Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Dialogic Developmental Movement in Mathematical Modelling: A Bakhtinian Vygotskian Analysis of Consolidation in Fermi Estimation

Michael Erotoma Omuvwie

m.e.omuvwie@hull.ac.uk

University of Hull

Research Article

Keywords: Dialogic learning, Mathematical modelling, Fermi estimation, Sociocultural theory, Phase-contrast indicator, Further Education

Posted Date: November 3rd, 2026

DOI: https://doi.org/10.21203/rs.3.rs-10611724/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

Abstract

Although dialogic interaction is widely regarded as beneficial for mathematical learning, research offers limited analytic means for determining when such interaction reorganises learners' reasoning rather than merely sustaining participation. This paper examines how dialogue becomes developmentally consequential within contextualised modelling activity, drawing on Bakhtinian dialogism and Vygotskian sociocultural theory. Through microanalysis of two *Critical Mathematical Incidents (CMIs)* from a single 90-minute Core Mathematics lesson on Fermi estimation involving three learners in an English Further Education college, the study traces shifts in learners' *contextual anchoring*, *stance*, *tool appropriation*, and *epistemic authority* across *pre-dialogue, in-dialogue*, and *post-dialogue* phases. The analysis introduces the *phase-contrast indicator (ΔΔ)* as an *interpretive heuristic*, not a quantitative measure, for distinguishing activation (the initial emergence of contextual and representational resources in dialogue) from consolidation (the subsequent reorganisation of those resources into structured, learner-owned modelling reasoning). Findings show that consolidation depends not on dialogue alone but on the intersection of dialogic challenge with access to cultural tools and representational structure. A counterexample demonstrates that engagement without representational structure produces activation but not consolidation. The analytic approach offers transferable conceptual tools for researchers examining how classroom dialogue contributes to the development of mathematical reasoning, within the interpretive scope appropriate to a single-case microanalytic design.

1. Introduction

Mathematics education increasingly positions modelling and real-world application as central to meaningful mathematical activity (Blum & Borromeo Ferri, 2009; Stillman et al., 2016). Within modelling-oriented classrooms, dialogue plays a critical role: learners must articulate assumptions, negotiate interpretations, and justify quantitative decisions in interaction with peers and teachers. Yet a persistent methodological challenge concerns identifying when and how dialogic interaction reorganises learners' mathematical reasoning, rather than merely eliciting participation or generating surface-level ideas (Mercer & Littleton, 2007).

Dialogic research in mathematics education has generated substantial insight into participation structures, interactional roles, and the conditions under which exploratory talk supports conceptual engagement (Barwell, 2023; Kazak, Mercer & Howe, 2012; Wegerif, 2011a; Wegerif & Fujita, 2015). Alongside this, sociocultural research has illuminated how cultural tools, units, diagrams, representational strategies, mediate reasoning as learners appropriate them through interaction (Bartolini Bussi & Mariotti, 2008; Vygotsky, 1978; Wertsch, 1998). Both traditions, however, face a shared limitation: they offer rich accounts of what happens during productive interaction but fewer analytic resources for determining whether and when that interaction produces durable reorganisation of reasoning. This distinction, between transient engagement and developmental consolidation, is especially salient in contextualised modelling, where learners must coordinate everyday experience,

plausibility criteria, and representational tools across the temporal arc of activity (Ärlebäck, 2009; Blum & Leiss, 2007).

This paper addresses that gap through *microanalysis* of classroom interaction in a Core Mathematics lesson on Fermi estimation. Drawing on Bakhtinian dialogism and Vygotskian sociocultural theory, it introduces a *Bakhtinian Vygotskian (BV) analytic framework* and the *phase-contrast indicator (ΔΔ)*, an *interpretive heuristic* for tracing shifts in learners' reasoning across *pre-dialogue*, *in-dialogue*, and *post-dialogue phases*. The evidential basis is a single lesson involving three focal learners, selected through theoretically purposeful sampling to provide the depth of analysis the framework requires. Two Critical Mathematical Incidents are examined in depth, illustrating distinct mechanisms through which dialogue may or may not produce developmental consolidation. A counterexample is also examined, demonstrating that activation without representational structure does not yield developmental movement.

The study makes two principal contributions. Methodologically, it offers a structured interpretive procedure for distinguishing *activation* from *consolidation* in *dialogic modelling activity*, with a counterexample providing initial evidence of the procedure's discriminative capacity. Empirically, it provides microanalytic evidence of how learners' *contextual anchoring*, *stance, tool appropriation*, and *epistemic authority* shift within interaction, and how these shifts constitute the emergence of *critical mathematical agency* as a collective, interactional achievement. These contributions are grounded in a single-case design; the claims are analytic rather than representative, and the scope of generalisation is explicitly limited throughout.

Two research questions organise the study:

RQ1 (*Methodological*): What analytic framework and procedure can distinguish *activation* from *consolidation* within *dialogic mathematical modelling episodes*, and what evidence supports its discriminative validity within this case?

RQ2 (*Empirical*): Through what interactional mechanisms does dialogue produce *developmental consolidation* of learners' modelling reasoning in this context, and what conditions appear to determine whether *consolidation* occurs?

2. Theoretical Background and Analytic Framework

2.1 Dialogic perspectives on mathematical meaning-making

Dialogic approaches conceptualise learning as emerging through encounters among multiple voices, stances, and interpretive resources (Bakhtin, 1981, 1986). In Bakhtin's account, utterances are always oriented toward prior and anticipated responses; meaning is therefore inherently relational and open-ended, produced through the interplay of different evaluative positions rather than residing in any single voice. Heteroglossia, the simultaneous presence of multiple social languages, registers, and evaluative

orientations within any utterance, is the condition of all meaning-making in Bakhtin's account: speakers do not simply express pre-formed meanings but produce them through orientation to, and against, the evaluative positions of others. Outsideness, a related concept, refers to the analytically productive distance between a speaker and the object of discourse, the capacity to perceive something from a position that is not fully absorbed into it. In classroom interaction, these concepts characterise the interactional conditions under which learners maintain ironic or non-committal distance from a problem, and under which the encounter of different evaluative positions generates the definitional work that mathematical reasoning requires.

These specific Bakhtinian concepts, heteroglossia and outsideness, are selected because they map directly onto observable interactional phenomena in modelling activity: the co-presence of everyday and mathematical registers in learners' speech, and the movement from ironic distance to committed mathematical engagement. They are not invoked as a developmental theory; Bakhtin offers no account of how individual learners internalise or reorganise understanding. This is the crucial limitation that requires a Vygotskian supplement, as *Section 2.2* explains.

