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Adaptive Lagrangian Attention for Constrained Multimodal Multi-objective Optimization

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Abstract

Recently, constrained multimodal multi-objective optimization problems (CMMOPs) have attracted increasing attention due to their applications in engineering optimization, resource scheduling, and decision-making. However, they are challenging due to conflicting objectives, complex constraints, and multimodal solution structures, making it difficult for existing evolutionary algorithms to balance feasibility, diversity, and convergence, especially under uneven constraint distributions. To address these issues, this paper proposes an adaptive attention-driven Lagrangian relaxation evolutionary algorithm (AALR-CMMOEA). A dual-population co-evolution framework is developed to separately enhance exploration and convergence. A Lagrangian relaxation mechanism is introduced to dynamically adjust constraint pressure based on feasibility and constraint violation, improving exploration in infeasible regions. An adaptive attention strategy is further designed to allocate computational resources to promising and sparse modal regions, together with a two-stage optimization scheme to balance exploration and exploitation. Experimental results on CMMOP benchmark problems show that the proposed method achieves competitive performance in IGD, HV, and IGDp, and effectively improves both constrained Pareto set discovery and solution quality. The results confirm the effectiveness and robustness of the proposed approach.

Keywords : Constrained multimodal multi-objective optimization; Lagrangian relaxation; Dual-population co-evolution; Constraint relaxation

1. Introduction

Driven by the advancements in intelligent decision-making systems and intricate engineering processes, constrained multi-objective optimization problems (CMOPs) are increasingly utilized across diverse domains, including path planning[1], resource allocation[2], and financial portfolio optimization[3]. In practical scenarios, optimization tasks commonly necessitate the simultaneous reconciliation of trade-off objectives and the fulfillment of intricate constraints. This dual requirement imposes stringent performance criteria on the feasibility exploration and convergence speed of optimization algorithms. In general terms, a standard CMOP can be mathematically formulated as follows[4]:

$$\begin{aligned} & \text{Minimize } F(x) = \{f_1(x), f_2(x), \dots, f_M(x)\} \\ & \text{subject to} \\ & \quad g_i(x) \leq 0, \quad i = 1, \dots, p \\ & \quad h_j(x) = 0, \quad j = p + 1, \dots, q. \end{aligned} \tag{1}$$

In this formulation, $F(x)$ represents the vector of M trade-off objective functions targeted for minimization. Meanwhile, $g(x)$ and $h(x)$ represent the sets of inequality and equality constraint functions, respectively.

For constrained multi-objective optimization paradigms, the extent of a solution's failure to satisfy constraints is commonly quantified using a metric known as constraint violation (CV). When dealing with optimization problems bound by both equality and inequality constraints, the corresponding CV value is formulated as follows:

$$CV(x) = \sum_{i=1}^m \max(0, g_i(x)) + \sum_{j=1}^p \max(0, |h_j(x)| - \xi) \tag{2}$$

Specifically, the first and second terms quantify the violation extent of inequality constraints and the deviation margin of equality constraints, respectively. The threshold parameter ξ defines the tolerance level introduced for equality constraint relaxation, which is typically assigned a minimal value. An evaluation of $CV(x) = 0$ implies that all restrictions are fully met, meaning the corresponding solution lies within the feasible region; conversely, any non-zero value designates it as infeasible. Within the context of CMMOPs, diverse Pareto-optimal sets are often scattered across areas characterized by different levels of constraint severity. Consequently, the constraint violation metric serves a broader purpose than merely distinguishing feasible solutions from infeasible ones. It actively shapes the selection mechanism and the trajectory of population evolution throughout the search, heavily influencing overall optimization efficiency.

Tackling CMMOPs presents significant difficulties, primarily due to their distinct threefold nature: being constrained, multi-objective, and multi-modal. As a result, these problems exert substantial pressure on existing multi-objective evolutionary algorithms (MOEAs). To address them successfully, MOEAs must find a

proper equilibrium between feasibility, population diversity, and convergence within both the variable and objective spaces.

Historically, researchers have mainly explored two distinct domains: multi-objective optimization under constraints (CMO) and those possessing multimodal characteristics (MMO). Within the domain of CMO, prevalent techniques widely adopted in the literature encompass the Constraint Dominance Principle (CDP) [5], the ϵ -constraint approach [6], as well as strategies utilizing penalty functions [7]. While these approaches succeed in directing the search population toward the feasible portion of the search space, they frequently overemphasize solutions that are already feasible. Such a bias often results in ignoring highly valuable infeasible solutions, which ultimately degrades both algorithmic convergence and population diversity.

