

Truncated QLindley Distribution: Different Estimation Approaches with Application to Healthcare Data

Anas F. I. Alharshan

Prince Sattam Bin Abdulaziz University

Abdulrahman M. A. Aldawsari

Prince Sattam Bin Abdulaziz University

Ahmed W. Shawki

Central Agency for Public Mobilization & Statistics

Mohammed Elgarhy

`mohammed.elgarhy@ecu.edu.eg`

Egyptian Chinese University

Research Article

Keywords: QLindley distribution, Truncated distributions, Entropy measures, Maximum likelihood estimation, Monte Carlo simulation, Healthcare data

Posted Date: September 24th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-10619204/v1>

License:   This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: No competing interests reported.

Truncated QLindley Distribution: Different Estimation Approaches with Application to Healthcare Data

Anas F. I. Alharshan¹, Abdulrahman M. A. Aldawsari¹, Ahmed W. Shawki², and Mohammed Elgarhy^{3,4,*}

¹ *Department of Mathematics, College of Sciences and Humanities, Prince Sattam Bin Abdulaziz University, Al-Kharj 16273, Saudi Arabia; af.alharshan@psau.edu.sa; abd.aldawsari@psau.edu.sa*

² *Central Agency for Public Mobilization & Statistics (CAPMAS), Cairo, Egypt; ahmed23484@hotmail.com*

³ *Faculty of Computers and Information Systems, Egyptian Chinese University, Nasr City, Egypt; dr.moelgarhy@gmail.com*

⁴ *Department of Computer Engineering, Biruni University, 34010, Istanbul, Turkey*

**Corresponding author: Mohammed.Elgarhy@ecu.edu.eg*

Abstract

Bounded continuous data frequently arise in healthcare, reliability analysis, environmental sciences, engineering, survival studies, and other applied sciences, requiring probability models with flexible distributional properties on the unit interval. In this article, we introduce and study a new one-parameter bounded truncated distribution, called the truncated Q-Lindley distribution (TQLD), constructed by truncating the suggested Q-Lindley distribution to the unit interval. Numerous important statistical and mathematical properties of the suggested TQLD are computed in closed form, including the quantile function, moments, mean, variance, moment generating function, extropy measure, and weighted extropy measure. Parameter estimation is conducted employing nine classical estimation approaches, namely maximum likelihood, Cramér-von Mises, maximum product of spacings, ordinary least squares, Anderson-Darling, right-tail Anderson-Darling, left-tail Anderson-Darling, percentile, and Kolmogorov estimators. Their performance is assessed through an extensive Monte Carlo simulation study under different values of the parameter θ and sample sizes. The practical and importance of the suggested TQLD is further illustrated by analyzing a healthcare dataset and comparing its performance with numerous competing well-known distributions. The numerical outcomes indicate that the TQLD provides an excellent fit to the data.

Keywords: QLindley distribution; Truncated distributions; Extropy measures; Maximum likelihood estimation; Monte Carlo simulation; Healthcare data.

1 Introduction

Mixture statistical distributions are an important class of statistical models that arise when a population contains several types, each following its own probability distribution. These statistical distributions, offered as weighted combinations of component statistical models, provide great flexibility in capturing complicated data properties like skewness, heavy tails, and numerous hazard rate shapes. Their importance is evident across numerous fields, including environmental science, finance, physics, reliability engineering, survival analysis, and medicine. Mixture distributions are a fundamental component of contemporary statistical methods due to their theoretical simplicity and practical versatility.

The construction of novel mixture statistical distributions has been a focus area of research, with numerous extensions proposed to improve the flexibility of modeling and address specific empirical challenges. A well-known contribution is the Lindley distribution, originally investigated by [2] as a one-parameter alternative to the exponential distribution, which has since been examined and used widely in survival analysis and reliability. Based on this basis, numerous authors have proposed new mixture continuous distributions to improve upon the classical Lindley distribution. For instance, [3] proposed the Shanker distribution, [4] studied the Akash distribution, [5] investigated Ishita distribution, [6] discussed Pranav distribution, [7] proposed Shukla distribution, [8] proposed

am Awadh distribution, [9] proposed Rama distribution, [10] proposed Xgamma distribution, [11] proposed Chris-Jerry distribution, [12] proposed XLindley distribution, [13] proposed Zeghdoudi distribution, [14] proposed Haq distribution, [15] proposed Garhy distribution, [16] proposed Hamza distribution, [17] proposed xRama distribution, [18] proposed Komal distribution.

Recently, [1] proposed a novel one-parameter lifetime model known as the Q-Lindley distribution (QLD). This distribution is characterized by a straightforward probability density function (PDF) that improves its ability to model right-skewed data while offering increased flexibility compared to many well-known one-parameter lifetime statistical models. The PDF and cumulative distribution function (CDF) of the QLD are given by:

$$g(x; \theta) = \frac{\theta(\theta^2 x + 1)e^{-\theta x}}{1 + \theta}, \quad x > 0, \theta > 0, \quad (1)$$

and

$$G(x; \theta) = 1 - \left(1 + \frac{\theta^2 x}{1 + \theta}\right)e^{-\theta x}, \quad x > 0, \theta > 0. \quad (2)$$

A truncated random variable X on the unit interval $(0, 1)$ was proposed and studied by Mahdavi and Silva [19], with its CDF and PDF defined respectively as below:

$$F(x) = \frac{G(x) - G(0)}{G(1) - G(0)}, \quad 0 < x < 1, \quad (3)$$

and

$$f(x) = \frac{g(x)}{G(1) - G(0)}, \quad 0 < x < 1, \quad (4)$$

where $G(\cdot)$ and $g(\cdot)$ denote the CDF and PDF of any baseline (generator) distribution, while $G(0)$ and $G(1)$ signify the values of $G(x)$ investigated at 0 and 1, respectively. The functions $F(\cdot)$ and $f(\cdot)$ correspond to the CDF and PDF of the new truncated distribution.

Several authors have formulated truncated-generated families and truncated classical distributions according to the increasing demand for adaptable models that can analyze bounded and abbreviated data across a variety of scientific disciplines. Significant contributions in this strategy include the truncated Cauchy distribution [20], the truncated inverse Topp-Leone distribution [21], truncated inverse power Lindley distribution [22], truncated Lindley distribution [23], truncated Weibull distribution [24], truncated Fréchet-G class of distributions [25], truncated inverted generalized exponential distribution [26], truncated Burr-G class of distributions [27], truncated inverted Kumaraswamy-G class of distributions [28], new truncated Zubair-G class of distributions [29], truncated Lomax-G class of distributions [30], truncated inverse Lomax distribution [31], new truncated Muth-G class of distributions [32], truncated power Lomax-G class of distributions [33, 34], truncated Shukla distribution [35], truncated Gumbel distribution [36], truncated XLindley distribution [37], truncated inverted arctan power distribution [38], truncated inverse power Ailamujia distribution [39], new truncated Lindley-G class of distributions [40], and truncated xgamma-G class of distributions [41].

The current article is mainly motivated by the desire for increased flexibility and superior fitting performance when modeling unit-interval data. Specifically, this article aims to accomplish the following objectives:

- To introduce a novel truncated distribution, termed the Truncated Q-Lindley Distribution (TQLD), which offers greater modeling flexibility compared to the baseline Q-Lindley distribution (QLD).
- To derive and examine several important statistical properties of the TQLD, including quantile function, moments, mean, variance, moment generating function, extropy measure and weighted extropy measure..
- To estimate the model parameters via nine different methods of estimation and to assess the performance of the estimators through extensive Monte Carlo simulation studies.
- To demonstrate the practical superiority of the TQLD over a range of well-established continuous distributions by applying it to one real-world dataset related to healthcare science.

The rest of this article is structured as follows. Section 2 presents the formulation of the TQLD and some important reliability measures. In Section 3, we calculate several statistical and mathematical properties of the proposed TQLD. Section 4 is devoted to parameter estimation using nine different estimation approaches. The simulation

study and numerical results are reported in Section 5. In Section 6, we illustrate the applicability and superiority of the TQLD through the analysis of one real data set related to healthcare, using various model selection criteria and nine different methods of estimation. Finally, concluding remarks are mentioned in Section 7.

2 The Truncated QLindley Distribution and Reliability Measures

In this section, we formulate the TQLD, as a truncated extension of the QLD, and study some important reliability measures. The CDF of TQLD can be obtained by inserting (2) in CDF (3) as follows

$$F(x; \theta) = \omega(\theta) \left[1 - \left(1 + \frac{\theta^2 x}{1 + \theta} \right) e^{-\theta x} \right], \quad 0 < x < 1, \theta > 0, \quad (5)$$

where $\omega(\theta) = \frac{1}{1 - \left(1 + \frac{\theta^2}{1 + \theta} \right) e^{-\theta}}$. The PDF of the TQLD is constructed by inserting PDF (1) and CDF (2) into PDF (4) as follows:

$$f(x; \theta) = \frac{\theta \omega(\theta) (\theta^2 x + 1) e^{-\theta x}}{1 + \theta}, \quad 0 < x < 1, \theta > 0. \quad (6)$$

The survival function and HRF can be computed as follows

$$S(x; \theta) = 1 - F(x; \theta) = 1 - \omega(\theta) \left[1 - \left(1 + \frac{\theta^2 x}{1 + \theta} \right) e^{-\theta x} \right].$$

and

$$h(x; \theta) = \frac{f(x; \theta)}{S(x; \theta)} = \frac{\omega(\theta) \theta (\theta^2 x + 1) e^{-\theta x}}{(1 + \theta) \left(1 - \omega(\theta) \left[1 - \left(1 + \frac{\theta^2 x}{1 + \theta} \right) e^{-\theta x} \right] \right)}.$$

The reversed HRF and cumulative HRF for the TQLD are given by

$$\tau(x; \theta) = \frac{f(x; \theta)}{F(x; \theta)} = \frac{\theta (\theta^2 x + 1) e^{-\theta x}}{(1 + \theta) \left[1 - \left(1 + \frac{\theta^2 x}{1 + \theta} \right) e^{-\theta x} \right]},$$

and

$$H(x; \theta) = -\ln[S(x; \theta)] = -\ln \left[1 - \omega(\theta) \left[1 - \left(1 + \frac{\theta^2 x}{1 + \theta} \right) e^{-\theta x} \right] \right].$$

The Mills ratio and odds ratio for the TQLD are given by

$$M(x; \theta) = \frac{S(x; \theta)}{f(x; \theta)} = \frac{(1 + \theta) \left(1 - \omega(\theta) \left[1 - \left(1 + \frac{\theta^2 x}{1 + \theta} \right) e^{-\theta x} \right] \right)}{\omega(\theta) \theta (\theta^2 x + 1) e^{-\theta x}}.$$

and

$$O(x; \theta) = \frac{F(x; \theta)}{S(x; \theta)} = \frac{\omega(\theta) \left[1 - \left(1 + \frac{\theta^2 x}{1 + \theta} \right) e^{-\theta x} \right]}{1 - \omega(\theta) \left[1 - \left(1 + \frac{\theta^2 x}{1 + \theta} \right) e^{-\theta x} \right]}.$$

Figure 1 indicates the PDF and HRF plots for the TQLD. From Figure 1, we can note that the PDF can be decreasing, unimodal and right-skewed. However, the HRF can increase or be J-shaped.

3 Statistical Properties

In this section, some important statistical and mathematical properties of the TQLD are computed.

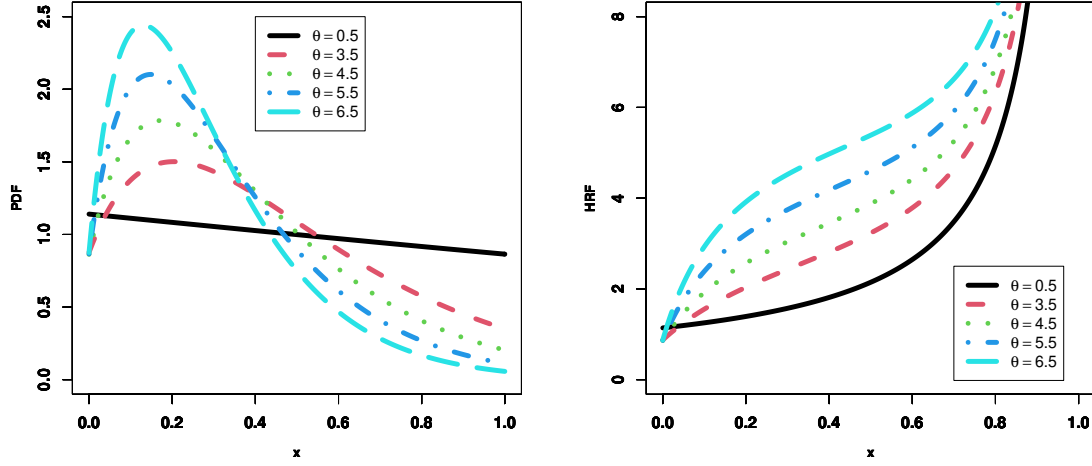


Figure 1: Plots of PDF and HRF for the TQLD

3.1 Quantile Function

Theorem 1. Let $X \sim \text{TQLD}(\theta)$ with CDF given in (5). Then the quantile function of X , denoted by $Q(p) = F^{-1}(p)$ for $0 < p < 1$, is given by

$$Q(p) = -\frac{1}{\theta} - \frac{1}{\theta^2} - \frac{1}{\theta} W_{-1} \left(-\left(\frac{1+\theta}{\theta} \right) \left(1 - \frac{p}{\omega(\theta)} \right) e^{-\left(\frac{1+\theta}{\theta} \right)} \right), \quad (7)$$

where $W_{-1}(\cdot)$ denotes the negative branch of the Lambert W function.

Proof. To obtain the quantile function $Q(p) = F^{-1}(p)$, we solve the equation $F(x; \theta) = p$. That is,

$$\omega(\theta) \left[1 - \left(1 + \frac{\theta^2 x}{1 + \theta} \right) e^{-\theta x} \right] = p.$$

Then,

$$\left(1 + \frac{\theta^2 x}{1 + \theta} \right) e^{-\theta x} = 1 - \frac{p}{\omega(\theta)}.$$

Multiplying both sides by $(1 + \theta)$, we obtain

$$(1 + \theta + \theta^2 x) e^{-\theta x} = (1 + \theta) \left(1 - \frac{p}{\omega(\theta)} \right).$$

Let us define $k = \frac{(1+\theta)(1-\frac{p}{\omega(\theta)})}{\theta}$, $c = 1 + \frac{1}{\theta}$. Then the equation becomes:

$$(\theta x + c) e^{-\theta x} = k.$$

Then

$$(\theta x + c) e^{-(\theta x + c)} = k e^{-c}.$$

Multiply both sides by -1 to obtain the standard form of the Lambert W function:

$$-(\theta x + c) e^{-(\theta x + c)} = -k e^{-c}.$$

Thus,

$$-(\theta x + c) = W_{-1}(-k e^{-c}),$$

where W_{-1} is the negative branch of the Lambert W function, appropriate because $0 < p < 1$, so the argument is in the domain of W_{-1} . Then,

$$Q(p) = x = -\frac{1}{\theta} - \frac{1}{\theta^2} - \frac{1}{\theta} W_{-1} \left(-\left(\frac{1+\theta}{\theta} \right) \left(1 - \frac{p}{\omega(\theta)} \right) e^{-\left(\frac{1+\theta}{\theta} \right)} \right).$$

This completes the proof. $\square$

Corollary 1. *The median of the TQLD can be obtained by setting $p = 0.5$ in (7):*

$$Q(0.5) = -\frac{1}{\theta} - \frac{1}{\theta^2} - \frac{1}{\theta} W_{-1} \left(-\left(\frac{1+\theta}{\theta} \right) \left(1 - \frac{0.5}{\omega(\theta)} \right) e^{-\left(\frac{1+\theta}{\theta} \right)} \right).$$

3.2 Ordinary Moments and Related Measures

Theorem 2. *Let $X \sim \text{TQLD}(\theta)$ with PDF (6). Then the r -th ordinary moment of X is given by*

$$\mu'_r = \mathbb{E}(X^r) = \frac{\omega(\theta)}{\theta^r(1+\theta)} [\theta\gamma(r+2, \theta) + \gamma(r+1, \theta)], \quad (8)$$

where $\gamma(a, x) = \int_0^x u^{a-1} e^{-u} du$ denotes the lower incomplete gamma function.

