

Supplementary Information for

Reconstructing non-Abelian braiding and fusion without anyon transport

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Supplementary Note 1: Dense reduction from a four-qudit plaquette to two qutrits

The experimental protocol is enabled by a sequence of exact reductions of the observables used to reconstruct the $D(S_3)$ braiding and fusion data. We begin with the minimal four-qudit plaquette, reduce the relevant expectation values to operators acting on two qudits, and subsequently restrict these operators to a two-qutrit subspace. Finally, we show that the required overlaps can be evaluated on a simple product state rather than on the original entangled plaquette state. Each reduction preserves the squared braiding phases and fusion amplitudes extracted by the protocol, while substantially lowering the resources required for implementation on quantum hardware.

From four qudits to two qudits

We first show that the four-qudit expectation values used to reconstruct the braiding and fusion data can be reduced exactly to expectation values on two qudits. AAs described in the Methods subsection “Dense reduction from four qudits to two qutrits” of the main text, the squared braiding phases of the non-Abelian G anyons are obtained from

$$(R^i)^2 = N_i \langle \eta | F_{\rho_2}^G F_{\rho_1}^G A^i(v) F_{\rho_2}^G F_{\rho_1}^G | \eta \rangle, \quad (\text{S1})$$

with $i \in \{+, G\}$, while the squared fusion amplitudes $|F_{ij}|^2$ are obtained from two sets of measured expectation values

$$\begin{aligned} \langle \eta | O_{m_{ji}} | \eta \rangle = \\ \langle \eta | F_{\rho_2}^G F_{\rho_2}^G A^j(v) F_{\rho_1}^G A^G(v) F_{\rho_2}^G A^i(v) F_{\rho_2}^G F_{\rho_1}^G | \eta \rangle, \end{aligned} \quad (\text{S2})$$

and

$$\begin{aligned} \langle \eta | O_{n_j} | \eta \rangle = \\ \langle \eta | F_{\rho_2}^G F_{\rho_2}^G A^j(v) F_{\rho_1}^G A^G(v) F_{\rho_1}^G A^j(v) F_{\rho_2}^G F_{\rho_2}^G | \eta \rangle, \end{aligned} \quad (\text{S3})$$

where $i, j \in \{+, G\}$ and N_i is the corresponding normalisation factor. Here $|\eta\rangle$ denotes the vacuum ground state of the minimal four-qudit $D(S_3)$ plaquette shown in Fig. ??(a), with each qudit having local dimension $d = 6$.

The operator products in Eqs. (S1), (S2) and (S3) act non-trivially only on qudits 3 and 4 of the plaquette [1]. Consequently, any such product can be written as

$$O = \mathbf{1}_{36} \otimes o, \quad (\text{S4})$$

where o acts on the Hilbert space of the remaining two qudits. Its expectation value in the four-qudit ground state then reduces to

$$\langle \eta | O | \eta \rangle = 6 \sum_{g \in S_3} \langle \psi_g | o | \psi_g \rangle, \quad (\text{S5})$$

where

$$|\psi_g\rangle = \frac{1}{\sqrt{216}} \sum_{\substack{g_1, g_2 \in S_3 \\ g_1 g_2 = g}} |g_1, g_2\rangle. \quad (\text{S6})$$

Note that, with this convention, the states $|\psi_g\rangle$ are not individually normalised. Their normalisation is incorporated into the prefactor in Eq. (S5). Thus, all four-qudit plaquette expectation values required by the protocol can be evaluated using operators acting on only two $d = 6$ qudits.

From two qudits to two qutrits

We next exploit the semidirect-product structure $S_3 \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ to reduce each local six-dimensional qudit to a qutrit [2]. Ordering the group-element basis as

$$\{e, c, c^2\} \oplus \{t, tc, tc^2\}, \quad (\text{S7})$$

separates the $\mathbb{Z}_3$ subgroup from its complementary coset. This decomposition also underlies efficient constructions of $D(S_3)$ states on quantum hardware [3–5].

For the braiding and fusion processes considered here, the relevant operators in the $\{+, G\}$ sector have support only on the $\mathbb{Z}_3 = \{e, c, c^2\}$ block. Here

$$|+\rangle \equiv \frac{|A\rangle + |B\rangle}{\sqrt{2}} \quad (\text{S8})$$

denotes the normalised Abelian-sector superposition accessed by the reduced protocol. Each term in the corresponding two-qudit operator products can be written in the form

$$(\mathcal{B} \oplus \mathbf{0}_3) \otimes (\mathcal{C} \oplus \mathbf{0}_3), \quad (\text{S9})$$

where $\mathcal{B}$ and $\mathcal{C}$ act on the $\mathbb{Z}_3$ subspace. The complementary $\{t, tc, tc^2\}$ block therefore does not contribute to the group-averaged expectation values. In particular,

$$\begin{aligned} \sum_{g \in S_3} \langle \psi_g | (\mathcal{B} \oplus \mathbf{0}_3) \otimes (\mathcal{C} \oplus \mathbf{0}_3) | \psi_g \rangle \\ = \frac{1}{72} \sum_{g \in \mathbb{Z}_3} \langle \Psi_g | \mathcal{B} \otimes \mathcal{C} | \Psi_g \rangle, \end{aligned} \quad (\text{S10})$$

where the normalised two-qutrit states

$$|\Psi_g\rangle = \frac{1}{\sqrt{3}} \sum_{\substack{g_1, g_2 \in \mathbb{Z}_3 \\ g_1 g_2 = g}} |g_1, g_2\rangle \quad (\text{S11})$$

have definite total $\mathbb{Z}_3$ flux g . The factor $1/72$ follows from the normalisation of the two-qudit states $|\psi_g\rangle$ in Eq. (S6).

Consequently, all expectation values entering Eqs. (S1), (S2) and (S3) can be evaluated using the reduced two-qutrit operators

$$\mathcal{F}_{\rho_1}^G = Z \otimes X + Z^2 \otimes X^2, \quad (\text{S12})$$

$$\mathcal{F}_{\rho_2}^G = X \otimes Z^2 + X^2 \otimes Z, \quad (\text{S13})$$

$$\mathcal{A}^+(v) = \frac{1}{3} (\mathbf{1}_9 + X \otimes X^2 + X^2 \otimes X), \quad (\text{S14})$$

$$\mathcal{A}^G(v) = \frac{1}{3} (2\mathbf{1}_9 - X \otimes X^2 - X^2 \otimes X). \quad (\text{S15})$$

Here

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}, \quad \omega = e^{2\pi i/3}, \quad (\text{S16})$$

are the qutrit shift and clock operators, satisfying $X^3 = Z^3 = \mathbf{1}_3$ and $ZX = \omega XZ$.

The operators $\mathcal{F}_{\rho_1}^G$ and $\mathcal{F}_{\rho_2}^G$ retain the action of the two intersecting G -anyon ribbons, while $\mathcal{A}^+(v)$ and $\mathcal{A}^G(v)$ resolve the corresponding Abelian and non-Abelian charge sectors. Thus, the four-qudit observables required to reconstruct the braiding and fusion data are reduced exactly to operators on two qutrits. In the subsection ‘‘Anyonic content of the reduced two-qutrit operators’’ below, we verify explicitly that these reduced operators retain the fusion rules and braiding phases of the relevant $\{A, B, G\}$ subcategory of $D(S_3)$.

From entangled to product states

The final reduction replaces the ensemble of entangled two-qutrit states $|\Psi_g\rangle$ defined in Eq. (S11) by a single, experimentally accessible product state. The states $|\Psi_g\rangle$ have definite total $\mathbb{Z}_3$ flux g , and therefore their expectation values depend only on matrix elements that preserve this flux.