In the field of MMO, researchers have introduced mechanisms such as microhabitats, clustering, or species partitioning to maintain solution diversity; however, due to the lack of effective constraint handling strategies, such methods struggle to reliably obtain multiple feasible Pareto solution sets in complex constraint environments. For CMMOPs, a small number of studies have attempted to combine constraint handling with multimodality preservation. Examples include frameworks based on co-evolution or subspace partitioning strategies; however, under complex constraints, achieving an effective balance among feasibility, convergence, and multimodal diversity remains a challenging research problem.

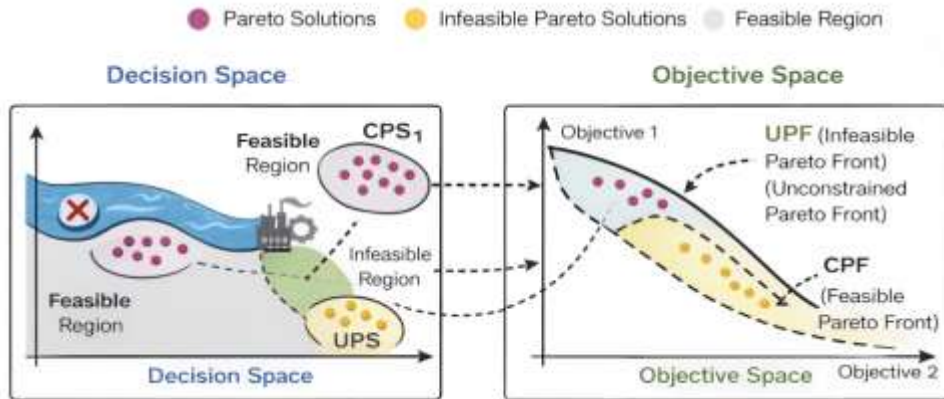


Fig. 1. Constrained Multimodal Multi-objective Optimization Problems

As illustrated in **Fig. 1**, constraints partition the decision space into feasible and infeasible regions. Within the feasible region, multiple disconnected constrained Pareto sets (CPSs) may coexist, each representing a distinct decision-space modality while yielding equivalent objective performance. Meanwhile, some high-quality solutions may be located in infeasible regions and constitute the unconstrained Pareto set (UPS)[10]. Although these solutions exhibit promising objective values, they cannot be accepted as final solutions due to constraint violations.

Upon projection into the objective space, candidate solutions arising from distinct CPSs converge onto a single constrained Pareto front (CPF). This phenomenon demonstrates that completely different parameter-space configurations can yield identical performance in the objective space. Such a many-to-one relationship dramatically compounds the complexity of CMMOPs, given that a solver is tasked

with both capturing the CPF and locating all corresponding CPSs simultaneously. As a result, preserving a proper mixture of population diversity, constraint satisfaction, and convergence across generations represents a formidable hurdle.

Accordingly, the defining trait of CMMOPs lies in the co-existence of several non-identical yet equivalent Pareto-optimal subsets within the variable space, all of which correspond to a single, identical Pareto front in the objective space. Hence, any metaheuristic designed for these tasks must be capable of both tracking down the global Pareto front and identifying these multiple, feasible equivalent sets across the parameter landscape. This dual capability is crucial to establish a well-synchronized harmony among constraint adherence, convergence, and diverse multimodal solution capture.

To overcome these hurdles, we present an evolutionary algorithm designed for constrained multimodal multi-objective optimization, which integrates adaptive attention with Lagrangian relaxation (AALR-CMMOE). The developed algorithm leverages a self-adaptive attention module that on-the-fly monitors the population's evolutionary state. By distributing search efforts based on this information, the approach bolsters search capacity in highly promising but insufficiently explored areas. Additionally, we design a constraint-handling scheme based on Lagrangian relaxation, where dual variables are updated dynamically to adaptively calibrate constraint boundaries. Such an approach empowers the search agent to efficiently negotiate intricate infeasible regions, leading to a better detection of latent constrained Pareto sets.

Furthermore, the optimization process is structured into two distinct stages to maintain a balanced synergy between exploration and exploitation. Specifically, the initial phase prioritizes wide-ranging multimodal exploration to help locate varied equivalent solution subsets. In contrast, the subsequent phase shifts focus toward local refinement and feasibility-guided convergence to elevate solution accuracy. By uniting adaptive attention, Lagrangian relaxation, and a two-phase search protocol, AALR-CMMOE achieves a superior trade-off among convergence, constraint adherence, and multimodal diversity preservation.