Proof. By definition,

$$\mu'_r = \mathbb{E}(X^r) = \int_0^1 x^r f(x; \theta) dx. \quad (9)$$

Substituting the PDF of TQLD (6) in (9), we get

$$\mu'_r = \frac{\theta\omega(\theta)}{1+\theta} \int_0^1 x^r (\theta^2 x + 1) e^{-\theta x} dx.$$

Expanding the integrand,

$$\mu'_r = \frac{\theta\omega(\theta)}{1+\theta} \left[\theta^2 \int_0^1 x^{r+1} e^{-\theta x} dx + \int_0^1 x^r e^{-\theta x} dx \right].$$

Let $u = \theta x$, then,

$$\mu'_r = \frac{\omega(\theta)}{1+\theta} \left[\theta^{-r+1} \int_0^\theta u^{r+1} e^{-u} du + \theta^{-r} \int_0^\theta u^r e^{-u} du \right].$$

Using the definition of the lower incomplete gamma function, we obtain

$$\mu'_r = \frac{\omega(\theta)}{1+\theta} [\theta^{-r+1} \gamma(r+2, \theta) + \theta^{-r} \gamma(r+1, \theta)].$$

Simplifying,

$$\mu'_r = \frac{\omega(\theta)}{\theta^r(1+\theta)} [\theta\gamma(r+2, \theta) + \gamma(r+1, \theta)].$$

This completes the proof. $\square$

Corollary 2. *The mean of the TQLD is obtained by setting $r = 1$ in (8):*

$$\mu = \mathbb{E}(X) = \frac{\omega(\theta)}{\theta(1+\theta)} [\theta\gamma(3, \theta) + \gamma(2, \theta)].$$

Corollary 3. *The second moment of the TQLD is obtained by setting $r = 2$ in (8):*

$$\mu'_2 = \mathbb{E}(X^2) = \frac{\omega(\theta)}{\theta^2(1+\theta)} [\theta\gamma(4, \theta) + \gamma(3, \theta)].$$

Corollary 4. *The variance of the TQLD is*

$$\text{Var}(X) = \mu'_2 - \mu^2 = \frac{\omega(\theta)}{\theta^2(1+\theta)} [\theta\gamma(4, \theta) + \gamma(3, \theta)] - \left[\frac{\omega(\theta)}{\theta(1+\theta)} [\theta\gamma(3, \theta) + \gamma(2, \theta)] \right]^2.$$

Corollary 5. *The third and fourth moments of the TQLD is obtained by setting $r = 3$ and 4 in (8):*

$$\mu'_3 = \mathbb{E}(X^3) = \frac{\omega(\theta)}{\theta^3(1+\theta)} [\theta\gamma(5, \theta) + \gamma(4, \theta)],$$

and

$$\mu'_4 = \mathbb{E}(X^4) = \frac{\omega(\theta)}{\theta^4(1+\theta)} [\theta\gamma(6, \theta) + \gamma(5, \theta)].$$

3.3 Moment Generating Function

Theorem 3. *Let $X \sim \text{TQLD}(\theta)$ with PDF (6). Then the moment generating function of X , is given by*

$$M_X(t) = \frac{\theta\omega(\theta)}{1+\theta} \left[\theta^2 \left(\frac{e^{(t-\theta)}}{t-\theta} - \frac{e^{(t-\theta)} - 1}{(t-\theta)^2} \right) + \frac{e^{(t-\theta)} - 1}{t-\theta} \right], \quad t \neq \theta \quad (10)$$

Proof. By definition,

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^1 e^{tx} f(x; \theta) dx.$$

Substituting the TQLD density yields

$$M_X(t) = \frac{\theta\omega(\theta)}{1+\theta} \int_0^1 (\theta^2 x + 1) e^{(t-\theta)x} dx.$$

Expanding the integrand,

$$M_X(t) = \frac{\theta\omega(\theta)}{1+\theta} \left[\theta^2 \int_0^1 x e^{(t-\theta)x} dx + \int_0^1 e^{(t-\theta)x} dx \right] = \frac{\theta\omega(\theta)}{1+\theta} [\theta^2 I_1 + I_2].$$

For the first integral, we use integration by parts. Let $u = x$ and $dv = e^{(t-\theta)x} dx$. Then $du = dx$ and $v = \frac{e^{(t-\theta)x}}{t-\theta}$. Thus,

$$I_2 = \int_0^1 x e^{(t-\theta)x} dx = \frac{e^{(t-\theta)}}{t-\theta} - \frac{e^{(t-\theta)} - 1}{(t-\theta)^2}.$$

For the second integral,

$$I_1 = \int_0^1 e^{(t-\theta)x} dx = \frac{e^{(t-\theta)} - 1}{t-\theta}.$$

Then

$$M_X(t) = \frac{\theta\omega(\theta)}{1+\theta} \left[\theta^2 \left(\frac{e^{(t-\theta)}}{t-\theta} - \frac{e^{(t-\theta)} - 1}{(t-\theta)^2} \right) + \frac{e^{(t-\theta)} - 1}{t-\theta} \right].$$

This completes the proof. □

Corollary 6. *The cumulant generating function of the TQLD is given by*

$$K_X(t) = \ln(M_X(t)) = \ln \left(\frac{\theta\omega(\theta)}{1+\theta} \left[\theta^2 \left(\frac{e^{(t-\theta)}}{t-\theta} - \frac{e^{(t-\theta)} - 1}{(t-\theta)^2} \right) + \frac{e^{(t-\theta)} - 1}{t-\theta} \right] \right),$$

and the cumulants can be obtained as $\kappa_r = \frac{d^r}{dt^r} K_X(t) \Big|_{t=0}$.

3.4 Extropy Measure

Theorem 4. Let $X \sim \text{TQLD}(\theta)$ with PDF (6). Then the extropy of X , denoted by $J(X)$, is given by

$$J(X) = -\frac{\theta^2 \omega^2(\theta)}{2(1+\theta)^2} \left[\frac{\theta(2 - e^{-2\theta}(4\theta^2 + 4\theta + 2))}{8} + \frac{1 - e^{-2\theta}(1 + 2\theta)}{2} + \frac{1 - e^{-2\theta}}{2\theta} \right]. \quad (11)$$

Proof. By definition, the extropy is

$$J(X) = -\frac{1}{2} \int_0^1 f^2(x; \theta) dx. \quad (12)$$

Substituting the PDF of TQLD in Equation (6) into Equation (12), we get

$$J(X) = -\frac{\theta^2 \omega^2(\theta)}{2(1+\theta)^2} \left[\theta^4 \int_0^1 x^2 e^{-2\theta x} dx + 2\theta^2 \int_0^1 x e^{-2\theta x} dx + \int_0^1 e^{-2\theta x} dx \right] = -\frac{\theta^2 \omega^2(\theta)}{2(1+\theta)^2} [\theta^4 J_3 + 2\theta^2 J_2 + J_1].$$

Now, using the standard integral formulas:

$$J_1 = \int_0^1 e^{-2\theta x} dx = \frac{1 - e^{-2\theta}}{2\theta},$$

$$J_2 = \int_0^1 x e^{-2\theta x} dx = \frac{1 - e^{-2\theta}(1 + 2\theta)}{4\theta^2},$$

and

$$J_3 = \int_0^1 x^2 e^{-2\theta x} dx = \frac{2 - e^{-2\theta}(4\theta^2 + 4\theta + 2)}{8\theta^3}.$$

Substituting these integrals, we obtain

$$J(X) = -\frac{\theta^2 \omega^2(\theta)}{2(1+\theta)^2} \left[\theta^4 \left(\frac{2 - e^{-2\theta}(4\theta^2 + 4\theta + 2)}{8\theta^3} \right) + 2\theta^2 \left(\frac{1 - e^{-2\theta}(1 + 2\theta)}{4\theta^2} \right) + \left(\frac{1 - e^{-2\theta}}{2\theta} \right) \right].$$

Simplifying,

$$J(X) = -\frac{\theta^2 \omega^2(\theta)}{2(1+\theta)^2} \left[\frac{\theta(2 - e^{-2\theta}(4\theta^2 + 4\theta + 2))}{8} + \frac{1 - e^{-2\theta}(1 + 2\theta)}{2} + \frac{1 - e^{-2\theta}}{2\theta} \right].$$

This completes the proof. $\square$

3.5 Weighted Extropy Measure

Theorem 5. Let $X \sim \text{TQLD}(\theta)$ with PDF given in (6). Then the weighted extropy of X under the linear weight function $w(x) = x$, denoted by $J_w(X)$, is given by

$$J_w(X) = -\frac{\theta^2 \omega^2(\theta)}{2(1+\theta)^2} \left[\frac{6 - e^{-2\theta}(8\theta^3 + 12\theta^2 + 12\theta + 6)}{16} + \frac{2 - e^{-2\theta}(4\theta^2 + 4\theta + 2)}{4\theta} + \frac{1 - e^{-2\theta}(1 + 2\theta)}{4\theta^2} \right]. \quad (13)$$

Proof. By definition, the weighted extropy under the weight function $w(x) = x$ is

$$J_w(X) = -\frac{1}{2} \int_0^1 x f^2(x; \theta) dx. \quad (14)$$

Substituting the PDF of TQLD from (6) into (14), we obtain

$$J_w(X) = -\frac{\theta^2 \omega^2(\theta)}{2(1+\theta)^2} \left[\theta^4 \int_0^1 x^3 e^{-2\theta x} dx + 2\theta^2 \int_0^1 x^2 e^{-2\theta x} dx + \int_0^1 x e^{-2\theta x} dx \right] = -\frac{\theta^2 \omega^2(\theta)}{2(1+\theta)^2} [\theta^4 J_4 + 2\theta^2 J_3 + J_2].$$

Now, using the standard integral formula:

$$J_4 = \int_0^1 x^3 e^{-2\theta x} dx = \frac{6 - e^{-2\theta}(8\theta^3 + 12\theta^2 + 12\theta + 6)}{16\theta^4}.$$

Substituting these integrals J_4 , J_3 and J_2 , we obtain

$$J_w(X) = -\frac{\theta^2 \omega^2(\theta)}{2(1+\theta)^2} \left[\theta^4 \left(\frac{6 - e^{-2\theta}(8\theta^3 + 12\theta^2 + 12\theta + 6)}{16\theta^4} \right) + 2\theta^2 \left(\frac{2 - e^{-2\theta}(4\theta^2 + 4\theta + 2)}{8\theta^3} \right) + \left(\frac{1 - e^{-2\theta}(1 + 2\theta)}{4\theta^2} \right) \right].$$

Simplifying

$$J_w(X) = -\frac{\theta^2 \omega^2(\theta)}{2(1+\theta)^2} \left[\frac{6 - e^{-2\theta}(8\theta^3 + 12\theta^2 + 12\theta + 6)}{16} + \frac{2 - e^{-2\theta}(4\theta^2 + 4\theta + 2)}{4\theta} + \frac{1 - e^{-2\theta}(1 + 2\theta)}{4\theta^2} \right].$$

This completes the proof. $\square$

4 Estimation Methods

In this section, we examine the estimation of the parameter θ of the TQLD using nine different estimation methods. The proposed methods are the maximum likelihood estimator (MLE), Cramér–von Mises estimator (CME), maximum product of spacings estimator (MPS), ordinary least-squares estimator (LSE), Anderson–Darling estimator (ADE), right-tail Anderson–Darling estimator (RADE), left-tail Anderson–Darling estimator (LADE), percentile estimator (PE), and Kolmogorov estimator (KM).

4.1 Maximum Likelihood Method (MLE)

The maximum likelihood method is one of the most popular estimation techniques due to its desirable asymptotic properties, including consistency, efficiency, and asymptotic normality. The log-likelihood function based on the observed sample is given by:

$$\ell(\theta) = \sum_{i=1}^n \log f(x_i; \theta).$$

Substituting the TQLD PDF from (6) yields:

$$\ell(\theta) = n \log \theta + n \log \omega(\theta) + \sum_{i=1}^n \log(\theta^2 x_i + 1) - n \log(1 + \theta) - \theta \sum_{i=1}^n x_i.$$

Differentiating with respect to θ and setting the derivative to zero gives the score equation:

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + n \frac{\omega'(\theta)}{\omega(\theta)} + \sum_{i=1}^n \frac{2\theta x_i}{\theta^2 x_i + 1} - \frac{n}{1 + \theta} - \sum_{i=1}^n x_i = 0,$$

where $\omega'(\theta) = \frac{d}{d\theta} \omega(\theta)$. As this equation cannot be solved analytically, the MLE $\hat{\theta}_{MLE}$ is obtained using numerical optimization techniques.

4.2 Cramér–von Mises Method (CME)

The Cramér–von Mises method, discussed by [44], is based on minimizing the squared distance between the theoretical CDF and the empirical CDF. The estimate $\hat{\theta}_{CME}$ is obtained by minimizing the following function:

$$C(\theta) = \frac{1}{12n} + \sum_{i=1}^n \left\{ F(x_{i:n}; \theta) - \frac{2i-1}{2n} \right\}^2,$$

where $F(\cdot; \theta)$ is given in (5). The minimizer is found by solving the corresponding normal equation numerically.

4.3 Maximum Product of Spacings Method (MPS)

The maximum product of spacings method, proposed by [45], is based on maximizing the product of the spacings between consecutive CDF values. Define the spacings as:

$$D_i(\theta) = F(x_{i:n}; \theta) - F(x_{i-1:n}; \theta), \quad i = 1, \dots, n+1,$$

with $F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$. The MPS estimator $\hat{\theta}_{\text{MPS}}$ is obtained by maximizing:

$$M(\theta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i(\theta),$$

or equivalently, by minimizing $-M(\theta)$.

4.4 Ordinary Least Squares Estimator (LSE)

Following [46], the ordinary least squares estimator $\hat{\theta}_{\text{LSE}}$ is obtained by minimizing:

$$V(\theta) = \sum_{i=1}^n \left[F(x_{i:n}; \theta) - \frac{i}{n+1} \right]^2,$$

where $F(\cdot; \theta)$ is the TQLD CDF from (5).

4.5 Anderson–Darling Methods

This class includes three estimators based on the Anderson–Darling goodness-of-fit test: ADE, RADE, and LADE.

4.5.1 Anderson–Darling Estimator (ADE)

The ADE $\hat{\theta}_{\text{ADE}}$ is obtained by minimizing:

$$A(\theta) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \{ \log F(x_{i:n}; \theta) + \log [1 - F(x_{n-i+1:n}; \theta)] \}.$$

4.5.2 Right-Tail Anderson–Darling Estimator (RADE)

The RADE $\hat{\theta}_{\text{RADE}}$ is obtained by minimizing:

$$R(\theta) = \frac{n}{2} - 2 \sum_{i=1}^n F(x_{i:n}; \theta) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log [1 - F(x_{n-i+1:n}; \theta)].$$

4.5.3 Left-Tail Anderson–Darling Estimator (LADE)

The LADE $\hat{\theta}_{\text{LADE}}$ is obtained by minimizing:

$$L(\theta) = -\frac{3n}{2} + 2 \sum_{i=1}^n F(x_{i:n}; \theta) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log F(x_{i:n}; \theta).$$

All three estimators are computed numerically by solving the respective partial derivative equations.

4.6 Percentile Estimator (PE)

The percentile estimator, also known as the least-squares quantile estimator, is obtained by minimizing the squared differences between the observed order statistics and their corresponding quantiles from the fitted model. The estimator $\hat{\theta}_{\text{PE}}$ is obtained by minimizing:

$$P(\theta) = \sum_{i=1}^n \left\{ x_{i:n} - Q\left(\frac{i}{n+1}; \theta\right) \right\}^2,$$

where $Q(u; \theta) = F^{-1}(u; \theta)$ is the quantile function of the TQLD.