To make this structure explicit, write a general two-qutrit operator $\mathcal{O}$ as

$$\begin{aligned} \mathcal{O} = & \sum_{\substack{g_1 g_2 = g_3 g_4 \\ g_1, \dots, g_4 \in \mathbb{Z}_3}} p_{g_1 g_2 g_3 g_4} |g_1, g_2\rangle \langle g_3, g_4| \\ & + \sum_{\substack{g_1 g_2 \neq g_3 g_4 \\ g_1, \dots, g_4 \in \mathbb{Z}_3}} q_{g_1 g_2 g_3 g_4} |g_1, g_2\rangle \langle g_3, g_4|, \end{aligned} \quad (\text{S17})$$

where the coefficients $p_{g_1 g_2 g_3 g_4}$ and $q_{g_1 g_2 g_3 g_4}$ multiply the flux-preserving and flux-changing matrix elements, respectively. Since $|\Psi_g\rangle$ has fixed total flux, matrix elements connecting different flux sectors do not contribute, and hence

$$\sum_{g \in \mathbb{Z}_3} \langle \Psi_g | \mathcal{O} | \Psi_g \rangle = \frac{1}{3} \sum_{\substack{g_1 g_2 = g_3 g_4 \\ g_1, \dots, g_4 \in \mathbb{Z}_3}} p_{g_1 g_2 g_3 g_4}. \quad (\text{S18})$$

Consider the normalised equal-superposition state

$$|\xi\rangle = \frac{1}{3} \sum_{g_1, g_2 \in \mathbb{Z}_3} |g_1, g_2\rangle. \quad (\text{S19})$$

This state factorizes as

$$|\xi\rangle = \left(\frac{1}{\sqrt{3}} \sum_{g_1 \in \mathbb{Z}_3} |g_1\rangle \right) \otimes \left(\frac{1}{\sqrt{3}} \sum_{g_2 \in \mathbb{Z}_3} |g_2\rangle \right), \quad (\text{S20})$$

and is therefore a product state of the two qutrits. Its expectation value contains both the flux-preserving and flux-changing components of $\mathcal{O}$:

$$\begin{aligned} 9 \langle \xi | \mathcal{O} | \xi \rangle = & \sum_{\substack{g_1 g_2 = g_3 g_4 \\ g_1, \dots, g_4 \in \mathbb{Z}_3}} p_{g_1 g_2 g_3 g_4} \\ & + \sum_{\substack{g_1 g_2 \neq g_3 g_4 \\ g_1, \dots, g_4 \in \mathbb{Z}_3}} q_{g_1 g_2 g_3 g_4}. \end{aligned} \quad (\text{S21})$$

For the particular operator products entering the braiding and fusion overlaps in Eqs. (S1), (S2) and (S3), direct evaluation shows that the total flux-changing contribution cancels

$$\sum_{\substack{g_1 g_2 \neq g_3 g_4 \\ g_1, \dots, g_4 \in \mathbb{Z}_3}} q_{g_1 g_2 g_3 g_4} = 0. \quad (\text{S22})$$

Equations (S18)-(S22) then imply

$$\sum_{g \in \mathbb{Z}_3} \langle \Psi_g | \mathcal{O} | \Psi_g \rangle = 3 \langle \xi | \mathcal{O} | \xi \rangle. \quad (\text{S23})$$

Combining Eq. (S23) with the four-to-two-qudit reduction in Eq. (S5) and the subsequent restriction to the $\mathbb{Z}_3$ block in Eq. (S10), including the normalization factors absorbed into the definitions of the reduced operators, gives

$$\langle \eta | O | \eta \rangle = \frac{1}{4} \langle \xi | \mathcal{O} | \xi \rangle. \quad (\text{S24})$$

Thus, every four-qudit plaquette overlap required to reconstruct the braiding phases and fusion amplitudes can be evaluated using reduced operators acting on two qutrits prepared in the simple product state $|\xi\rangle$. This exact product-state reduction is what makes the protocol directly implementable on the quantum hardware described in the Methods.

Anyonic content of the reduced two-qutrit operators

The dense reduction is useful only if the resulting two-qutrit operators retain the fusion and braiding data of the $\{A, B, G\}$ fusion subcategory of $D(S_3)$. A detailed derivation of the corresponding ribbon algebra in the full quantum-double construction was given in Ref. [6]. Here we record only the identities needed to establish the consistency of the reduced description used in the present experiment.

Fusion relations

In the ordered group basis $\{e, c, c^2, t, tc, tc^2\}$, the ribbon operators creating the Abelian charges A and B can be written as

$$F^A = \mathbf{1}_6, \quad F^B = \text{diag}(1, 1, 1, -1, -1, -1). \quad (\text{S25})$$

Their sum therefore has support only on the $\mathbb{Z}_3 = \{e, c, c^2\}$ block. In the reduced two-qutrit description, the corresponding composite ribbon is

$$\mathcal{F}^+ \equiv \mathcal{F}^A + \mathcal{F}^B = 2\mathbf{1}_3 \otimes \mathbf{1}_3, \quad (\text{S26})$$

where $+$ denotes the unnormalised composite charge $A + B$.

Together with the reduced G -ribbon operators in Eqs. (S12) and (S13), direct multiplication gives

$$(\mathcal{F}^+)^2 = 2\mathcal{F}^+, \quad (\text{S27})$$

$$\mathcal{F}^+ \mathcal{F}_{\rho_k}^G = 2\mathcal{F}_{\rho_k}^G, \quad k = 1, 2, \quad (\text{S28})$$

$$(\mathcal{F}_{\rho_k}^G)^2 = \mathcal{F}^+ + \mathcal{F}_{\rho_k}^G, \quad k = 1, 2. \quad (\text{S29})$$

These identities reproduce

$$+ \times + = 2+, \quad + \times G = 2G, \quad G \times G = A + B + G, \quad (\text{S30})$$

respectively. The factors of two in the first two relations arise because $+$ denotes the unnormalised composite $A + B$. Thus, the reduced ribbon operators retain the fusion algebra of the relevant $D(S_3)$ anyons.

Braiding phases

The braiding information is encoded in the non-commutativity of the two intersecting G -ribbon operators [6]. For every non-zero matrix element in the two-qutrit group basis, direct multiplication gives

$$\begin{aligned} & \langle g_1, g_2 | \mathcal{F}_{\rho_1}^G \mathcal{F}_{\rho_2}^G | h_1, h_2 \rangle \\ &= \begin{cases} \bar{\omega} \langle g_1, g_2 | \mathcal{F}_{\rho_2}^G \mathcal{F}_{\rho_1}^G | h_1, h_2 \rangle, & g_1 g_2 = h_1 h_2, \\ \omega \langle g_1, g_2 | \mathcal{F}_{\rho_2}^G \mathcal{F}_{\rho_1}^G | h_1, h_2 \rangle, & g_1 g_2 \neq h_1 h_2, \end{cases} \end{aligned} \quad (\text{S31})$$

where $\omega = e^{2\pi i/3}$.

The condition $g_1 g_2 = h_1 h_2$ preserves the plaquette flux and therefore corresponds to the trivial-flux fusion channels A and B . These channels acquire the common squared braiding phase $\bar{\omega}$. By contrast, $g_1 g_2 \neq h_1 h_2$ changes the plaquette flux and corresponds to the non-trivial-flux channel G , which acquires the phase ω . Hence,

$$(R^A)^2 = (R^B)^2 = \bar{\omega}, \quad (R^G)^2 = \omega, \quad (\text{S32})$$

in agreement with the $D(S_3)$ braiding data in Eq. (3) of the main text. The reduced two-qutrit operators therefore preserve both the fusion algebra and the squared braiding phases reconstructed experimentally.

Supplementary Note 2: Recovery of the anyonic data

The preceding section established an exact reduction of the four-qudit plaquette overlaps to expectation values of operators acting on two qutrits. We now show how the experimentally accessible measurements in the $\{+, G\}$ sectors recover the braiding and fusion data in the physical anyon basis $\{A, B, G\}$. The reduced protocol does not resolve the A and B sectors independently, but instead accesses their coherent superposition

$$|+\rangle \equiv \frac{|A\rangle + |B\rangle}{\sqrt{2}}. \quad (\text{S33})$$

For braiding, this is sufficient because the A and B fusion channels have the same squared braiding phase. For fusion, the measured $+$ - and G -sector overlaps, together with the symmetry and unitarity of the $D(S_3)$ F -symbol, determine the squared amplitudes in the full $\{A, B, G\}$ basis.