It is worth noting that heteroglossia, as Bakhtin develops it in The Dialogic Imagination (1981), is a concept worked out primarily through the analysis of novelistic prose rather than face-to-face classroom interaction. The application of heteroglossia to classroom discourse involves a theoretical translation that Bakhtin himself did not make. This translation is justified here on the grounds that classrooms share with the novel the essential feature that makes heteroglossia analytically productive: they are sites where multiple social languages – everyday, mathematical, institutional, personal – coexist in tension, and where the evaluative orientations carried by those languages are in productive conflict. It is the tension, not the stability of these languages, that the BV framework draws on.

Prominent scholars working across dialogic and sociocultural perspectives are all concerned with development, not only participation. Radford's (2018) theory of objectification treats learning as transformation achieved through joint activity and the appropriation of cultural objects. Sfard's (2008) commognitive framework is explicitly developmental, treating mathematical learning as change in the nature and structure of mathematical discourse over time. Roth and Radford (2011) draw on dialectical materialism to theorise learning as irreversible change produced through the contradictions inherent in joint activity. The methodological gap this study addresses is therefore not that dialogic research ignores development, but that it offers limited systematic tools for tracing developmental movement at the level of fine-grained classroom interaction within bounded modelling episodes, the turn-by-turn level at which *consolidation* either occurs or does not. Kazak, Wegerif and Fujita (2015) examine dialogic processes and conceptual development at a comparably fine-grained level; the BV framework's distinctive contribution is its structured *three-phase procedure* for distinguishing *activation* from *consolidation* within bounded episodes.

2.2 Sociocultural accounts of mediated development

Vygotskian sociocultural theory conceptualises learning as transformation mediated through cultural tools and participation in socially organised activity (Vygotsky, 1978; Wertsch, 1998). Learners develop as they appropriate tools, measurement systems, representational strategies, modelling structures, which restructure reasoning through use rather than merely facilitating access to pre-existing knowledge. The Zone of Proximal Development (ZPD) highlights the relational character of this process: what a learner can do in collaboration with a more knowledgeable other, or with a culturally provided tool, defines the space within which development is possible (Rogoff, 1990). Crucially, the ZPD is not simply a gap to be filled by instruction; it is a space of potential reorganisation, within which tools introduced at the right moment can become genuinely appropriated, repurposed for new contexts and extended beyond the conditions of their introduction. It is worth noting that Vygotsky himself was not entirely consistent on developmental direction: in Thinking and Speech (1987), he sometimes writes as if scientific concepts descend toward spontaneous ones, acquiring experiential meaning through contact with the everyday, rather than simply replacing them in a single upward direction. This internal tension in the primary text strengthens the case for a Bakhtinian supplement: if the developmental endpoint is less pre-given than the standard reading of the ZPD implies, an account of the interactional conditions that make development possible becomes all the more necessary.

Research on scaffolding and semiotic mediation has documented how teacher assistance supports engagement with tools in ways that can, over time, become internalised as learner-owned strategies (Bakker, Smit & Wegerif, 2015; Bartolini Bussi & Mariotti, 2008). The limitation is that sociocultural analyses have been more successful at describing how tools are introduced and negotiated than at identifying when tool use results in consolidated, learner-owned reasoning that endures beyond the scaffolded moment. This temporal question, whether dialogue leaves traces in subsequent reasoning, is precisely what the BV framework addresses.

2.3 Contextualised modelling and critical mathematical agency

Contextualised mathematical modelling requires learners to coordinate everyday experience, plausibility criteria, representational tools, and quantitative structure (Blum & Borromeo Ferri, 2009; Stillman et al., 2016). Fermi problems, open estimation tasks requiring order-of-magnitude reasoning without determinate answers, are particularly productive contexts for studying this coordination because they cannot be solved by routine application. Learners must instead activate contextual knowledge, negotiate what counts as a reasonable assumption, and construct a representational structure capable of organising that knowledge into a quantitative argument (Albarracín & Ärlebäck, 2025; Ärlebäck, 2009; Albarracín & Gorgoró, 2014). The movement from activated contextual resources to an organised modelling chain is therefore directly observable in the transcript, making Fermi problems analytically productive for the purposes of this study.

Work in Critical Mathematics Education foregrounds learners' capacity to interrogate real-world phenomena, justify assumptions, and challenge taken-for-granted interpretations (Andersson & Barwell, 2021; Skovsmose, 2023). Skovsmose (2023) distinguishes between learners who exercise mathemacy,

the capacity to use mathematics to read, interpret, and act on the world, and those who merely execute procedures. This distinction is analytically productive here because Fermi estimation, precisely because it lacks a determinate answer, requires learners to exercise judgement about what counts as a reasonable assumption and a plausible result. Following Radford's (2018) argument that mathematical agency manifests not as an individual trait but as a collective, negotiated practice emerging through joint activity, the present study treats critical mathematical agency as an interactional achievement: visible when learners collectively interrogate assumptions, evaluate plausibility, and produce mathematically warranted interpretations of everyday situations. The empirical question addressed by RQ2 is whether and through what mechanisms dialogue produces the conditions for this agency to emerge.

2.4 The Bakhtinian Vygotskian (BV) framework and the Phase-Contrast Indicator (ΔΔ)

The BV framework synthesises these perspectives to examine dialogic developmental movement: the process by which learners' participation in heteroglossic interaction, mediated by cultural tools, produces reorganisation of mathematical reasoning. Bakhtin's concepts, specifically heteroglossia and outsideness, as justified in *Section 2.1*, provide an analytic vocabulary for the interactional conditions of dialogue: the co-presence of multiple evaluative positions, and the movement from ironic distance to committed engagement. Vygotskian theory accounts for the developmental consequences of this interaction when it is mediated by cultural tools: the reorganisation of reasoning through appropriation. The framework does not claim full theoretical compatibility between these two traditions; it draws on each for the specific analytic work it can do, and the boundary between them is marked where it matters: *activation* is what Bakhtinian heteroglossia produces; *consolidation* is what Vygotskian tool-mediated development accounts for.

The framework distinguishes *three analytic phases* of interaction. These are *interpretive lenses* for organising analysis, not *discrete chronological stages*; a single episode may involve movement within and between phases, and the researcher's task is to characterise the quality of reasoning at different moments rather than to assign turns to categories:

Pre-dialogue captures learners' initial stances, contextual references, and tool use prior to sustained interaction. It establishes the evidential baseline from which developmental movement is interpreted. The pre-dialogue phase often displays outsideness (Bakhtin, 1981): deliberate distance from the problem maintained through humour or deflection, preceding genuine dialogic investment.

In-dialogue captures the interactional space in which ideas are proposed, contested, revoiced, and negotiated among participants. This is where heteroglossia becomes analytically visible: everyday and mathematical registers collide, different evaluative positions are articulated and contested, and the definitional work required for modelling begins. *Activation* – the emergence of contextual anchors, boundary conditions, and redistributed epistemic authority – is necessary but not sufficient for *consolidation*.