4.7 Kolmogorov Method (KM)

The Kolmogorov method minimizes the maximum absolute distance between the theoretical CDF and the empirical CDF. The estimate $\hat{\theta}_{\text{KM}}$ is obtained by minimizing the following function:

$$K(\theta) = \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - F(x_{i:n}; \theta), F(x_{i:n}; \theta) - \frac{i-1}{n} \right\}.$$

Due to the non-differentiable nature of the Kolmogorov distance, this minimization is carried out using derivative-free optimization algorithms.

5 Simulation Study

A Monte Carlo simulation study was conducted to evaluate the finite-sample performance of the nine estimation methods introduced in Section 4. The study investigates the effects of both the true parameter value and the sample size on the accuracy and stability of the resulting estimators.

Random samples were generated from the TQLD using inverse-transform sampling for six parameter values,

$$\theta \in \{0.5, 1, 1.5, 2, 3, 5\},$$

and five sample sizes,

$$n \in \{50, 100, 150, 200, 250\}.$$

For each (θ, n) combination, $R = 1000$ independent Monte Carlo replications were performed. All nine estimation methods were applied to the same simulated samples, and numerical optimization was carried out under the constraint $\theta > 0$.

Let $\hat{\theta}_r$ denote the estimate obtained from the r th replication ($r = 1, \dots, R$). The performance of the estimators was assessed using the empirical Bias, absolute bias (ABias), relative bias (RBias), mean squared error (MSE), and root mean squared error (RMSE).

The detailed numerical results for $\theta = 0.5, 1, 1.5, 2, 3$, and 5 are reported in Tables 1–6, while the corresponding MSE and RMSE comparisons are displayed in Figures 2 and 3.

Table 1: Simulation values for $\theta = 0.5$

n	Est.	MLE	ADE	CME	MPS	LSE	RADE	LADE	PE	KM
50	RBias	0.5858 ^[6]	0.5548 ^[5]	0.5933 ^[7]	0.0188 ^[1]	0.4197 ^[4]	0.8153 ^[9]	0.0999 ^[2]	0.7949 ^[8]	0.2009 ^[3]
	Bias	0.2929 ^[6]	0.2774 ^[5]	0.2967 ^[7]	0.0094 ^[1]	0.2098 ^[4]	0.4076 ^[9]	0.0500 ^[2]	0.3974 ^[8]	0.1005 ^[3]
	MSE	1.3114 ^[5]	1.3126 ^[6]	1.3495 ^[7]	0.7333 ^[2]	1.1272 ^[4]	1.5851 ^[9]	0.6969 ^[1]	1.5780 ^[8]	0.8715 ^[3]
	RMSE	1.1452 ^[5]	1.1457 ^[6]	1.1617 ^[7]	0.8563 ^[2]	1.0617 ^[4]	1.2590 ^[9]	0.8348 ^[1]	1.2562 ^[8]	0.9335 ^[3]
	$\sum Ranks$	22 ^[5.5]	22 ^[5.5]	28 ^[7]	6 ^[1.5]	16 ^[4]	36 ^[9]	6 ^[1.5]	32 ^[8]	12 ^[3]
100	RBias	0.5064 ^[6]	0.5402 ^[7]	0.5060 ^[5]	0.0379 ^[1]	0.4353 ^[4]	0.6783 ^[9]	0.1287 ^[2]	0.5911 ^[8]	0.2102 ^[3]
	Bias	0.2532 ^[6]	0.2701 ^[7]	0.2530 ^[5]	0.0189 ^[1]	0.2177 ^[4]	0.3391 ^[9]	0.0643 ^[2]	0.2956 ^[8]	0.1051 ^[3]
	MSE	0.8975 ^[5]	0.9743 ^[7]	0.9417 ^[6]	0.5273 ^[1]	0.8715 ^[4]	1.0621 ^[9]	0.5496 ^[2]	1.0106 ^[8]	0.6298 ^[3]
	RMSE	0.9473 ^[5]	0.9871 ^[7]	0.9704 ^[6]	0.7262 ^[1]	0.9336 ^[4]	1.0306 ^[9]	0.7413 ^[2]	1.0053 ^[8]	0.7936 ^[3]
	$\sum Ranks$	22 ^[5.5]	28 ^[7]	22 ^[5.5]	4 ^[1]	16 ^[4]	36 ^[9]	8 ^[2]	32 ^[8]	12 ^[3]
150	RBias	0.4753 ^[4]	0.6053 ^[7]	0.5473 ^[6]	0.0685 ^[1]	0.4972 ^[5]	0.6589 ^[9]	0.1677 ^[2]	0.6261 ^[8]	0.2613 ^[3]
	Bias	0.2377 ^[4]	0.3026 ^[7]	0.2737 ^[6]	0.0342 ^[1]	0.2486 ^[5]	0.3294 ^[9]	0.0839 ^[2]	0.3131 ^[8]	0.1306 ^[3]
	MSE	0.7256 ^[4]	0.8928 ^[8]	0.8154 ^[6]	0.4244 ^[1]	0.7728 ^[5]	0.9056 ^[9]	0.4914 ^[2]	0.8860 ^[7]	0.5297 ^[3]
	RMSE	0.8518 ^[4]	0.9449 ^[8]	0.9030 ^[6]	0.6515 ^[1]	0.8791 ^[5]	0.9516 ^[9]	0.7010 ^[2]	0.9413 ^[7]	0.7278 ^[3]
	$\sum Ranks$	16 ^[4]	30 ^[7.5]	24 ^[6]	4 ^[1]	20 ^[5]	36 ^[9]	8 ^[2]	30 ^[7.5]	12 ^[3]
200	RBias	0.4484 ^[4]	0.6491 ^[8]	0.5904 ^[6]	0.0800 ^[1]	0.5149 ^[5]	0.6878 ^[9]	0.2718 ^[3]	0.6485 ^[7]	0.2363 ^[2]
	Bias	0.2242 ^[4]	0.3246 ^[8]	0.2952 ^[6]	0.0400 ^[1]	0.2575 ^[5]	0.3439 ^[9]	0.1359 ^[3]	0.3243 ^[7]	0.1181 ^[2]
	MSE	0.6097 ^[4]	0.8466 ^[9]	0.7780 ^[6]	0.3458 ^[1]	0.7141 ^[5]	0.8338 ^[8]	0.5502 ^[3]	0.8298 ^[7]	0.4619 ^[2]
	RMSE	0.7808 ^[4]	0.9201 ^[9]	0.8820 ^[6]	0.5881 ^[1]	0.8450 ^[5]	0.9131 ^[8]	0.7417 ^[3]	0.9109 ^[7]	0.6796 ^[2]
	$\sum Ranks$	16 ^[4]	34 ^[8.5]	24 ^[6]	4 ^[1]	20 ^[5]	34 ^[8.5]	12 ^[3]	28 ^[7]	8 ^[2]
250	RBias	0.3881 ^[4]	0.5354 ^[7]	0.5136 ^[6]	0.0976 ^[1]	0.4654 ^[5]	0.5783 ^[9]	0.2296 ^[2]	0.5385 ^[8]	0.2434 ^[3]
	Bias	0.1941 ^[4]	0.2677 ^[7]	0.2568 ^[6]	0.0488 ^[1]	0.2327 ^[5]	0.2891 ^[9]	0.1148 ^[2]	0.2692 ^[8]	0.1217 ^[3]
	MSE	0.5046 ^[4]	0.6737 ^[9]	0.6461 ^[6]	0.3135 ^[1]	0.6083 ^[5]	0.6686 ^[8]	0.4528 ^[3]	0.6628 ^[7]	0.4117 ^[2]
	RMSE	0.7103 ^[4]	0.8208 ^[9]	0.8038 ^[6]	0.5599 ^[1]	0.7800 ^[5]	0.8177 ^[8]	0.6729 ^[3]	0.8141 ^[7]	0.6417 ^[2]
	$\sum Ranks$	16 ^[4]	32 ^[8]	24 ^[6]	4 ^[1]	20 ^[5]	34 ^[9]	10 ^[2.5]	30 ^[7]	10 ^[2.5]

Table 2: Simulation values for $\theta = 1$

n	Est.	MLE	ADE	CME	MPS	LSE	RADE	LADE	PE	KM
50	RBias	0.2342 ^[9]	0.0613 ^[4]	0.1079 ^[5]	0.1448 ^[7]	0.0110 ^[1]	0.1356 ^[6]	0.0430 ^[3]	0.1654 ^[8]	0.0231 ^[2]
	Bias	0.2342 ^[9]	0.0613 ^[4]	0.1079 ^[5]	0.1448 ^[7]	0.0110 ^[1]	0.1356 ^[6]	0.0430 ^[3]	0.1654 ^[8]	0.0231 ^[2]
	MSE	1.1675 ^[5]	1.1604 ^[4]	1.2955 ^[7]	0.8177 ^[2]	1.2064 ^[6]	1.4701 ^[9]	0.6589 ^[1]	1.3607 ^[8]	0.8677 ^[3]
	RMSE	1.0805 ^[5]	1.0772 ^[4]	1.1382 ^[7]	0.9043 ^[2]	1.0984 ^[6]	1.2125 ^[9]	0.8118 ^[1]	1.1665 ^[8]	0.9315 ^[3]
	$\sum Ranks$	28 ^[7]	16 ^[4]	24 ^[6]	18 ^[5]	14 ^[3]	30 ^[8]	8 ^[1]	32 ^[9]	10 ^[2]
100	RBias	0.1135 ^[8]	0.0108 ^[1]	0.0689 ^[5]	0.1528 ^[9]	0.0156 ^[3]	0.0527 ^[4]	0.0113 ^[2]	0.0881 ^[7]	0.0729 ^[6]
	Bias	0.1135 ^[8]	0.0108 ^[1]	0.0689 ^[5]	0.1528 ^[9]	0.0156 ^[3]	0.0527 ^[4]	0.0113 ^[2]	0.0881 ^[7]	0.0729 ^[6]
	MSE	0.7595 ^[4]	0.8114 ^[5]	0.9279 ^[7]	0.5650 ^[2]	0.8803 ^[6]	1.0026 ^[9]	0.4639 ^[1]	0.9579 ^[8]	0.6340 ^[3]
	RMSE	0.8715 ^[4]	0.9008 ^[5]	0.9633 ^[7]	0.7516 ^[2]	0.9383 ^[6]	1.0013 ^[9]	0.6811 ^[1]	0.9787 ^[8]	0.7962 ^[3]
	$\sum Ranks$	24 ^[6.5]	12 ^[2]	24 ^[6.5]	22 ^[5]	18 ^[3.5]	26 ^[8]	6 ^[1]	30 ^[9]	18 ^[3.5]
150	RBias	0.0777 ^[6]	0.0232 ^[2]	0.0834 ^[8]	0.1387 ^[9]	0.0195 ^[1]	0.0489 ^[4]	0.0307 ^[3]	0.0804 ^[7]	0.0551 ^[5]
	Bias	0.0777 ^[6]	0.0232 ^[2]	0.0834 ^[8]	0.1387 ^[9]	0.0195 ^[1]	0.0489 ^[4]	0.0307 ^[3]	0.0804 ^[7]	0.0551 ^[5]
	MSE	0.6434 ^[4]	0.7572 ^[5]	0.8720 ^[7]	0.4881 ^[2]	0.8492 ^[6]	0.8862 ^[9]	0.4014 ^[1]	0.8763 ^[8]	0.5769 ^[3]
	RMSE	0.8021 ^[4]	0.8702 ^[5]	0.9338 ^[7]	0.6986 ^[2]	0.9215 ^[6]	0.9414 ^[9]	0.6335 ^[1]	0.9361 ^[8]	0.7595 ^[3]
	$\sum Ranks$	20 ^[5]	14 ^[2.5]	30 ^[8.5]	22 ^[6]	14 ^[2.5]	26 ^[7]	8 ^[1]	30 ^[8.5]	16 ^[4]
200	RBias	0.0607 ^[7]	0.0069 ^[1]	0.0411 ^[5]	0.1079 ^[9]	0.0071 ^[2]	0.0161 ^[3]	0.0184 ^[4]	0.0434 ^[6]	0.0652 ^[8]
	Bias	0.0607 ^[7]	0.0069 ^[1]	0.0411 ^[5]	0.1079 ^[9]	0.0071 ^[2]	0.0161 ^[3]	0.0184 ^[4]	0.0434 ^[6]	0.0652 ^[8]
	MSE	0.5357 ^[4]	0.6090 ^[5]	0.6896 ^[7]	0.4090 ^[2]	0.6737 ^[6]	0.7142 ^[9]	0.3573 ^[1]	0.6955 ^[8]	0.4426 ^[3]
	RMSE	0.7319 ^[4]	0.7804 ^[5]	0.8304 ^[7]	0.6395 ^[2]	0.8208 ^[6]	0.8451 ^[9]	0.5977 ^[1]	0.8340 ^[8]	0.6653 ^[3]
	$\sum Ranks$	22 ^[5]	12 ^[2]	24 ^[7.5]	22 ^[5]	16 ^[3]	24 ^[7.5]	10 ^[1]	28 ^[9]	22 ^[5]
250	RBias	0.0344 ^[5]	0.0060 ^[2]	0.0372 ^[6]	0.1308 ^[9]	0.0041 ^[1]	0.0227 ^[3]	0.0324 ^[4]	0.0425 ^[7]	0.0620 ^[8]
	Bias	0.0344 ^[5]	0.0060 ^[2]	0.0372 ^[6]	0.1308 ^[9]	0.0041 ^[1]	0.0227 ^[3]	0.0324 ^[4]	0.0425 ^[7]	0.0620 ^[8]
	MSE	0.4859 ^[4]	0.5792 ^[5]	0.6350 ^[7]	0.3626 ^[2]	0.6307 ^[6]	0.6468 ^[9]	0.3486 ^[1]	0.6422 ^[8]	0.4073 ^[3]
	RMSE	0.6971 ^[4]	0.7611 ^[5]	0.7969 ^[7]	0.6021 ^[2]	0.7941 ^[6]	0.8042 ^[9]	0.5904 ^[1]	0.8014 ^[8]	0.6382 ^[3]
	$\sum Ranks$	18 ^[4]	14 ^[2.5]	26 ^[8]	22 ^[5.5]	14 ^[2.5]	24 ^[7]	10 ^[1]	30 ^[9]	22 ^[5.5]

