o(1)}{2J(t)}.$$

The translation also leaves the complex lower block estimate and the saddle argument unchanged.

Ordinary addition and XOR have the same parity. Every exponent in $D_x(q)$ is therefore congruent to x modulo two. Hence $A_x(-q) = (-1)^x A_x(q)$. The second saddle is multiplied by $(-1)^{N+x}$, and the two saddles add when $N \equiv x \pmod 2$. When the parities differ, the coefficient is zero. Proposition 5.3 and the replacement of t_N by τ_N give (6.9). □

6.5 Two complementary checks

The dyadic hierarchy also gives fixed total size approximations.

Remark 6.5 Fix $r \geq 1$. For even N,

$$\frac{a_N}{p(N)} = \frac{2\mathcal{H}_r(t_N, J_N^*)}{J_N^*}(1 + o(1)). \tag{6.10}$$

The hierarchy determines the Boltzmann component to any prescribed algebraic order. The unquantified conditioning error still determines the final precision at fixed total size. The same statement holds for every fixed target nim-sum on its admissible parity class.

There is also a proof of the order of magnitude that does not use the complex saddle argument.

Remark 6.6 Put $t_n = \pi/(2\sqrt{3n})$ and $\mathcal{R}_n = R(e^{-t_n})$. Appendix B proves that, for all sufficiently large n,

$$\frac{e^{-21}}{8}\mathcal{R}_n \leq \frac{a_{2n}}{p(2n)} \leq 32\mathcal{R}_n.$$

Together with Theorem A.1, this gives an independent proof that

$$\frac{a_N}{p(N)} = \Theta\left(\frac{1}{\sqrt{N}\log N}\right).$$

This proof does not use Theorem 4.4 or Proposition 5.3.

7 Exact computation

The formulas above are asymptotic. We now compare them with exact counts.

The computation serves three purposes. It extends the known sequence, checks the exact formulas, and separates the finite-size errors into their two sources. One source is the dyadic approximation in the Boltzmann model. The other is the passage from the Boltzmann model to a fixed total size.

7.1 The exact algorithm

The Walsh formula can be reduced before it is evaluated.

Proposition 7.1 *Let M be a power of two and suppose that $0 \le n < 2M$. Every zero-XOR partition of n has all its parts below M. Consequently,*

$$a_n = \frac{1}{M} \sum_{u=0}^{M-1} [q^n] \prod_{j=1}^{M-1} \frac{1}{1-(-1)^{\langle u,j\rangle} q^j}. \tag{7.1}$$

Proof A partition of $n < 2M$ contains at most one part of size at least M. Such a part has its M-bit equal to one. Every smaller part has that bit equal to zero. The XOR cannot therefore vanish.

All relevant parts lie below M. Character orthogonality on $\mathbb{F}_2^{\log_2 M}$ gives (7.1). □

There is a second reduction for even coefficients. Define

$$G_u(q) = \prod_{j=1}^{M-1} \frac{1}{1-(-1)^{\langle u,j\rangle} q^j}.$$

Toggling the least significant bit of u changes the character sign exactly for odd j. Hence $G_{u\oplus 1}(q) = G_u(-q)$, and therefore $[q^{2k}]G_{u\oplus 1} = [q^{2k}]G_u$. Thus, for $2k < 2M$,

$$a_{2k} = \frac{1}{M/2} \sum_{\substack{0 \le u < M \\ u \equiv 0 \pmod 2}} [q^{2k}] G_u(q). \tag{7.2}$$

We use $M = 16{,}384$ to calculate all even coefficients through $N = 20{,}000$. The pairing reduces the Walsh sum from 16,384 characters to 8,192 representatives.

For each representative character, the coefficient vector is obtained from the signed partition recurrence. Positive and negative character factors correspond to multiplication by $(1-q^j)^{-1}$ and $(1+q^j)^{-1}$, respectively.

The calculation is performed modulo seventeen distinct primes. The integer coefficients are reconstructed by the Chinese remainder theorem. The product modulus has 527 bits, while $p(20{,}000)$ has 507 bits. Since $0 \le a_n \le p(n)$, the reconstruction is unique.

The calculation agrees with every term in the preceding public OEIS b-file through index 750, corresponding to total size $N = 1{,}500$. The calculation extends OEIS A048833, whose index is $k = N/2$, through index 10,000. All 10,001 reconstructed values are also checked modulo an eighteenth prime that is not used in the reconstruction. A complete rerun with the released set of seventeen prime moduli reproduced all 10,001 values.

We use a different algorithm for a further check at $N = 1{,}500$. It first computes the distinct-part XOR counts by a knapsack recursion and then applies the convolution

in Proposition 2.3. The resulting density agrees with the Walsh and CRT calculation to a relative discrepancy of

$$1.2 \times 10^{-11}$$

in double precision. This floating-point calculation is only a cross-check. Exactness follows from the CRT bound.

7.2 Computational implementation

The supplied C++ program is compiled with

```
g++ -O3 -march=native -mavx2 -fopenmp.
```

Representative characters are assigned to OpenMP threads. The coefficient updates are processed in blocks of eight using AVX2. Prime moduli and character ranges can be run independently.

For one prime, the recurrence performs approximately 1.585×10^{12} scalar coefficient updates before vector packing. The full calculation performs approximately 2.69×10^{13} updates.

The benchmark was run on an AMD Ryzen 9 9950X using sixteen OpenMP threads. At $N = 20{,}000$, the seventeen reconstruction passes over all 8,192 representative characters took 111 seconds in aggregate, with a median of 6.44 seconds per prime. The independent check-prime pass took 6.45 seconds. Including compilation, CRT reconstruction, output, and verification, one complete run took 122 seconds.

The benchmark describes this implementation on one machine. It is not used in any mathematical claim. The source code, the coefficient table, the CRT scripts, the independent checks, and the figure scripts are included in the computational supplement.

7.3 The first-order approximation

The computed range crosses the boundaries $B_N \approx 256$ at $N = 6{,}328$ and $B_N \approx 512$ at $N = 19{,}584$. The exact saddle choice changes from $J_N^* = 256$ at $N = 11{,}052$ to $J_N^* = 512$ at $N = 11{,}054$.

