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Developing Conceptual Understanding of Measurement Division Through Multiple Representations: A Design-Based Research Study

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Abstract

Measurement division is an essential conceptual foundation for understanding fraction division, yet classroom instruction often emphasizes partitive division, providing students with limited opportunities to develop flexible reasoning about repeated grouping. This study investigated how a redesigned instructional sequence supported Grade 5 students' conceptual understanding of measurement division through multiple representations and identified instructional design principles for teaching measurement division.

The study reports the second iteration of an ongoing design-based research (DBR) project conducted in a Norwegian Grade 5 classroom. Building on findings from the first iteration, the instructional sequence was redesigned to establish measurement division as an explicit object of inquiry before introducing fraction division. Students engaged in exploratory activities using manipulatives, drawings, repeated addition, repeated subtraction, multiplication, and symbolic measurement division, supported by collaborative problem solving and whole-class mathematical discussions.

The findings indicate that students developed greater representational flexibility, increasingly coordinated multiple representations of the same mathematical structure, transferred their reasoning across different contexts, and successfully extended measurement division from whole-number to fraction situations. Many students also demonstrated a conceptual understanding that division by a fraction may produce a quotient larger than the dividend, suggesting that the redesigned sequence challenged the common misconception that division always makes numbers smaller.

The study contributes three design principles for teaching measurement division: establishing measurement division as an explicit object of inquiry before fraction division, coordinating multiple representations of the same mathematical structure, and preserving that structure while gradually varying contexts and extending the numerical domain from whole numbers to fractions. These findings contribute design knowledge for supporting students' conceptual understanding of measurement division and fraction division.

1 Introduction

The Norwegian mathematics curriculum (LK20) states that by the end of Grade 4, students should *explore and use measurement and partitive division in practical situations*. Thus, before students encounter formal fraction division, they are expected to have developed conceptual understanding of both interpretations of division. Research has nevertheless shown that students' experiences with measurement (quotative) division are often limited, while classroom instruction tends to emphasize partitive division (Bell et al., 1981; Carpenter & Moser, 1982; Fischbein et al., 1985; Gaidoschik et al., 2025; Sanjuán Cremades, 2021). As a consequence, many students develop a view of division primarily as sharing, making it difficult to interpret situations in which division represents determining how many groups of a given size can be formed.

One possible explanation for this gap between curricular expectations and students' understanding is the nature of the learning opportunities available in school (Alia et al., 2022; Walkowiak et al., 2017). In a recent textbook analysis currently under review, we found that the Grade 4 textbook used by the participating class devoted relatively little attention to measurement division despite its explicit inclusion in the Norwegian curriculum. Most division tasks were presented as sharing situations, providing students with limited opportunities to develop flexible reasoning about measurement division before entering Grade 5.

The present study builds on the first iteration of an ongoing design-based research (DBR) project aimed at developing exploratory approaches to teaching fraction operations. The first iteration demonstrated that Grade 5 students made substantial progress in solving contextualized problems involving multiplication and division with unit fractions and reduced common misconceptions such as *multiplication makes larger* and *division makes smaller*. However, an unexpected finding emerged during the implementation. Classroom observations and pre-test data suggested that many students had limited experience with measurement division, even when working with whole numbers. Rather than representing a difficulty specific to fractions, this appeared to be a missing conceptual prerequisite for understanding fraction division and was therefore identified as an important design challenge for subsequent iterations.

Following the principles of design-based research, the instructional sequence was therefore redesigned before being implemented in a new Grade 5 classroom. The redesign did not focus on introducing additional procedures for fraction division. Instead, it aimed to strengthen students' conceptual understanding of measurement division through exploratory activities, multiple representations, and mathematical discussions before extending this reasoning to fraction contexts.

The purpose of this study is to investigate how this redesigned instructional sequence supported students' development of measurement division and to identify design principles that may inform future teaching of measurement division and fraction division.