Thorough empirical evaluations on the CMMOP benchmark functions confirm that our proposed framework yields significantly better results than several state-of-the-art baselines under IGD, HV, and IGDp criteria. These findings validate its capacity to accurately capture multiple constrained Pareto sets and yield high-quality solutions when facing challenging, constrained multimodal environments.

To address the difficulty of simultaneously handling constraints, preserving multimodal diversity, and improving convergence in CMMOPs, this paper proposes an Adaptive Attention and Lagrangian Relaxation-based Constrained Multimodal Multi-objective Evolutionary Algorithm (AALR-CMMOE). The proposed algorithm integrates adaptive attention, Lagrangian relaxation, and a two-stage evolutionary search mechanism to dynamically allocate search resources, regulate constraint pressure, and coordinate global exploration with local exploitation. The main contributions of this work are summarized as follows:

(1) An adaptive attention-based search resource allocation strategy is proposed. By dynamically sensing the feasibility ratio, constraint violation, and evolutionary state of the population, the proposed strategy adaptively adjusts the search focus at different evolutionary stages. In this way, search resources can be allocated more effectively to promising Pareto sets and sparsely distributed modal regions, thereby enhancing the exploratory capability of the algorithm.

(2) A Lagrangian relaxation-based adaptive constraint-handling mechanism is designed. By constructing a dynamic update model for the Lagrangian multipliers, the constraint pressure is adaptively adjusted according to the feasibility ratio and the average constraint violation of the population. This mechanism preserves the ability to search around feasible regions while alleviating premature convergence, thereby improving the search efficiency of the algorithm in complex constrained environments.

(3) A dual-population co-evolutionary framework is developed for constrained multimodal optimization. Through the collaborative evolution of a relaxed population and a feasible population, the proposed framework strengthens multimodal exploration in the early stage and enhances convergence and feasibility in the later stage. As a result, global exploration and local exploitation can be effectively coordinated, improving the ability of the algorithm to identify multiple constrained Pareto sets.

The remainder of this paper is organized as follows. Section II reviews the related work on constrained multi-objective optimization and constrained multimodal multi-objective optimization. Section III presents the proposed AALR-CMMOEA in detail. Section IV reports the experimental setup and comparative results. Finally, Section V concludes this paper and outlines several directions for future research.

2. Related work and motivation

2.1. Related work

Historically, optimization under constraints (CMO) and multimodal optimization with multiple objectives (MMO) have been explored extensively, focusing on managing constraints and maintaining diversity within the decision space, respectively. Nevertheless, simply merging these two methodologies fails to provide a viable solution for multi-objective optimization challenges that are simultaneously constrained and multimodal (CMMOPs). As highlighted in prior literature, successfully tackling CMMOPs demands that a search method achieve two distinct targets: it must capture the constrained Pareto front (CPF) within the objective space while simultaneously maintaining multiple identical-performing constrained Pareto sets (CPSs) throughout the variable space. Consequently, rather than focusing on a single aspect, a solver must concurrently address convergence, constraint adherence, and diversity preservation across both the parameter and objective environments. This multi-faceted requirement renders CMMOPs significantly more complex than

standard CMOPs or MMOPs.

In the field of constrained multi-objective optimization, existing methods can be roughly classified into four categories. The first category separates constraints from objectives, such as CDP, the ϵ -constraint method, and stochastic ranking, which usually guide the population toward feasible regions. The second category is penalty function-based methods, where constraint violations are incorporated into the objective evaluation as penalty terms. The third category reformulates constraints as additional objectives, thereby jointly optimizing feasibility, convergence, and diversity in a unified multi-objective manner. The fourth category relies on problem transformation strategies, such as co-evolutionary frameworks or two-stage mechanisms, where unconstrained populations, auxiliary tasks, or stage switching are introduced to provide additional search information for the original problem. Although these approaches have achieved promising performance on CMOPs, most of them mainly focus on approximating the CPF in the objective space and pay limited attention to preserving multiple equivalent solution sets in the decision space. Consequently, when the feasible regions are disconnected, narrow, or highly isolated, these methods may retain only one CPS or merely partial segments of multiple CPSs.

Within the domain of multimodal multi-objective optimization (MMO), investigators have primarily focused on preserving population diversity within the parameter space. This is commonly achieved through techniques including niching, clustering, speciation, self organizing strategies, and dual archive frameworks. The core concept underlying these methods involves partitioning ecological niches or estimating solution density. Doing so helps mitigate direct competition between distinct groups of solutions, thereby maintaining multiple equivalent Pareto optimal sets throughout the variable landscape.