Squared braiding phases from qutrit measurements

We first define the states whose overlap gives the squared braiding phase. Let

$$|\tilde{\phi}_{12}(i)\rangle \equiv A^i(v) F_{\rho_1}^G F_{\rho_2}^G |\eta\rangle, \quad (\text{S34})$$

$$|\tilde{\phi}_{21}(i)\rangle \equiv A^i(v) F_{\rho_2}^G F_{\rho_1}^G |\eta\rangle, \quad (\text{S35})$$

where $i \in \{A, B, G\}$. Here $|\eta\rangle$ is the vacuum state of the minimal four-qudit plaquette, the two ribbon operators create the same pair of G anyons in opposite orders, and $A^i(v)$ resolves their total fusion channel at the shared endpoint. We denote the corresponding normalised states by

$$|\phi_\mu(i)\rangle = \frac{|\tilde{\phi}_\mu(i)\rangle}{\sqrt{\langle \tilde{\phi}_\mu(i) | \tilde{\phi}_\mu(i) \rangle}}. \quad (\text{S36})$$

with $\mu \in \{12, 21\}$.

The two states differ by a double exchange of the G anyons. With the orientation convention used in Fig. 1(b), their relation in a fixed fusion channel is

$$|\phi_{21}(i)\rangle = (R^i)^2 |\phi_{12}(i)\rangle, \quad (\text{S37})$$

and therefore

$$(R^i)^2 = \langle \phi_{12}(i) | \phi_{21}(i) \rangle. \quad (\text{S38})$$

The reduced protocol accesses the normalised Abelian-sector superpositions

$$|\phi_\mu(+)\rangle \equiv \frac{1}{\sqrt{2}} [|\phi_\mu(A)\rangle + |\phi_\mu(B)\rangle]. \quad (\text{S39})$$

with $\mu \in \{12, 21\}$. Using Eq. (S37), the state obtained after the double exchange is

$$|\phi_{21}(+)\rangle = \frac{1}{\sqrt{2}} \left[(R^A)^2 |\phi_{12}(A)\rangle + (R^B)^2 |\phi_{12}(B)\rangle \right]. \quad (\text{S40})$$

On the other hand, the reduced $+$ sector transforms with a single measured phase,

$$\begin{aligned} |\phi_{21}(+)\rangle &= (R^+)^2 |\phi_{12}(+)\rangle \\ &= \frac{(R^+)^2}{\sqrt{2}} [|\phi_{12}(A)\rangle + |\phi_{12}(B)\rangle]. \end{aligned} \quad (\text{S41})$$

Since the A and B fusion channels are orthogonal, comparison of Eqs. (S40) and (S41) gives

$$(R^A)^2 = (R^B)^2 = (R^+)^2. \quad (\text{S42})$$

Using the dense reduction derived in Supplementary Note 1, the common Abelian-sector phase can be evaluated on the two-qutrit product state $|\xi\rangle$ as

$$(R^+)^2 = \mathcal{N}_+ \langle \xi | \mathcal{F}_{\rho_2}^G \mathcal{F}_{\rho_1}^G \mathcal{A}^+(v) \mathcal{F}_{\rho_2}^G \mathcal{F}_{\rho_1}^G | \xi \rangle, \quad (\text{S43})$$

where $\mathcal{N}_+ = 1/2$ includes the normalisation associated with the composite $+$ projection and the scaled implementation of the ribbon operators. This is the non-unitary expectation value reconstructed by the adapted Hadamard test described in the Methods subsection “Experimental extraction of the braiding phases” of the main text.

The G -sector phase is obtained from the separately measured identity-channel overlap rather than from an additional controlled G -projection circuit. With the normalisation conventions of the protocol,

$$(R^G)^2 = (R^1)^2 - (R^+)^2, \quad (\text{S44})$$

in agreement with Eq. (5) of the main text. Thus, the two reduced measurements $(R^+)^2$ and $(R^1)^2$ recover all three squared braiding phases in the physical $\{A, B, G\}$ basis.

Squared fusion amplitudes from qutrit measurements

We now show how the experimentally accessible two-qutrit overlaps determine the squared fusion amplitudes in the physical anyon basis $\{A, B, G\}$. We write

$$F \equiv F_{GGG}^G, \quad \Phi_{ij} \equiv |F_{ij}|^2, \quad (\text{S45})$$

with $i, j \in \{A, B, G\}$. The four-qudit plaquette construction relating overlaps between alternative fusion histories to the corresponding F -symbols was derived in Ref. [1]. We denote the unnormalised overlap associated with intermediate fusion channels i and j by

$$\tilde{m}_{ji} = \langle \eta | O_{m_{ji}} | \eta \rangle, \quad (\text{S46})$$

with $i, j \in \{A, B, G\}$, where

$$O_{m_{ji}} = F_{\rho_2}^G F_{\rho_2}^G A^j(v) F_{\rho_1}^G A^G(v) F_{\rho_2}^G A^i(v) F_{\rho_2}^G F_{\rho_1}^G. \quad (\text{S47})$$

The corresponding normalisation of the j -resolved fusion history is

$$\tilde{n}_j = \langle \eta | O_{n_j} | \eta \rangle, \quad (\text{S48})$$

with $j \in \{A, B, G\}$ and

$$O_{n_j} = F_{\rho_2}^G F_{\rho_2}^G A^j(v) F_{\rho_1}^G A^G(v) F_{\rho_1}^G A^j(v) F_{\rho_2}^G F_{\rho_2}^G. \quad (\text{S49})$$

The resulting overlap ratio is related to the fusion matrix element by

$$(F_{ij})^2 = \left(R_G^{Gj}\right)^* \frac{\tilde{m}_{ji}}{\tilde{n}_j}. \quad (\text{S50})$$

Here

$$R_G^{Gj} = \begin{cases} 1, & j = A, B, \\ \omega, & j = G, \end{cases} \quad \omega = e^{2\pi i/3}, \quad (\text{S51})$$

is the known phase generated by the crossing of the ribbon histories. Removing this phase and taking the modulus gives the squared fusion amplitude $\Phi_{ij} = |F_{ij}|^2$.

We next apply the dense reduction of Supplementary Note 1. Introducing the combined charge projection $A^+(v) = A^A(v) + A^B(v)$, the experimentally accessible sectors are $i, j \in \{+, G\}$. For the overlap quantities, the four-qudit expectation values reduce to

$$\tilde{m}_{ji} = \langle \eta | \mathcal{O}_{m_{ji}} | \eta \rangle = \frac{1}{4} \langle \xi | \mathcal{O}_{m_{ji}} | \xi \rangle \equiv \frac{1}{4} m_{ji}, \quad (\text{S52})$$

with $i, j \in \{+, G\}$, where $\mathcal{O}_{m_{ji}}$ denotes the corresponding product of reduced two-qudit ribbon and projection operators. Similarly,

$$\tilde{n}_j = \langle \eta | \mathcal{O}_{n_j} | \eta \rangle = \frac{1}{4} \langle \xi | \mathcal{O}_{n_j} | \xi \rangle \equiv \frac{1}{4} n_j, \quad (\text{S53})$$

with $j \in \{+, G\}$. The reduced quantities m_{++} , m_{+G} , m_{G+} , m_{GG} , n_+ and n_G can therefore be evaluated using the two-qudit product state $|\xi\rangle$. The common factor of $1/4$ cancels from every normalised overlap,

$$\frac{\tilde{m}_{ji}}{\tilde{n}_j} = \frac{m_{ji}}{n_j}. \quad (\text{S54})$$

It is therefore sufficient in what follows to work with the reduced two-qudit quantities m_{ji} and n_j .

We parameterise the F -symbol in the real symmetric gauge used throughout this work as

$$F = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_2 & f_4 & f_5 \\ f_3 & f_5 & f_6 \end{pmatrix}, \quad (\text{S55})$$

where the rows and columns are ordered as A, B, G . The corresponding matrix of elementwise squared magnitudes is

$$\Phi = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_2 & \phi_4 & \phi_5 \\ \phi_3 & \phi_5 & \phi_6 \end{pmatrix}, \quad \phi_k \equiv |f_k|^2. \quad (\text{S56})$$

Since the chosen gauge is real, $\phi_k = f_k^2$, although the squared-magnitude notation makes explicit the quantities reconstructed by the protocol.