The study addresses the following research questions:

RQ1. *How did the redesigned instructional sequence support Grade 5 students' conceptual understanding of measurement division as reflected in their use of multiple solution strategies?*

RQ2. *What design principles for teaching measurement division were developed through the iterative design-based research process?*

2 Background

2.1 Measurement Division as a Distinct Interpretation of Division

Division is commonly understood as involving two related but conceptually distinct interpretations: partitive (sharing) division and measurement (quotative) division (Bell et al., 1981; Carpenter & Moser, 1982). In partitive division, a known quantity is shared equally among a given number of groups, and the unknown is the size of each group. In measurement division, the size of each group is known, and the unknown is the number of groups that can be formed.

Although these interpretations are mathematically equivalent, they place different cognitive demands on learners. Previous research has shown that classroom instruction frequently emphasizes partitive situations, while measurement division receives considerably less attention (Bell et al., 1981; Carpenter & Moser, 1982; Fischbein et al., 1985; Gaidoschik et al., 2025; Sanjuán Cremades, 2021). Consequently, many students develop a view of division primarily as sharing rather than as repeated grouping. This imbalance becomes particularly problematic when students later encounter fraction division, where measurement interpretations often provide the most natural conceptual explanation.

2.2 Conceptual Understanding Through Multiple Representations

Conceptual understanding develops when students recognize relationships among different representations of the same mathematical idea rather than learning isolated procedures (Hankeln & Prediger, 2025; Latifa et al., 2025; Skemp, 1976). From this perspective, meaningful learning involves connecting concrete experiences, visual representations, symbolic notation, and mathematical reasoning.

Bruner (1966) described learning as progressing through enactive, iconic, and symbolic forms of representation, while Lesh et al. (1987) argued that mathematical understanding is strengthened when learners are able to translate flexibly among multiple representational systems. Rather than treating manipulatives, drawings, symbolic notation, and arithmetic reasoning as separate instructional tools, these perspectives emphasize the importance of coordinating representations so that each illuminates the same underlying mathematical structure.

This theoretical perspective is particularly relevant for measurement division, where the same grouping process can be represented through concrete manipulation, drawings, repeated addition or subtraction, multiplication, and symbolic division. Developing connections among these representations may therefore support students in constructing a relational understanding of measurement division rather than relying on memorized computational procedures.

2.3 Measurement Division as a Foundation for Fraction Division

Fraction division is widely recognized as one of the most challenging topics in elementary mathematics. Many students learn procedural algorithms without developing a conceptual understanding of why these procedures work. Difficulties are further reinforced by common misconceptions, such as the belief that

division always produces a smaller result (Copur-Gencturk, 2021; Christou & Prokopou, 2020; Fischbein et al., 1985; Wahyu et al., 2020).

The present study is based on the assumption that these difficulties may not originate primarily in students' understanding of fractions, but in an insufficient understanding of measurement division itself. If students already understand division as determining how many groups of a given size can be formed, extending this reasoning from whole-number group sizes to fractional group sizes becomes a conceptual generalization rather than the introduction of a fundamentally new operation.

From this perspective, instruction should focus not only on teaching fraction division procedures but also on strengthening the conceptual foundations upon which those procedures are built. This assumption motivated the redesign of the instructional sequence investigated in the present study.

3 Methodology

3.1 Research Design

This study adopts a *design-based research* (DBR) methodology, which aims to improve educational practice while generating theoretical knowledge through iterative cycles of design, implementation, analysis, and redesign (Design-Based Research Collective, 2003; Anderson & Shattuck, 2012).

The paper reports on the second iteration of an ongoing DBR project investigating exploratory approaches to teaching fraction operations in primary school. Based on findings from the first iteration, the instructional sequence was redesigned to place greater emphasis on developing students' conceptual understanding of measurement division before introducing fraction division. The redesigned sequence was then implemented in a new Grade 5 classroom and analyzed to examine both students' learning and the instructional design.