Post-dialogue captures stabilised reasoning, coordinated tool use, and authorial positioning emerging following sustained interaction. This is where *consolidation*, if it has occurred, becomes analytically visible. Consolidation is visible when contextual anchors function as modelling parameters, stance is authorial, tool use is coordinated, and epistemic authority is grounded in the group's own reasoning.

Dialogic developmental movement is examined across four analytic dimensions. Each dimension is characterised at each phase using three qualitative descriptors – *Absent, Emerging*, and *Coordinated or Authorial* – defined in full in *Online Resource 1*. The following descriptions explain what each descriptor captures within each dimension. The dimensions are not independent: in practice, movement in one dimension is often accompanied by movement in others, and the researcher attends to their coordination as much as to each in isolation:

Contextual Anchoring (CA) captures how learners draw on everyday, disciplinary, or quantifiable contexts to ground modelling assumptions. Pre-dialogue anchoring is typically absent or purely narrative, a reference to personal experience that has not yet been connected to any mathematical structure. Movement along this dimension is characterised by the shift from anecdotal reference toward parametric, mathematically structured grounding: an everyday context becomes a quantifiable variable, and a quantifiable variable becomes an explicit modelling parameter that the group can operate on.

Voice and Stance (VS) captures how learners position themselves in relation to mathematical claims, including whether they offer, defend, challenge, or justify propositions. Pre-dialogue stance is typically characterised by deferral, learners wait for the teacher to validate, speculate without committing, or maintain ironic distance. Movement is characterised by the shift from deferral and speculation toward authorial, warranted assertion: learners begin to propose claims as their own and to defend them with explicit reasoning rather than seeking external validation.

Tool Appropriation (TA) captures how learners introduce, take up, coordinate, or extend cultural tools and representational resources within the activity. Tools include measurement systems, units, multiplicative structures, decomposition strategies, and domain-specific schemas such as pacing. Pre-dialogue tool use is typically absent or invoked without integration into a broader representational structure. Movement is characterised by the shift from absent or isolated tool use toward coordinated, generative deployment: tools are used to do mathematical work, connected to other elements of the emerging model, and at the fullest extent of consolidation extended to new cases or sub-problems that the group generates independently.

Epistemic Authority (EA) captures how authority for knowledge claims is negotiated, distributed, or stabilised within interaction. Pre-dialogue authority is typically teacher-centred by default: the teacher poses the question, and the implicit frame is that correctness is determined externally. Movement is characterised by the shift from teacher-centred or unclaimed authority toward collectively grounded, learner-owned epistemic positioning. *Consolidation* on this dimension is visible when learners hold each other’s claims to account, demanding justification, challenging figures, evaluating plausibility, rather than

deferring to an external arbiter. VS and EA tend to co-move in *consolidating episodes*, and the researcher attends to both as aspects of the same underlying shift in interactional positioning.

The phase-contrast indicator (ΔΔ) is an interpretive heuristic for summarising movement across phases within each dimension and across the episode as a whole. It draws on two contrasting characterisations:

Activation [Δ(pre to in)] refers to the shift from *pre-dialogue* to *in-dialogue*, capturing the initial emergence of contextual, conceptual, or representational resources during interaction. Activation is necessary, but not sufficient, for developmental movement.

Consolidation [Δ(in to post)] refers to the shift from *in-dialogue* to *post-dialogue*, capturing the reorganisation and stabilisation of activated resources into structured, learner-owned reasoning. Consolidation reflects what dialogue produces that endures beyond the moment of interaction: contextual anchors that become modelling parameters, stances that become authorial, tools that become coordinated and transferable, and authority that becomes collectively grounded. Consolidation is the primary analytic object, because it is consolidation that bears most directly on when dialogic participation becomes developmental.

ΔΔ is the interpretive contrast between these two movements. When *consolidation* is stronger than *activation*, when reasoning does not merely emerge in dialogue but becomes structured and learner-owned in its aftermath, ΔΔ is positive. When activation does not yield consolidation, ΔΔ is approximately zero or negative. Crucially, ΔΔ is not an arithmetic operation on ordinal scores; it is a *structured interpretive judgement* grounded in transcript evidence about the relative degree of movement between phases. Its value lies in discipline: by requiring the researcher to characterise both activation and consolidation, it prevents the common move of treating any rich interactional moment as evidence of learning. Discriminative validity is addressed through the counterexample in *Section 4.3*.

3. Methodology

3.1 Research context and design

The study draws on data collected during the author's PhD research, funded by Mathematics for Education and Industry (MEI) and conducted at the University of Manchester. The research site was a Further Education college in England offering Core Mathematics, a post-16 qualification designed to develop applied quantitative reasoning for learners not pursuing A-level Mathematics (Noyes, Wake & Drake, 2017). The lesson examined was a 90-minute session on Fermi estimation involving a class of sixteen learners working in small groups on a series of open estimation problems.

The study adopts a qualitative microanalytic case study design (Erickson, 1992; Yin, 2018). This design is appropriate for research questions concerning fine-grained interactional mechanisms through which reasoning develops within bounded episodes. The goal is analytic generalisation, the development and illustration of conceptual tools that may prove transferable to other contexts, rather than population level

generalisation (Cobb & Gravemeijer, 2008). The single-lesson focus permits the depth of analysis the research questions require. The limitation this entails, that mechanisms identified here cannot be assumed to be representative or typical across contexts, is acknowledged explicitly and consistently throughout.

3.2 Participant selection and ethical considerations

One group of three learners, referred to as Callum, Jane, and Jack (pseudonyms), was selected as the focal triad for sustained microanalysis. Selection was theoretically purposeful (Miles, Huberman & Saldaña, 2014): this group was identified during preliminary data review as producing the most analytically varied episodes, including both consolidating interactions and an episode where activation did not yield consolidation. The selection criterion is important: the group was chosen not because it was typical or high-performing, but because it offered the clearest within-case contrast for developing and testing the discriminative capacity of ΔΔ. This criterion is documented in the analytic audit trail available in *Online Resource 2*.

The small number of participants is a *genuine and significant limitation* of the study. The analysis is not intended to be representative of how learners engage in Fermi estimation across contexts or populations. The methodological contribution depends on the depth and internal consistency of the evidence, not its breadth; claims about mechanism are made within the case, and the reader is invited to assess their analytic plausibility rather than their statistical representativeness.

Ethical approval was granted by the University of Manchester Research Ethics Committee (ID: 2064918). All participants, and parents or guardians where required, provided informed consent for audio recording and anonymised reporting. Pseudonyms are used throughout. Transcripts have been lightly edited for readability, false starts and non-lexical fillers are removed where they do not affect analytic interpretation, and this editorial process is documented in the analytic memos available in the *Online Resource 3*. CMI identification and phase boundary determination were made from the full unedited transcript prior to editing; the excerpts presented in the Findings are representative of the reasoning observable across each phase as a whole, not selected post-hoc for analytic convenience.