Let $r_N = a_N/p(N)$ and define $\widehat{r}_N = (1 + e^{-2\Lambda_N^*})/J_N^*$.

Table 1 reports six values. They include the previously published endpoint, the two dyadic boundaries, one point on each side of the branch switch, and the new endpoint.

The error at $N = 10{,}000$ is not a discontinuity in the exact density. It occurs just before the nearest dyadic scale changes from 256 to 512. The block above 512 is still relevant but is omitted from the one-block formula.

The two-level formula includes all three blocks in $[J, 4J)$. At $N = 10{,}000$, it reduces the Boltzmann approximation error from 11.09% to 0.016%.

7.4 The dyadic approximation error

We now separate the approximation error from the fixed-size conditioning error. Define $g_N = 2R(e^{-t_N})$ and $h_N^{(2)} = 2\mathcal{H}_2(t_N, J_N^*)/J_N^*$.

Table 1 Exact density and first-order prediction. The final column is $100(\widehat{r}_N/r_N - 1)\%$.

N	J_N^*	r_N	$\widehat{r}_N$	signed error
1,500	128	0.010760619781	0.011006685108	2.29%
6,328	256	0.004252024871	0.004421652536	3.99%
10,000	256	0.003532861381	0.003916981208	10.87%
12,000	512	0.003227790295	0.003222276143	−0.17%
19,584	512	0.002159193967	0.002213282158	2.51%
20,000	512	0.002127725034	0.002183078655	2.60%

The ratios $\widehat{r}_N/g_N$, $h_N^{(2)}/g_N$, and g_N/r_N measure the one-block, two-level, and fixed-size errors. The values of g_N are calculated by a finite Walsh–Hadamard transform. The omitted Bernoulli tail has mass below 4×10^{-16}.

Table 2 separates the total error into these three components at four representative values of N.

Table 2 The finite-size error. The middle columns report $100(\widehat{r}_N/g_N - 1)\%$, $100(h_N^{(2)}/g_N - 1)\%$, and $100(g_N/r_N - 1)\%$. The final column reports $100(h_N^{(2)}/r_N - 1)\%$.

N	one block	two levels	fixed size	total
10,000	11.094%	0.016%	−0.199%	−0.183%
12,000	0.084%	−0.027%	−0.255%	−0.282%
19,584	0.822%	0.00008%	1.669%	1.670%
20,000	0.922%	0.00010%	1.664%	1.664%

The one-block error is large near a branch switch. The two-level formula removes almost all of it. At the two largest values in the table, the remaining error comes almost entirely from the fixed-size step.

Theorem 3.7 gives two error scales. The low-block nonuniformity is $\beta_J(t)$, while the first omitted upper block has scale $\rho_{2,J}(t) = e^{-4tJ}/t$. Table 3 compares the observed two-level residual with these two scales.

Table 3 The two-level error budget. The first numerical column is the observed absolute relative error. The last two columns are the two error scales in Theorem 3.7.

N	observed error	$100\beta_{J_N^*}(t_N)$	$100e^{-4t_N J_N^*}/t_N$
10,000	0.016%	1.79×10^{-9}%	0.0159%
12,000	0.027%	0.0272%	3.45×10^{-7}%
19,584	0.00008%	5.16×10^{-7}%	0.000079%
20,000	0.00010%	2.76×10^{-7}%	0.000097%

The two sources alternate during a dyadic epoch. When

$$\frac{J_N^*}{L_N^*}$$

is close to $2^{1/2}$, the lower block error is larger. When the ratio is close to $2^{-1/2}$, the omitted upper block is larger. The sharp dips in the observed error occur near the points at which the two scales cross. This is consistent with cancellation between the two corrections.

Figures 3 and 4 use the exact saddle phase $\log_2 L_N^*$. The first shows the gain from the two-level approximation. The second separates its two error sources. The final figure in this section uses $\log_2 B_N$. Since $L_N^*/B_N \to 1$, the two horizontal coordinates differ by $o(1)$.

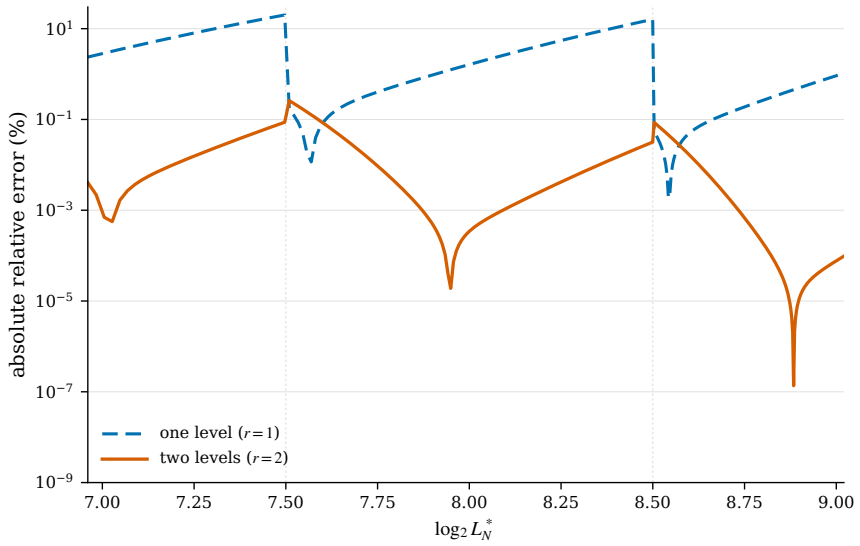


Fig. 3 The dyadic hierarchy. The figure compares the absolute relative errors of the one-block and two-level Boltzmann formulas. The vertical lines mark changes in the nearest dyadic scale.