While these methods are generally successful at identifying multiple Pareto sets in unconstrained MMO scenarios, they struggle when constraints are introduced. Specifically, because they lack proper constraint handling procedures, these techniques often keep a vast pool of infeasible candidate solutions trapped in non feasible areas. This limitation hinders stable convergence toward feasible optimal sets. Consequently, existing literature explicitly highlights that the direct integration of constraint handling schemes into standard MMOEAs does not suffice to resolve multi-objective optimization problems that are both constrained and multimodal (CMMOPs).

Recently, several notable works have been introduced to address CMMOPs. For instance, CMMOCEA[11] utilizes a coevolutionary structure incorporating two distinct populations: one that actively processes constraints and another that overlooks them. This setup is designed to strike a well balanced trade off among constraint adherence, convergence, and diversity preservation within both the parameter and objective environments. Furthermore, the same study introduced the CMMOP1 to CMMOP14 benchmark suite, establishing a foundational testbed for subsequent investigations.

Concurrently, researchers have designed numerous complex test suites with multiple objectives under constraints to assess the efficacy of CMOEAs. Prominent

test functions, exemplified by the CTP[12], C-DTLZ[13], and MW[14] problem sets, feature distinct landscapes in their objective spaces. These include discontinuous, restricted, or asymmetric feasible areas, alongside degenerate Pareto fronts.

In the recent past, several key methodologies addressing CMMOPs have emerged in the literature. Of these, CMMODE[15] presents a differential evolution model incorporating niching mechanisms. This algorithm preserves multiple feasible Pareto optimal sets by utilizing speciation and a refined strategy for environmental selection. This study underscores how crucial self governing niche evolution is for maintaining diverse constrained Pareto sets (CPSs). Additionally, DCMMEA[16] develops a dynamic coevolutionary scheme utilizing two separate populations. It incorporates a time varying epsilon constraint technique along with self adjusting population distribution to establish a proper trade off between local exploitation and global exploration across multiple equivalent CPSs.

SCCMMO[17] addresses the problem from the perspective of subspace decomposition, and improves the ability to locate multiple CPSs by incorporating subspace generation and maintenance, subspace-type awareness, and a two-stage co-evolutionary mechanism. More recently, CoEA-LPD[18] recognizes that different CPSs may be associated with varying constraint intensities. It introduces a local auxiliary population together with a local Pareto dominance principle to mitigate the bias of traditional “constraint-first” mechanisms, which tend to suppress CPSs located in highly constrained regions.

Feng and Liu introduced MTGA-CMMO[19], a method designed around a multitasking genetic framework that operates with a single primary task alongside a pair of secondary tasks. These secondary processes modify constraint limits dynamically. This allows the primary task to break out of non feasible areas. Meanwhile, to bolster wide ranging search performance, they implement a mating selection strategy driven by probabilistic leader identification.

Wang et al. designed DTCMMO-RL[20], which combines evolutionary multitasking paradigms with reinforcement learning. In this approach, several subordinate tasks are established, each targeting distinct objectives: fulfilling constraints, exploring the objective space, or searching the decision landscape. Furthermore, a Q learning mechanism is utilized for the dynamic selection of suitable secondary tasks. This increases the algorithm's self adjusting search efficiency across various phases of the evolutionary process.

Despite the progress made by existing approaches, several common limitations remain in current studies on CMMOPs. First, many methods still rely on a “constraint-priority” principle or dynamic ϵ -constraint strategies. When significant differences exist in constraint intensities around different CPSs, these approaches tend to prematurely discard individuals with high constraint violations but potential convergence ability, thereby neglecting promising CPSs located in highly constrained regions.

Second, although dual-population frameworks, local auxiliary populations, and subspace-based strategies can partially alleviate search bias, the regulation of when to emphasize feasibility, convergence, or multimodal preservation is often determined by

pre-defined rules. This results in a lack of fine-grained adaptive control during the evolutionary process.

Third, most existing constraint-handling techniques are still based on CDP, ϵ -constraint, or penalty functions. While these strategies are effective, the adjustment of constraint pressure is largely heuristic, and their ability to adapt to heterogeneous constraint distributions across different modal regions remains limited.