Table 3: Simulation values for $\theta = 1.5$

n	Est.	MLE	ADE	CME	MPS	LSE	RADE	LADE	PE	KM
50	RBias	0.0147 ^[2]	0.0458 ^[3]	0.0053 ^[1]	0.1328 ^[9]	0.0582 ^[5]	0.1040 ^[8]	0.0641 ^[7]	0.0607 ^[6]	0.0570 ^[4]
	Bias	0.0220 ^[2]	0.0688 ^[3]	0.0079 ^[1]	0.1991 ^[9]	0.0873 ^[5]	0.1560 ^[8]	0.0962 ^[7]	0.0910 ^[6]	0.0856 ^[4]
	MSE	1.2450 ^[2]	1.4651 ^[6]	1.5673 ^[7]	1.1905 ^[1]	1.6116 ^[9]	1.5715 ^[8]	1.2987 ^[4]	1.4381 ^[5]	1.2804 ^[3]
	RMSE	1.1158 ^[2]	1.2104 ^[6]	1.2519 ^[7]	1.0911 ^[1]	1.2695 ^[9]	1.2536 ^[8]	1.1396 ^[4]	1.1992 ^[5]	1.1315 ^[3]
	$\sum Ranks$	8 ^[1]	18 ^[4]	16 ^[3]	20 ^[5]	28 ^[8]	32 ^[9]	22 ^[6.5]	22 ^[6.5]	14 ^[2]
100	RBias	0.0587 ^[3]	0.0779 ^[4]	0.0487 ^[2]	0.1220 ^[8]	0.0819 ^[6]	0.1238 ^[9]	0.0297 ^[1]	0.0830 ^[7]	0.0798 ^[5]
	Bias	0.0880 ^[3]	0.1168 ^[4]	0.0731 ^[2]	0.1830 ^[8]	0.1228 ^[6]	0.1856 ^[9]	0.0446 ^[1]	0.1244 ^[7]	0.1197 ^[5]
	MSE	0.8672 ^[3]	1.0298 ^[6]	1.0541 ^[7]	0.7870 ^[1]	1.0779 ^[8]	1.0854 ^[9]	0.8529 ^[2]	1.0021 ^[5]	0.9509 ^[4]
	RMSE	0.9312 ^[3]	1.0148 ^[6]	1.0267 ^[7]	0.8871 ^[1]	1.0382 ^[8]	1.0418 ^[9]	0.9235 ^[2]	1.0011 ^[5]	0.9751 ^[4]
	$\sum Ranks$	12 ^[2]	20 ^[6]	18 ^[4]	18 ^[4]	28 ^[8]	36 ^[9]	6 ^[1]	24 ^[7]	18 ^[4]
150	RBias	0.0657 ^[3]	0.0760 ^[5]	0.0709 ^[4]	0.0873 ^[6]	0.0938 ^[8]	0.1124 ^[9]	0.0265 ^[1]	0.0917 ^[7]	0.0577 ^[2]
	Bias	0.0986 ^[3]	0.1140 ^[5]	0.1063 ^[4]	0.1310 ^[6]	0.1407 ^[8]	0.1686 ^[9]	0.0398 ^[1]	0.1375 ^[7]	0.0866 ^[2]
	MSE	0.7072 ^[4]	0.8104 ^[5]	0.8471 ^[7]	0.6045 ^[1]	0.8641 ^[9]	0.8582 ^[8]	0.6386 ^[2]	0.8114 ^[6]	0.7004 ^[3]
	RMSE	0.8410 ^[4]	0.9002 ^[5]	0.9204 ^[7]	0.7775 ^[1]	0.9296 ^[9]	0.9264 ^[8]	0.7991 ^[2]	0.9008 ^[6]	0.8369 ^[3]
	$\sum Ranks$	14 ^[3.5]	20 ^[5]	22 ^[6]	14 ^[3.5]	34 ^[8.5]	34 ^[8.5]	6 ^[1]	26 ^[7]	10 ^[2]
200	RBias	0.1044 ^[5]	0.1119 ^[7]	0.0905 ^[2]	0.0917 ^[3]	0.1119 ^[8]	0.1339 ^[9]	0.0058 ^[1]	0.1116 ^[6]	0.1028 ^[4]
	Bias	0.1566 ^[5]	0.1678 ^[7]	0.1358 ^[2]	0.1375 ^[3]	0.1678 ^[8]	0.2009 ^[9]	0.0087 ^[1]	0.1675 ^[6]	0.1542 ^[4]
	MSE	0.6504 ^[4]	0.7436 ^[7]	0.7344 ^[6]	0.5082 ^[1]	0.7604 ^[9]	0.7544 ^[8]	0.5697 ^[2]	0.7150 ^[5]	0.6410 ^[3]
	RMSE	0.8065 ^[4]	0.8623 ^[7]	0.8570 ^[6]	0.7129 ^[1]	0.8720 ^[9]	0.8686 ^[8]	0.7548 ^[2]	0.8456 ^[5]	0.8006 ^[3]
	$\sum Ranks$	18 ^[5]	28 ^[7]	16 ^[4]	8 ^[2]	34 ^[8.5]	34 ^[8.5]	6 ^[1]	22 ^[6]	14 ^[3]
250	RBias	0.1188 ^[6]	0.1304 ^[8]	0.0944 ^[4]	0.0904 ^[2]	0.1150 ^[5]	0.1467 ^[9]	0.0003 ^[1]	0.1188 ^[7]	0.0909 ^[3]
	Bias	0.1781 ^[6]	0.1956 ^[8]	0.1416 ^[4]	0.1356 ^[2]	0.1726 ^[5]	0.2201 ^[9]	0.0004 ^[1]	0.1783 ^[7]	0.1364 ^[3]
	MSE	0.6158 ^[4]	0.7371 ^[9]	0.6872 ^[5]	0.4455 ^[1]	0.7153 ^[7]	0.7272 ^[8]	0.4856 ^[2]	0.6879 ^[6]	0.6140 ^[3]
	RMSE	0.7847 ^[4]	0.8585 ^[9]	0.8290 ^[5]	0.6675 ^[1]	0.8457 ^[7]	0.8527 ^[8]	0.6969 ^[2]	0.8294 ^[6]	0.7836 ^[3]
	$\sum Ranks$	20 ^[5]	34 ^[8.5]	18 ^[4]	6 ^[1.5]	24 ^[6]	34 ^[8.5]	6 ^[1.5]	26 ^[7]	12 ^[3]

Table 4: Simulation values for $\theta = 2$

n	Est.	MLE	ADE	CME	MPS	LSE	RADE	LADE	PE	KM
50	RBias	0.0549 ^[3]	0.0715 ^[4]	0.0523 ^[2]	0.1552 ^[9]	0.0847 ^[6]	0.1063 ^[8]	0.0272 ^[1]	0.0904 ^[7]	0.0843 ^[5]
	Bias	0.1098 ^[3]	0.1431 ^[4]	0.1046 ^[2]	0.3103 ^[9]	0.1693 ^[6]	0.2127 ^[8]	0.0544 ^[1]	0.1808 ^[7]	0.1686 ^[5]
	MSE	1.2763 ^[1]	1.4756 ^[5]	1.4914 ^[6]	1.3526 ^[2]	1.5838 ^[9]	1.5376 ^[8]	1.3907 ^[3]	1.4183 ^[4]	1.5198 ^[7]
	RMSE	1.1297 ^[1]	1.2147 ^[5]	1.2212 ^[6]	1.1630 ^[2]	1.2585 ^[9]	1.2400 ^[8]	1.1793 ^[3]	1.1909 ^[4]	1.2328 ^[7]
	$\sum Ranks$	8 ^[1.5]	18 ^[4]	16 ^[3]	22 ^[5.5]	30 ^[8]	32 ^[9]	8 ^[1.5]	22 ^[5.5]	24 ^[7]
100	RBias	0.0882 ^[4]	0.0941 ^[7]	0.0702 ^[2]	0.1201 ^[9]	0.0921 ^[6]	0.0994 ^[8]	0.0168 ^[1]	0.0885 ^[5]	0.0725 ^[3]
	Bias	0.1765 ^[4]	0.1882 ^[7]	0.1404 ^[2]	0.2401 ^[9]	0.1843 ^[6]	0.1987 ^[8]	0.0336 ^[1]	0.1770 ^[5]	0.1450 ^[3]
	MSE	0.9023 ^[5]	0.9843 ^[9]	0.9019 ^[4]	0.8025 ^[2]	0.9623 ^[8]	0.9399 ^[7]	0.7834 ^[1]	0.8694 ^[3]	0.9130 ^[6]
	RMSE	0.9499 ^[5]	0.9921 ^[9]	0.9497 ^[4]	0.8958 ^[2]	0.9810 ^[8]	0.9695 ^[7]	0.8851 ^[1]	0.9324 ^[3]	0.9555 ^[6]
	$\sum Ranks$	18 ^[4.5]	32 ^[9]	12 ^[2]	22 ^[6]	28 ^[7]	30 ^[8]	4 ^[1]	16 ^[3]	18 ^[4.5]
150	RBias	0.0851 ^[6]	0.1022 ^[9]	0.0676 ^[2]	0.0997 ^[8]	0.0828 ^[5]	0.0948 ^[7]	0.0243 ^[1]	0.0826 ^[4]	0.0684 ^[3]
	Bias	0.1703 ^[6]	0.2043 ^[9]	0.1352 ^[2]	0.1994 ^[8]	0.1656 ^[5]	0.1896 ^[7]	0.0486 ^[1]	0.1652 ^[4]	0.1368 ^[3]
	MSE	0.6861 ^[3]	0.8525 ^[9]	0.7119 ^[5]	0.6026 ^[2]	0.7493 ^[7]	0.7716 ^[8]	0.5990 ^[1]	0.7042 ^[4]	0.7232 ^[6]
	RMSE	0.8283 ^[3]	0.9233 ^[9]	0.8437 ^[5]	0.7763 ^[2]	0.8656 ^[7]	0.8784 ^[8]	0.7740 ^[1]	0.8392 ^[4]	0.8504 ^[6]
	$\sum Ranks$	18 ^[4.5]	36 ^[9]	14 ^[2]	20 ^[6]	24 ^[7]	30 ^[8]	4 ^[1]	16 ^[3]	18 ^[4.5]
200	RBias	0.0814 ^[4]	0.1101 ^[9]	0.0724 ^[2]	0.1020 ^[7]	0.0840 ^[5]	0.1050 ^[8]	0.0454 ^[1]	0.0880 ^[6]	0.0796 ^[3]
	Bias	0.1628 ^[4]	0.2202 ^[9]	0.1448 ^[2]	0.2040 ^[7]	0.1680 ^[5]	0.2100 ^[8]	0.0908 ^[1]	0.1760 ^[6]	0.1592 ^[3]
	MSE	0.4565 ^[2]	0.6436 ^[9]	0.4997 ^[4]	0.4385 ^[1]	0.5249 ^[6]	0.5937 ^[8]	0.4643 ^[3]	0.5060 ^[5]	0.5676 ^[7]
	RMSE	0.6756 ^[2]	0.8022 ^[9]	0.7069 ^[4]	0.6622 ^[1]	0.7245 ^[6]	0.7705 ^[8]	0.6814 ^[3]	0.7114 ^[5]	0.7534 ^[7]
	$\sum Ranks$	12 ^[2.5]	36 ^[9]	12 ^[2.5]	16 ^[4]	22 ^[6.5]	32 ^[8]	8 ^[1]	22 ^[6.5]	20 ^[5]
250	RBias	0.0393 ^[2]	0.0523 ^[4]	0.0458 ^[3]	0.0598 ^[8]	0.0547 ^[6]	0.0711 ^[9]	0.0277 ^[1]	0.0547 ^[7]	0.0543 ^[5]
	Bias	0.0787 ^[2]	0.1047 ^[4]	0.0917 ^[3]	0.1196 ^[8]	0.1094 ^[6]	0.1421 ^[9]	0.0554 ^[1]	0.1094 ^[7]	0.1087 ^[5]
	MSE	0.3740 ^[2]	0.4677 ^[6]	0.4631 ^[4]	0.3708 ^[1]	0.4855 ^[7]	0.5333 ^[9]	0.4664 ^[5]	0.4555 ^[3]	0.5071 ^[8]
	RMSE	0.6116 ^[2]	0.6839 ^[6]	0.6805 ^[4]	0.6090 ^[1]	0.6967 ^[7]	0.7303 ^[9]	0.6829 ^[5]	0.6749 ^[3]	0.7121 ^[8]
	$\sum Ranks$	8 ^[1]	20 ^[5.5]	14 ^[3]	18 ^[4]	26 ^[7.5]	36 ^[9]	12 ^[2]	20 ^[5.5]	26 ^[7.5]

Table 5: Simulation values for $\theta = 3$

n	Est.	MLE	ADE	CME	MPS	LSE	RADE	LADE	PE	KM
50	RBias	0.0272 ^[3]	0.0312 ^[6]	0.0129 ^[2]	0.0716 ^[9]	0.0295 ^[4]	0.0383 ^[8]	0.0034 ^[1]	0.0372 ^[7]	0.0302 ^[5]
	Bias	0.0815 ^[3]	0.0936 ^[6]	0.0388 ^[2]	0.2149 ^[9]	0.0884 ^[4]	0.1149 ^[8]	0.0102 ^[1]	0.1117 ^[7]	0.0907 ^[5]
	MSE	1.0861 ^[7]	1.0908 ^[8]	0.9718 ^[2]	1.0541 ^[4]	1.0611 ^[5]	1.0612 ^[6]	1.0134 ^[3]	0.9342 ^[1]	1.1544 ^[9]
	RMSE	1.0421 ^[7]	1.0444 ^[8]	0.9858 ^[2]	1.0267 ^[4]	1.0301 ^[5]	1.0301 ^[6]	1.0067 ^[3]	0.9665 ^[1]	1.0744 ^[9]
	$\sum Ranks$	20 ^[5]	28 ^[8]	8 ^[1.5]	26 ^[6]	18 ^[4]	28 ^[8]	8 ^[1.5]	16 ^[3]	28 ^[8]
100	RBias	0.0172 ^[3]	0.0175 ^[4]	0.0129 ^[2]	0.0436 ^[9]	0.0188 ^[5]	0.0224 ^[6]	0.0091 ^[1]	0.0242 ^[7]	0.0256 ^[8]
	Bias	0.0517 ^[3]	0.0526 ^[4]	0.0388 ^[2]	0.1308 ^[9]	0.0565 ^[5]	0.0673 ^[6]	0.0273 ^[1]	0.0725 ^[7]	0.0768 ^[8]
	MSE	0.4477 ^[3]	0.4425 ^[2]	0.4529 ^[4]	0.4667 ^[6]	0.4716 ^[7]	0.4605 ^[5]	0.5079 ^[8]	0.4290 ^[1]	0.6002 ^[9]
	RMSE	0.6691 ^[3]	0.6652 ^[2]	0.6730 ^[4]	0.6832 ^[6]	0.6868 ^[7]	0.6786 ^[5]	0.7126 ^[8]	0.6550 ^[1]	0.7747 ^[9]
	$\sum Ranks$	12 ^[2]	12 ^[2]	12 ^[2]	30 ^[8]	24 ^[7]	22 ^[6]	18 ^[5]	16 ^[4]	34 ^[9]
150	RBias	0.0108 ^[5]	0.0089 ^[3]	0.0076 ^[2]	0.0266 ^[9]	0.0106 ^[4]	0.0109 ^[6]	0.0053 ^[1]	0.0133 ^[7]	0.0150 ^[8]
	Bias	0.0323 ^[5]	0.0266 ^[3]	0.0228 ^[2]	0.0797 ^[9]	0.0319 ^[4]	0.0328 ^[6]	0.0158 ^[1]	0.0399 ^[7]	0.0450 ^[8]
	MSE	0.2942 ^[7]	0.2759 ^[3]	0.2838 ^[5]	0.2807 ^[4]	0.2879 ^[6]	0.2713 ^[2]	0.3394 ^[8]	0.2619 ^[1]	0.3799 ^[9]
	RMSE	0.5424 ^[7]	0.5253 ^[3]	0.5328 ^[5]	0.5298 ^[4]	0.5365 ^[6]	0.5208 ^[2]	0.5826 ^[8]	0.5118 ^[1]	0.6164 ^[9]
	$\sum Ranks$	24 ^[7]	12 ^[1]	14 ^[2]	26 ^[8]	20 ^[6]	16 ^[3.5]	18 ^[5]	16 ^[3.5]	34 ^[9]
200	RBias	0.0002 ^[2]	0.0018 ^[4]	0.0006 ^[3]	0.0146 ^[9]	0.0028 ^[6]	0.0028 ^[5]	0.0001 ^[1]	0.0054 ^[8]	0.0045 ^[7]
	Bias	0.0007 ^[2]	0.0055 ^[4]	0.0019 ^[3]	0.0439 ^[9]	0.0084 ^[6]	0.0083 ^[5]	0.0004 ^[1]	0.0161 ^[8]	0.0135 ^[7]
	MSE	0.1744 ^[3]	0.1866 ^[5]	0.1963 ^[6]	0.1809 ^[4]	0.1982 ^[7]	0.1708 ^[1]	0.2509 ^[9]	0.1733 ^[2]	0.2447 ^[8]
	RMSE	0.4176 ^[3]	0.4320 ^[5]	0.4431 ^[6]	0.4254 ^[4]	0.4452 ^[7]	0.4132 ^[1]	0.5009 ^[9]	0.4163 ^[2]	0.4947 ^[8]
	$\sum Ranks$	10 ^[1]	18 ^[3.5]	18 ^[3.5]	26 ^[7.5]	26 ^[7.5]	12 ^[2]	20 ^[5.5]	20 ^[5.5]	30 ^[9]
250	RBias	0.0129 ^[1]	0.0162 ^[4]	0.0154 ^[2]	0.0247 ^[9]	0.0171 ^[6]	0.0159 ^[3]	0.0169 ^[5]	0.0179 ^[7]	0.0185 ^[8]
	Bias	0.0388 ^[1]	0.0486 ^[4]	0.0462 ^[2]	0.0740 ^[9]	0.0514 ^[6]	0.0476 ^[3]	0.0508 ^[5]	0.0536 ^[7]	0.0555 ^[8]
	MSE	0.1503 ^[3]	0.1563 ^[4]	0.1621 ^[6]	0.1595 ^[5]	0.1641 ^[7]	0.1485 ^[1]	0.2027 ^[8]	0.1500 ^[2]	0.2061 ^[9]
	RMSE	0.3877 ^[3]	0.3954 ^[4]	0.4027 ^[6]	0.3994 ^[5]	0.4051 ^[7]	0.3854 ^[1]	0.4503 ^[8]	0.3873 ^[2]	0.4540 ^[9]
	$\sum Ranks$	8 ^[1.5]	16 ^[3.5]	16 ^[3.5]	28 ^[8]	26 ^[6.5]	8 ^[1.5]	26 ^[6.5]	18 ^[5]	34 ^[9]