Expanding the experimentally accessible $+$ - and G -sector overlaps using Eq. (S50) gives

$$\phi_6 = \frac{m_{GG}}{\omega n_G}, \quad (\text{S57})$$

together with

$$m_{++} = n_A(\phi_1 + \phi_2) + n_B(\phi_2 + \phi_4), \quad (\text{S58})$$

$$m_{+G} = n_A\phi_3 + n_B\phi_5, \quad (\text{S59})$$

$$m_{G+} = \omega n_G(\phi_3 + \phi_5). \quad (\text{S60})$$

To expose the information contained in the unresolved Abelian sector, we complete the experimentally accessible $\{+, G\}$ sectors by introducing the orthonormal combinations

$$|+\rangle = \frac{|A\rangle + |B\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|A\rangle - |B\rangle}{\sqrt{2}}. \quad (\text{S61})$$

Correspondingly, we define

$$A^\pm(v) = A^A(v) \pm A^B(v). \quad (\text{S62})$$

Here $A^+(v)$ is the combined Abelian-sector operation used in the experiment, whereas $A^-(v)$ is an auxiliary signed combination introduced only to analyse the unresolved A and B channels.

To determine the mixed $+$ and $-$ overlaps, define

$$O_{1,i} = A^G(v) F_{\rho_2}^G A^i(v) F_{\rho_2}^G F_{\rho_1}^G, \quad (\text{S63})$$

$$O_{2,j} = A^G(v) F_{\rho_1}^G A^j(v) F_{\rho_2}^G F_{\rho_2}^G, \quad (\text{S64})$$

so that

$$O_{m_{ji}} = O_{2,j}^\dagger O_{1,i}. \quad (\text{S65})$$

Under the decomposition $\mathcal{H}_{S_3} = \mathcal{H}_{\mathbb{Z}_3} \oplus \mathcal{H}_{t\mathbb{Z}_3}$, direct expansion shows that $O_{1,-}$ and $O_{2,+}$ have support on orthogonal blocks, as do $O_{1,+}$ and $O_{2,-}$. Consequently,

$$m_{+-} = m_{-+} = 0. \quad (\text{S66})$$

The corresponding mixed normalisation overlap also vanishes,

$$n_{+-} = 4 \langle \eta | O_{2,+}^\dagger O_{2,-} | \eta \rangle, \quad (\text{S67})$$

$$= n_A - n_B = 0, \quad (\text{S68})$$

and hence

$$n_A = n_B = \frac{n_+}{2}. \quad (\text{S69})$$

In terms of the entries of Φ , the two mixed overlaps are

$$m_{+-} = n_A(\phi_1 - \phi_2) + n_B(\phi_2 - \phi_4) = 0, \quad (\text{S70})$$

$$m_{-+} = n_A(\phi_1 + \phi_2) - n_B(\phi_2 + \phi_4) = 0. \quad (\text{S71})$$

Using Eq. (S69), either relation gives

$$\phi_1 = \phi_4. \quad (\text{S72})$$

The additional equality $\phi_2 = \phi_1$ is the repeated A – B block of the exact $D(S_3)$ F -symbol in the chosen gauge. Using this model-specific property, the Abelian block therefore satisfies

$$\phi_1 = \phi_2 = \phi_4. \quad (\text{S73})$$

Equation (S58) then reduces to

$$m_{++} = 2n_+\phi_1, \quad (\text{S74})$$

and hence

$$\phi_1 = \frac{m_{++}}{2n_+}. \quad (\text{S75})$$

Unitarity of F implies that the squared magnitudes in each row sum to unity. The first two rows of Eq. (S56) therefore satisfy

$$\phi_1 + \phi_2 + \phi_3 = \phi_2 + \phi_4 + \phi_5 = 1. \quad (\text{S76})$$

Together with Eq. (S73), this gives

$$\phi_3 = \phi_5. \quad (\text{S77})$$

The two equations (S59) and (S60) provide independent estimates of the squared amplitudes,

$$\phi_3^{(+G)} = \frac{m_{+G}}{n_+}, \quad \phi_3^{(G+)} = \frac{m_{G+}}{2\omega n_G}. \quad (\text{S78})$$

As they are obtained from distinct circuit realisations and may differ because of finite sampling we impose the symmetrisation

$$\phi_3 = \frac{1}{2} [\phi_3^{(+G)} + \phi_3^{(G+)}] = \frac{1}{2} \left(\frac{m_{+G}}{n_+} + \frac{m_{G+}}{2\omega n_G} \right). \quad (\text{S79})$$

The squared fusion-amplitude matrix can therefore be written in terms of three independent quantities as

$$\Phi = \begin{pmatrix} a & a & b \\ a & a & b \\ b & b & c \end{pmatrix}, \quad (\text{S80})$$

with $a = \phi_1$, $b = \phi_3$, $c = \phi_6$, where

$$a = \frac{m_{++}}{2n_+}, \quad b = \frac{1}{2} \left(\frac{m_{+G}}{n_+} + \frac{m_{G+}}{2\omega n_G} \right), \quad c = \frac{m_{GG}}{\omega n_G}. \quad (\text{S81})$$

Thus, the six reduced quantities m_{++} , m_{+G} , m_{G+} , m_{GG} , n_+ and n_G determine the three independent squared amplitudes a , b and c .

The matrix Φ determines only the magnitudes of the entries of F . Since the fusion spaces V_{GG}^i are one-dimensional for $i \in \{A, B, G\}$, a change of vertex gauge amounts to rephasing each trivalent vertex $G \times G \rightarrow i$. Restricting to gauges in which F remains real, these rephasings reduce to independent signs $s_i = \pm 1$. Introducing

$$D_s = \text{diag}(s_A, s_B, s_G), \quad (\text{S82})$$

the real symmetric representatives compatible with a reference matrix F_0 can be written as

$$F_{s,\epsilon} = \epsilon D_s F_0 D_s, \quad \epsilon = \pm 1. \quad (\text{S83})$$

The diagonal matrices implement the corresponding sign changes of the fusion-channel basis on the two fusion trees, while ϵ represents their relative overall sign. Since D_s and $-D_s$ generate the same transformation $D_s F_0 D_s$, the three vertex signs produce four distinct channel-sign patterns. Including the overall sign ϵ gives eight admissible sign assignments, grouped into four pairs that differ only by an overall sign. These matrices are related by fusion-vertex gauge transformations and therefore represent gauge-equivalent real forms of the same $D(S_3)$ F -symbol. We apply the corresponding gauge-related sign patterns to the experimentally reconstructed magnitudes, although finite sampling means that the resulting matrices are not exactly unitary.

Supplementary Note 3: Circuit design for encoded qutrit operations

We now describe the circuit decompositions used to implement the reduced two-qutrit protocol in the three-qubit encoding introduced in Eq. (S90). The required operations comprise the qutrit Fourier transform, preparation of the product state $|\xi\rangle$, and the ancilla-assisted implementations of the charge projections and ribbon operators. Unitary qutrit transformations are compiled into single-qubit rotations and two-qubit Givens rotations that preserve the unit-Hamming-weight subspace, whereas the non-unitary primitives are realised through unitary dilations followed by heralding on ancillary outcomes.

Qutrit Fourier transform

In the logical basis $\{|e\rangle, |c\rangle, |c^2\rangle\}$, the qutrit Fourier transform is

$$U_Q = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \omega = e^{2\pi i/3}. \quad (\text{S84})$$

To implement U_Q in the three-qubit encoding, we decompose it using the Reck construction [7], which expresses a general $U(3)$ transformation as a sequence of $U(2)$ rotations acting on pairs of logical basis states. Within the unit-Hamming-weight encoding, each such rotation is realised by a two-qubit gate acting on the subspace spanned by $|01\rangle$ and $|10\rangle$, as described by the Givens rotation in Eq. (S86).

The resulting decomposition is shown in Fig. 1. It uses two types of two-qubit operation. The first is the phase gate

$$U_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{S85})$$

written in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The second is the Givens rotation

$$G(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{S86})$$

which mixes the single-excitation states $|01\rangle$ and $|10\rangle$ while leaving $|00\rangle$ and $|11\rangle$ unchanged. Consequently, both operations preserve the encoded qutrit subspace.

Combining these gates with single-qubit R_z rotations implements U_Q exactly on the logical qutrit. The decomposition uses three Givens rotations with

$$\theta_1 = \frac{\pi}{4}, \quad \theta_2 = \arccos\left(\frac{1}{\sqrt{3}}\right), \quad (\text{S87})$$

together with the phase gate U_ϕ and the indicated single-qubit rotations. After compilation for the H2 processors, each encoded qutrit Fourier transform contains 17 single-qubit and 7 two-qubit gates.