3.3 Data collection

Data sources included: whole-class audio of plenary discussion; continuous small-group audio of the focal triad's problem solving; learners' written artefacts collected at the end of the lesson; researcher field notes taken during and immediately after the session; and short semi-structured post-lesson interviews conducted individually with the three focal learners and with the teacher. The interviews were used to check the plausibility of analytic interpretations, specifically, to verify that the interactional moves identified as consolidating were experienced as such by participants, rather than to generate new data. Audio was transcribed verbatim and then lightly edited as described above, using conversation analytic conventions (Jefferson, 2004). Thick description was used throughout to represent interactional detail, enabling readers to assess the interpretive claims independently (Erickson, 1992; Geertz, 1973).

3.4 Critical Mathematical Incidents and analytic procedure

The primary analytic unit is the Critical Mathematical Incident (CMI): a bounded interactional episode in which a notable shift in reasoning, stance, or tool use is observable. CMIs were identified when three conditions were met: a contextual or mathematical resource was introduced or revoiced in a way that changed the trajectory of the group's reasoning; a change in stance, modelling orientation, or tool coordination was observable across at least two participants; and interactional tension, negotiation, or reframing occurred rather than simple acceptance or rejection. Full identification criteria and the complete list of CMIs identified in the lesson are available in the *Online Resource 2*.

Analysis proceeded in four phases: identification of CMIs across the full transcript; turn-by-turn microanalysis of each CMI; interpretive coding of each analytic dimension across pre-dialogue, in-dialogue, and post-dialogue phases; and cross-incident synthesis. Approximately 20% of the transcript was double-coded by a colleague familiar with sociocultural approaches to classroom discourse, with agreement assessed on the qualitative descriptor, *absent, emerging*, or *coordinated/authorial*, assigned to each dimension at each phase, yielding initial agreement of 78–85% across dimensions; discrepancies were resolved through negotiated consensus and the resolution rationale is documented in the analytic memos. The counterexample (CMI 4) was identified and coded using the same procedure as the consolidating episodes; the near-zero ΔΔ result emerged from the analysis rather than being predetermined.

Two CMIs are examined in depth in the Findings. CMI 1 (A4 Pages) illustrates learner-driven consolidation without teacher intervention, including self-correction of an error within the group's own modelling chain. CMI 3 (Pacing Analogy) illustrates ZPD-informed tool introduction followed by learner-owned consolidation. CMI 4 (Media Use) is examined as a counterexample, essential to addressing RQ1.

4. Findings

This section presents microanalytic accounts of two Critical Mathematical Incidents and one counterexample, directly addressing RQ2. For each CMI, the analysis moves through the three analytic phases, tracing shifts across the four dimensions. ΔΔ interpretations are offered as summative characterisations grounded in the transcript evidence.

4.1 CMI 1 – A4 Pages (Q5): From competing definitions to a self-corrected modelling chain

Pre-dialogue and early in-dialogue: Heteroglossic dispute over what counts as "average"

The question posed was: "Estimate roughly how many A4 pages is a 30,000-word essay?" The triad's opening exchange was immediately contentious. Jane proposed grounding the estimate in "the average words per line," prompting Callum's challenge: "Define average…" Jane's retort, "A lot better than you…", and Callum's further push, "Define a lot then…", constitute genuine heteroglossic sparring: competing

evaluative positions about what counts as a legitimate basis for estimation, with no contextual anchor yet established (CA: none) and stance adversarial rather than collaborative (VS: contested).

Jack's intervention escalates the dispute productively: "I challenge… about the average number of pages… as the words could have been typed… handwritten…" This introduces a genuine confound the group had not yet considered. Callum extends it ("even with typing, it depends on the type of font and size as well…") and Jane adds a further variable ("How about if the text is in bold?"). This sequence shows the group collectively discovering that their initial framing of the problem is underspecified, a strong instance of *dialogic activation* generating definitional work (TA: emerging, as multiple candidate variables are surfaced without yet being coordinated; EA: distributed, as no single voice dominates the framing).

In-dialogue: From recollection to contested calculation to empirical pivot

Jane offers an experiential anchor: "I have kind of written 30,000 words before and that was 26, 27 pages… that was typing though, in Times New Roman…" (CA: narrative/everyday). Callum challenges this with a plausibility check: "…that will make it over a 1000 words per page roughly… that's a lot… maybe a bit more." Jane revises in response: "mmm… you're right there… maybe it's even twice that number of pages… am not too sure though… but could be nearer 57 pages." This exchange shows real-time mutual correction and stance revision grounded in plausibility reasoning (VS: tentative but responsive to challenge), a productive form of contextual accountability operating before any tool has been agreed.

Jack then proposes a font-based calculation: "if it is typed and using font Calibri with size 12… then it's how many words on a line… we will be looking at… which could be 15 may be 17?" Callum immediately raises a further confound ("What about the margins? Wide or the standard one could affect the number of words…"), and at this point Jane makes the decisive move of the episode: "All this could affect how accurate our estimates will be… why don't we do it… handwritten instead? We could use our notebooks here to make it as realistic as possible…" This is the pivotal tool-appropriation moment in the episode (TA: shift from contested abstraction to a concrete, agreed empirical procedure). Callum and Jack's immediate uptake ("Yeah… good idea…" / "Yep…") confirms the proposal as a collective decision rather than an individual one (EA: collectively grounded).

The group then counts lines and words directly from their own notebooks: "if we count mine… mine is about 11" (Jane); "Mine is about 8, 9…" (Callum); "If we count the lines on this notebook we get… say 32" (Jack), refined by Callum to "a couple more" for a full A4 sheet, settling at 35 lines. Jane closes the in-dialogue phase by proposing the working parameters explicitly: "So we work with 9–11 words/line and 35 lines on a page… let's work it out." Activation across all four dimensions is now strong: CA is empirically grounded rather than merely recollected, VS has shifted from adversarial to collaborative, TA has moved from contested abstraction to an agreed measurement procedure, and EA is collectively held.

Post-dialogue: Collective construction and self-corrected modelling chain

The group's subsequent written working, photographed by the researcher as field evidence (see Online Resource 3, Figure 1), documents the consolidation phase directly. The group records their agreed parameters, 9–11 words per line, 35 lines per A4 sheet, with an adjustment of 8 lines "missed on average" to account for partially filled lines at paragraph breaks, yielding a words-per-page range of 315 to 385.

The working then shows a revealing self-correction. An initial step multiplies the words-per-page figures directly by the 30,000-word target (315 × 30,000 and 385 × 30,000), producing the dimensionally incorrect values of 9,450,000 and 11,550,000, a unit error, since multiplying words-per-page by total words does not yield pages. A further line in the working, partially crossed out, shows the group beginning a second erroneous step ("385 × 9,450,000") before abandoning it. The group then performs the correct operation: dividing the total word count by the words-per-page range, 30,000 ÷ 315 and 30,000 ÷ 385, yielding a final estimate of approximately 77.92 to 95.24 pages.