2.2. Motivation

From the analysis of existing studies, it can be observed that the core difficulty of CMMOPs lies in the significant imbalance among different Pareto solution sets, which is manifested as a coupling between constraint heterogeneity and convergence difficulty. On the one hand, certain CPSs are located in regions with strong constraints or extremely narrow feasible domains, making them substantially more difficult to locate than those in relatively relaxed regions. On the other hand, traditional constraint-prioritized selection mechanisms tend to favor individuals with smaller constraint violations, which gradually leads to the neglect of promising solution sets located in highly constrained regions during the evolutionary process.

Such imbalance is one of the main reasons why existing algorithms struggle to comprehensively identify all CPSs, particularly when feasible regions are disconnected, irregular, or highly unevenly distributed.

In addition, the scheduling mechanisms employed in existing methods remain relatively coarse. Although dual-population co-evolution and two-stage optimization strategies have improved algorithm performance to some extent, their resource allocation and strategy-switching processes are largely governed by pre-defined rules. As a result, they lack real-time awareness and responsiveness to dynamic population states, such as feasibility ratio, constraint violation degree, and diversity distribution. This limitation significantly restricts the adaptability of existing algorithms in complex constrained environments.

To address the above issues, this paper improves CMMOP solving from two aspects: constraint handling and search scheduling. First, a Lagrangian relaxation mechanism is introduced, where constraint information is incorporated into the optimization process through Lagrangian multipliers. The multipliers are dynamically updated based on the feasibility ratio (FR) and constraint violation (CV) of the population, enabling adaptive regulation of constraint pressure and alleviating the limitations of overly strict or overly relaxed constraint handling strategies.

Second, an adaptive attention mechanism is developed to model the search status of different modal regions. Based on this mechanism, search resources and evolutionary operators are dynamically allocated to enhance the discovery of multiple Pareto solution sets. Furthermore, a two-stage optimization strategy is incorporated, which emphasizes multimodal exploration in the early stage and strengthens feasibility and convergence in the later stage, thereby achieving an effective balance between global exploration and local exploitation.

Overall, the proposed method aims to develop a unified framework that

simultaneously maintains feasibility, convergence, and multimodal diversity under complex constrained environments, thereby improving the overall performance on CMMOPs.

3. Proposed Algorithm

For clarity, the proposed AALR-CMMOEa is decomposed into three components. Algorithm 1 presents the overall framework of the proposed method, including initialization, stage determination, offspring generation, archive update, and population update. Algorithm 2 is responsible for adaptively updating the Lagrangian multipliers based on the feasibility ratio (FR) and constraint violation (CV), thereby enabling adaptive adjustment of constraint intensity. Algorithm 3 performs dual-population co-evolutionary environmental selection, where the relaxed population maintains exploration capability while the feasible population enhances convergence performance.

These three components work in a coordinated manner to realize a synergistic optimization process consisting of adaptive constraint handling, multimodal exploration, and feasibility refinement. The flowchart of the proposed AALR-CMMOEa is shown in Fig. 2.

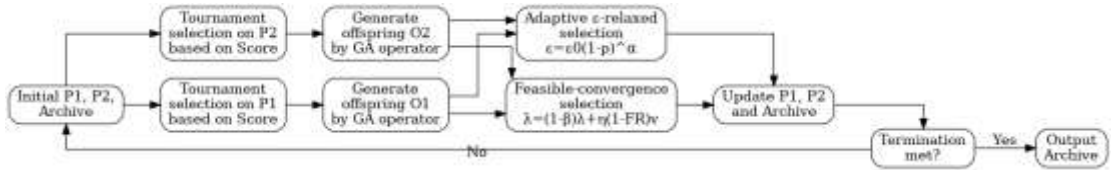


Fig. 2. Framework of the proposed AALR-CMMOEa

3.1. Framework of AALR-CMMOEa

Algorithm 1 presents the overall framework of the proposed AALR-CMMOEa. The algorithm begins by initializing the relaxed population, the feasible population, and the feasible archive. Based on the initial population's constraint violations, the Lagrangian multipliers and the initial relaxation threshold are constructed.

During each generation, the evolutionary stage is first identified according to the current progress, and the constraint relaxation level is adaptively adjusted. Subsequently, the Lagrangian multipliers are updated using the feasibility ratio (FR) and the average constraint violation (CV), enabling adaptive regulation of constraint pressure throughout the search process.

Then, offspring are generated through selection and variation operators applied to both populations. High-quality feasible solutions are stored in the feasible archive to preserve promising candidates. Finally, a relaxation-based selection strategy and a feasibility-driven selection strategy are employed to update the relaxed population and feasible population, respectively, achieving a coordinated balance between

multimodal exploration and convergence toward feasible regions.