Consistent with the aims of DBR, the study seeks not only to investigate how the redesigned learning sequence supported students' understanding of measurement division, but also to generate design principles that may inform future teaching of measurement division and fraction division.

3.2 Iteration 2: Redesign of the Instructional Sequence

The instructional sequence implemented in the second iteration was developed as a direct response to findings from the first DBR cycle. Rather than making minor adjustments to individual tasks, the redesign focused on strengthening students' conceptual understanding of measurement division before introducing fraction division. The underlying assumption was that a well-developed understanding of measurement division with whole numbers would provide a stronger foundation for reasoning about measurement division involving fractions.

Three main design considerations informed the redesign. First, measurement division was made an explicit object of inquiry rather than being treated as one of several interpretations of division. Second,

students were encouraged to explore and compare multiple solution strategies and representations. Third, the instructional sequence was organized to support a gradual transition from whole-number contexts to fraction contexts. These considerations informed the concrete design decisions implemented in the second iteration and were subsequently abstracted into the three design principles presented in the findings.

The redesigned sequence consisted of exploratory classroom activities in which students worked collaboratively to solve contextualized measurement division tasks. Whole-class discussions were used throughout the lessons to compare strategies, justify reasoning, and establish connections between different representations. These design features were intended to promote conceptual understanding while providing opportunities to examine how students developed and communicated their mathematical reasoning.

3.3 Participants and Context

The study was conducted in a Grade 5 class at a Norwegian primary school. The class consisted of 25 students aged 10–11 years. The class represented a typical Norwegian classroom setting and primarily had experience with traditional, textbook-driven mathematics instruction. The implementation took place at an ordinary Norwegian school with no special emphasis on alternative pedagogical approaches.

To facilitate active participation and classroom discussions, the instructional sequence was implemented with half of the class at a time. Consequently, the same learning sequence was taught twice, with each session involving approximately half of the students. Both groups participated in identical activities following the same instructional design.

The lessons were conducted by the researchers, while the classroom teachers were not actively involved in the implementation of the instructional sequence. This ensured that the learning sequence was carried out consistently across both groups and allowed the researchers to closely observe students' mathematical reasoning and interactions during the exploratory activities.

The second iteration was conducted with a class of 25 Grade 5 students. Pre-test responses were available for 23 students, while post-test responses were available for 21 students. The descriptive analyses of pre- and post-test performance were based on the responses available for each test. The qualitative analysis drew on the first author's observation notes and students' written work collected during the instructional sequence.

3.4 Data analyses

Data analysis was conducted iteratively in relation to the two cycles of the design-based research process. It included descriptive analysis of students' pre- and post-test responses and qualitative analysis of observation notes and students' written work.

Following the first implementation, the authors discussed their observations of students' responses and participation. These discussions identified challenges that informed the redesign. Many students were

not familiar with measurement division with whole numbers, although this had been assumed in the original design. Students also used different representations and solution strategies without necessarily connecting them, and needed support to recognize that the same strategies could be used across contexts. These observations informed the second implementation.

The second implementation was analyzed using the first author's observation notes and students' written work. Notes were written during and immediately after teaching sessions and supplemented by later recollection. The analysis involved reviewing the notes and written work and identifying episodes and student reasoning relevant to the research questions. Attention was given to students' use of representations and solution strategies, connections between representations and contexts, use of the same strategies across whole-number and fraction measurement-division tasks, and changes in reasoning across the instructional sequence. Concrete student products were selected to illustrate patterns in the data. The analysis was guided by the intended focus of the redesigned sequence while remaining open to emerging patterns. Group work and mathematical discussions provided evidence of students' reasoning during the sequence, while the post-test provided evidence of individual strategy use following instruction.