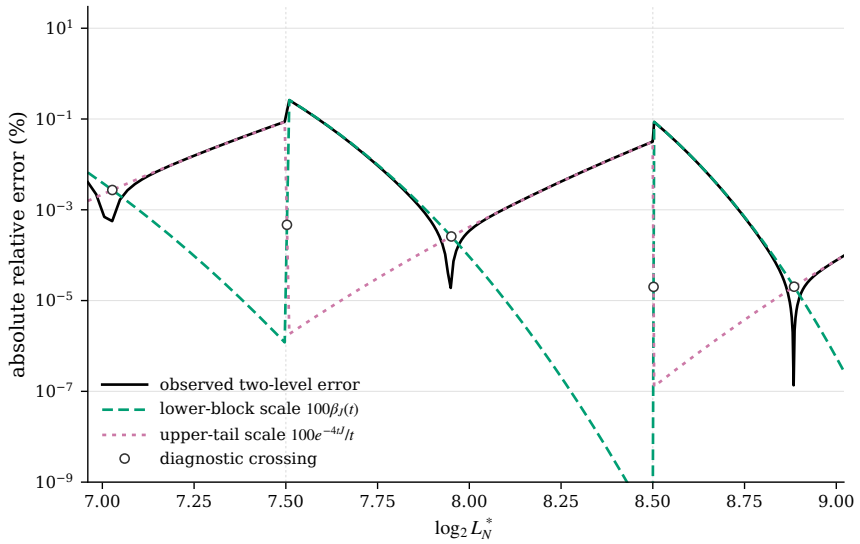


Fig. 4 The sources of the two-level error. The observed error is compared with the lower-block scale $100\beta_{J_N^*}(t_N)\%$ and the upper-tail scale $100e^{-4t_N J_N^*}/t_N\%$. Open circles mark crossings of the two scales. The nearby dips are consistent with cancellation.

7.5 The fixed-size correction

The remaining error comes from averaging the Boltzmann probability over the saddle arc.

Put $F(t) = R(e^{-t})$. Let σ_N^2 and $\kappa_{3,N}$ be the second and third cumulants of the total size under the partition Boltzmann measure at $t = t_N$. Define

$$\Delta_N^{(2)} = \frac{F''(t_N)}{2\sigma_N^2 F(t_N)} + \frac{\kappa_{3,N} F'(t_N)}{2\sigma_N^4 F(t_N)}. \tag{7.3}$$

This expression follows from a formal saddle expansion. On the central arc,

$$\frac{P(e^{-t_N+i\theta})}{P(e^{-t_N})} e^{-iN\theta} = \exp\left(-\frac{\sigma_N^2\theta^2}{2} - \frac{i\kappa_{3,N}\theta^3}{6} + \cdots\right).$$

Also,

$$F(t_N - i\theta) = F(t_N) - i\theta F'(t_N) - \frac{\theta^2}{2} F''(t_N) + \cdots.$$

The Gaussian moments are $\langle\theta^2\rangle = \sigma_N^{-2}$ and $\langle\theta^4\rangle = 3\sigma_N^{-4}$. The pure partition corrections cancel between the numerator and the denominator. Formally,

$$\frac{a_N}{p(N)} = 2F(t_N)\left[1 - \Delta_N^{(2)} + \cdots\right]. \tag{7.4}$$

Thus $g_N/(a_N/p(N)) - 1 = \Delta_N^{(2)} + \cdots$.

This calculation identifies the first derivative-dependent correction. It is not a second-order theorem. Such a theorem would require uniform bounds for the higher derivatives and for the Edgeworth remainder on the saddle arc.

The Walsh–Hadamard evaluator computes the derivatives without finite differences. Let

$$M_{\mathrm{W}} = 2^s$$

be the transform dimension. Applying the transform to

$$w_j(t) = \log\tanh(tj/2)$$

and to its first two derivatives gives

$$\ell_u, \qquad \ell'_u, \qquad \ell''_u$$

for each $u < M_{\mathrm{W}}$. The derivatives of F are then

$$F'(t) = \frac{1}{M_{\mathrm{W}}} \sum_{u<M_{\mathrm{W}}} e^{\ell_u}\ell'_u$$

and

$$F''(t) = \frac{1}{M_{\mathrm{W}}} \sum_{u<M_{\mathrm{W}}} e^{\ell_u}\left[(\ell'_u)^2 + \ell''_u\right].$$

Table 4 compares the observed fixed-size error with the formal second-order correction at four representative values of N.

Table 4 The fixed-size correction. Both numerical columns are signed percentages.

N	observed error	second-order term
10,000	−0.199%	−0.075%
12,000	−0.255%	−0.187%
19,584	1.669%	1.533%
20,000	1.664%	1.530%

Over the grid $N = 2{,}000, 2{,}050, \ldots, 20{,}000$, the observed correction and the second-order expression have correlation 0.996. Their mean absolute difference is 0.121 percentage points. Figure 5 compares the two curves over the full grid.

7.6 A contour check

The fixed-size ratio also has an exact contour representation. For every $t > 0$,

$$\frac{a_N}{p(N)} = \frac{\displaystyle\int_{-\pi}^{\pi} F(t - i\theta) P(e^{-t+i\theta}) e^{-iN\theta}\, d\theta}{\displaystyle\int_{-\pi}^{\pi} P(e^{-t+i\theta}) e^{-iN\theta}\, d\theta}. \tag{7.5}$$

For even N, the numerator integrand has period π, since $A(-q) = A(q)$. For a cutoff $C > 0$, we evaluate the two-saddle major-arc truncation

$$2\frac{\displaystyle\int_{-C/\sigma_N}^{C/\sigma_N} F(t_N - i\theta) P(e^{-t_N+i\theta}) e^{-iN\theta}\, d\theta}{\displaystyle\int_{-C/\sigma_N}^{C/\sigma_N} P(e^{-t_N+i\theta}) e^{-iN\theta}\, d\theta}. \tag{7.6}$$

Equation (7.5) is exact. Equation (7.6) replaces the numerator half-contour and the denominator full contour with the same central window.

This check tests three parts of the computation. It tests the complex Walsh evaluator, the factor of two from the second saddle, and the smallness of the omitted arcs. It does not test the separate first-order replacement

$$F(t_N - i\theta) = F(t_N)(1 + o(1)).$$

We evaluate $F(t_N - i\theta)$ by the complex Walsh–Hadamard formula. With $C = 5$, Gauss–Legendre quadrature using 256 nodes gives relative errors of order 10^{-6}. With $C = 10$, 512 nodes reduce every discrepancy below 10^{-13}.