To enable adaptive variation of the constraint relaxation level during evolution, a dynamic relaxation threshold is constructed based on the ratio between the current number of function evaluations and the maximum number of function evaluations. Specifically, a relatively large ε is maintained in the early stage to allow near-feasible solutions to participate in the evolution, thereby enhancing the ability of the population to traverse infeasible regions and discover potential CPSs. As the evolutionary process proceeds, the value of ε gradually decreases, guiding the search toward feasible regions and improving the feasibility and convergence of the final solution set. The updating strategy is defined as follows:

$$\varepsilon_t = \varepsilon_0(1 - progress)^{epsPow} \quad (3)$$

Considering the different requirements of constraint pressure at different evolutionary stages, the relaxation threshold is further reduced in the second stage to strengthen feasibility-guided search.

$$\varepsilon_t = \begin{cases} \varepsilon_0(1 - progress)^{epsPow}, & stage = 1, \\ 0.1\varepsilon_0(1 - progress)^{epsPow}, & stage = 2. \end{cases} \quad (4)$$

Furthermore, when the evolutionary process enters the final stage, the relaxation threshold is set to zero to ensure that the final solution set strictly satisfies all constraints.

$$\varepsilon_t = 0, \quad progress > 0.95 \quad (5)$$

Algorithm 1 Main Framework of AALR-CMMOEA

Input: $N, \tau, \lambda_{\max}$

Output: Final population P

- 1: Initialize relaxed population P_1 and feasible population P_2 ;
 - 2: Initialize feasible archive A ;
 - 3: Initialize Lagrangian multipliers λ and relaxation threshold ε_0 ;
 - 4: while termination condition is not met do
 - 5: Determine current stage according to FE/maxFE;
 - 6: Update adaptive relaxation threshold ε ;
 - 7: Calculate FR and mean constraint violation;
 - 8: Update Lagrangian multipliers λ ;
 - 9: Calculate diversity-based parent scores;
 - 10: Select parents from P_1 and P_2 ;
 - 11: Generate offspring O_1 and O_2 ;
 - 12: Update feasible archive A ;
 - 13: Construct candidate population $R=[P_1, P_2, O_1, O_2, A]$;
 - 14: Update P_1 using relaxed selection;
 - 15: Update P_2 using feasible-convergence selection;
 - 16: Update output population P ;
 - 17: end while
 - 18: return P ;
-

3.2. Lagrangian multiplier update

Algorithm 2 is designed to achieve adaptive updating of the Lagrangian multipliers. First, it checks whether constraint information is present in the current problem. If no constraints exist, the original multipliers are directly returned without modification. Then, different learning rates are assigned according to the current evolutionary stage. A smaller learning rate is used in the exploration stage to avoid excessively strong constraint pressure, while a larger learning rate is adopted in the refinement stage to enhance feasibility guidance. The update signal is jointly determined by the feasibility ratio (FR) and the average constraint violation (meanViol). When the feasibility ratio is low and the constraint violation is high, the multipliers increase more significantly, thereby strengthening the penalty imposed on infeasible solutions.

Specifically, let λ_j^{t+1} denote the Lagrangian multiplier corresponding to the (j)-th constraint at generation (t+1), and let $\bar{v}_j^t$ represent the average violation degree of the population with respect to the (j)-th constraint at generation (t). A smoothed updating scheme is adopted to adaptively adjust the Lagrangian multipliers. The update rule is defined as follows:

$$\lambda_j^{t+1} = (1 - \beta)\lambda_j^t + \eta_t(1 - FR_t)\bar{v}_j^t \quad (6)$$

where β is a smoothing factor, η_t denotes the stage-dependent learning rate, and FR_t represents the feasibility ratio of the current population. When FR_t is low and $\bar{v}_j^t$ is large, the increment of the Lagrangian multiplier becomes more significant, thereby strengthening the penalty imposed on infeasible solutions. Conversely, when the population exhibits a higher feasibility level, the update magnitude of the multipliers is reduced to prevent excessively strong constraint pressure from degrading population diversity.

To prevent the Lagrangian multipliers from growing without bound or taking negative values, a boundary truncation strategy is further applied to the updated multipliers, ensuring that they remain within a reasonable range:

$$\lambda_j^{t+1} = \min(\lambda_{\max}, \max(0, \lambda_j^{t+1})) \quad (7)$$

where $\lambda_{\max}$ denotes the upper bound of the Lagrangian multipliers. Through the above updating and truncation operations, the algorithm can adaptively regulate the constraint pressure according to the population state, thereby improving the stability of the search process in complex constrained environments.