The three design principles presented in the findings were developed through this iterative process. They were not formulated in their final generalized form before the second implementation. Rather, concrete design decisions made after the first implementation were subsequently abstracted into general principles. The first principle, establish measurement division as an explicit object of inquiry before introducing fraction division, was based on students' limited familiarity with whole-number measurement division. The redesign therefore established whole-number measurement division and different solution strategies before introducing fraction measurement division. The second principle, coordinate multiple representations of the same mathematical situation, was based on students' use of different representations without necessarily connecting them. The redesign therefore provided opportunities to compare and connect representations and solution strategies. The third principle, preserve the underlying mathematical structure while gradually varying context and numerical domain, was based on students' need for support in recognizing the same strategies across contexts. The redesign varied contexts and extended the numerical domain from whole numbers to fractions while maintaining the underlying measurement-division structure. Following the second implementation, these concrete design decisions were abstracted into the three principles.

For the pre- and post-tests, student responses were examined individually using the same five predefined solution-strategy categories: drawing, repeated addition, repeated subtraction, multiplication, and measurement division. The categories were not mutually exclusive; when multiple strategies were used, each was recorded. Measurement division was coded only when the student explicitly represented the solution as a division expression. Correctness was determined from both the answer and the mathematical reasoning. For the post-test item concerning whether division can make a quantity larger, a response was correct when the student provided an example and a mathematically correct explanation. The pre- and post-test responses were coded by the first author.

4 Results

4.1 Pre-Test

Table 1
Accuracy on the pre-test

Task	Correct	Incorrect
Water ($8 \div 2$)	22 (96%)	1 (4%)
Buns ($20 \div 5$)	21 (91%)	2 (9%)
Biscuits ($900 \div 300$)	17 (74%)	6 (26%)

Table 2
Solution strategies identified in the pre-test

Strategy	Used by number of students
Drawing the groups	18 (78%)
Repeated addition	8 (35%)
Repeated subtraction	2 (9%)
Multiplication	7 (30%)
Measurement division	5 (22%)

Table 3
Number of solution strategies used per student in the pre-test

Number of strategies used	5	4	3	2	1	0
Number of students	0 (0%)	0 (0%)	4 (17%)	11 (48%)	6 (26%)	2 (9%)

The pre-test was designed to examine students' initial understanding of measurement division and identify the solution strategies they used before the instructional sequence. The complete pre-test tasks are provided in Appendix A.

Table 1 shows that nearly all students correctly solved the two whole-number measurement division tasks. Twenty-two of the 23 students correctly solved the water task, and 21 students correctly solved the buns task. Performance declined on the biscuits task, where 17 students provided correct solutions while six answered incorrectly. These results suggest that students generally possessed an intuitive understanding of measurement division in familiar whole-number contexts.

The written responses revealed considerable variation in how students expressed their thinking. As shown in Table 2, drawing the groups was the most common strategy, appearing in the work of 18 students. Repeated addition and multiplication were used by eight and seven students, respectively, while repeated subtraction appeared in two responses. Five students expressed the solution directly using symbolic measurement division, whereas two provided only a written answer without documenting their reasoning. Thus, students entered the intervention with access to a range of representations, although visual representations dominated. An example combining drawings and arithmetic representations is provided in Appendix B (Figure B1).

Table 3 further illustrates the diversity of students' mathematical thinking. No student employed all five identified solution strategies, and none used four strategies. Four students demonstrated three strategies, while most students (11) used two and six relied on a single strategy. Although no individual student displayed the full range of representations identified across the class, every strategy was represented collectively. This variation provided opportunities for classroom discussions in which students could compare and connect alternative representations and explain their reasoning.

4.2 Developing Multiple Representations of Measurement Division

The first classroom discussion centered on the opening task from the pre-test: *Lea has 8 liters of water that she pours into 2-litre bottles. How many 2-litre bottles does she need?* The purpose was to establish measurement division as a mathematical idea that could be represented in different ways.