Table 6: Simulation values for $\theta = 5$

n	Est.	MLE	ADE	CME	MPS	LSE	RADE	LADE	PE	KM
50	RBias	0.0071 ^{7}	0.0005 ^{1}	0.0031 ^{5}	0.0232 ^{9}	0.0006 ^{2}	0.0009 ^{3}	0.0064 ^{6}	0.0139 ^{8}	0.0010 ^{4}
	Bias	0.0354 ^{7}	0.0026 ^{1}	0.0156 ^{5}	0.1160 ^{9}	0.0030 ^{2}	0.0043 ^{3}	0.0319 ^{6}	0.0695 ^{8}	0.0051 ^{4}
	MSE	0.6125 ^{4}	0.6580 ^{5}	0.7050 ^{7}	0.6070 ^{2}	0.7044 ^{6}	0.6121 ^{3}	0.8552 ^{9}	0.5772 ^{1}	0.7606 ^{8}
	RMSE	0.7826 ^{4}	0.8112 ^{5}	0.8397 ^{7}	0.7791 ^{2}	0.8393 ^{6}	0.7823 ^{3}	0.9247 ^{9}	0.7597 ^{1}	0.8721 ^{8}
	$\sum Ranks$	22 ^{5.5}	12 ^{1.5}	24 ^{7.5}	22 ^{5.5}	16 ^{3}	12 ^{1.5}	30 ^{9}	18 ^{4}	24 ^{7.5}
100	RBias	0.0015 ^{3}	0.0003 ^{2}	0.0020 ^{4}	0.0143 ^{9}	0.0001 ^{1}	0.0025 ^{5}	0.0067 ^{7}	0.0094 ^{8}	0.0029 ^{6}
	Bias	0.0077 ^{3}	0.0014 ^{2}	0.0099 ^{4}	0.0716 ^{9}	0.0005 ^{1}	0.0126 ^{5}	0.0336 ^{7}	0.0468 ^{8}	0.0147 ^{6}
	MSE	0.2746 ^{3}	0.2968 ^{5}	0.3176 ^{7}	0.2745 ^{2}	0.3173 ^{6}	0.2784 ^{4}	0.3892 ^{9}	0.2720 ^{1}	0.3426 ^{8}
	RMSE	0.5240 ^{3}	0.5448 ^{5}	0.5636 ^{7}	0.5239 ^{2}	0.5633 ^{6}	0.5277 ^{4}	0.6238 ^{9}	0.5215 ^{1}	0.5853 ^{8}
	$\sum Ranks$	12 ^{1}	14 ^{2.5}	22 ^{6.5}	22 ^{6.5}	14 ^{2.5}	18 ^{4.5}	32 ^{9}	18 ^{4.5}	28 ^{8}
150	RBias	0.0061 ^{8}	0.0041 ^{3}	0.0055 ^{7}	0.0047 ^{6}	0.0043 ^{5}	0.0031 ^{2}	0.0069 ^{9}	0.0015 ^{1}	0.0042 ^{4}
	Bias	0.0306 ^{8}	0.0204 ^{3}	0.0277 ^{7}	0.0234 ^{6}	0.0216 ^{5}	0.0157 ^{2}	0.0343 ^{9}	0.0077 ^{1}	0.0211 ^{4}
	MSE	0.1936 ^{3}	0.2101 ^{5}	0.2234 ^{7}	0.1905 ^{1}	0.2230 ^{6}	0.1972 ^{4}	0.2704 ^{9}	0.1923 ^{2}	0.2438 ^{8}
	RMSE	0.4400 ^{3}	0.4584 ^{5}	0.4727 ^{7}	0.4365 ^{1}	0.4722 ^{6}	0.4440 ^{4}	0.5200 ^{9}	0.4385 ^{2}	0.4938 ^{8}
	$\sum Ranks$	22 ^{5.5}	16 ^{4}	28 ^{8}	14 ^{3}	22 ^{5.5}	12 ^{2}	36 ^{9}	6 ^{1}	24 ^{7}
200	RBias	0.0012 ^{4}	0.0006 ^{2}	0.0016 ^{5}	0.0070 ^{9}	0.0007 ^{3}	0.0004 ^{1}	0.0032 ^{7}	0.0041 ^{8}	0.0021 ^{6}
	Bias	0.0059 ^{4}	0.0032 ^{2}	0.0080 ^{5}	0.0350 ^{9}	0.0034 ^{3}	0.0020 ^{1}	0.0161 ^{7}	0.0204 ^{8}	0.0106 ^{6}
	MSE	0.1400 ^{3}	0.1503 ^{5}	0.1603 ^{7}	0.1397 ^{2}	0.1602 ^{6}	0.1411 ^{4}	0.1937 ^{9}	0.1394 ^{1}	0.1702 ^{8}
	RMSE	0.3741 ^{3}	0.3877 ^{5}	0.4004 ^{7}	0.3737 ^{2}	0.4002 ^{6}	0.3757 ^{4}	0.4401 ^{9}	0.3734 ^{1}	0.4126 ^{8}
	$\sum Ranks$	14 ^{2.5}	14 ^{2.5}	24 ^{7}	22 ^{6}	18 ^{4.5}	10 ^{1}	32 ^{9}	18 ^{4.5}	28 ^{8}
250	RBias	0.0014 ^{4}	0.0009 ^{2}	0.0018 ^{5}	0.0051 ^{9}	0.0010 ^{3}	0.0001 ^{1}	0.0028 ^{7}	0.0030 ^{8}	0.0021 ^{6}
	Bias	0.0071 ^{4}	0.0043 ^{2}	0.0088 ^{5}	0.0257 ^{9}	0.0051 ^{3}	0.0005 ^{1}	0.0142 ^{7}	0.0149 ^{8}	0.0103 ^{6}
	MSE	0.1168 ^{2}	0.1306 ^{5}	0.1395 ^{7}	0.1164 ^{1}	0.1394 ^{6}	0.1196 ^{4}	0.1714 ^{9}	0.1173 ^{3}	0.1583 ^{8}
	RMSE	0.3418 ^{2}	0.3613 ^{5}	0.3735 ^{7}	0.3411 ^{1}	0.3733 ^{6}	0.3458 ^{4}	0.4141 ^{9}	0.3425 ^{3}	0.3979 ^{8}
	$\sum Ranks$	12 ^{2}	14 ^{3}	24 ^{7}	20 ^{5}	18 ^{4}	10 ^{1}	32 ^{9}	22 ^{6}	28 ^{8}

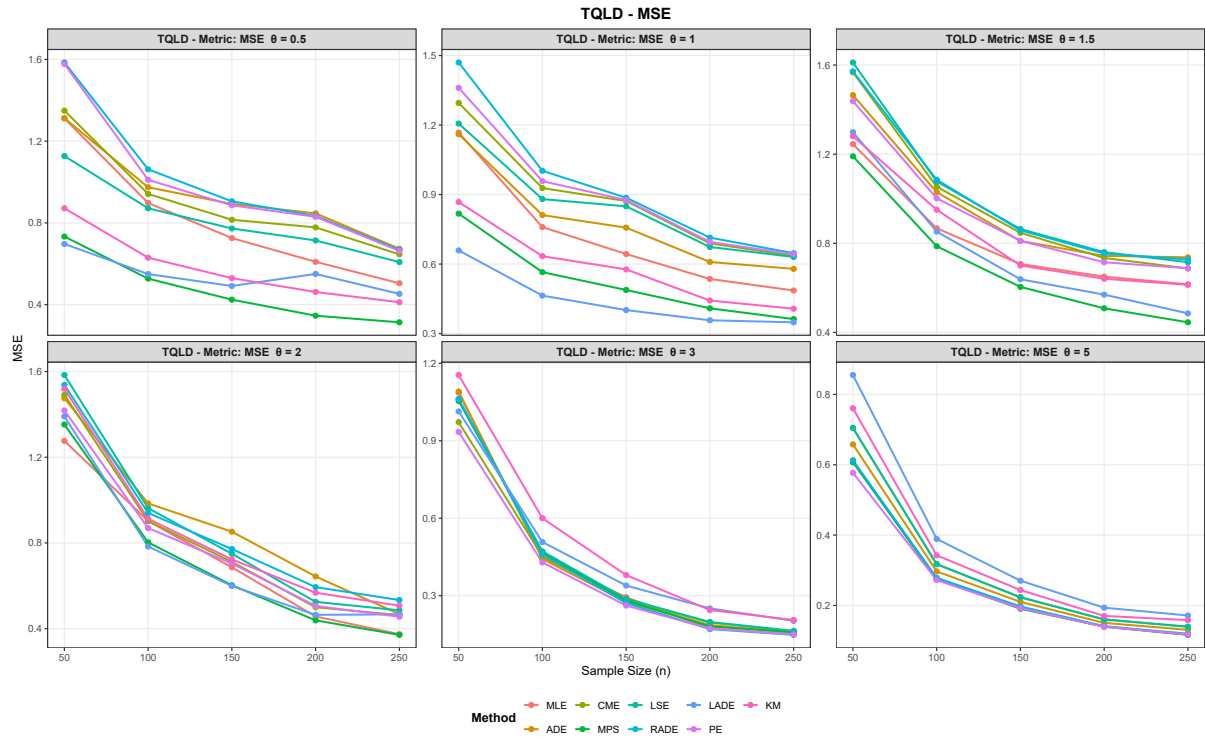


Figure 2: Graphical representation for MSE values.

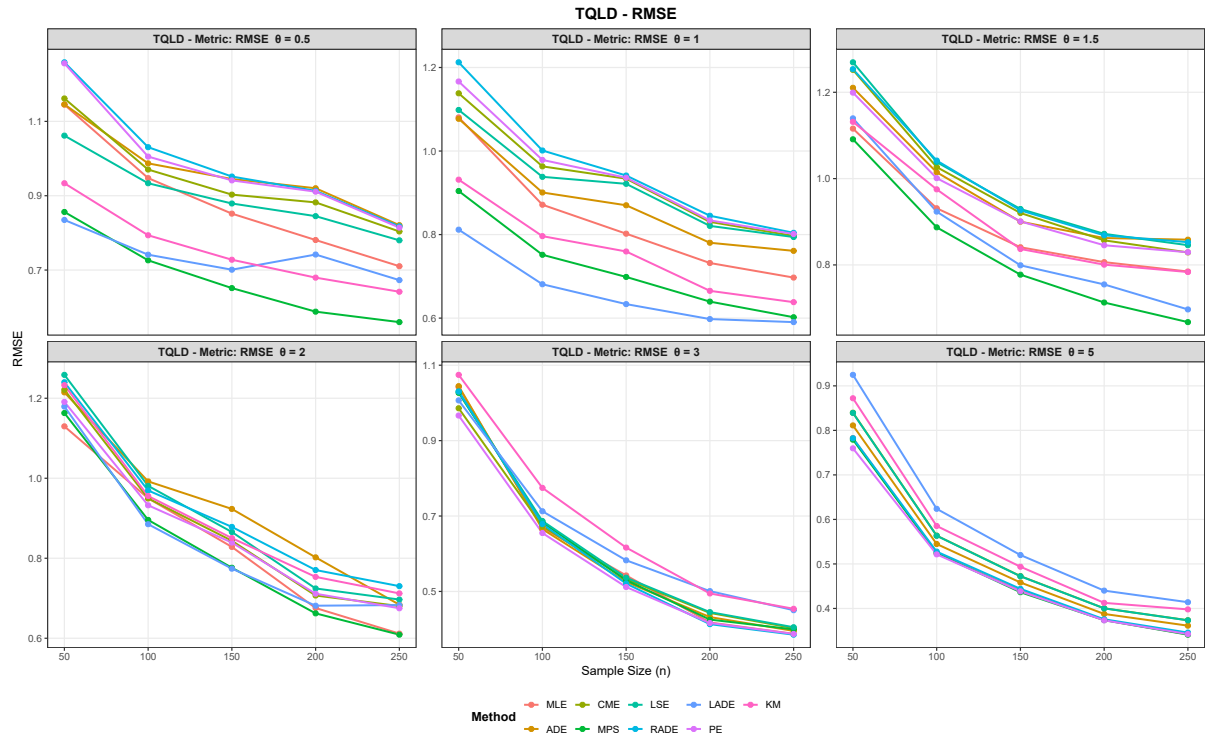


Figure 3: Graphical representation for RMSE values.

These simulation results expose a few general trends present across the considered scenarios. First, the estimation error tends to go down as the sample size increases for all methods, reflecting better finite-sample performance

with larger samples.

Second, the relative performance of the estimators depends on the value of θ . For small parameters, especially $\theta = 0.5$, the MPS and LADE often have the smallest bias and prediction error. For moderate values ($\theta = 1, 1.5, 2$), LADE, MPS, MLE, ADE, and CME all deliver competitive results, with marginal differences between the top estimators. For larger parameter values ($\theta = 3, 5$), MLE, ADE, and MPS typically offer the best trade-off between bias and variance.

The KM and PE estimators display comparatively less stable behaviour across the simulation design, especially for smaller sample sizes where their MSE and RMSE are frequently larger than that of the front runners.

In general, no estimator dominates across all simulation scenarios. However, MLE, MPS, ADE, and LADE regularly rank as the top performing methods over a large range of different parameter values and sample sizes.

The average ranking of the nine estimation methods appears in Table 7. Based on the combined rank scores, LADE has the top overall performance due to its low bias in multiple simulation settings. The MLE, MPS, and ADE estimators also perform excellently and stably, offering a nice compromise between estimation bias and variance.

As a result, the simulation study indicates that MLE, MPS, and ADE are good general-purpose estimation procedures for the TQLD, while LADE is appealing when bias minimization is paramount. In contrast, the KM and PE estimators exhibit relatively less stable finite-sample performance across the simulation settings considered.

Table 7: Partial and overall ranks (9 methods).