Table 5 reports the direct major-arc check at four representative values of N.

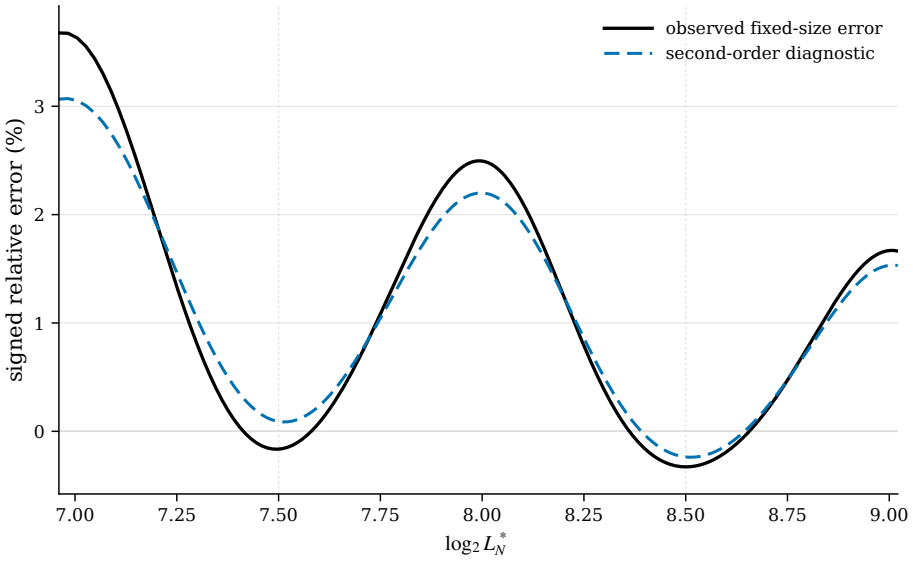


Fig. 5 The fixed-size correction. The solid line is the observed saddle-averaging error. The dashed line is the formal second-order correction $100\Delta_N^{(2)}\%$.

Table 5 The contour check. The third column is the signed relative error of (7.6). All errors in the last column are below 10^{-13}.

N	exact $a_N/p(N)$	window $5/\sigma_N$	window $10/\sigma_N$
10,000	0.003532861381	$+7.24\times10^{-7}$	$<10^{-13}$
12,000	0.003227790295	$+4.60\times10^{-8}$	$<10^{-13}$
19,584	0.002159193967	-9.28×10^{-7}	$<10^{-13}$
20,000	0.002127725034	-9.58×10^{-7}	$<10^{-13}$

The fixed-size column in Table 2 is therefore not numerical noise. It is the finite-N averaging of

$$F(t_N - i\theta)$$

by the partition saddle. The numerical truncation error in (7.6) is much smaller.

7.7 The visible range

The computed range is still far from the uniform first-order regime. At $N = 20{,}000$, we have $t_N = 0.0090565\ldots$, and the worst-phase one-block scale is $t_N^{\sqrt{2}-1} = 0.142\ldots$. Ignoring the unspecified constant, this scale falls to one percent only when N is of order 10^{10}.

The analytic saddle window is also narrow at the computed sizes. With

$$\delta = t_N^{11/8},$$

we have

$$\delta\sigma_N = 3.26\ldots$$

at $N = 20{,}000$. This quantity tends to infinity, but at the largest computed size it spans only a few standard deviations.

These numbers explain why the full sawtooth is slow to appear. They also explain why the dyadic hierarchy is useful at finite N. Figure 6 compares the exact normalized density with the first-order prediction and the limiting dyadic sawtooth.

The calculations confirm the predicted scale and the dyadic organization. They also show that the finite-size correction depends on the phase. None of the numerical calculations is used in the proof. The complete exact table through $N = 20{,}000$ is included in the computational supplement.

8 Conclusion

The main result has a simple explanation. Small parts make the lower binary digits nearly uniform. Very large parts are rare. Between these two ranges lies one dyadic block whose parity is not yet randomized. That parity determines the first-order probability of XOR zero.

This mechanism leaves a permanent binary signature in a random partition. As the natural partition scale crosses a power of two, the normalized density rises from 1

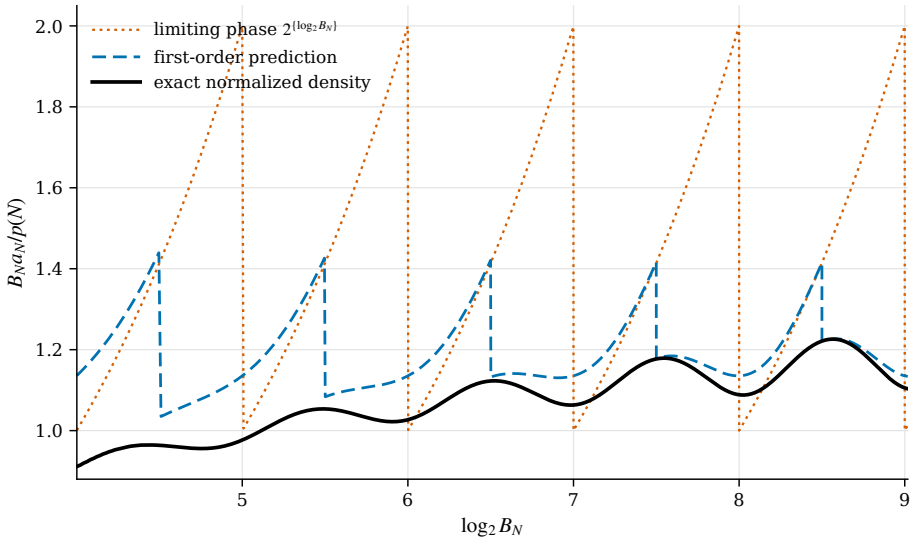


Fig. 6 Exact density and first-order prediction. The solid line is the exact normalized density $B_N a_N/p(N)$. The dashed line is the first-order formula. The dotted line is the limiting dyadic sawtooth.

to 2 and then returns to 1. The return appears as a jump on the outer scale. On the smaller boundary scale, it is resolved by the parity law of a Poisson random variable.