The lesson began with a concrete demonstration using a container representing 8 liters of water and several 2-litre bottles. Rather than focusing immediately on a numerical calculation, the discussion centered on the question, *What are we trying to find?* Students distinguished the task from a sharing situation and recognized that it involved determining how many groups of 2 liters could be formed.

Students then suggested different ways of representing the same situation. Together, the class identified six representations: physical manipulatives, drawing, repeated addition, repeated subtraction, multiplication, and measurement division. The discussion focused on how these representations described the same grouping process. For example, students recognized that repeated addition counted groups of 2 liters, repeated subtraction removed one group at a time, multiplication expressed the relationship between the number and size of the groups, and symbolic measurement division represented the same relationship.

The analysis of the observation notes and student responses showed that attention shifted from the individual procedures to relationships between representations. Students were encouraged to explain how one representation could be translated into another and whether they answered the same question. The activity thus provided evidence of students beginning to view measurement division as a mathematical relationship that could be expressed in multiple ways.

4.3 Exploring Multiple Solution Strategies Across Contexts

Students next worked collaboratively on a second measurement division task: *Sivert has 20 buns that he places into bags. Each bag contains 5 buns. How many bags does he need?* Each group received 20 cardboard buns and plastic bags and solved the problem before exploring alternative representations.

The purpose was to examine whether students could apply the representations developed in the water-bottle task to a new context. During the whole-class discussion, the teacher directed attention to similarities between the two tasks rather than to the numerical answers. Students compared what was known and unknown and considered how the grouping process could be represented.

The same representations identified in the previous activity—drawings, repeated addition, repeated subtraction, multiplication, and symbolic measurement division—were used for the buns task. The discussion therefore shifted from the surface features of the contexts to the mathematical structure shared by the two problems. Students were encouraged to explain how a representation used in one situation could be applied to the other.

Analysis of the observation notes and student work indicated that students increasingly attended to the underlying mathematical structure rather than only to the context. The activity provided evidence that the same measurement-division strategies could be transferred across contexts, preparing the transition from whole-number to fraction measurement division.

4.4 Extending Measurement Division from Whole Numbers to Fractions

Students were then presented with the task: There are 20 buns. If each bag holds 2 1/2 buns, how many bags can be filled? They worked collaboratively with cardboard buns and scissors, partitioning buns into halves when necessary.

The mathematical challenge now concerned the size of the group rather than a new context. Students first modelled the situation concretely and then compared it with the earlier whole-number tasks. The discussion focused on whether the mathematical structure had changed when the group size was no longer a whole number.

Students used the representations developed in the previous activities—including manipulatives, drawings, repeated addition, repeated subtraction, multiplication, and symbolic measurement division—to represent the fraction situation. An example of student work demonstrating several representations is provided in Appendix B (Figure B2).

Analysis of the observation notes and student work indicated that students treated the fraction task as an extension of the measurement-division situations they had already explored. Comparing whole-number and fraction tasks kept attention on the grouping structure while the numerical domain changed.

4.5 Challenging the Misconception that Division Always Makes Smaller

The final activity addressed the idea that division always produces a result smaller than the original quantity. Students were presented with the problem: Lea has 4 liters of water. She pours the water into bottles that each hold ½ liter. How many bottles does she need? As in the earlier activities, students first modelled the situation using manipulatives before discussing alternative representations.

Students repeatedly filled half-liter bottles from the 4-litre container and then related the task to the earlier measurement-division activities. The discussion focused on what had remained the same and what had changed. Students identified that the underlying grouping structure was unchanged, while the size of each group was smaller.

The same representations used throughout the sequence were used again. Students explained why dividing 4 liters into groups of ½ liter produces 8 groups without relying on a memorized algorithm. Analysis of the observation notes, group discussions, and student work indicated that students increasingly interpreted the quotient as the number of groups that can be formed from a fixed quantity. Within this interpretation, a quotient larger than the dividend became meaningful because smaller groups produce more groups.