Parameter	n	MLE	ADE	CME	MPS	LSE	RADE	LADE	PE	KM
0.5	50	5.5	5.5	7	1.5	4	9	1.5	8	3
	100	5.5	7	5.5	1	4	9	2	8	3
	150	4	7.5	6	1	5	9	2	7.5	3
	200	4	8.5	6	1	5	8.5	3	7	2
	250	4	8	6	1	5	9	2.5	7	2.5
1	50	7	4	6	5	3	8	1	9	2
	100	6.5	2	6.5	5	3.5	8	1	9	3.5
	150	5	2.5	8.5	6	2.5	7	1	8.5	4
	200	5	2	7.5	5	3	7.5	1	9	5
	250	4	2.5	8	5.5	2.5	7	1	9	5.5
1.5	50	1	4	3	5	8	9	6.5	6.5	2
	100	2	6	4	4	8	9	1	7	4
	150	3.5	5	6	3.5	8.5	8.5	1	7	2
	200	5	7	4	2	8.5	8.5	1	6	3
	250	5	8.5	4	1.5	6	8.5	1.5	7	3
2	50	1.5	4	3	5.5	8	9	1.5	5.5	7
	100	4.5	9	2	6	7	8	1	3	4.5
	150	4.5	9	2	6	7	8	1	3	4.5
	200	2.5	9	2.5	4	6.5	8	1	6.5	5
	250	1	5.5	3	4	7.5	9	2	5.5	7.5
3	50	5	8	1.5	6	4	8	1.5	3	8
	100	2	2	2	8	7	6	5	4	9
	150	7	1	2	8	6	3.5	5	3.5	9
	200	1	3.5	3.5	7.5	7.5	2	5.5	5.5	9
	250	1.5	3.5	3.5	8	6.5	1.5	6.5	5	9
5	50	5.5	1.5	7.5	5.5	3	1.5	9	4	7.5
	100	1	2.5	6.5	6.5	2.5	4.5	9	4.5	8
	150	5.5	4	8	3	5.5	2	9	1	7
	200	2.5	2.5	7	6	4.5	1	9	4.5	8
	250	2	3	7	5	4	1	9	6	8
Σ Ranks		114	148	149	137	163	198.5	102	180	158.5
Overall Rank		2	4	5	3	7	9	1	8	6

6 Application to Healthcare Data

This section illustrates the practical applicability of the proposed TQLD through a real-world dataset on private healthcare premiums. This dataset was originally analyzed by [47]. The data consist of $n = 38$ observations representing the average annual percent change in private healthcare premiums for the calendar years 1969–2007, as reported below:

{14.4, 14.0, 15.4, 9.4, 11.7, 15.0, 24.9, 20.7, 12.5, 14.9, 12.6, 16.7, 13.8, 11.0, 12.9, 10.1, 1.9, 8.5, 16.5, 15.3, 13.3, 9.8, 8.4, 7.9, 3.7, 5.1, 4.6, 4.4, 5.4, 6.1, 8.0, 10.0, 11.2, 10.1, 6.4, 6.7, 5.7, 5.8}.

Since all distributions under consideration are defined on the unit interval $(0, 1)$, we apply the min–max transformation, which preserves skewness and relative dispersion while ensuring support on $(0, 1)$. The transformed data are:

{0.5435, 0.5261, 0.5870, 0.3261, 0.4261, 0.5696, 1.0000, 0.8174, 0.4609, 0.5652, 0.4652, 0.6435, 0.5174, 0.3957, 0.4783, 0.3565, 0.0000, 0.2870, 0.6348, 0.5826, 0.4957, 0.3435, 0.2826, 0.2609, 0.0783, 0.1391, 0.1174, 0.1087, 0.1522, 0.1826, 0.2652, 0.3522, 0.4043, 0.3565, 0.1957, 0.2087, 0.1652, 0.1696}.

Descriptive statistics for the transformed data are presented in Table 8. The positive skewness and moderate excess kurtosis indicate right-skewed data with slightly heavier tails than normal.

Table 8: Descriptive statistics for the transformed healthcare premium data.

N	Mean	Median	Variance	Std. Dev.	Skewness	Kurtosis	Sum
38	0.38055	0.35652	0.04603	0.21455	0.57851	3.33224	14.46087

To assess the performance of the proposed TQLD, we compare it with six commonly used distributions on $(0, 1)$: QLD, truncated exponential distribution (TED) [48], unit-Lindley distribution (ULD) [51], unit-exponential distribution (UED) [49], truncated new XLindley distribution (TNXLD) [50], and truncated Lomax distribution (TL0D) [52]. These cover light-tailed (TED, UED), medium-tailed (QLD, ULD, TNXLD), and heavy-tailed (TL0D) alternatives. For each distribution, the unknown parameter is estimated using the nine estimation methods described in Section 5: MLE, MPS, LSE, CME, ADE, RADE, LADE, PE, and KM. This wide range of estimators allows us to evaluate the robustness of model selection under different objective functions.

We primarily compare models using the Kolmogorov–Smirnov (KS) statistic and its p -value; smaller KS and larger p indicate better fit. The TQLD consistently achieves the smallest KS statistic and highest p -value across all nine methods. Tables 9–17 shown here for MLE, MPS, LSE, CME, ADE, RADE, LADE, PE, and KM illustrate this; the TQLD is ranked first for every method. The estimated parameter remains remarkably stable across all estimation methods, varying only from 3.54 to 4.19. This narrow range indicates that the proposed TQLD is robust with respect to the choice of estimation criterion. Although the estimated standard errors differ across estimation methods, the resulting parameter estimates remain highly consistent, suggesting that the fitted TQLD is relatively insensitive to the choice of estimation criterion.

Table 9: Estimation results for the MLE method.

Distribution	$\hat{\theta}$	$SE(\hat{\theta})$	KS-statistic	p -value	Rank
TQLD	4.1849	0.7980	0.1061	0.7857	1
QLD	4.9272	0.6295	0.1385	0.4598	3
TED	1.4853	0.5928	0.1517	0.3459	4
UED	0.6984	0.1133	0.2124	0.0649	7
TNXLD	3.0492	0.7874	0.1327	0.5153	2
ULD	3.2466	0.4427	0.1963	0.1070	6
TL0D	0.8936	0.8185	0.1709	0.2168	5

Table 18 summarizes the performance ranking of the TQLD across all nine estimation methods. Across the nine estimation methods, the KS statistic ranges only from 0.1058 to 0.1443, indicating that the fitted TQLD is stable

Table 10: Estimation results for the MPS method.

Distribution	$\hat{\theta}$	SE($\hat{\theta}$)	KS-statistic	p -value	Rank
TQLD	4.0119	5.0314	0.1160	0.6861	1
QLD	4.7409	3.8342	0.1237	0.6063	2
TED	1.4386	3.6470	0.1558	0.3148	4
UED	0.6590	0.6769	0.1995	0.0971	7
TNXLD	2.9401	4.8835	0.1347	0.4959	3
ULD	3.1543	2.6768	0.1844	0.1510	6
TLoD	0.8673	5.0475	0.1726	0.2077	5

Table 11: Estimation results for the LSE method.

Distribution	$\hat{\theta}$	SE($\hat{\theta}$)	KS-statistic	p -value	Rank
TQLD	3.7886	1.8297	0.1292	0.5502	1
QLD	4.3524	1.3292	0.1465	0.3882	2
TED	1.2785	1.1205	0.1702	0.2208	4
UED	0.7009	0.2446	0.2132	0.0632	7
TNXLD	2.6220	1.6081	0.1555	0.3167	3
ULD	2.7502	0.7725	0.1982	0.1012	6
TLoD	0.7641	1.5339	0.1791	0.1743	5

Table 12: Estimation results for the CME method.

Distribution	$\hat{\theta}$	SE($\hat{\theta}$)	KS-statistic	p -value	Rank
TQLD	3.8098	1.8176	0.1279	0.5630	1
QLD	4.3714	1.3270	0.1448	0.4028	2
TED	1.2833	1.1173	0.1698	0.2234	4
UED	0.7026	0.2443	0.2137	0.0621	7
TNXLD	2.6335	1.6019	0.1548	0.3225	3
ULD	2.7582	0.7720	0.1969	0.1049	6
TLoD	0.7678	1.5292	0.1789	0.1755	5

Table 13: Estimation results for the ADE method.

Distribution	$\hat{\theta}$	SE($\hat{\theta}$)	KS-statistic	p -value	Rank
TQLD	3.8736	0.8574	0.1241	0.6019	1
QLD	4.4613	0.6370	0.1369	0.4747	2
TED	1.2966	0.5541	0.1686	0.2305	4
UED	0.7078	0.1236	0.2154	0.0588	7
TNXLD	2.6692	0.7831	0.1524	0.3407	3
ULD	2.8025	0.3914	0.1902	0.1277	6
TLoD	0.7743	0.7619	0.1785	0.1775	5

Table 14: Estimation results for the RADE method.

Distribution	$\hat{\theta}$	SE($\hat{\theta}$)	KS-statistic	p -value	Rank
TQLD	4.1094	1.0792	0.1104	0.7435	1
QLD	4.6785	0.8139	0.1188	0.6566	2
TED	1.5630	0.7349	0.1450	0.4009	4
UED	0.6362	0.1584	0.1920	0.1215	7
TNXLD	2.9940	0.9999	0.1312	0.5295	3
ULD	3.0151	0.5236	0.1687	0.2298	6
TLoD	1.1618	1.0123	0.1544	0.3251	5

Table 15: Estimation results for the LADE method.

Distribution	$\hat{\theta}$	SE($\hat{\theta}$)	KS-statistic	p -value	Rank
TQLD	3.5408	1.3160	0.1443	0.4077	1
QLD	4.1436	0.9229	0.1662	0.2448	2
TED	1.0041	0.7482	0.1963	0.1071	4
UED	0.7725	0.1692	0.2360	0.0290	7
TNXLD	2.2683	1.1300	0.1797	0.1717	3
ULD	2.5480	0.5083	0.2311	0.0345	6
TLoD	0.3675	1.0239	0.2053	0.0814	5

Table 16: Estimation results for the PE method.

Distribution	$\hat{\theta}$	SE($\hat{\theta}$)	KS-statistic	p -value	Rank
TQLD	4.1751	2.2924	0.1067	0.7804	1
QLD	5.0719	1.4260	0.1499	0.3604	3
TED	1.6574	1.5258	0.1511	0.3511	4
UED	0.6332	0.3056	0.1927	0.1189	6
TNXLD	3.0970	2.1031	0.1345	0.4971	2
ULD	3.6494	0.8760	0.2455	0.0205	7
TLoD	1.3004	2.0969	0.1588	0.2936	5

Table 17: Estimation results for the KM method.

Distribution	$\hat{\theta}$	SE($\hat{\theta}$)	KS-statistic	p -value	Rank
TQLD	4.1902	0.5976	0.1058	0.7886	1
QLD	4.6793	0.5483	0.1188	0.6572	2
TED	1.5633	0.3299	0.1450	0.4009	4
UED	0.6352	0.0865	0.1916	0.1227	7
TNXLD	3.0011	0.5514	0.1308	0.5337	3
ULD	2.9792	0.3642	0.1652	0.2505	6
TLoD	1.1869	0.3323	0.1529	0.3365	5

regardless of the estimation criterion. The KM estimator obtains the smallest KS statistic and largest p -value and therefore the best overall rank.

Table 18: Performance ranking of the TQLD across the estimation methods.

Method	KS-statistic	p -value	Rank
KM	0.1058	0.7886	1
MLE	0.1061	0.7857	2
PE	0.1067	0.7804	3
RADE	0.1104	0.7435	4
MPS	0.1160	0.6861	5
ADE	0.1241	0.6019	6
CME	0.1279	0.5630	7
LSE	0.1292	0.5502	8
LADE	0.1443	0.4077	9

Figures 4–9 present the histogram with fitted densities, the empirical and fitted CDFs, survival and hazard functions, together with the PP and QQ plots for the nine estimation methods. The fitted curves closely follow the empirical distribution over the entire support. In particular, the histogram and fitted density show excellent agreement, while the empirical and fitted CDFs almost overlap. The PP and QQ plots also indicate that the fitted quantiles closely match the empirical quantiles, confirming the adequacy of the proposed model.

The observed superiority of the TQLD may be attributed to three characteristics:

1. **Flexibility** in capturing both skewness and moderate excess kurtosis.
2. **Natural handling** of the boundary points (0, 1).
3. Ability to accommodate moderate tail behaviour while preserving analytical simplicity.

For practitioners, the KM method gives the best fit ($KS = 0.1058$), followed closely by MLE. Overall, the application demonstrates that the proposed TQLD provides an accurate, stable, and robust model for bounded positively skewed healthcare data. The consistency of the results across nine estimation procedures further confirms that the distribution is not sensitive to the estimation criterion. Although the KM estimator achieves the best goodness-of-fit according to the KS statistic, the MLE provides nearly identical performance while preserving its desirable asymptotic properties. These findings highlight the practical usefulness of the proposed TQLD as a competitive alternative for modelling bounded lifetime and healthcare-related data.

7 Concluding Remarks

This article introduced and studied a new one-parameter bounded distribution called the truncated Q-Lindley distribution (TQLD), formulated by truncating the Q-Lindley distribution to the unit interval. The proposed TQLD extends the applicability of the original Q-Lindley distribution to bounded continuous data. Several mathematical and statistical properties of the TQLD were presented and computed. Closed-form expressions were calculated for different important reliability measures, such as; the survival, hazard rate, reversed hazard rate, and cumulative hazard rate functions, together with the Mills ratio and odds function. Additional important statistical properties, like the quantile function, moments, mean, variance, moment generating function, entropy, and weighted entropy, were also calculated explicitly, demonstrating the analytical tractability of the proposed TQLD.

To estimate the unknown parameter θ , nine classical estimation approaches were conducted and compared through an extensive Monte Carlo simulation study. The numerical outcomes indicated that estimation accuracy improves as the sample size increases for all competing methods. Although no single estimator uniformly dominates across all parameter combinations, the left-tail Anderson-Darling, maximum likelihood, and maximum product of spacings estimators consistently demonstrate favorable performance with respect to bias, relative bias, mean squared error, and root mean squared error. The importance and practicality of the proposed TQLD were illustrated using a healthcare dataset on normalized private healthcare premium changes. Comparative goodness-of-fit analyses indicated that the TQLD provides a competitive and, in different cases, superior fit relative to some well-known

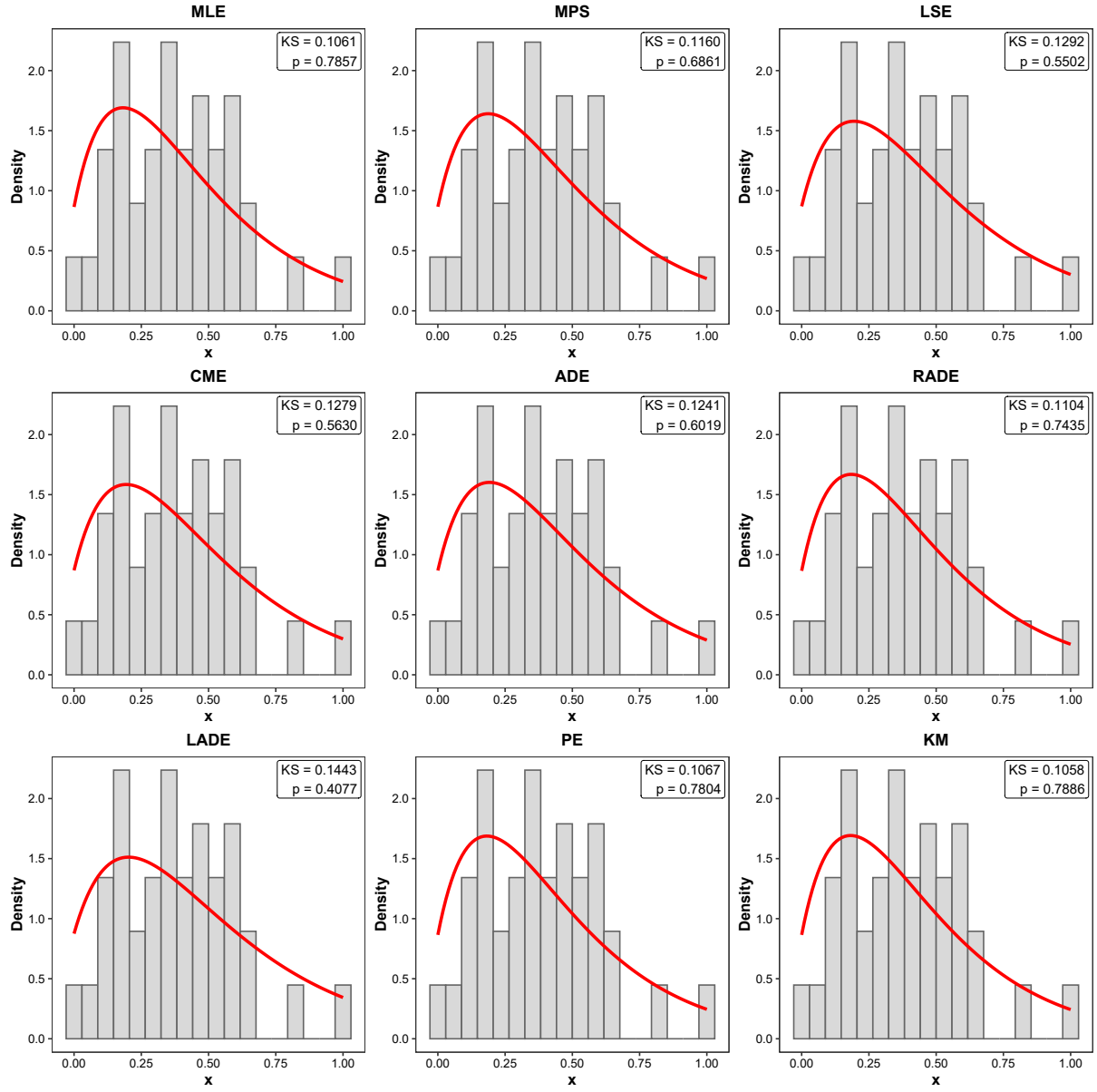


Figure 4: Histogram and fitted PDF for the TQLD under the nine estimation methods for healthcare data.