Problem E15 in Nowakowski (2025) asks for the number of losing Nim positions with a fixed total number of counters. This paper gives two exact representations of that number. It also gives a complete first-order asymptotic description. In particular,

$$\frac{a_N}{p(N)} = \Theta\left(\frac{1}{\sqrt{N}\log N}\right)$$

along even N, but the normalized density does not converge. Its set of subsequential limits is the entire interval $[1, 2]$.

The result is not specific to losing positions. Every fixed target nim-sum has the same first-order profile on its admissible parity class. Thus the dyadic oscillation describes the local mass of the whole XOR distribution near every fixed value.

The one-block formula is also not the end of the Boltzmann analysis. The dyadic hierarchy retains more upper binary digits and determines the Boltzmann probability to any prescribed algebraic order. The computations show why this refinement matters. At accessible values of N, the first omitted block can be visible even when the first-order theorem is already valid.

The main remaining analytic question concerns the passage to a fixed total size. The formal expansion in (7.4) identifies the first correction terms, and the contour calculation reproduces the finite-size averaging with high accuracy. A uniform second-order theorem would require higher-derivative estimates for $R(e^{-t+i\theta})$ and a uniform control of the Edgeworth remainder. The Walsh representation expresses the derivatives of R as character averages of transformed log-weights. This suggests differentiating the spectral estimates of Section 4 and combining them with a uniform Edgeworth expansion for the partition saddle.

Other questions arise naturally. One may let the target nim-sum grow with N. One may choose the depth of the dyadic hierarchy as a function of N. The distinct-part sequence recorded as OEIS A307506 has its own fixed-size asymptotic problem. Similar questions can also be asked for partitions with restrictions on their parts or on the number of heaps.

The broader point is that a smooth random structure can retain a sharp arithmetic feature. Here the feature is binary. It survives because one block remains visible after all lower information has been randomized and all higher information has become too rare to matter.

Supplementary information. Online Resource 1 contains the exact counts through $N = 20{,}000$, source code, verification scripts, and figure code.

Statements and Declarations

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Author contribution. The author conducted the research and wrote the manuscript.

Data availability. The exact counts through $N = 20{,}000$ are included in Online Resource 1. The main data file records the OEIS index k and total size $N = 2k$.

Code availability. The source code, verification scripts, and figure code are included in Online Resource 1.

Appendix A An explicit bound

The main result contains an unspecified $o(1)$. This appendix gives a weaker estimate with explicit constants. It uses only the real Boltzmann model and Walsh orthogonality.

Theorem A.1 *For $0 < t \le e^{-20}$, we have*

$$\frac{t}{12\log(1/t)} \le \mathbb{P}(Y_t = 0) \le \frac{5t}{\log(1/t)}. \tag{A1}$$

Moreover,

$$\frac{1}{4} - o(1) \le \mathbb{P}(Y_t = 0)\frac{\log(1/t)}{t} \le 2 + o(1). \tag{A2}$$

The proof uses two facts. First, the event that the XOR is zero can be approximated by congruence modulo a large power of two. Second, the Walsh coefficients are controlled by the position of their lowest nonzero binary digit.

A.1 The Walsh spectrum

Put $q_j = \tanh(tj/2)$. For a nonnegative integer u, define

$$\widehat{\nu}_t(u) = \prod_{\substack{j \ge 1 \\ \langle u,j\rangle = 1}} q_j.$$

Here $\langle u, j\rangle$ is the parity of the binary digits shared by u and j, and $\widehat{\nu}_t(0) = 1$.

Lemma A.2 *For every integer $r \ge 1$,*

$$\mathbb{P}\left(Y_t \equiv 0 \pmod{2^r}\right) = 2^{-r} \sum_{u=0}^{2^r-1} \widehat{\nu}_t(u). \tag{A3}$$

Also,

$$0 \le \mathbb{P}\left(Y_t \equiv 0 \pmod{2^r}\right) - \mathbb{P}(Y_t = 0) \le \frac{2}{t} e^{-t2^r}. \tag{A4}$$

Proof The first identity follows from character orthogonality on $\mathbb{F}_2^r$.

Suppose that all selected integers are below 2^r. If their XOR is divisible by 2^r, then the XOR is zero. The difference between the two probabilities in (A4) is therefore bounded by the probability that some selected integer is at least 2^r.

A union bound gives

$$\mathbb{P}(\exists j \geq 2^r : I_j = 1) \leq \sum_{j \geq 2^r} e^{-tj} = \frac{e^{-t2^r}}{1 - e^{-t}}.$$

For sufficiently small t,

$$1 - e^{-t} \geq \frac{t}{2}.$$

This proves (A4). □

For $m \geq 0$, let $B_m = \{2^m, \ldots, 2^{m+1} - 1\}$. For $u \neq 0$, let $v(u)$ be the position of its lowest nonzero binary digit.

Lemma A.3 *Let $u \neq 0$ and put $v = v(u)$. Then $\langle u, j \rangle = 0$ for $1 \leq j < 2^v$, while $\langle u, j \rangle = 1$ for every $j \in B_v$. For every $m > v$, exactly 2^{m-1} integers in B_m satisfy $\langle u, j \rangle = 1$.*

Proof An integer below 2^v has no nonzero binary digit in common with the lowest nonzero digit of u. Every integer in B_v has that digit equal to one and has no higher digit in common with u.

For $m > v$, the map

$$j \longmapsto \langle u, j \rangle$$

is a nonzero linear functional on the lower m binary digits. It takes each value on exactly half of the vectors. □

For $v \geq 0$, define

$$\mathcal{M}_v = \sup_{\substack{u \neq 0 \\ v(u) = v}} \widehat{\nu}_t(u).$$

The block B_v gives the first bound. Since $q_j \leq \tanh(t2^v)$ for $j \in B_v$, the inequality $\log \tanh x \leq -e^{-2x}$ gives

$$\mathcal{M}_v \leq \exp\left(-2^v e^{-2t2^v}\right). \tag{A5}$$

Small values of v admit a stronger common bound. Put $m_0 = \lceil \log_2(1/(2t)) \rceil$. If $v < m_0$, Lemma A.3 shows that exactly $2^{m_0 - 1}$ integers in B_{m_0} contribute to the product. Since

$$\frac{1}{2} \leq t2^{m_0} < 1,$$

we obtain

$$\mathcal{M}_v \leq \exp\left(-\frac{1}{4e^2 t}\right). \tag{A6}$$

We now collect these estimates. Put $\ell = \log(1/t)$ and $Y_1 = (\ell - 3\log \ell)/2$.