Algorithm 2 Adaptive Lagrangian Multiplier Updating

Input: λ , meanViol, FR, stage, η_1 , η_2 , $\lambda_{\max}$

Output: Updated λ

```

1: if  $\lambda$  is empty or meanViol is empty then
2:   return  $\lambda$ ;
3: end if
4:  $\eta = \eta_1$ ;

5: if stage = 2 then  $\eta = \eta_2$ ;

6: signal =  $(1 - FR) \cdot \text{meanViol}$ ;
7:  $\lambda = 0.8 \cdot \lambda + \eta \cdot \text{signal}$ ;
8:  $\lambda = \min(\lambda, \lambda_{\max})$ ;
9:  $\lambda = \max(\lambda, 0)$ ;
10: return  $\lambda$ ;

```

3.3. Dual-Population Environmental Selection

Algorithm 3 describes the dual-population environmental selection process. The candidate population R is constructed by combining the parent population, offspring population, and the feasible archive. For the relaxed population P_1 , more

near-feasible solutions are preserved during the exploration stage to enhance the ability to search potential CPS regions, while in the refinement stage, greater emphasis is placed on feasibility and high-quality solutions near constraint boundaries.

For the convergence-oriented feasible population P_2 , feasible non-dominated solutions are preferentially selected. When feasible solutions are insufficient, ε -feasible solutions or individuals with smaller constraint violations are used as a supplement. This strategy enables the two populations to respectively focus on exploring difficult regions and enhancing feasible convergence.

Algorithm 3 Dual-Population Environmental Selection

Input: $R, N_1, N_2, \lambda, \varepsilon, \text{stage}$

Output: P_1, P_2

- 1: Calculate CV, diversity Div and Lagrangian score Lag of R ;
 - 2: Identify feasible and ε -feasible solutions;
 - 3: if stage = 1 then preserve feasible and near-feasible solutions for P_1 ;
 - 4: else preserve feasible and ε -feasible solutions for P_1 ;
 - 5: Rank remaining solutions by FrontNo, Lag, CV and Div;
 - 6: Fill P_1 until its size reaches N_1 ;
 - 7: Select feasible solutions with nondominated sorting for P_2 ;
 - 8: if feasible solutions are insufficient then add ε -feasible solutions;
 - 9: Fill P_2 by feasibility-convergence-diversity ranking;
 - 10: return P_1 and P_2 ;
-

4. Experimental Results and Analysis

In this section, extensive experiments are conducted to evaluate the performance of the proposed AALR-CMMOEa. The experimental studies are presented in terms of experimental settings, comparative results, and performance analysis.

4.1. Experimental Settings

1) Comparative Algorithms

To evaluate the performance of the proposed AALR-CMMOEa on multimodal multi-objective optimization problems, six representative state-of-the-art MMOEAs are selected for comparison, including PPS[21], C3M[22], ToP[23], MOEAD[24], MOCGDE[25], and DTCMMORL[20].

2) Benchmark Problems and Performance Metrics

In this study, the CMMOP benchmark suite is adopted as the test problems. The selected benchmark set consists of multiple instances with diverse characteristics and

varying levels of difficulty, which can simulate complex scenarios such as small feasible regions, infeasible barriers, overlapping solution sets, inconsistencies between constrained and unconstrained Pareto fronts, and high-dimensional decision spaces. Therefore, it provides a comprehensive testbed for evaluating the effectiveness and robustness of the proposed algorithm on constrained multimodal multi-objective optimization problems.

In this study, three widely used performance metrics are employed for evaluation, namely IGD[26], HV[27], and IGDp[28]. The IGD metric measures the convergence degree of the obtained solution set toward the true Pareto front in the objective space. The HV metric evaluates the volume of the objective space dominated by the obtained non-dominated solutions. The IGDp metric assesses the coverage of the obtained solution set over the true Pareto set in the decision space. The definitions of these three metrics are given as follows:

$$IGD(X) = \frac{1}{|Y^*|} \sum_{y \in Y^*} \min_{x \in X} ED(f(x), y) \quad (8)$$

$$HV(X) = VOL \left(\bigcup_{x \in X} [f_1(x), r_1] \times [f_2(x), r_2] \times \dots \times [f_m(x), r_m] \right) \quad (9)$$

$$IGD_p(X) = \frac{1}{|X^*|} \sum_{y \in X^*} \min_{x \in X} ED(x, y) \quad (10)$$

where X represents the approximation set generated by the proposed algorithm, whereas Y^* and X^* correspond to the reference sets situated on the actual Pareto front and Pareto optimal set, respectively. Additionally, ED stands for Euclidean distance, while r_m acts as the specified reference point for computing the hypervolume (HV).