This self-correction is the most analytically significant feature of the post-dialogue phase. The artefact does not capture the group's verbal reasoning about why the initial step was abandoned, and the claim made here is accordingly bounded: the written record demonstrates that an erroneous operation was identified and replaced with a dimensionally correct one, within a chain constructed and corrected entirely by the group with no teacher involvement at any point in the episode. This is a strong instance of learner-owned epistemic authority (EA: learner-owned) operating not only in the assertion of a result but in the monitoring and correction of the group's own mathematical process (TA: coordinated and self-correcting). The final answer is expressed as a defensible range rather than false precision, consistent with the plausibility-oriented stance the group adopted throughout (VS: authorial).

ΔΔ interpretation: CMI 1

Activation [Δ(pre to in)] was substantial: the dispute over "average" generated multiple competing framings, contextual anchoring moved from absent to narrative-experiential, and tool appropriation moved from absent to a clearly proposed empirical procedure, though authority remained contested through much of this phase. Consolidation [Δ(in to post)] was strong on all four dimensions: contextual anchoring became fully parametric and empirically derived, stance became authorial and self-correcting rather than merely confident, tool use became a coordinated multi-step chain that the group itself audited and corrected, and epistemic authority became fully learner-owned, evidenced not only in dialogue but in the written record of the group's independent mathematical work. ΔΔ is clearly positive, with consolidation substantially exceeding activation. The CMI demonstrates that the productive resolution of heteroglossic tension, the dispute over what counts as a legitimate basis for estimation, can generate not merely a modelling chain but a self-monitoring one, in which the group identifies and corrects its own error without external prompting.

4.2 CMI 3 – Pacing Analogy: ZPD-informed tool introduction and learner-owned appropriation

Pre-dialogue: Representational difficulty

The question concerned estimating distances walked using pacing. Initial responses revealed a representational difficulty: learners had contextual knowledge of pacing from sport and orienteering but could not connect it to mathematical estimation. Callum stated: “I don’t really know how to measure that.” Jane added: “I mean, I know roughly, but I can’t put a number on it.” Jack was silent. The pre-dialogue baseline differed from CMI 1 in an analytically important way: learners were not maintaining ironic distance but were genuinely unable to proceed. Contextual knowledge existed (they knew what pacing was) but the representational bridge between that knowledge and a mathematical structure was absent. CA was absent as a functioning anchor, VS was one of deferral, TA was absent, and EA was teacher-centred.

In-dialogue: ZPD-informed tool introduction

The teacher, observing the group’s difficulty, introduced pacing as a culturally familiar measurement tool:

Teacher: *Do you know how pacing works in rugby, or in Orienteering? You count your steps and you know roughly how long each step is.*

Callum: *Oh – like 500 steps is roughly a certain distance?*

Teacher: *Exactly. A step for most adults is about 0.7 to 0.8 metres. So, if you walk 1,000 steps, how far is that?*

Jane: *So, about 700 to 800 metres. Nearly a kilometre.*

Teacher: *Right. Now can you use that to estimate your daily walking distance?*

The teacher’s intervention illustrates ZPD-informed scaffolding in a precise sense. The tool introduced, pacing as a unit of measurement, is within the learners’ cultural reach: they have encountered it in sport and Orienteering contexts that are part of their lived experience. But it has not previously been connected to mathematical estimation, which places it outside their current mathematical repertoire. The teacher’s intervention creates exactly the bridge that Vygotskian theory anticipates: access to a culturally available tool at the moment when the learner is positioned to appropriate it (Vygotsky, 1978; Bakker et al., 2015). Callum’s immediate uptake (“Oh – like 500 steps is roughly a certain distance?”) is the analytic marker of recognition: the tool is meaningful to him, not merely received as information. The particle “Oh” signals a genuine moment of connection between prior cultural knowledge and new mathematical application. Jane’s calculation (700 to 800 metres) confirms that the tool is not just recognised but is already being used: this is partial appropriation within the in-dialogue phase itself.

At the close of the in-dialogue phase, CA is strong (the pacing schema provides a structured mapping between steps and distance), TA is partially appropriated, EA is distributed between teacher and learners, and VS remains exploratory. Activation in this CMI is stronger at the close of the in-dialogue phase than in CMI 1, because the teacher’s intervention provided immediate representational structure, but the authority and stance dimensions remain partially teacher-anchored.

Post-dialogue: Appropriation and learner-owned consolidation

Following the teacher's withdrawal from the group, the learners extended and applied the pacing tool independently:

Callum: *Okay so if a step is like 0.75 metres, and I take about 1,500 steps to college…*

Jack: *How d'you know it's 1,500?*

Callum: *Because it takes me about fifteen minutes to walk there, and I reckon I do about 100 steps a minute.*

Jane: *That's quite fast.*

Callum: *Okay, maybe 90. So, 90 times 15 is 1,350 steps.*

Jack: *Times 0.75 metres. That's… just over a kilometre.*

Jane: *Which is about right for that walk. So, then we do that, times two for the return, times five days…*

Jack: *And then add weekends. We can use the same method.*

The post-dialogue phase is characterised by learner-owned appropriation and extension of the teacher-introduced tool. Wertsch's (1998) distinction between mastery and appropriation is directly relevant here: mastery is the ability to use a tool when directed; appropriation is the capacity to repurpose it for new contexts. The post-dialogue exchange demonstrates appropriation rather than mastery. Callum does not simply apply the pacing formula the teacher provided; he constructs two new sub-estimates, steps per minute (revised from 100 to 90 under challenge) and duration of the walk, that make the formula workable for his specific situation. These sub-estimates were not present in the teacher's scaffolding and were not prompted by the teacher.

Jack's challenge ("How d'you know it's 1,500?") is an epistemic move of the same kind as Callum's plausibility check in CMI 1: learners are holding each other's claims to account rather than deferring to the teacher (EA: learner-owned). Callum's response provides a full derivation, steps per minute, times minutes of walking, demonstrating authorial stance under dialogic pressure (VS: authorial). Jane's plausibility check ("which is about right for that walk") mirrors the plausibility-checking practice seen in CMI 1, confirming that contextual accountability operates as a consistent practice across both consolidating episodes. Most analytically significant is Jack's unprompted generalisation: "We can use the same method" for weekends. This is the fullest indicator of consolidation, the tool has become a transferable modelling resource, not a single-use procedure, and this transfer is learner-initiated rather than teacher-directed.