Lemma A.4 *For $\ell \geq 20$,*

$$\sum_{v \geq 0} 2^{-v-1} \mathcal{M}_v \leq e^{-1/(4e^2 t)} + \ell t e^{-\ell^3/2} + \frac{4t}{\ell}. \tag{A7}$$

Proof We divide the indices v into three groups.

For $v < m_0$, use (A6). Since $\sum_{v<m_0} 2^{-v-1} < 1$, this contribution is at most $e^{-1/(4e^2 t)}$.

Consider next the indices for which

$$\frac{1}{2} \leq y_v \leq Y_1, \qquad y_v = t2^v.$$

The function

$$y \longmapsto y e^{-2y}$$

is decreasing for $y \geq 1/2$. Therefore

$$2^v e^{-2t2^v} = \frac{y_v}{t} e^{-2y_v} \geq \frac{Y_1}{t} e^{-2Y_1}.$$

By definition, $e^{-2Y_1} = t\ell^3$. Thus

$$2^v e^{-2t2^v} \geq \frac{\ell^3}{2}$$

for $\ell \geq 20$. Equation (A5) gives

$$\mathcal{M}_v \leq e^{-\ell^3/2}.$$

There are fewer than ℓ such indices, and $2^{-v-1} \leq t$, so their contribution is at most $\ell t e^{-\ell^3/2}$.

Finally, consider the indices with $y_v > Y_1$. Let v_1 be the first such index. Then

$$\sum_{v \geq v_1} 2^{-v-1} = 2^{-v_1} < \frac{t}{Y_1}.$$

For $\ell \geq 20$, we have $1/Y_1 \leq 4/\ell$. This proves (A7). □

A.2 Proof of the bound

Proof of Theorem A.1 We first prove the upper bound. Choose $r = \lceil 2\log_2(1/t) \rceil$. Lemma A.2 gives

$$\mathbb{P}(Y_t = 0) \leq 2^{-r} \sum_{u=0}^{2^r - 1} \widehat{\nu}_t(u).$$

There are 2^{r-v-1} characters $u < 2^r$ whose lowest nonzero binary digit is v. Therefore

$$\mathbb{P}(Y_t = 0) \leq 2^{-r} + \sum_{v=0}^{r-1} 2^{-v-1} \mathcal{M}_v.$$

Using Lemma A.4,

$$\mathbb{P}(Y_t = 0) \leq 2^{-r} + e^{-1/(4e^2 t)} + \ell t e^{-\ell^3/2} + \frac{4t}{\ell}.$$

When $\ell \geq 20$, each of the first three terms is at most $t/(3\ell)$. Hence

$$\mathbb{P}(Y_t = 0) \leq \frac{5t}{\ell}.$$

For the lower bound, put $X = 2\ell + \log \ell + 6$, choose $r = \lceil \log_2(X/t) \rceil$, and set $x = t2^r$. Then $X \leq x < 2X$.

The trivial character in (A3) gives

$$\mathbb{P}\left(Y_t \equiv 0 \pmod{2^r}\right) \geq 2^{-r}.$$

Lemma A.2 therefore gives

$$\mathbb{P}(Y_t = 0) \geq 2^{-r} - \frac{2}{t} e^{-t2^r}.$$

Since $2^{-r} = t/x$, we obtain

$$\mathbb{P}(Y_t = 0) \geq \frac{t}{x} - \frac{2}{t} e^{-x}.$$

The choice of X implies $(2/t)e^{-x} \leq t/(2x)$. Hence

$$\mathbb{P}(Y_t = 0) \geq \frac{t}{2x}.$$

Since $x < 2X < 6\ell$ when $\ell \geq 20$,

$$\mathbb{P}(Y_t = 0) > \frac{t}{12\ell}.$$

This proves (A1).

For the asymptotic upper bound, retain

$$\frac{t}{Y_1} = (2 + o(1))\,\frac{t}{\ell}$$

instead of replacing it by $4t/\ell$. The other terms are $o(t/\ell)$. Thus

$$\mathbb{P}(Y_t = 0) \leq (2 + o(1))\,\frac{t}{\ell}.$$

For the asymptotic lower bound, replace X by $2\ell + \log \ell + \omega(\ell)$, where $\omega(\ell) \to \infty$ and $\omega(\ell) = o(\ell)$. The tail term is then $o(t/\ell)$. Since $x < 2X$,

$$\mathbb{P}(Y_t = 0) \geq \left(\frac{1}{4} - o(1)\right)\frac{t}{\ell}.$$

This proves (A2). □

Appendix B A real-variable proof

This appendix proves the order of the fixed-size density without complex analysis. It does not use the sharp dyadic formula in Theorem 6.1. The proof uses the convolution in (2.2), the explicit bound in Theorem A.1, and standard estimates for partition numbers.

Put $\mathcal{R}(t) = R(e^{-t}) = \mathbb{P}(Y_t = 0)$. For $n \geq 1$, define

$$t_n = \frac{\pi}{2\sqrt{3n}} \tag{B1}$$

and $\mathcal{R}_n = \mathcal{R}(t_n)$.

The main result compares the fixed-size density with its real Boltzmann counterpart.

Theorem B.1 *There is an integer n_0 such that, for every $n \geq n_0$,*

$$\frac{e^{-21}}{8}\mathcal{R}_n \leq \frac{a_{2n}}{p(2n)} \leq 32\mathcal{R}_n. \tag{B2}$$

Consequently, along even N,

$$\frac{a_N}{p(N)} = \Theta\left(\frac{1}{\sqrt{N}\log N}\right). \tag{B3}$$

B.1 Partition bounds

Put $c_p = \pi\sqrt{2/3}$. The Hardy–Ramanujan exponential scale is $\log p(m) \sim c_p\sqrt{m}$.