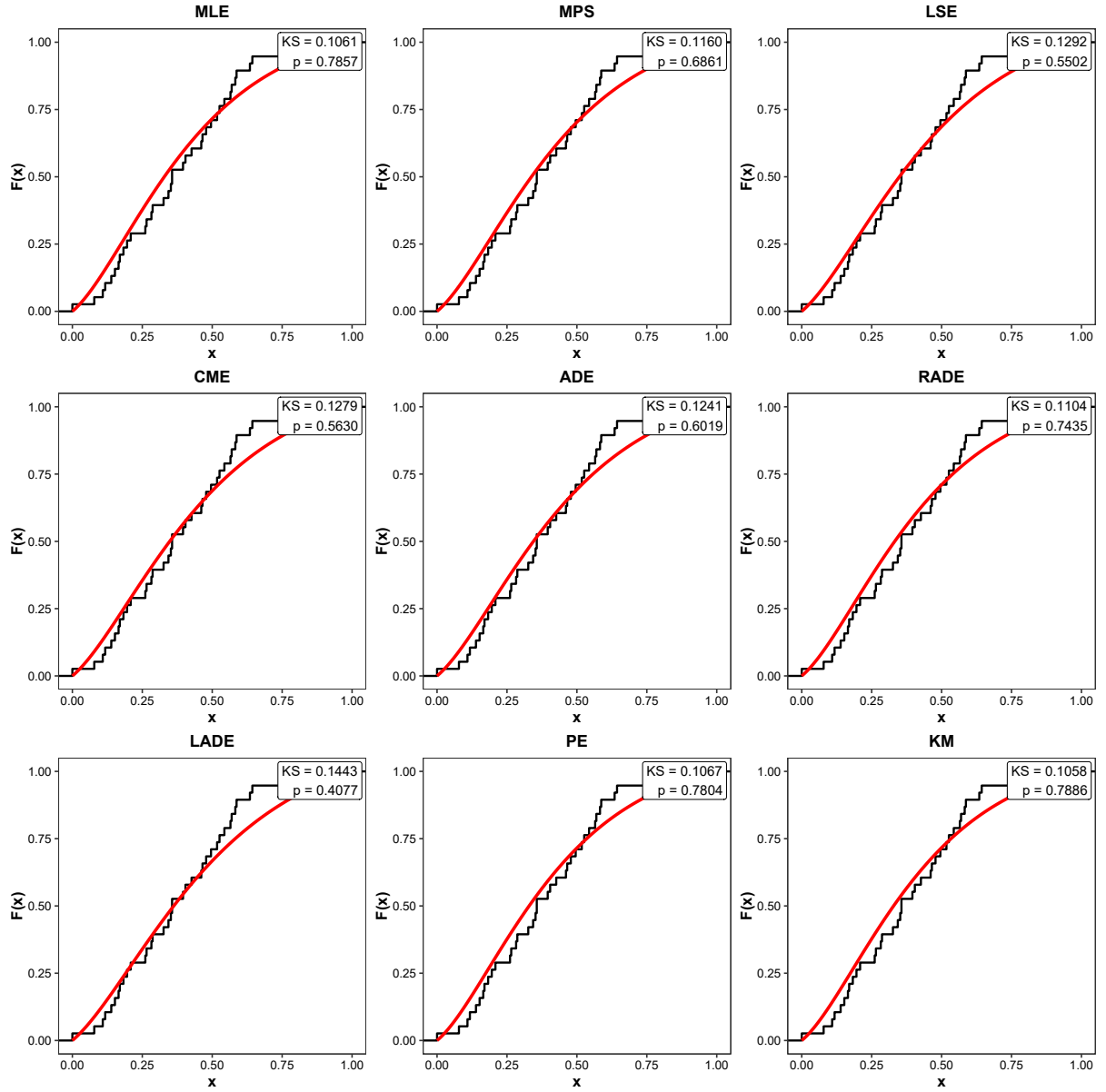


Figure 5: Empirical and fitted CDF for the TQLD under the nine estimation methods for healthcare data.

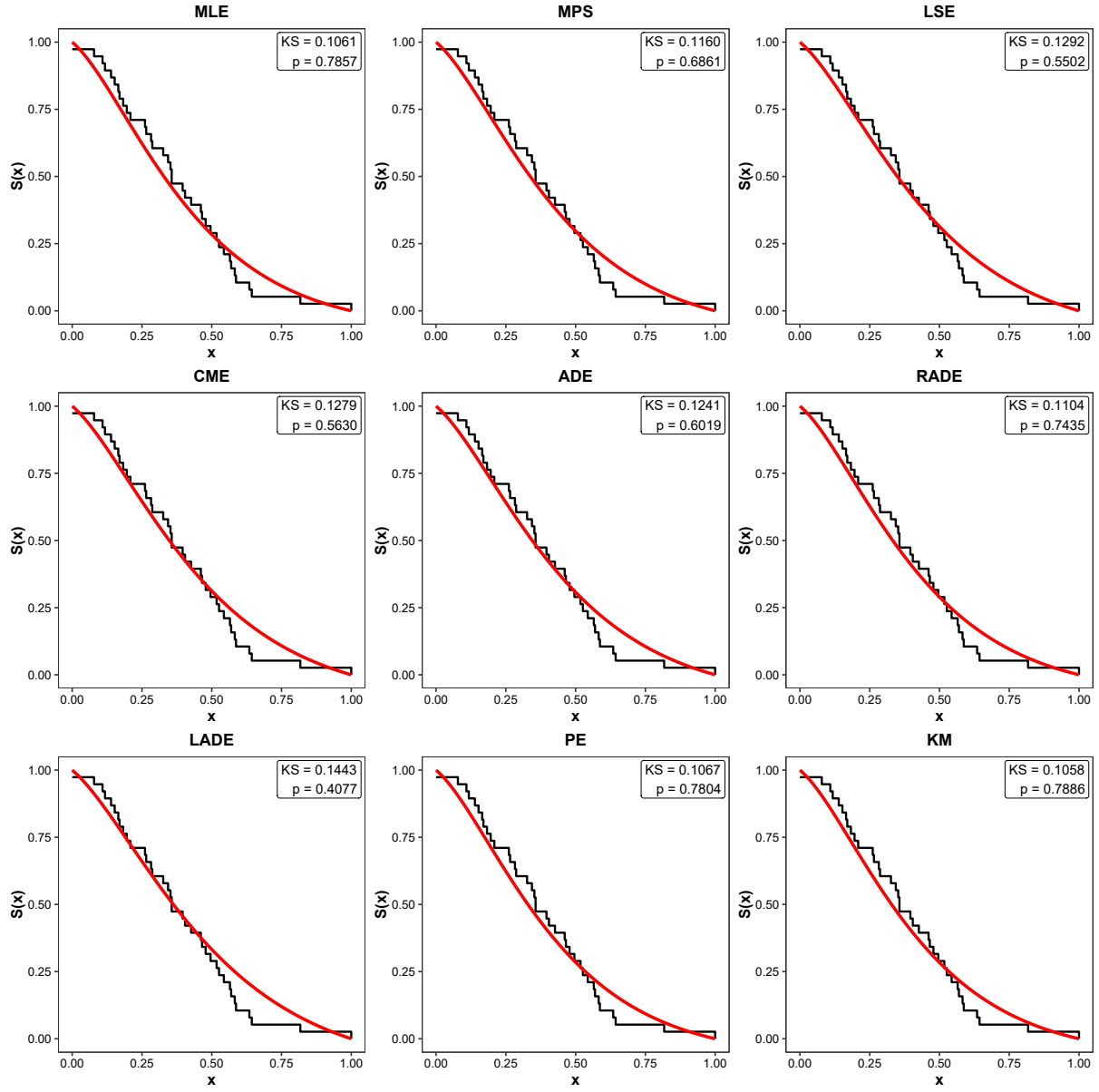


Figure 6: Empirical and fitted SFs for the TQLD under the nine estimation methods for healthcare data.

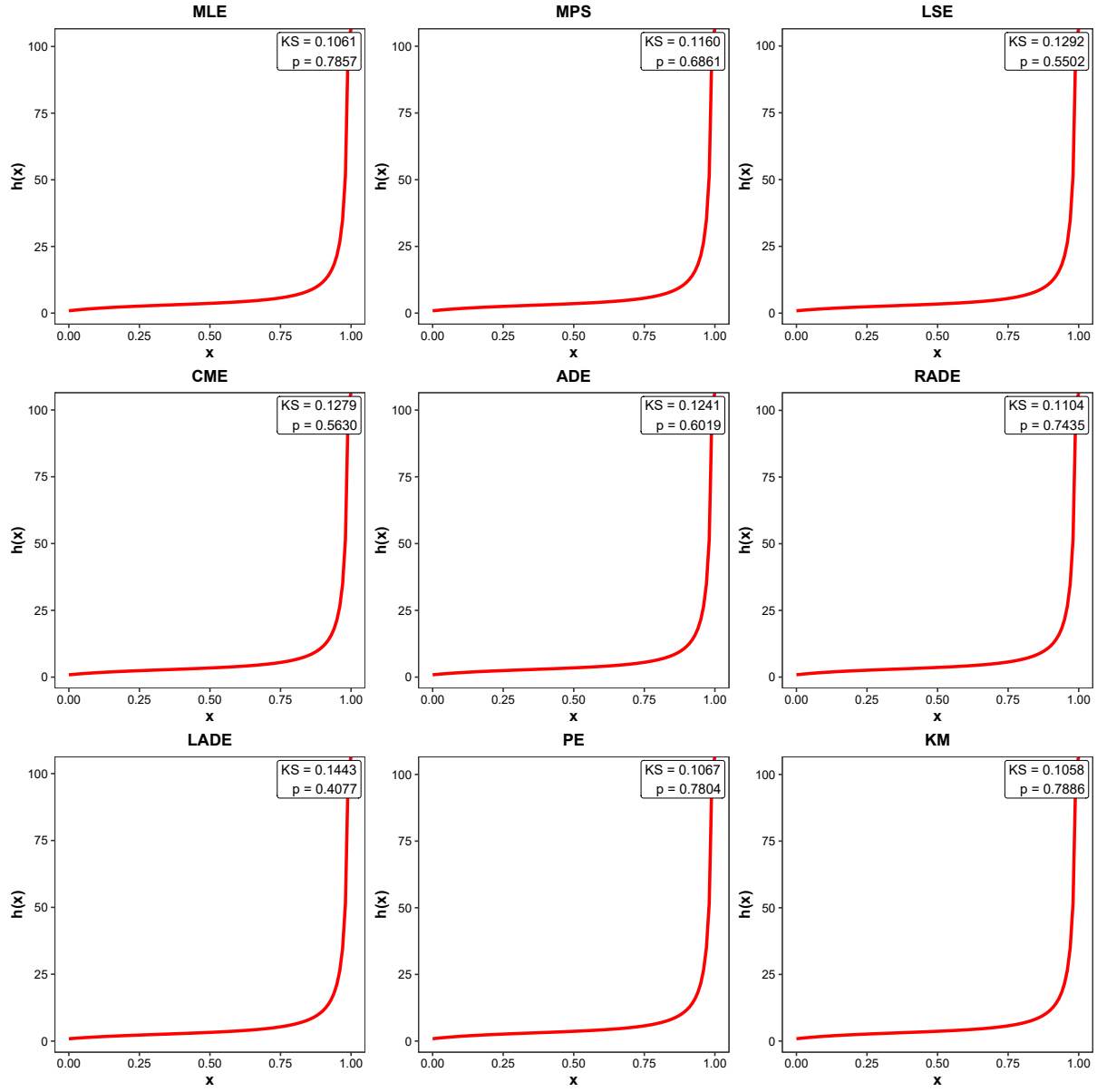


Figure 7: Empirical and fitted HRFs for the TQLD under the nine estimation methods for healthcare data.

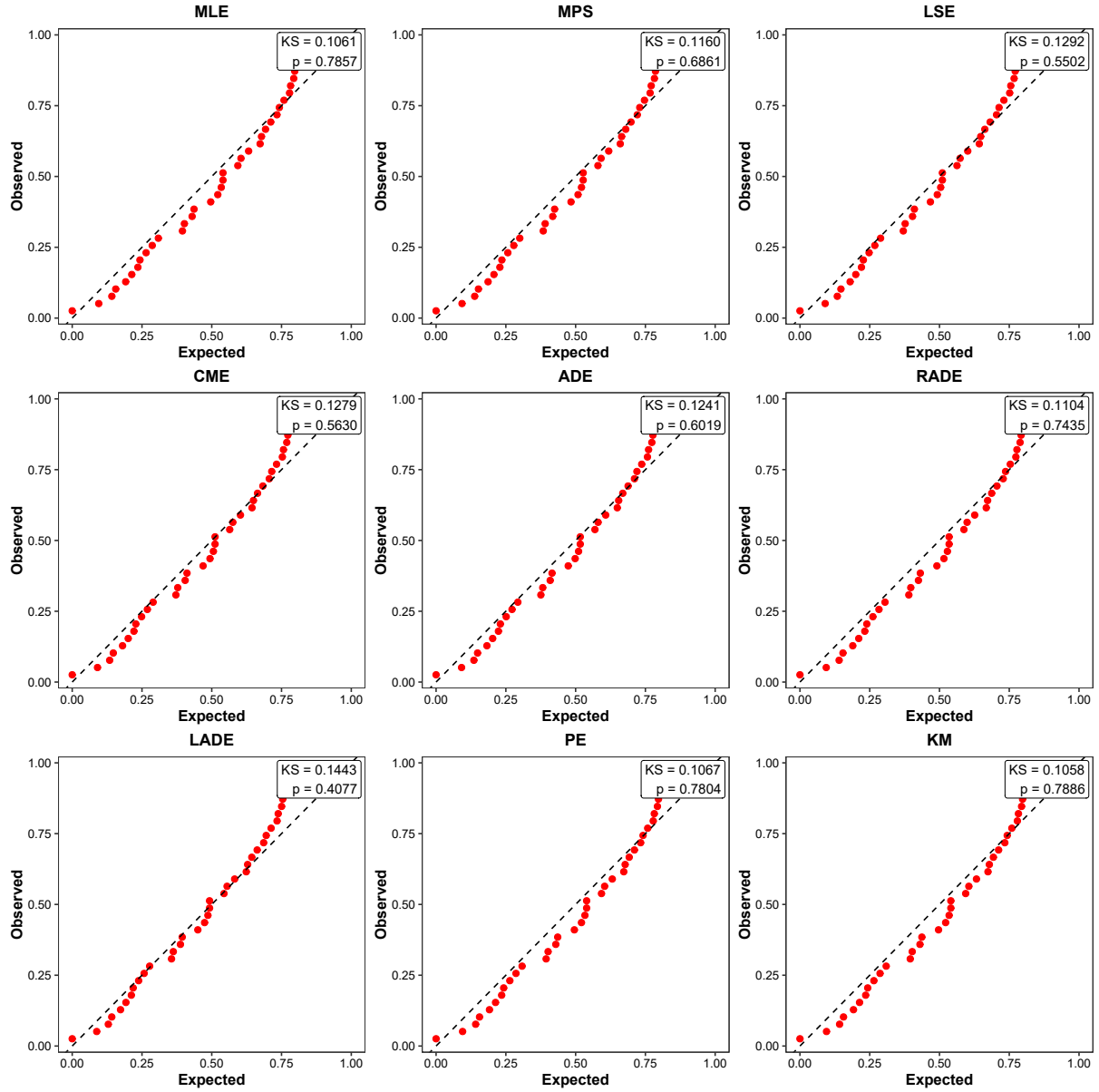


Figure 8: PP plots for the TQLD under the nine estimation methods for healthcare data.