Initial-state preparation

The dense reduction of Supplementary Note 1 expresses all braiding and fusion observables required by the protocol as expectation values on the two-qutrit product state

$$|\xi\rangle = \frac{1}{3} \sum_{g_1, g_2 \in \mathbb{Z}_3} |g_1, g_2\rangle. \quad (\text{S88})$$

Since the state factorises into identical equal superpositions on the two qutrits, it can be prepared as

$$|\xi\rangle = (U_Q \otimes U_Q) |e, e\rangle, \quad (\text{S89})$$

where U_Q is the qutrit Fourier transform defined in Eq. (S84). In the three-qubit encoding

$$|e\rangle \equiv |100\rangle, \quad |c\rangle \equiv |010\rangle, \quad |c^2\rangle \equiv |001\rangle, \quad (\text{S90})$$

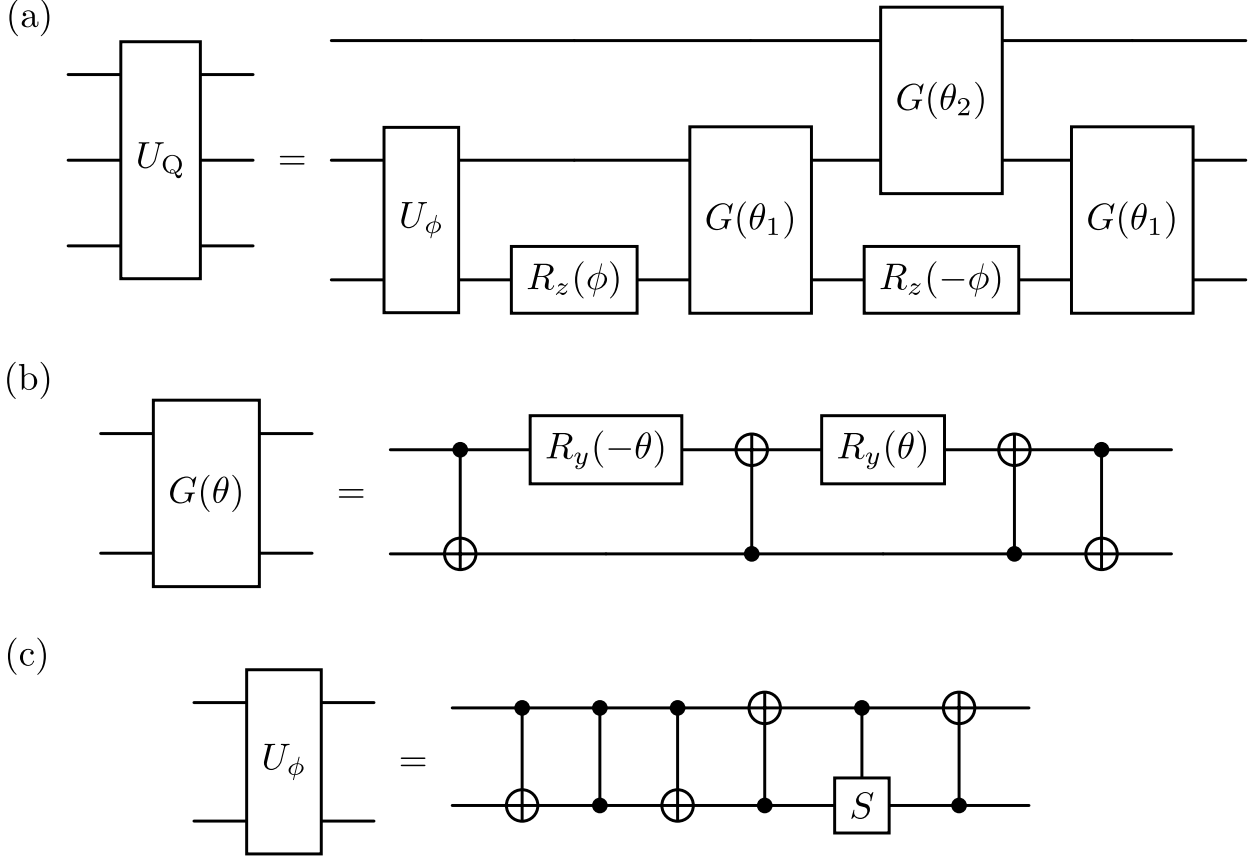
the single-qutrit equal superposition is the three-qubit W state,

$$|W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle). \quad (\text{S91})$$

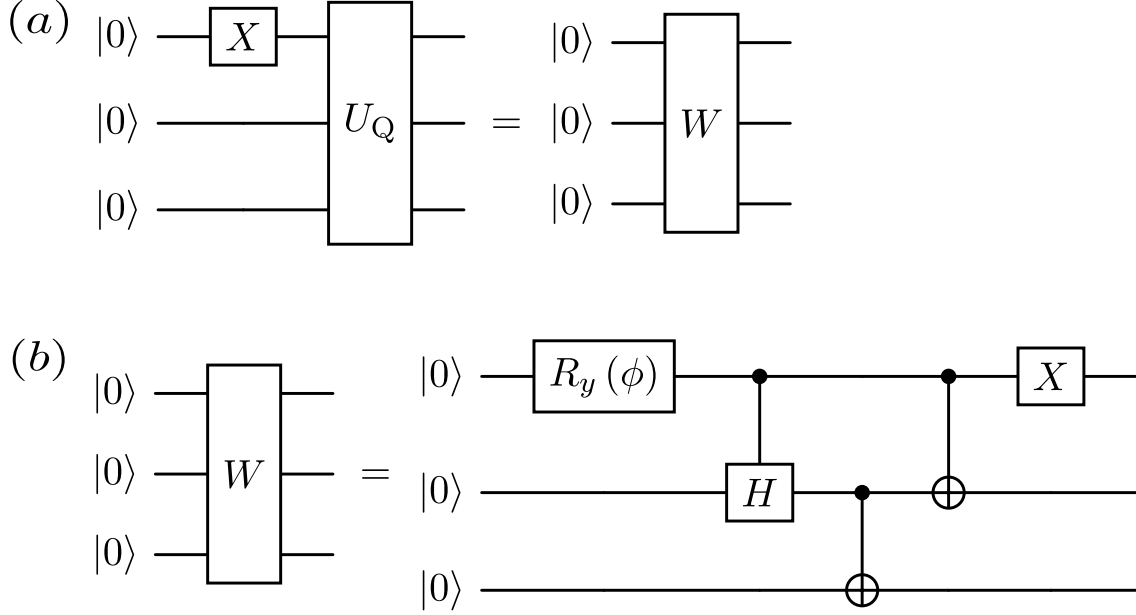
Consequently,

$$|\xi\rangle = |W\rangle \otimes |W\rangle. \quad (\text{S92})$$

Rather than implementing the complete encoded Fourier transform solely for state preparation, we compile a shorter circuit that prepares $|W\rangle$ directly from $|000\rangle$. As shown in Fig. 2, this circuit has the same action on the specified input as first preparing $|e\rangle = |100\rangle$ with an X gate and then applying U_Q . After compilation for the H2 processors, each three-qubit W -state preparation requires four single-qubit and three two-qubit gates. Preparing the full state $|\xi\rangle$ therefore uses two identical W circuits acting independently on the two encoded qutrits.



Supplementary Figure 1: **Encoded qutrit Fourier transform.** (a) Decomposition of the qutrit Fourier transform U_Q in the three-qubit unit-Hamming-weight encoding. The circuit consists of single-qubit rotations $R_z(\theta)$, three two-qubit Givens rotations $G(\theta)$ and the phase gate U_ϕ , obtained using the Reck decomposition. The rotation angles are $\theta_1 = \pi/4$ and $\theta_2 = \arccos(1/\sqrt{3})$. (b) Two-qubit Givens rotation $G(\theta)$ acting on the single-excitation subspace. (c) Two-qubit phase gate U_ϕ .



Supplementary Figure 2: **Preparation of the two-qutrit input state.** (a) On the input $|000\rangle$, preparing $|e\rangle = |100\rangle$ with an X gate and subsequently applying the encoded qutrit Fourier transform U_Q produces the three-qubit W state. This state-specific transformation can be replaced by the shorter direct preparation denoted by W . (b) Circuit decomposition of the direct W -state preparation used in the experiment. Applying this circuit independently to the two three-qubit registers prepares $|\xi\rangle = |W\rangle \otimes |W\rangle$.