We first collect the estimates used below.

Lemma B.2 *For every integer $m \geq 0$ and every $t > 0$,*

$$p(m) \leq e^{c_p\sqrt{m}} \tag{B4}$$

and

$$\log P(e^{-t}) \leq \frac{\pi^2}{6t}.$$

Also,

$$\frac{1}{2}e^{\pi^2/(12t)} \leq D(e^{-t}) \leq e^{\pi^2/(12t)}. \tag{B5}$$

Proof The functions $f(x) = -\log(1 - e^{-x})$ and $g(x) = \log(1 + e^{-x})$ are positive and decreasing, with

$$\int_0^\infty f(x)\,dx = \frac{\pi^2}{6}, \qquad \int_0^\infty g(x)\,dx = \frac{\pi^2}{12}.$$

A comparison between the corresponding Riemann sums and the integrals gives

$$\log P(e^{-t}) \leq \frac{\pi^2}{6t}$$

and

$$\frac{\pi^2}{12t} - \log 2 \leq \log D(e^{-t}) \leq \frac{\pi^2}{12t}.$$

This proves (B5).

For any $t > 0$,

$$p(m) \leq e^{tm}P(e^{-t}) \leq \exp\left(tm + \frac{\pi^2}{6t}\right).$$

For $m \geq 1$, choose $t = \pi/\sqrt{6m}$. This gives (B4). The case $m = 0$ is immediate. □

We will also use a crude uniform version of the Hardy–Ramanujan formula.

Lemma B.3 *There is an integer m_0 such that, for every $m \geq m_0$,*

$$\frac{1}{8\sqrt{3}\,m}e^{c_p\sqrt{m}} \leq p(m) \leq \frac{1}{2\sqrt{3}\,m}e^{c_p\sqrt{m}}. \tag{B6}$$

Proof The Hardy–Ramanujan formula gives

$$p(m) \sim \frac{e^{c_p\sqrt{m}}}{4\sqrt{3}\,m}.$$

See Hardy and Ramanujan (1918). □

The explicit Boltzmann bound now gives a corresponding estimate for the zero-XOR distinct core.

Corollary B.4 *Let $0 < t \le e^{-20}$ and put $\ell = \log(1/t)$. Then*

$$\frac{t}{24\ell} e^{\pi^2/(12t)} \le D_0(e^{-t}) \le \frac{5t}{\ell} e^{\pi^2/(12t)}. \tag{B7}$$

Proof The Boltzmann identity gives $D_0(e^{-t}) = \mathcal{R}(t) D(e^{-t})$. Combine Theorem A.1 with (B5). □

B.2 Localization

The convolution in (2.2) involves the coefficients d_k. We now show that their Boltzmann mass is concentrated near one point.

Define a probability measure on $\mathbb{Z}_{\ge 0}$ by

$$\nu_t(k) = \frac{d_k e^{-2kt}}{D_0(e^{-t})}. \tag{B8}$$

Its natural center is $k_*(t) = \pi^2/(24t^2)$.

Lemma B.5 *Let $0 < t \le (4/5)e^{-20}$ and $0 < \beta \le 1$. Then*

$$\nu_t\left(\{k \,|\, |k - k_*(t)| \ge \beta k_*(t)\}\right) \le 420 \exp\left(-\frac{\beta^2}{8t}\right). \tag{B9}$$

Proof We estimate the two tails by changing the temperature.

For the right tail, put $t_- = t(1 - \beta/4)$ and $K_+ = (1 + \beta)k_*(t)$. For $k \ge K_+$,

$$e^{-2kt} \le e^{-2kt_-} e^{-2K_+(t - t_-)}.$$

Hence

$$\nu_t(k \ge K_+) \le e^{-2K_+(t-t_-)} \frac{D_0(e^{-t_-})}{D_0(e^{-t})}. \tag{B10}$$

Corollary B.4, applied at t and t_-, gives

$$\frac{D_0(e^{-t_-})}{D_0(e^{-t})} \le 120 \exp\left[\frac{\pi^2}{12}\left(\frac{1}{t_-} - \frac{1}{t}\right)\right].$$

The exponent in (B10) is at most

$$(t - t_-)\frac{\pi^2}{12t^2}\left[\frac{1}{1 - \beta/4} - (1 + \beta)\right].$$

Using $(1 - x)^{-1} \le 1 + x + 2x^2$ for $0 \le x \le 1/4$, we obtain

$$\nu_t\left(k \ge (1 + \beta)k_*(t)\right) \le 120 e^{-\beta^2/(8t)}.$$

For the left tail, put $t_+ = t(1 + \beta/4)$ and $K_- = (1 - \beta)k_*(t)$. For $k \le K_-$,

$$e^{-2kt} \le e^{-2kt_+} e^{2K_-(t_+ - t)}.$$

Corollary B.4, applied at t and t_+, gives

$$\nu_t(k \le K_-) \le 300 e^{E_-},$$

where

$$E_- = (t_+ - t)\frac{\pi^2}{12t^2}\left[(1-\beta) - \frac{1}{1+\beta/4}\right].$$

Here we use $t_+/t \le 5/4$, $\log(1/t)/\log(1/t_+) \le 2$, and $(1+x)^{-1} \ge 1 - x$. Hence

$$E_- \le -\frac{\beta^2}{8t}.$$

Thus

$$\nu_t\left(k \le (1-\beta)k_*(t)\right) \le 300 e^{-\beta^2/(8t)}.$$

Adding the two bounds proves (B9). □

B.3 The saddle weight

We now specialize to $t = t_n$. The definition in (B1) gives

$$t_n^2 = \frac{\pi^2}{12n}, \qquad k_*(t_n) = \frac{n}{2}, \qquad 2nt_n = \frac{\pi^2}{6t_n}. \tag{B11}$$

If $\ell_n = \log(1/t_n)$, then

$$\frac{1}{2}\log n \le \ell_n \le \log n \tag{B12}$$

for $n \ge 2$.

Define

$$h_n(m) = c_p\sqrt{m} + 2t_n(n-m), \qquad 0 \le m \le n. \tag{B13}$$

This function combines the exponential growth of $p(m)$ with the Boltzmann weight in the convolution.