ΔΔ interpretation: CMI 3

Activation [Δ(pre to in)] was strong: the teacher's ZPD-informed intervention immediately provided a structured representational tool that learners recognised and partially appropriated within the in-dialogue phase itself. Consolidation [Δ(in to post)] was also strong, but the mechanism differs from CMI 1: because activation was higher at the close of the in-dialogue phase, consolidation in CMI 3 represents extension and independent ownership rather than construction from scratch. The shift in EA from distributed (teacher-and-learner) to learner-owned is the clearest marker of consolidation here. ΔΔ is positive, but the analytic interest lies in the different mechanism: in CMI 1, consolidation was produced through the resolution of heteroglossic tension; in CMI 3, it was produced through the learner-initiated extension of a teacher-provided tool. Both mechanisms produce positive ΔΔ, but neither reduces to the other.

4.3 CMI 4 (counterexample) – Media Use: Activation without consolidation

The counterexample is essential to addressing RQ1, because a discriminative heuristic must be capable of yielding a null result when applied to a non-consolidating episode. When asked to estimate daily media use, the triad generated contextually plausible contributions: Callum suggested nine hours, Jane proposed six, Jack offered a higher figure. Talk was engaged and conversational: learners drew on personal experience (CA: narrative), positioned their claims tentatively (VS: speculative), and distributed authority informally (EA: partially distributed). However, no representational structure emerged across the episode, no agreed variable, no decomposition strategy, no modelling chain, no attempt to connect personal estimates into a structured calculation (TA: absent throughout). When the teacher moved the class on, the group's reasoning was in the same state it had been at the close of the in-dialogue phase.

Post-dialogue codes were unchanged from in-dialogue codes on all four dimensions. ΔΔ is approximately zero. The absence of consolidation is not attributable to low engagement, the group was fully participative, nor to learner incapacity, since the same three learners consolidated their reasoning in CMIs 1 and 3. The differentiating variable, visible in the transcript, is the absence of a representational structure capable of organising the contextual resources that dialogue activated. In the media use context, there is no natural decomposition strategy available to the learners, no equivalent of the multiplicative chain of CMI 1 or the pacing schema of CMI 3, and neither the learners nor the teacher introduced one. Dialogue without representational structure produces activation but not consolidation; the counterexample demonstrates this within the same case, with the same participants, in the same lesson, providing the most controlled contrast the design allows.

4.4 Cross-incident synthesis

Table 1 summarises the ΔΔ characterisations across the three episodes, addressing both research questions in combination: it shows the analytic framework operating across contrasting cases (RQ1) and the differential mechanisms it reveals (RQ2).

Table 1. ΔΔ interpretive summary across Critical Mathematical Incidents

CMI	Δ(pre to in): Activation	Δ(in to post): Consolidation	ΔΔ	Mechanism
1. A4 Pages	Substantial: dispute over what counts as "average" generates competing framings; contextual anchoring shifts from absent to narrative-experiential; tool appropriation moves toward an agreed empirical procedure	Strong: parameters empirically derived from measured handwriting; full multiplicative chain constructed and self-corrected by the group; stance authorial; authority learner-owned	Positive	Resolution of heteroglossic tension (dispute about what counts as a legitimate basis for estimation) generates an empirical procedure; group identifies and corrects its own unit error without teacher involvement
3. Pacing Analogy	Strong: teacher-introduced tool is culturally familiar and immediately recognised; partial appropriation occurs within the in-dialogue phase; CA moves from absent to quantifiable	Strong: tool extended beyond teacher scaffolding; new sub-estimates constructed; unprompted generalisation to new cases; EA shifts from distributed to learner-owned	Positive (distinct mechanism)	ZPD-informed tool introduction provides representational bridge; learner-initiated extension and generalisation constitute appropriation rather than mastery
4. Media Use (counter-example)	Modest: narrative personal anchors offered; conversational engagement; informal authority distribution; no modelling operation introduced	Negligible: no representational structure emerges; no modelling chain; all four dimension codes unchanged from in-dialogue phase	≈0	Dialogue without representational structure produces activation but not consolidation; participation is necessary but insufficient for developmental movement

Two findings from the synthesis warrant emphasis. First, the two positive-ΔΔ episodes show that consolidation can occur through qualitatively different mechanisms, one driven by the internal resolution of dialogic tension and culminating in self-corrected, learner-constructed measurement, the other by the external provision of a culturally familiar tool. Yet they produce the same pattern of reorganised reasoning in the post-dialogue phase. The mechanisms are not interchangeable: CMI 1 could not have been resolved by a teacher-introduced tool in the same way, because the definitional work about what counts as a legitimate basis for estimation was generative precisely because learners did it themselves, down to constructing and correcting their own modelling chain unaided. Equally, CMI 3 could not have resolved through dialogic tension alone, because the learners had no representational resource from which to construct a chain without the pacing schema.

Second, critical mathematical agency is visible in both consolidating episodes as a consistent set of collective interactional practices: interrogating the boundary conditions of estimates; generating and defending contextual warrants; evaluating plausibility against lived experience; and extending modelling

tools unprompted to new cases. These practices are absent in CMI 4. Their presence or absence tracks the ΔΔ result directly, suggesting that critical mathematical agency and consolidation are not merely correlated in this data but are constitutively related: consolidation is the process through which agency becomes collectively available.

5. Discussion

5.1 Dialogue as a condition, not a guarantee, of developmental movement

The findings demonstrate that dialogic participation is necessary but not sufficient for developmental reorganisation of mathematical reasoning, directly answering RQ2 within the scope of this case. This is consistent with Mercer and Littleton's (2007) argument that productive talk does not automatically produce understanding, and with sociocultural accounts that developmental movement arises when dialogic openness intersects with tool-mediated reorganisation (Vygotsky, 1978; Wertsch, 1998). What the present analysis adds, within the limits of a single case, is a fine-grained account of two specific interactional conditions under which that intersection occurs, and one condition under which it does not.

In CMI 1, the productive condition was the retention and resolution of heteroglossic tension. Callum and Jane's opening dispute over what counts as "average", escalated by Jack's challenge about format and typography, was not a disruption of the group's modelling process but its generative core: the negotiation it forced produced an empirically grounded procedure that no single speaker's initial proposal would have generated. This is the mechanism that Bakhtinian heteroglossia illuminates most directly, not as a general claim about the value of multiple voices, but as a specific account of how the collision of different evaluative positions (recollection-based estimation versus typographically precise calculation versus empirical measurement) forces definitional precision, and ultimately a self-monitoring modelling chain. In CMI 3, the productive condition was ZPD-informed tool introduction at the moment of representational difficulty: the teacher provided a culturally familiar schema that learners could immediately recognise and subsequently appropriate beyond the terms of its introduction. The measure of consolidation in CMI 3 is not that learners applied the pacing formula but that they extended it to sub-problems the teacher had not specified (Wertsch, 1998).

The counterexample is essential to the argument and is not merely illustrative. It demonstrates that the same learners, engaging in apparently similar discussion in the same lesson, failed to produce developmental movement when no representational structure was available. This within-case contrast gives the activation/consolidation distinction empirical grounding it would lack if only consolidating episodes were examined.