4.6 Post-Test

Table 4
Accuracy on the post-test

Task	Correct	Incorrect
Water (2 ÷ 1/2)	19 (90%)	2 (10%)
Bun (10 ÷ 2 1/2)	17 (81%)	4 (19%)
Division makes larger	15 (71%)	6 (29%)

Table 5
Solution strategies identified in the post-test

Strategy	Used by number of students
Drawing the groups	18 (86%)
Repeated addition	18 (86%)
Repeated subtraction	7 (33%)
Multiplication	11 (52%)
Measurement division	12 (57%)

Table 6
Number of solution strategies used per student in the post-test

Number of strategies used	5	4	3	2	1
Number of students	3 (14%)	8 (38%)	4 (19%)	3 (14%)	3 (14%)

The post-test was administered after the instructional sequence to examine students' understanding following the intervention. It consisted of five tasks; however, the present study analyzes only the two measurement division tasks and the conceptual task addressing the misconception that division always produces a smaller result. The complete post-test and an explanation of the tasks included in the present analysis are provided in Appendix A.

As shown in Table 4, 19 of the 21 students correctly solved the water task, while 17 correctly solved the bun task. The conceptual task was more challenging, with 15 students answering correctly.

Table 5 shows that students used a variety of solution strategies. Drawings and repeated addition were most common, each used by 18 students. Twelve students used symbolic measurement division, 11 used multiplication, and seven used repeated subtraction. Most students used more than one strategy in their responses.

Table 6 further shows that three students used all five strategies, eight used four, and four used three. Only three students relied on a single strategy. Thus, most students represented measurement division in multiple ways. An example is provided in Appendix B (Figure B3a).

The misconception task was less successful than the two measurement division tasks, but 15 students demonstrated an understanding that division by a fraction can produce a quotient larger than the dividend. An example of a student using numerical examples to justify this reasoning is provided in Appendix B (Figure B3b).

5 Discussion

5.1 Findings in Light of the Research Question

The findings provide evidence that the redesigned instructional sequence supported students' conceptual understanding of measurement division in several ways. First, students developed a broader repertoire of solution strategies, progressing from primarily using drawings in the pre-test to employing multiple representations in the post-test. Classroom activities encouraged students to move between concrete manipulatives, drawings, repeated addition, repeated subtraction, multiplication, and symbolic measurement division. The post-test further showed that most students represented measurement division using several strategies, suggesting increased representational flexibility.

Second, the findings indicate that students increasingly recognized the common mathematical structure underlying different measurement division situations. Rather than treating the water, buns, and fraction

tasks as unrelated problems, classroom discussions encouraged students to identify the same grouping process across contexts. This progression extended from whole-number to fraction measurement division without introducing a formal algorithm for fraction division. In the final activity, many students were also able to explain why dividing by one-half produces a quotient larger than the dividend, suggesting a conceptual interpretation of division.

Together, these findings address the first research question by indicating that the redesigned instructional sequence supported students' conceptual understanding through exploratory activities, coordinated representations, and opportunities to compare and discuss alternative solution strategies.

The second research question concerned the development of design principles through the DBR process. The iterative analysis resulted in three principles: (1) establish measurement division as an explicit object of inquiry before introducing fraction division; (2) coordinate multiple representations of the same mathematical situation; and (3) preserve the underlying mathematical structure while gradually varying context and numerical domain. These principles provide the basis for the following discussion of the study's contribution to teaching and learning of measurement division.

5.2 Relation to Previous Research

The findings of this study are consistent with previous research suggesting that students often have limited experience with measurement division compared with partitive division (Bell et al., 1981; Carpenter & Moser, 1982; Fischbein et al., 1985; Gaidoschik et al., 2025; Sanjuán Cremades, 2021). The pre-test indicated that most students could obtain correct answers in familiar contexts but relied primarily on drawings or repeated addition, while only a few represented situations using symbolic measurement division. These findings support earlier observations that classroom experiences with measurement division are often insufficient for developing flexible conceptual understanding.