Lemma B.6 *The function h_n is strictly concave on $(0, n]$. Its unique maximum is at $m = n/2$, and*

$$h_n(n/2) + \frac{\pi^2}{12t_n} = \frac{\pi^2}{3t_n} = c_p\sqrt{2n}. \tag{B14}$$

If $0 < \Delta \le n/6$ and $|m - n/2| \le \Delta$, then

$$h_n(m) \ge h_n(n/2) - \frac{3\pi}{4\sqrt{2}}\frac{\Delta^2}{n^{3/2}}. \tag{B15}$$

If $0 \le m \le n/4$, then

$$h_n(m) \le h_n(n/2) - \frac{\sqrt{n}}{13}. \tag{B16}$$

Proof We have

$$h_n'(m) = \frac{\pi}{\sqrt{6m}} - 2t_n, \qquad h_n''(m) = -\frac{c_p}{4}m^{-3/2}.$$

The function is strictly concave. Its unique critical point is $m = n/2$, which is therefore its maximum. Direct substitution gives (B14).

Suppose that $|m - n/2| \le \Delta \le n/6$. Then $m \in [n/3, 2n/3]$. On this interval,

$$|h_n''(m)| \le \frac{3\pi}{2\sqrt{2}}n^{-3/2}.$$

Taylor's theorem at $n/2$ gives (B15).

Finally, h_n is increasing on $[0, n/2]$. Hence, for $m \le n/4$,

$$h_n(n/2) - h_n(m) \ge h_n(n/2) - h_n(n/4).$$

Direct calculation gives

$$h_n(n/2) - h_n(n/4) = \pi\sqrt{n}\left(\frac{\sqrt{3}}{4} - \frac{1}{\sqrt{6}}\right).$$

The coefficient is larger than $1/13$. This proves (B16). □

B.4 The transfer

We now prove Theorem B.1.

Proof of Theorem B.1 Choose n_0 large enough that all preceding estimates apply at every parameter used below.

We first prove the upper bound. The convolution (2.2), with $m = n - k$, gives

$$a_{2n} \le \left[\sup_{0\le m\le n} p(m)e^{2(n-m)t_n}\right] D_0(e^{-t_n}).$$

Suppose first that $n/4 \le m \le n$. Lemmas B.3 and B.6 give

$$p(m)e^{2(n-m)t_n} \le \frac{2}{\sqrt{3}\,n} e^{h_n(n/2)}.$$

Suppose next that $0 \le m < n/4$. Lemmas B.2 and B.6 give the same bound for all sufficiently large n.

Since

$$D_0(e^{-t_n}) = \mathcal{R}_n D(e^{-t_n}) \le \mathcal{R}_n e^{\pi^2/(12t_n)},$$

equation (B14) gives

$$a_{2n} \le \frac{2}{\sqrt{3}\,n} e^{c_p\sqrt{2n}}\mathcal{R}_n.$$

The lower bound in (B6), applied at $2n$, gives

$$p(2n) \ge \frac{1}{16\sqrt{3}\,n} e^{c_p\sqrt{2n}}.$$

Therefore

$$\frac{a_{2n}}{p(2n)} \le 32\mathcal{R}_n. \tag{B17}$$

We now prove the lower bound. Put

$$\beta_n = 7n^{-1/4}, \qquad \Delta_n = \frac{7}{2}n^{3/4}, \qquad W_n = \left\{k \in \mathbb{Z}_{\ge 0} \,\middle|\, |k - n/2| \le \Delta_n\right\}.$$

Lemma B.5 and $k_*(t_n) = n/2$ give

$$\nu_{t_n}(W_n^c) \le 420e^{-49\sqrt{3}/(4\pi)} < \frac{1}{2}.$$

Consequently,

$$\sum_{k\in W_n} d_k e^{-2kt_n} \ge \frac{1}{2}D_0(e^{-t_n}) \ge \frac{1}{4}\mathcal{R}_n e^{\pi^2/(12t_n)}. \tag{B18}$$

Fix $k \in W_n$ and put $m = n - k$. Lemmas B.3 and B.6 give

$$p(n-k)e^{2kt_n} \ge \frac{e^{-21}}{8\sqrt{3}\,n} e^{h_n(n/2)}.$$

Indeed,

$$\frac{3\pi}{4\sqrt{2}}\frac{\Delta_n^2}{n^{3/2}} < 21.$$

Retaining only the terms with $k \in W_n$ in (2.2), and using (B18), gives

$$a_{2n} \geq \frac{e^{-21}}{32\sqrt{3}\,n} e^{c_p\sqrt{2n}}\mathcal{R}_n.$$

The upper bound in (B6), applied at $2n$, gives

$$p(2n) \leq \frac{1}{4\sqrt{3}\,n} e^{c_p\sqrt{2n}}.$$

Therefore

$$\frac{a_{2n}}{p(2n)} \geq \frac{e^{-21}}{8}\mathcal{R}_n. \tag{B19}$$

Equations (B17) and (B19) prove (B2).

Finally, Theorem A.1 and (B12) give

$$\mathcal{R}_n = \Theta\left(\frac{1}{\sqrt{n}\log n}\right).$$

Substituting this estimate into (B2), and writing $N = 2n$, proves (B3). □

B.5 An interpretation

Let $q_d(m)$ be the number of partitions of m into distinct parts. The odd-core decomposition gives

$$p(2n) = \sum_{k=0}^{n} q_d(2k)p(n-k), \qquad \frac{a_{2n}}{p(2n)} = \sum_{k=0}^{n} \pi_n(k)\rho(k),$$

where $\pi_n(k) = q_d(2k)p(n-k)/p(2n)$ and $\rho(k) = d_k/q_d(2k)$.

The value $\rho(k)$ is the zero-XOR density among distinct partitions of $2k$. The weights $\pi_n(k)$ concentrate near the saddle $k = n/2$. The fixed-size losing density is therefore a saddle-weighted average of the local densities $\rho(k)$.

Theorem B.1 controls this average without assuming that $\rho(k)$ is smooth. In particular, the argument allows the local density to oscillate.

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