Charge-projection operators

We next describe the ancilla-assisted implementation of the diagonal projectors P^+ and $P^G = \mathbf{1}_9 - P^+$. They are related to the reduced charge projections by

$$\mathcal{A}^+(v) = (U_Q \otimes U_Q) P^+ (U_Q^\dagger \otimes U_Q^\dagger), \quad (\text{S93})$$

$$\mathcal{A}^G(v) = (U_Q \otimes U_Q) P^G (U_Q^\dagger \otimes U_Q^\dagger). \quad (\text{S94})$$

In the three-qubit unit-Hamming-weight encoding of Eq. (S90), two logical qutrits are equal if and only if their three corresponding physical qubits agree pairwise.

The elementary parity-check circuit Π is shown in Fig. 3(a). For two computational qubits $|x, y\rangle$ and an ancilla initialised in $|0\rangle_A$, it performs

$$|x, y\rangle |0\rangle_A \mapsto |x, y\rangle |x \oplus y\rangle_A. \quad (\text{S95})$$

Thus, the ancilla records $|0\rangle_A$ when the two qubits have equal occupations and $|1\rangle_A$ when they differ.

Applying Π to each pair of corresponding physical qubits produces a three-bit parity syndrome. Within the logical qutrit code space, equal qutrits give

$$000, \quad (\text{S96})$$

whereas unequal qutrits give one of the three syndromes

$$110, \quad 101, \quad 011. \quad (\text{S97})$$

The syndrome 000 therefore identifies the equal-qutrit subspace and implements the diagonal projector P^+ . Conjugating this operation by the qutrit Fourier transforms gives the charge-sector

operation $\mathcal{A}^+(v)$, as in Eq. (S93). The probability of obtaining the 000 syndrome is the weight of the input state in the equal-qutrit subspace. After compilation for the H2 processors, this implementation uses six computational qubits and three parity ancillas, together with 66 single-qubit and 34 two-qubit gates.

A direct measurement of the three parity ancillas would distinguish the syndromes in Eq. (S97). This would resolve more information than the projector $P^G = \mathbf{1}_9 - P^+$ and would therefore destroy coherence between the three components of the unequal subspace. Such a measurement cannot be used when the charge projection appears coherently inside a longer ribbon sequence.

To implement P^G without revealing which unequal syndrome occurred, we reversibly compress the parity information into a two-qubit sector register. A Toffoli-based Boolean circuit maps the syndrome 000 to the flag $|00\rangle_T$ and all three allowed unequal syndromes to the common flag $|11\rangle_T$. The original parity register is then uncomputed by applying the inverse parity checks. On logical basis states, the complete coherent operation therefore acts as

$$\begin{aligned} & |g_1, g_2\rangle |000\rangle_A |00\rangle_T \\ & \mapsto \begin{cases} |g_1, g_2\rangle |000\rangle_A |00\rangle_T, & g_1 = g_2, \\ |g_1, g_2\rangle |000\rangle_A |11\rangle_T, & g_1 \neq g_2. \end{cases} \end{aligned} \quad (\text{S98})$$

Here A denotes the three parity ancillas and T the two sector-flag ancillas. Because the parity register is returned to $|000\rangle_A$, the three unequal syndromes remain coherent and only the distinction between the $+$ and G sectors is retained.

After conjugation by the Fourier transforms, post-selection on $|00\rangle_T$ implements $\mathcal{A}^+(v)$, while post-selection on $|11\rangle_T$ implements $\mathcal{A}^G(v)$. The full coherent construction uses six computational qubits and five ancillas. After compilation for H2, it contains 84 single-qubit and 52 two-qubit gates. The rearrangement of qubit lines shown in the circuit is purely diagrammatic: no SWAP gates are required because the trapped-ion processor provides all-to-all connectivity.

Ribbon operators

The reduced ribbon operators are implemented by diagonalising them with the qutrit Fourier transform. Since the resulting diagonal operator $\mathcal{D}_F$ is non-unitary, the hardware circuit realises the scaled map $\mathcal{D}_F/2$ by embedding it in a unitary dilation with ancillary qubits.

We first introduce the elementary two-qubit filter $d_F(\theta)$ shown in Fig. 4(a). For two logical qubits and an ancilla initialised in $|0\rangle_A$, the dilation acts as

$$U_{d_F}(\theta) |\psi\rangle |0\rangle_A = d_F(\theta) |\psi\rangle |0\rangle_A + |\Phi^\perp\rangle |1\rangle_A, \quad (\text{S99})$$

where

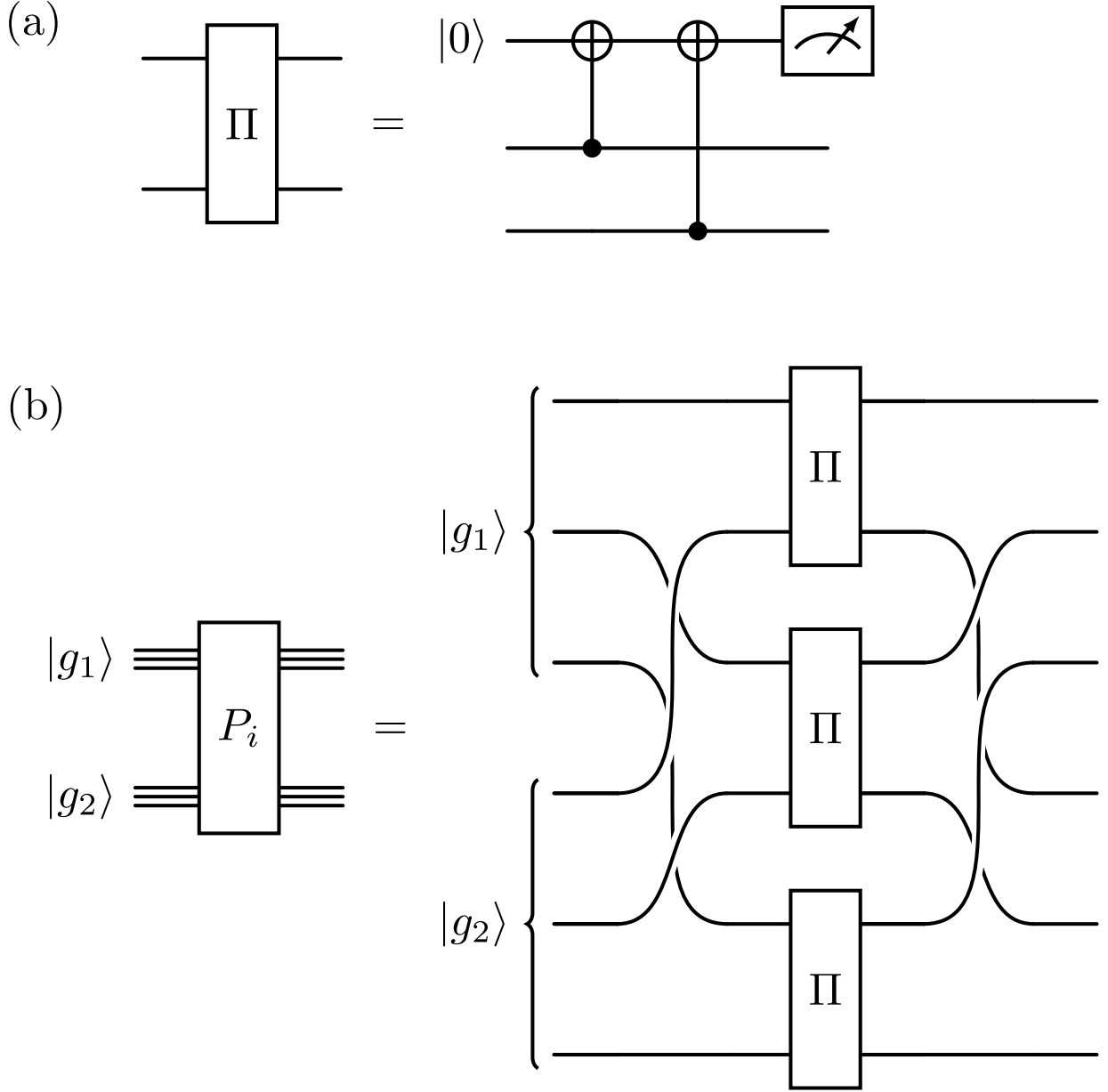
$$d_F(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{S100})$$

is written in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The successful branch is identified by the ancilla state $|0\rangle_A$, on which only the $|01\rangle$ component is multiplied by $\cos \theta$.