5.2 The BV framework and the contribution of ΔΔ

Addressing RQ1, the BV framework contributes a structured analytic vocabulary for examining the temporal trajectory of reasoning within bounded modelling episodes, a question that existing dialogic and sociocultural analyses, focused primarily on what happens during interaction, have been less

equipped to address (Mercer, 2008). The distinction between pre-dialogue, in-dialogue, and post-dialogue phases is not a developmental stage theory; it is an analytic device that enables the researcher to ask whether and how reasoning was reorganised across the episode, rather than merely whether dialogue was productive in the moment.

The ΔΔ indicator functions as a discipline of analytic attention rather than a measurement instrument. By requiring the characterisation of both activation and consolidation and their interpretive contrast, it prevents the common analytic move of treating any moment of rich interactional engagement as evidence of learning. Mercer's (2008) argument that classroom dialogue needs temporal analysis is directly relevant: ΔΔ is a practical procedure for implementing that analysis within the constraints of microanalytic case study research. Its discriminative validity within this case rests on the counterexample: the same four-dimension coding procedure, applied to CMI 4, produces a near-zero result rather than the positive result produced for CMIs 1 and 3. This is suggestive evidence of discriminative capacity, the indicator is not simply tracking interactional engagement, but it is evidence within a single case, and claims about validity beyond this context would require application across multiple groups, lessons, and modelling contexts.

5.3 Implications for research and pedagogy

For researchers in dialogic mathematics education, the BV framework offers analytic tools that are grounded in existing theoretical traditions but address a specific methodological gap: the examination of whether dialogue produces durable reorganisation of reasoning, not merely productive talk. The framework is applicable, in principle, to contexts beyond Fermi estimation and Further Education, because its four dimensions (*contextual anchoring, voice and stance, tool appropriation, epistemic authority*) are not curriculum specific. They track interactional processes that are present whenever learners engage with contextualised mathematical problems through dialogue. Future applications could examine ΔΔ patterns across different task types (geometry, statistical reasoning, algebraic modelling), different age groups, and different instructional designs, testing whether the mechanisms identified here, heteroglossic tension and ZPD-informed scaffolding, recur or whether other mechanisms emerge.

For practitioners and teacher educators, two pedagogical implications are grounded in the findings. First, tasks that require explicit negotiation of boundary conditions, what counts as the relevant variable, what is a reasonable assumption, what should be included or excluded from the mode, create dialogic tension that, when retained and resolved rather than deflected, can generate the definitional work that enables modelling structure to emerge without teacher intervention. The design of Fermi estimation tasks should therefore attend not only to their open-ended character but to the quality of the boundary conditions they generate: tasks that make learners negotiate what counts are more productive for consolidation than tasks that allow them to proceed with a single uncontested assumption. Second, teacher interventions that introduce culturally familiar tools at the moment of representational difficulty, not to resolve the difficulty for learners but to make it tractable, can provide the representational bridge that dialogue alone cannot construct. The timing and cultural grounding of such interventions appear to matter: the pacing

tool worked in CMI 3 because it was already meaningful to these learners in non-mathematical contexts. Tool introduction that lacks this cultural grounding may provide a formula without enabling appropriation.

5.4 Limitations and directions for further research

The study's single-case, single-lesson design is its principal and most consequential limitation. All claims about mechanism are made within this case; they are not claims about what typically happens in Fermi estimation classrooms, in Further Education, or in dialogic modelling activity more broadly. The transcript evidence, while detailed, involves three learners across three episodes in a single 90-minute lesson. The counterexample is instructive within the case but cannot substitute for comparative evidence across multiple groups, contexts, and lessons. A reader persuaded by the analytic distinctions the framework draws should treat the findings as a set of hypotheses about mechanism that require testing in wider empirical contexts rather than as established generalisations.

The interpretive nature of ΔΔ is both a strength and a limitation. It is a strength because it is honest about what microanalytic qualitative data can and cannot support; it does not impose a false precision on data that is inherently interpretive. It is a limitation because its application in new contexts will require careful calibration of the dimension definitions and careful training of coders, and there is no guarantee that the coding categories will travel without modification. Future research should involve multiple independent coders and test ΔΔ patterns across wider datasets and instructional contexts.

6. Conclusion

This study examined how classroom dialogue can reorganise mathematical reasoning within contextualised modelling activity, and what conditions determine whether dialogic participation produces developmental consolidation. Drawing on Bakhtinian concepts of heteroglossia and outsideness to characterise the interactional conditions of dialogue, and on Vygotskian sociocultural theory to account for developmental consequences, the BV framework conceptualised learning as dialogic developmental movement: shifts in contextual anchoring, stance, tool appropriation, and epistemic authority that emerge through heteroglossic interaction and are mediated by cultural tools. The phase-contrast indicator (ΔΔ) provided an interpretive heuristic for distinguishing *activation* from *consolidation*, grounded in microanalytic evidence from three contrasting episodes and structured around two research questions concerning the analytic procedure and the mechanisms it reveals.

The two consolidating CMIs revealed distinct mechanisms. In CMI 1, learners constructed a complete modelling chain through the productive retention and resolution of heteroglossic tension about boundary conditions, without teacher intervention. In CMI 3, a teacher-introduced cultural tool was appropriated and extended beyond the terms of its introduction into a learner-owned modelling strategy, demonstrating ZPD-informed scaffolding that enables appropriation rather than mere mastery. In both cases, consolidation was characterised by learners interrogating assumptions, evaluating plausibility against lived experience, defending modelling decisions under peer challenge, and generalising tools

unprompted to new cases. These practices constitute critical mathematical agency as a collective, interactional achievement, visible at the turn-by-turn level as mathemacy exercised through dialogue. The counterexample demonstrated that these outcomes are not guaranteed by participation: the same learners, in the same lesson, failed to produce developmental movement in the absence of representational structure.

The claims made in this paper are deliberately scoped. They concern mechanisms observed within a single case and their analytic intelligibility through the BV framework; they are not claims about the generality or typicality of those mechanisms. The discriminative validity of ΔΔ rests on within-case evidence and requires further testing. The broader question, when does participation become learning, is not resolved here but is given more specific analytic purchase than existing frameworks have provided for modelling contexts. The pursuit of that question, across wider and more varied datasets and with the benefit of collaborative coding and comparative design, is the direction this work opens.

Statements and Declarations

Competing Interests:

The author declares no financial or non-financial competing interests relevant to the work submitted for publication.

Funding:

This study was funded by *Mathematics for Education and Industry (MEI)*, which supported the author's PhD research at the University of Manchester, from which this manuscript was developed and subsequently extended during the author's appointment at the University of Hull.

Data Availability:

De-identified analytic materials are available in the Online Resources submitted with this manuscript.

Code Availability:

Not applicable.

Ethics Approval:

Ethical approval was granted by the *University of Manchester Research Ethics Committee* (ID: 2064918).