The redesigned instructional sequence sought to address this limitation by making measurement division an explicit object of inquiry. Rather than introducing formal procedures, students explored the same mathematical structure through concrete materials, visual representations, arithmetic reasoning, and symbolic notation. This approach aligns with Skemp's (1976) distinction between relational and instrumental understanding. Throughout the activities, students were encouraged to explain relationships among different representations rather than memorize procedures for obtaining correct answers. The post-test findings suggest that many students developed greater representational flexibility, indicating that they viewed different strategies as connected representations of the same mathematical idea.

The findings also support theoretical perspectives emphasizing the importance of multiple representations in mathematics learning (Bruner, 1966; Kokkonen & Schalk, 2021; Lesh et al., 1987; Liang & She, 2023; Latifa et al., 2025). Students were not introduced to each representation as an isolated strategy but were encouraged to move between concrete manipulatives, drawings, repeated addition, repeated subtraction, multiplication, and symbolic measurement division while discussing how each

described the same grouping process. This coordination of representations appeared to support students in recognizing the underlying mathematical structure across different contexts and, subsequently, extending this reasoning from whole numbers to fractions.

An important contribution of the present study is that students were able to reason about fraction measurement division before learning a formal algorithm for dividing by fractions. Rather than treating fraction division as a new computational procedure, the instructional sequence framed it as a natural extension of reasoning that had already been established with whole numbers. This finding suggests that conceptual understanding of measurement division may provide an important foundation for learning fraction division. In this respect, the study extends previous research by demonstrating how a carefully sequenced progression from concrete experiences to increasingly abstract representations can support students in generalizing measurement division across numerical domains.

5.3 Design Principles for Teaching Measurement Division

A central aim of design-based research is not only to examine whether an instructional design supports learning, but also to generate design principles that may inform future educational practice (Design-Based Research Collective, 2003). In the present study, the three principles were developed through the iterative redesign process. Concrete design decisions made after the first implementation were examined in light of the second implementation and subsequently abstracted into more general principles.

The first design principle is that **measurement division should be established as an explicit object of inquiry before introducing fraction division**. The first implementation revealed that many students were not familiar with measurement division with whole numbers, although this had been assumed in the original design. The instructional sequence was therefore redesigned to establish whole-number measurement division and different solution strategies before introducing fraction measurement division. In the second implementation, students were able to apply these strategies when working with fractional group sizes, supporting the rationale for this design decision.

The second design principle is that **instruction should support students in coordinating multiple representations of the same mathematical situation**. In the first implementation, students used different representations and solution strategies without necessarily connecting them. The redesign therefore created opportunities to compare and connect representations, including concrete, visual, arithmetic, and symbolic forms. The second implementation provided evidence that students could use and discuss these representations as different ways of expressing the same measurement-division structure.

The third design principle is that **instruction should preserve the underlying mathematical structure while gradually varying context and numerical domain**. The first implementation indicated that students needed support in recognizing that the same strategies could be used across different contexts. The redesigned sequence therefore maintained the measurement-division structure while varying contexts and extending the numerical domain from whole numbers to fractions. In the second implementation,

students applied familiar strategies across these changes, suggesting that maintaining the underlying structure while varying surface features supported transfer of their reasoning.

Taken together, the three principles emphasize a progression in which students first develop measurement division as an explicit mathematical object, then coordinate multiple representations of this structure, and finally extend the structure across contexts and numerical domains. The principles thus capture the design rationale developed through the iterative process rather than describing isolated features of the instructional sequence.

5.4 Implications, Limitations, and Future Research

The findings have several implications for mathematics education and classroom practice. First, they suggest that conceptual understanding of measurement division should be developed explicitly before students are introduced to formal procedures for fraction division. Rather than treating fraction division as a separate topic requiring a new algorithm, instruction may benefit from building on students' understanding of repeated grouping situations. The instructional sequence demonstrated that students could extend their reasoning from whole numbers to fractions when the underlying mathematical structure remained visible.