We choose

$$\theta = \frac{2\pi}{3}, \quad \cos \theta = -\frac{1}{2}. \quad (\text{S101})$$

The diagonal two-qutrit operation is then constructed by applying one such filter to each pair of corresponding physical qubits in the two encoded qutrits, as shown in Fig. 4(b). If the logical qutrit labels are equal, $g_1 = g_2$, the three physical-qubit pairs consist of one $|11\rangle$ pair and two $|00\rangle$ pairs. No filter therefore introduces an attenuation, and the state is unchanged. If $g_1 \neq g_2$, the three pairs consist of one $|10\rangle$ pair, one $|01\rangle$ pair and one $|00\rangle$ pair. Exactly one filter then contributes the



Supplementary Figure 3: **Ancilla-assisted charge projection.** (a) Parity-check circuit Π . The ancilla records $|0\rangle_A$ when the two computational qubits have equal occupations and $|1\rangle_A$ when they differ. (b) Three parity checks compare the corresponding physical qubits of the two encoded qutrits. Within the unit-Hamming-weight code space, equal qutrits produce the syndrome 000, whereas unequal qutrits produce 110, 101 or 011. In the coherent implementation, these three unequal syndromes are mapped to the common sector flag $|11\rangle_T$, while the equal syndrome is mapped to $|00\rangle_T$. The parity ancillas are subsequently uncomputed. Post-selection on the two flag outcomes therefore implements the $\pm$ - and G -sector charge projections without resolving the individual unequal-qutrit configurations. The reordering of the circuit lines is diagrammatic and does not require physical SWAP gates on the all-to-all connected H2 processors.

factor $-1/2$. Conditioning on all three operation ancillas occupying $|0\rangle_A$ consequently implements

$$\frac{1}{2}\mathcal{D}_F|g_1, g_2\rangle = \begin{cases} |g_1, g_2\rangle, & g_1 = g_2, \\ -\frac{1}{2}|g_1, g_2\rangle, & g_1 \neq g_2, \end{cases} \quad (\text{S102})$$

which is the scaled diagonal action required for the ribbon construction.

Conjugating this diagonal filter by the appropriate qutrit Fourier transforms gives scaled versions of the two reduced ribbon operators:

$$\frac{1}{2}\mathcal{F}_{\rho_1}^G = (\mathbf{1}_3 \otimes U_Q) \frac{\mathcal{D}_F}{2} (\mathbf{1}_3 \otimes U_Q^\dagger), \quad (\text{S103})$$

$$\frac{1}{2}\mathcal{F}_{\rho_2}^G = (U_Q^\dagger \otimes \mathbf{1}_3) \frac{\mathcal{D}_F}{2} (U_Q \otimes \mathbf{1}_3). \quad (\text{S104})$$

The circuits therefore implement $\mathcal{F}_{\rho_k}^G/2$, rather than the unscaled ribbon operator itself. The known factor of two is restored analytically through the normalization factors used when reconstructing the braiding and fusion overlaps.

Within the logical two-qutrit subspace, the probability of obtaining the successful ancilla branch is

$$p_{\text{succ}} = P_{\text{eq}} + \frac{1}{4}P_{\text{neq}}, \quad (\text{S105})$$

where P_{eq} and P_{neq} are the weights of the input state in the equal- and unequal-qutrit subspaces. Hence $1/4 \leq p_{\text{succ}} \leq 1$. In the fusion circuits, success is selected by measuring and post-selecting the operation ancillas. In the braiding Hadamard test, the same ancillas remain part of the coherent unitary dilation and are not measured.

Each ribbon primitive uses six computational qubits and three operation ancillas. After compilation for the H2 processors, $\mathcal{F}_{\rho_1}^G/2$ contains 55 single-qubit and 29 two-qubit gates, whereas $\mathcal{F}_{\rho_2}^G/2$ contains 61 single-qubit and 29 two-qubit gates.

Hadamard test

In the Methods subsection ‘‘Experimental extraction of the braiding phases’’ of the main text, the squared braiding phases were reconstructed from expectation values of the non-unitary two-qutrit maps $\mathcal{M}_+$ and $\mathcal{M}_1$. The map $\mathcal{M}_+$ contains the charge projection $\mathcal{A}^+(v)$, whereas $\mathcal{M}_1$ is obtained by replacing this projection with the identity. We denote their corresponding unitary dilations by M_+ and M_1 , respectively. The identity-channel quantity is an auxiliary overlap rather than a physical anyon braiding phase; it is used to obtain the G -sector result through

$$(R^G)^2 = (R^1)^2 - (R^+)^2, \quad (\text{S106})$$

as in Eq. (5) of the main text. Here we describe the state-specific controlled circuits used to perform the Hadamard tests for these dilations.

For a unitary operator M and an input state $|\Psi\rangle$, the standard Hadamard test is shown on the left-hand side of Fig. 5(a). After preparing the Hadamard ancilla in $(|0\rangle_H + |1\rangle_H)/\sqrt{2}$ and applying M conditional on $|1\rangle_H$, the joint state is

$$\frac{1}{\sqrt{2}} (|0\rangle_H |\Psi\rangle + |1\rangle_H M |\Psi\rangle). \quad (\text{S107})$$

Applying a final Hadamard gate and measuring the ancilla gives

$$p_H^{\text{Re}}(0) - p_H^{\text{Re}}(1) = \text{Re}(\langle \Psi | M | \Psi \rangle). \quad (\text{S108})$$

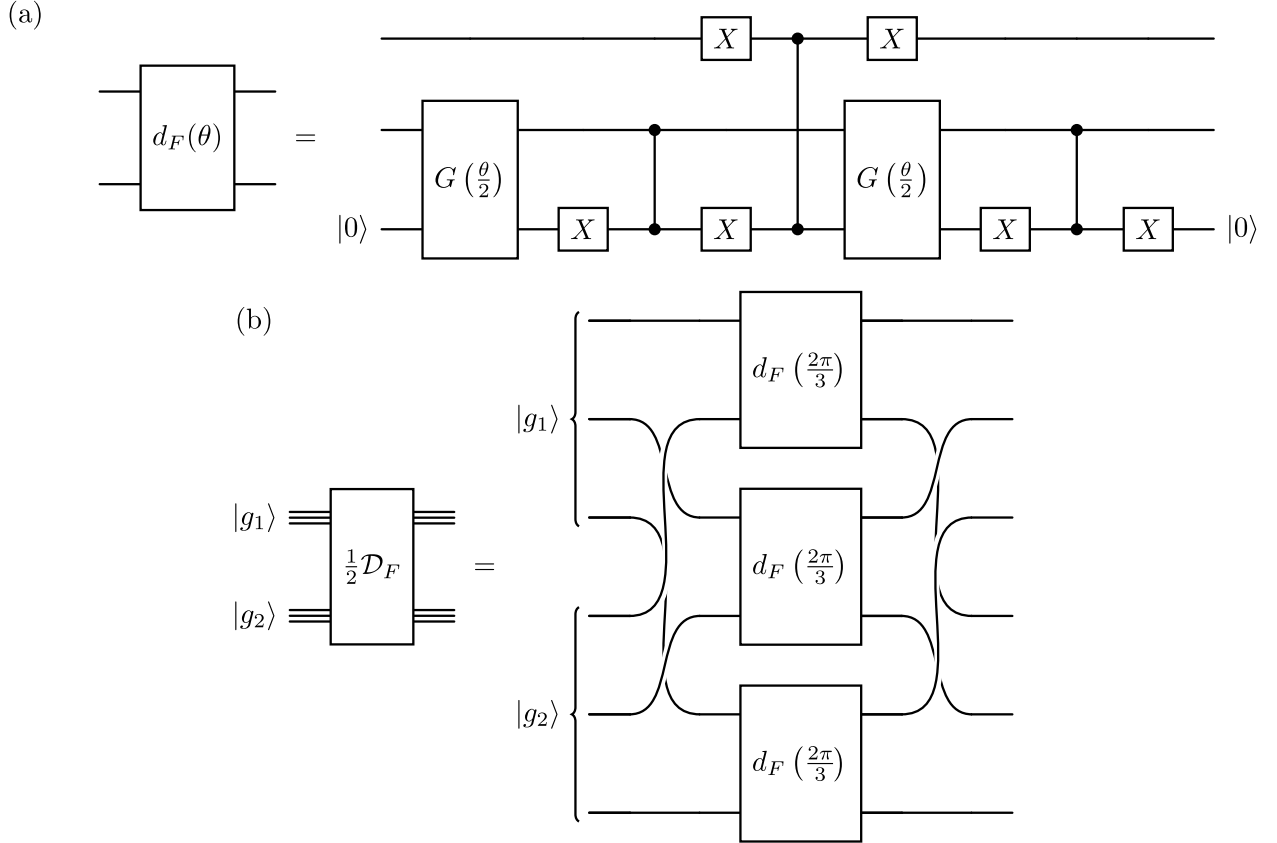
Inserting an $S^\dagger$ gate immediately before the final Hadamard gives the imaginary quadrature,

$$p_H^{\text{Im}}(0) - p_H^{\text{Im}}(1) = \text{Im}(\langle \Psi | M | \Psi \rangle), \quad (\text{S109})$$

with the phase convention used for the braiding reconstruction in the main text.