Consent to Participate and Publication:

All participants (and parents/guardians where required) provided informed consent to participate in the study. Participants consented to the anonymised use of their data for publication.

Authors' Contributions:

This manuscript is sole-authored. All conceptual, methodological, analytic, and writing work was conducted by the author.

Acknowledgements:

This study was funded by *Mathematics for Education and Industry (MEI)*, which supported the author's PhD research at the University of Manchester. The manuscript was subsequently developed and extended during the author's appointment at the University of Hull.

Clinical Trial Registration (if applicable):

Not applicable.

Country Affiliation:

United Kingdom (UK)

References

1. **Albarracín, L., & Ärlebäck, J. B. (2025).** Exploring the role of assumptions in mathematical modelling teacher training using Fermi problems. *ZDM–Mathematics Education, 57*(4), 213–228. https://doi.org/10.1007/s11858-025-01677-0
2. **Albarracín, L., & Gorgorió, N. (2014).** Devising a plan to solve Fermi problems involving large numbers. *Educational Studies in Mathematics, 86, 79–96.* https://doi.org/10.1007/s10649-013-9528-9
3. **Alexander, R. (2008).** *Towards dialogic teaching: Rethinking classroom talk* (4th ed.). Dialogos.
4. **Andersson, A., & Barwell, R. (Eds.). (2021).** *Applying critical mathematics education: An introduction.* Brill.
5. **Ärlebäck, J. B. (2009).** On the use of realistic Fermi problems in introducing mathematical modelling. *The Mathematics Enthusiast, 6*(3), 331–364.
6. **Ärlebäck, J. B., & Albarracín, L. (2024).** Lessons learned from research on Fermi problems and mathematical modelling: Implications for STEM education. In J. Anderson & K. Makar (Eds.), *The contribution of mathematics to school STEM education* (pp. xx–xx). Springer. https://doi.org/10.1007/978-981-97-2728-5_12
7. **Bakker, A., Smit, J., & Wegerif, R. (2015).** Scaffolding and dialogic teaching in mathematics education: Introduction and review. *ZDM–Mathematics Education, 47*(7), 1047–1065. https://doi.org/10.1007/s11858-015-0738-8
8. **Bakhtin, M. M. (1981).** *The dialogic imagination: Four essays.* University of Texas Press.
9. **Bakhtin, M. M. (1986).** *Speech genres and other late essays.* University of Texas Press.

10. **Barwell, R. (2023).** Sourcing mathematical meaning as a dialogic process. *ZDM–Mathematics Education, 55*, 535–550. https://doi.org/10.1007/s11858-023-01467-6
11. **Bartolini Bussi, M. G., & Mariotti, M. A. (2008).** Semiotic mediation in the mathematics classroom. In L. D. English et al. (Eds.), *Handbook of international research in mathematics education* (pp. 746–783). Routledge.
12. **Blum, W., & Borromeo Ferri, R. (2009).** Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application, 1*(1), 45–58.
13. **Blum, W., & Leiss, D. (2007).** How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling (ICTMA 12): Education, engineering and economics* (pp. 222–231). Horwood.
14.
15. **Cobb, P., & Gravemeijer, K. (2008).** Experimenting to support and understand learning processes. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education* (pp. 68–95). Routledge.
16. **Erickson, F. (1992).** Ethnographic microanalysis of interaction. In M. D. LeCompte, W. Millroy, & J. Preissle (Eds.), *Handbook of qualitative research in education* (pp. 201–225). Academic Press.
17. **Geertz, C. (1973).** *The interpretation of cultures.* Basic Books.
18. **Jefferson, G. (2004).** Glossary of transcript symbols. In G. H. Lerner (Ed.), *Conversation analysis* (pp. 13–31). John Benjamins.
19. **Kazak, S., Wegerif, R., & Fujita, T. (2015).** The importance of dialogic processes to conceptual development in mathematics. *Educational Studies in Mathematics, 90*(2), 105–120. https://doi.org/10.1007/s10649-015-9618-y
20. **Kieran, C., Forman, E., & Sfard, A. (2001).** Learning discourse: Sociocultural approaches to research in mathematics education. *Educational Studies in Mathematics, 46*(1–3), 1–12.
21. **Mercer, N. (2008).** The seeds of time: Why classroom dialogue needs a temporal analysis. *Journal of the Learning Sciences, 17*(1), 33–59. https://doi.org/10.1080/10508400701793182
22. **Mercer, N., & Howe, C. (2012).** Explaining dialogic processes of teaching and learning. *Learning, Culture and Social Interaction, 1*(1), 12–21. https://doi.org/10.1016/j.lcsi.2012.03.001
23. **Mercer, N., & Littleton, K. (2007).** *Dialogue and the development of children's thinking.* Routledge.
24. **Miles, M. B., Huberman, A. M., & Saldaña, J. (2014).** *Qualitative data analysis* (3rd ed.). Sage.
25. **Noyes, A., Wake, G., & Drake, P. (2017).** *Core Maths: Policy and practice.* University of Nottingham.
26. **Pais, A., & Valero, P. (2012).** Researching research: Mathematics education in the political. *Educational Studies in Mathematics, 80*(1–2), 9–24. https://doi.org/10.1007/s10649-012-9399-5
27. **Radford, L. (2018).** The emergence of symbolic algebraic thinking in primary school. In L. Radford (Ed.), *Semiotics in mathematics education.* Springer.
28. **Rogoff, B. (1990).** *Apprenticeship in thinking.* Oxford University Press.

29. **Roth, W. M., & Radford, L. (2011).** A cultural historical perspective on mathematics teaching and learning. Sense Publishers.
30. **Sfard, A. (2008).** Thinking as communicating: Human development, the growth of discourses, and mathematizing. Cambridge University Press.
31. **Skovsmose, O. (2023).** *Critical mathematics education* (2nd ed.). Springer.
32. **Stillman, G., Brown, J., Galbraith, P., & Ng, K. (2016).** Research into mathematical applications and modelling. In K. Makar et al. (Eds.), *Research in mathematics education in Australasia* (pp. 361–387). Springer. https://doi.org/10.1007/978-981-10-1419-2_14
33. **Vygotsky, L. S. (1978).** *Mind in society.* Harvard University Press.
34. **Wegerif, R. (2011a).** *Dialogic education and technology.* Springer.
35. **Wegerif, R. (2011b).** Towards a dialogic theory of how children learn to think. *Thinking Skills and Creativity, 6*(3), 179–190. https://doi.org/10.1016/j.tsc.2011.08.002
36. **Wertsch, J. V. (1998).** *Mind as action.* Oxford University Press.
37. **Yin, R. K. (2018).** *Case study research and applications* (6th ed.). Sage.

Supplementary Files

This is a list of supplementary files associated with this preprint. Click to download.

- OnlineResource1Codebook.docx
- OnlineResource2CMIAuditTable.docx
- OnlineResource3ExtendedTranscript.docx