Second, the findings highlight the importance of coordinating multiple representations during instruction. Encouraging students to move between concrete materials, visual models, arithmetic reasoning, and symbolic notation appeared to support conceptual understanding and recognition of relationships among solution strategies. Classroom discussions that compare representations rather than identify a single preferred method may therefore provide opportunities for developing relational understanding of division.

Several limitations should be considered. The study involved a single Grade 5 classroom, and the instructional sequence was implemented by the researchers rather than the classroom teachers. The findings therefore cannot be generalized directly to other educational settings. Furthermore, the study examined students' understanding immediately following the instructional sequence and does not provide evidence of long-term retention or transfer to subsequent learning of formal fraction division algorithms.

The design principles should therefore be regarded as tentative and open to further refinement through additional design iterations. Future research could investigate how they function with different student populations, teachers, and curricular contexts. It would also be valuable to examine whether students who develop a strong conceptual understanding of measurement division experience fewer difficulties when later learning formal fraction-division procedures. Longitudinal studies could provide evidence regarding this relationship.

The study also illustrates the value of design-based research for identifying unexpected learning challenges and using these insights to improve instructional design. The recognition that limited

understanding of measurement division represented an important prerequisite reshaped the instructional sequence and contributed to the development of the design principles.

5.5 Conclusion

This study investigated how a redesigned instructional sequence supported Grade 5 students' conceptual understanding of measurement division and identified design principles for teaching measurement division within a design-based research framework. Building on findings from the first DBR iteration, the sequence was redesigned to establish measurement division as a conceptual foundation before introducing fraction division. Students explored measurement division through multiple representations, collaborative problem solving, and classroom discussions.

The findings suggest that the redesigned sequence supported a more flexible understanding of measurement division. Students recognized connections among concrete, visual, arithmetic, and symbolic representations, transferred their reasoning across contexts, and extended their understanding from whole-number to fraction measurement division. Many also explained why division by a fraction may produce a quotient larger than the dividend.

The study contributes three design principles: establishing measurement division as an explicit object of inquiry before introducing fraction division, coordinating multiple representations of the same mathematical structure, and maintaining that structure while gradually extending contexts and numerical domains. Although further research is needed to examine their applicability across educational settings, the study illustrates how iterative design-based research can reveal conceptual prerequisites and use these insights to improve instruction.

Declarations

6.1 Competing Interests

The authors declare that they have no conflict of interest.

6.2 Funding

This research was supported by seed funding from the University School Partnership (*Universitetsskolesamarbeidet*) at Nord University. The funding scheme supports collaborative research and development projects involving teacher educators, partner schools, and student teachers.

6.3 Data Availability

The datasets generated and analyzed during the current study are not publicly available because they contain identifiable educational data but are available from the corresponding author on reasonable

request, subject to ethical approval and applicable regulations.

6.4 Code availability
No custom code or software was developed for this study.

6.5 Ethical Approval

The study was reported to Sikt – Norwegian Agency for Shared Services in Education and Research (reference number 952189). Separate approval by an institutional research ethics committee was not required for this study. The research was conducted in accordance with applicable Norwegian ethical guidelines and standards for research involving human participants.

6.6 Author Contributions

Marthe Måløy conceived the study, designed the instructional intervention, collected and analyzed the data, and prepared the first draft of the manuscript. Eirik Bergli contributed to the study design, led the implementation of the instructional intervention in the classroom, contributed to the interpretation of the findings, and critically revised the manuscript. Both authors read and approved the final manuscript.

6.7 Informed consent

Written informed consent was obtained from both the participating students and their parents or legal guardians prior to participation. Students and their parents or legal guardians were informed that participation was voluntary and that they could withdraw from the study.

8 Acknowledgments

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