The unitary dilations contain known scale factors arising from the implementation of the non-unitary ribbon and projection operators. Accounting for these factors gives

$$\begin{aligned} \text{Re}[(R^+)^2] &= 8 [p_H^{\text{Re}}(0) - p_H^{\text{Re}}(1)], \\ \text{Im}[(R^+)^2] &= 8 [p_H^{\text{Im}}(0) - p_H^{\text{Im}}(1)], \end{aligned} \quad (\text{S110})$$



Supplementary Figure 4: **Ancilla-assisted implementation of the ribbon filter.** (a) Unitary dilation of the elementary two-qubit filter $d_F(\theta)$. On the branch in which the ancilla returns to $|0\rangle_A$, the logical state $|01\rangle$ is multiplied by $\cos \theta$, while the other computational-basis states are unchanged. The gates $G(\theta/2)$ are the Givens rotations defined in Eq. (S86). (b) Implementation of the scaled diagonal ribbon operator $\mathcal{D}_F/2$. Three filters $d_F(2\pi/3)$ act on corresponding pairs of physical qubits in the two encoded qutrits. Equal qutrit labels are unchanged, whereas unequal labels acquire the factor $-1/2$. Successful operation is associated with all three ancillas occupying $|0\rangle_A$. The rearrangement of the circuit lines is diagrammatic and requires no SWAP gates on the all-to-all connected H2 processors.

equivalent to Eq. (4) of the main text because $p_H(0)+p_H(1) = 1$. The same normalization convention is used for the identity-channel overlap $(R^1)^2$.

For the $+$ -sector circuit, the input to the dilation is

$$|\Psi_+\rangle = |100, 100\rangle |0\rangle_A^{\otimes 15}, \quad (\text{S111})$$

where the first six qubits encode the two qutrits and the remaining 15 qubits are operation ancillas. A direct implementation of controlled- M_+ would require promoting every constituent gate of the dilation to a controlled operation. Since M_+ contains several qutrit Fourier transforms, ribbon filters and charge-projection blocks, this would produce a substantial two-qubit-gate overhead.

A general controlled implementation is unnecessary because the Hadamard test is applied only to the fixed input state in Eq. (S111). We instead construct a state-specific controlled circuit $\widetilde{M}_+$ satisfying

$$\widetilde{M}_+ \frac{|0\rangle_H |\Psi_+\rangle + |1\rangle_H |\Psi_+\rangle}{\sqrt{2}} = \frac{|0\rangle_H |\Psi_+\rangle + |1\rangle_H M_+ |\Psi_+\rangle}{\sqrt{2}}. \quad (\text{S112})$$

Thus, the $|1\rangle_H$ branch undergoes the complete dilation M_+ , whereas the $|0\rangle_H$ branch returns to the original state $|\Psi_+\rangle$. Equation (S112) establishes equivalence only for this specified input; $\widetilde{M}_+$ need not coincide with a general controlled- M_+ operation on arbitrary states.

The optimized construction is shown on the right-hand side of Fig. 5(a). Operations that already act as the identity on the input state $|100\rangle|100\rangle$ are left uncontrolled, and only the state-preparation and basis-change gates needed to distinguish the two Hadamard branches are conditioned on the Hadamard ancilla. In particular, the diagonal ribbon blocks D_F , the charge block D_+ and the indicated permutations are applied without an additional Hadamard control. Controlled operations are shown in red in the figure. In the decomposition used here, the dilation M_+ contains 311 gates, but only 29 operations need to be conditioned on the Hadamard ancilla.

The controlled state-preparation and Fourier-transform circuits are given in Fig. 5(b). Panel (b)(i) shows the controlled- W state-preparation circuit, while panel (b)(ii) shows the controlled qutrit Fourier transform U_Q . The latter is expressed using the phase gate U_ϕ and the three instances G_1 , G_2 and G_3 of the Givens rotation defined in Eqs. (S85) and (S86).

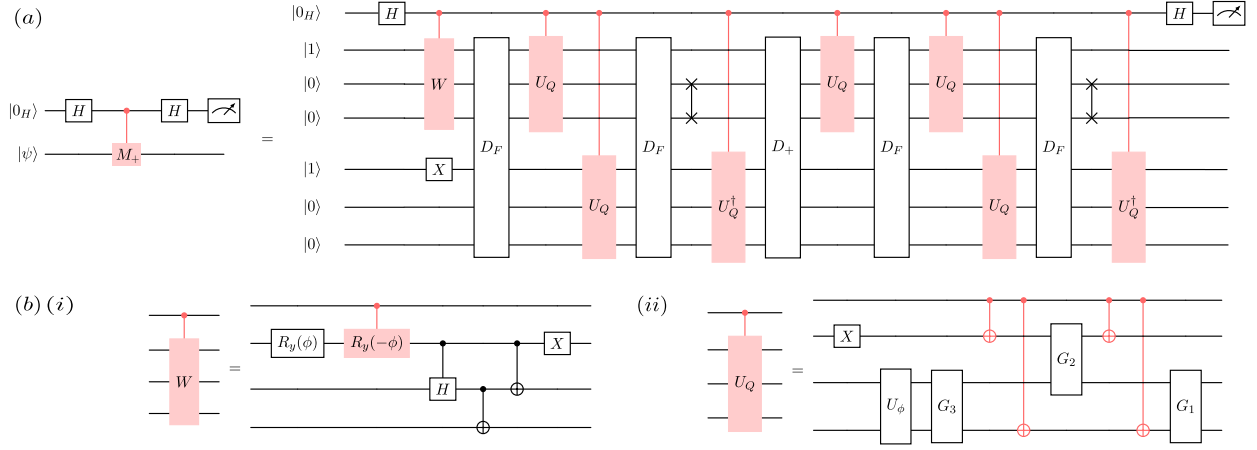
The same state-specific strategy is used for the identity-channel dilation M_1 . In this case the D_+ block is omitted and the input is

$$|\Psi_1\rangle = |100, 100\rangle |0\rangle_A^{\otimes 12}. \quad (\text{S113})$$

The operation ancillas form part of the coherent unitary dilations and are neither measured nor post-selected in the Hadamard-test experiments. Only the Hadamard ancilla is used in the reconstruction of the real and imaginary quadratures. Consequently, every experimental shot contributes to the measured expectation value.

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Supplementary Figure 5: **State-specific controlled Hadamard test.** (a) Left: standard Hadamard test for extracting $\text{Re}[\langle\Psi|M|\Psi\rangle]$ from the Hadamard-ancilla probabilities. Right: optimized implementation of the $+$ -sector dilation M_+ for the fixed input $|\Psi_+\rangle = |100, 100\rangle|0\rangle_A^{\otimes 15}$. Gates shown in red are conditioned on the Hadamard ancilla, whereas the diagonal ribbon blocks D_F , the charge block D_+ and the indicated permutations are applied unconditionally. The two circuits produce the same Hadamard-ancilla statistics for the specified input, but need not define the same controlled unitary on arbitrary states. Ancillas internal to the D_F and D_+ blocks are not shown. (b) Decompositions of the controlled operations used in panel (a): (i) controlled- W state preparation and (ii) controlled qutrit Fourier transform U_Q , implemented using the phase gate U_ϕ and Givens rotations G_1 , G_2 and G_3 .

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