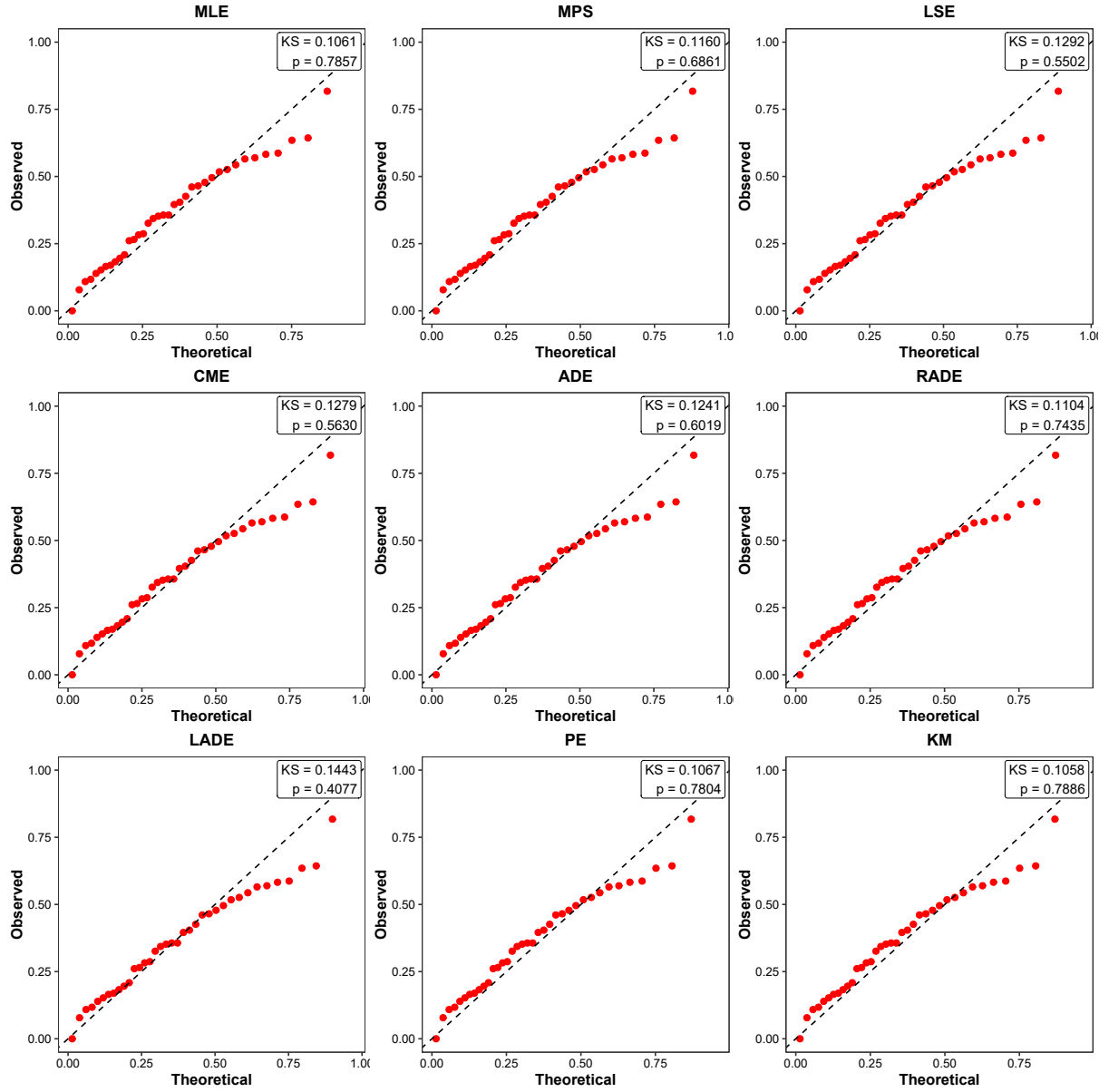


Figure 9: QQ plots for the TQLD under the nine estimation methods for healthcare data.

statistical distributions, highlighting its flexibility for modeling positively skewed observations on the unit interval. The suggested TQLD combines mathematical tractability with strong empirical performance, making it a useful addition to the family of bounded continuous probability distributions. The proposed truncated distribution provides a promising and practical framework for modeling bounded data in human health applications, offering the potential to capture important distributional features and support reliable statistical analysis in medical and healthcare research.

Data Availability Statement

The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

Conflicts of Interest

The authors declare no conflict of interest.

CRediT authorship contribution statement

Anas F. I. Alharshan: Writing-original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing-review, Investigation. **Abdulrahman M. A. Aldawsari:** Writing-original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing-review, Investigation. **Ahmed W. Shawki:** Writing-original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing-review, Investigation. **Mohammed Elgarhy:** Writing-original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing-review, Investigation.

Funding Statement

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through project number (PSAU/2026/01/45564).

References

- [1] Saaidia, N., Pakyari, R., and Zeghdoudi, H. (2026). The q-lindley distribution: Goodness-of-fit tests, modeling, inference, and applications. *Journal of Statistical Theory and Applications*, 25(1):6.
- [2] Lindley, D. V. (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society: Series B (Methodological)*, 20(1), 102-107.
- [3] Shanker, R. Shanker model and its applications. *Int J Stat Appl*. 2015a, 5(6), 338-348.
- [4] Shanker, R. (2015b). Akash Model and its Applications. *International Journal of Probability and Statistics*, 4(3), 65-75.
- [5] Shanker, R., & Shukla, K.K. (2017). Ishita Model And Its Application. *Biometric & Biostatistics International Journal*. 5(2), 39-46.
- [6] Shukla, K.K. (2018). Pranav Model with Properties and its Applications. *Biometrics & Bio- statistics International Journal*, 7(3), 244-254.
- [7] Shukla KK, Shanker R. Shukla model and its Application. *Reliability: theory & Applications*. Gnedenko Forum. 2019, 14(3), 46-55.
- [8] Shukla, K.K. Ram Awadh model with properties and applications, *Biometrics & Biostatistics International Journal*, 2018, 7(6), 515-523.
- [9] Shanker, R. (2017b). Rama Model and Its Application. *International Journal of Statistics and Applications*, 7(1), 26-35.

- [10] Sen, S., Maiti, S. & Chandra, N. (2016). The Xgamma Model: Statistical Properties and Application. *Journal of Modern Applied Statistical Methods*, 15(1), 774-788.
- [11] Onyekwere, C.K.; Obulezi, O.J. Chris-Jerry Model and Its Applications. *Asian J. Probab. Stat.* 2022, 20, 16-30.
- [12] Chouia S, Zeghdoudi H (2021) The XLindley model: properties and application. *J Stat Theory Appl* 20(2):318-327.
- [13] Messaadia, H.; Zeghdoudi, H. Zeghdoudi model and its applications. *Int. J. Comput. Sci. Math.* 2018, 9, 58-65.
- [14] Ahsan-ul-Haq, M. (2022). STATISTICAL ANALYSIS OF HAQ DISTRIBUTION: ESTIMATION AND APPLICATIONS. *Pak. J. Statist*, 38(4), 473-490.
- [15] M. Elgarhy, Garhy distribution with Different Estimation Methods and Applications to Engineering and Medical Data, *Rev. int. métodos numér. cálc. diseño ing.* (2026). Vol. 42, (3), 90.
- [16] Aijaz, A., Jallal, M., Ain, S. Q. U., & Tripathi, R. (2020). The Hamza distribution with statistical properties and applications. *Asian journal of probability and statistics*, 8(1), 28-42.
- [17] Etaga, H. O., Onyekwere, C. K., Lydia, O. I., Nwankwo, M. P., Oramulu, D. O., & Obulezi, O. J. (2023). Estimation of the Xrama distribution parameter under complete and progressive type-II censored schemes. *Sch J Phys Math Stat*, 10, 203-219.
- [18] Shanker, R. (2023). Komal distribution with properties and application in survival analysis. *Biometrics & Biostatistics International Journal*, 12(2), 40-44.
- [19] Mahdavi, A., and Silva, O. G. (2017). A method to expand family of continuous distributions based on truncated distributions. *Journal of Statistical Research of Iran (JSRI)*, 13(2), 231-247.
- [20] Nadarajah, S.; Kotz, S. A truncated Cauchy distribution. *Int. J. Math. Educ. Sci. Technol.* 2006, 37, 605-608.
- [21] Elgarhy, M.; Alsadat, N.; Hassan, A.S.; Chesneau, C.; Abdel-Hamid, A.H. (2023). A New Asymmetric Modified Topp-Leone Distribution: Classical and Bayesian Estimations Under Progressive Type-II Censored Data with Applications. *Symmetry* 2023, 15, 1396.
- [22] M. Elgarhy, A. Al Mutairi, A. S. Hassan, C. Chesneau, A. H. Abdel-Hamid; (2023). Bayesian and non-Bayesian estimations of truncated inverse power Lindley distribution under progressively type-II censored data with applications. *AIP Advances* 1 September 2023; 13 (9): 095130.
- [23] Singh, S. K.Singh, U. Sharma, V.K. (2014), The Truncated Lindley Distribution: Inference and Application, *Journal of Statistics Applications & Probability*, 3(2), 219-228.
- [24] Zhang, T. Xie, M. (2011), On the upper truncated Weibull distribution and its reliability implications, *Reliability Engineering and System Safety* (96) 194- 200
- [25] Abid, S. H. and Abdulrazak, R. K., [0, 1] truncated Fréchet-G generator of distributions, *Appl. Math.*, 7, 51-66 (2017).
- [26] Genc, A. I. (2017). Truncated inverted generalized exponential distribution and its properties. *Communications in Statistics Simulation and Computation*, 46(6): 4654-4670
- [27] Jamal, F., Bakouch, H.S. and Nasir, M.A., 2019. A truncated general-G class of distributions with application to truncated Burr-G family, *RevStat*, in press, <https://hal.archives-ouvertes.fr/hal-01956892/document>.
- [28] Bantan, R.A.; Jamal, F.; Chesneau, C.; Elgarhy, M. Truncated Inverted Kumaraswamy Generated Family of Distributions with Applications. *Entropy* 2019, 21, 108
- [29] Soliman, D., Hegazy, M., AL-Dayian, G., & EL-Helbawy, A. (2025). Statistical Properties and Applications of a New Truncated Zubair- Generalized Family of Distributions. *Computational Journal of Mathematical and Statistical Sciences*, 4(1), 222-257.
- [30] Hassan, A.S., Sabry, M., A. and Elsehetry, A. M. (2020). A New Family of Upper-Truncated Distributions: Properties and Estimation. *Thailand Statistician*, 18(2): 196-214.

- [31] Yadav, A.S., Shukla, S. and Kumari, A., 2019. Statistical inference for truncated inverse Lomax distribution and its application to survival data. *Annals of Data Science*, to appear, DOI:10.1007/s40745-019-00235-2.
- [32] Almarashi, A.M.; Jamal, F.; Chesneau, C.; Elgarhy, M. A New Truncated Muth Generated Family of Distributions with Applications. *Complexity* 2021.
- [33] Hassan, A.S., Sabry, M.A.H., and Elsehery, A.M. (2020). A New Probability Distribution Family Arising from Truncated power Lomax Distribution with Application to Weibull Model. *Pakistan Journal of Statistics and Operation Research*, 16(4), 661-674
- [34] Hassan, A.S., Sabry, M., A. and Elsehetry, A. M. (2020). Truncated Power Lomax Distribution with Application to Flood Data. *Journal of Statistics Applications & Probability*, 9(2), 347-359
- [35] Shukla, K.K. and Shanker, R. (2021). Truncated Shukla distribution: Properties and applications. *Reliability Theory & Applications*, 16 (3), 41-55.
- [36] Neamah, M.W.; Qasim, B.A. A New Left Truncated Gumbel Distribution: Properties and Estimation. *J. Phys. Conf. Ser.* 2021, 1897, 012015.
- [37] Khodja, N., Aiachi, H., Talhi, H. and Benatallah, I.N. (2022). The truncated XLindley distribution with classic and Bayesian inference under censored data. *Advances in Mathematics: Scientific Journal* 11 (2), 1191-1207
- [38] Semaary, H. E., Chesneau, C., Aldahlan, M. A., Elbatal, I., Elgarhy, M., Abdelwahab, M. M., & Almetwally, E. M. (2024). Univariate and bivariate extensions of the truncated inverted arctan power distribution with applications. *Alexandria Engineering Journal*, 100, 340-356.
- [39] El Gazar, A. M. ., Ramadan, D. A. ., ElGarhy, M. ., & El-Desouky, B. S. . (2025). Estimation of Parameters for Inverse Power Ailamujia and Truncated Inverse Power Ailamujia Distributions Based on Progressive Type-II Censoring Scheme. *Innovation in Statistics and Probability* , 1(1), 76-87.
- [40] Hussein, M.; Rodrigues, G.M.; Ortega, E.M.M.; Vila, R.; Elsayed, H. (2023). A New Truncated Lindley-Generated Family of Distributions: Properties, Regression Analysis, and Applications. *Entropy* 2023, 25, 1359. <https://doi.org/10.3390/e25091359>
- [41] Al Abbasi, J.N., Resen, I.A., Abdulwahab, A.M., Oguntunde, P.E., Al-Mofleh, H., Khaleel, M.A. (2023). The right truncated Xgamma-G family of distributions: Statistical properties and applications. *AIP Conf. Proc.* 2834, 080060 (2023), <https://doi.org/10.1063/5.0161683>
- [42] Hinkley, D. (1977). On quick choice of power transformation. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 26(1), 67-69.
- [43] Law, J. (1944). A statistical approach to the interstitial heterogeneity of sand reservoirs. *Transactions of the AIME*, 155(01), 202-222.
- [44] Macdonald, P. D. M. (1971). Comments on "An estimation procedure for mixtures of distributions" by Choi and Bulgren. *Journal of the Royal Statistical Society: Series B*, 33(3), 460-462.
- [45] Cheng, R. C. H., & Amin, N. A. K. (1983). Estimating parameters in continuous univariate distributions with a shifted origin. *Journal of the Royal Statistical Society: Series B*, 45(3), 394-403.
- [46] Swain, J. J., Venkatraman, S., & Wilson, J. R. (1988). Least-squares estimation of distribution functions in Johnson's translation system. *Journal of Statistical Computation and Simulation*, 29(4), 271-297.
- [47] Kibria, B. G., & Shakil, M. (2011). A new five-parameter Burr system of distributions based on generalized Pearson differential equation. *JSM Proceedings, Section on Physical and Engineering Sciences*, 866-880.
- [48] Wingo, D. R. (1988). Methods for Fitting the Right-Truncated Weibull Distribution to Life-Test and Survival Data. *Biometrical journal*, 30(5), 545-551.
- [49] Mazucheli, J.; Menezes, A. F. B.; Ghitany, M. E. The unit-Weibull distribution and associated inference. *J. Appl. Probab. Stat.* 2018,13, 1-22.

- [50] Kouadria¹, M., & Zeghdoudi, H. (2025). The truncated new-XLindley distribution with applications. *Journal of Computational Analysis and Applications*, 34(3).
- [51] Mazucheli, J., Menezes, A. F. B., & Chakraborty, S. (2019). On the one parameter unit-Lindley distribution and its associated regression model for proportion data. *Journal of Applied Statistics*, 46(4), 700-714.
- [52] Hadi, H.H., Al-Noor, N. H. (2023). Truncated exponential Marshall Olkin Lomax distribution: Properties, entropies, and applications. *AIP Conf. Proc.* 2414, 040054 (2023) <https://doi.org/10.1063/5.0115489>