For equitable comparisons, we configure the population size at 100, capping the total number of objective function evaluations at 70,000. Every candidate algorithm executes 30 independent trials on each benchmark instance. We then document the resulting mean performance along with the standard deviation. The Wilcoxon rank sum test, utilizing a significance threshold of 0.05, serves to evaluate the statistical significance of these results.

Within our performance tables, optimal outcomes are designated with bold font and gray shading. To denote comparison outcomes, “+”, “−”, and “=” are employed to show if a competing solver is superior, inferior, or statistically equivalent to the developed AALR-CMMOEA, respectively. We carried out all empirical simulations utilizing the PlatEMO platform[29]. To support academic reproduction and future extensions, the source code of our algorithm is openly accessible.

4.2. Comparative Experiments

To assess metaheuristics tailored for multi-objective optimization under constraints and multimodality, the CMMOP benchmark suite serves as a standard tool. This is because it concurrently evaluates constraint handling, convergence within the objective environment, and diverse solution preservation in the variable space. We present the empirical findings of both AALR-CMMOEa and the baseline methods across Table 1-4, evaluating them under the IGD, HV, and IGDp metrics.

Focusing on the IGD metric, AALR-CMMOEa outclasses other methods on numerous test functions, securing the top average rank of 2.29. This outcome verifies its superior ability to approach the true constrained Pareto front. When benchmarked against PPS, C3M, and ToP, our proposed approach demonstrates enhanced global exploration when navigating complex search spaces, which prevents the population from falling into premature convergence within sub-optimal feasible regions. At the same time, although DTCMMORL displays respectable outcomes on certain benchmark instances, its collective performance remains secondary to the developed AALR-CMMOEa.

From the perspective of the HV metric, the proposed AALR-CMMOEa achieves the best overall ranking. A larger HV value indicates that the obtained solution set not only approximates the true constrained Pareto front well, but also maintains a better coverage and diversity distribution in the objective space. In particular, on several test problems with complex feasible region structures, AALR-CMMOEa is able to effectively preserve population diversity and avoid excessive concentration of solutions in local regions.

In terms of the IGDp metric, the proposed AALR-CMMOEa further demonstrates clear superiority and achieves the lowest average rank (2.14). This result indicates that the proposed method is capable of discovering a larger number of equivalent Pareto solution sets distributed across different regions of the decision space, demonstrating strong multimodal preservation ability. This advantage is mainly attributed to the adaptive attention mechanism, which dynamically allocates search resources across the population and continuously focuses on sparsely distributed modal regions, thereby enhancing the ability to identify diverse constrained Pareto sets.

Furthermore, Fig. 3-4 present the final PF visualizations of different algorithms on CMMOP5 and CMMOP11, respectively. In the two-dimensional PF distribution of CMMOP5, the proposed AALR-CMMOEa can almost fully cover multiple segments of the Pareto front while maintaining a relatively uniform distribution of solutions. In contrast, some comparison algorithms suffer from incomplete coverage or local aggregation phenomena. For the three-objective PF visualization of CMMOP11, AALR-CMMOEa also maintains a well-structured distribution of the Pareto front, further demonstrating its strong convergence and diversity preservation capability under complex constrained multimodal environments.

Fig. 5 and Fig. 6 illustrate the PS visualizations of different algorithms on CMMOP5 and CMMOP11, respectively. The scattered points represent the distribution of feasible non-dominated solutions obtained by each algorithm in the decision space. It can be observed that the proposed AALR-CMMOEa maintains a

well-distributed solution set and covers multiple potential Pareto solution regions, indicating its strong capability in multimodal solution discovery. In contrast, some comparison algorithms exhibit more concentrated distributions and fail to adequately cover different regions of the decision space.

To provide a more intuitive comparison of the overall performance across different test problems, a ranking-based heatmap of IGD, HV, and IGDp metrics is further presented. In the heatmap, darker colors (corresponding to lower rank values) indicate better performance. As shown in **Fig. 7**, the proposed AALR-CMMOEA achieves consistently superior rankings on most CMMOP test instances, demonstrating its comprehensive performance in terms of convergence, objective-space coverage, and decision-space multimodal preservation.

Table 1. Average ranks of different algorithms on IGD, HV, and IGDp metrics over the CMMOP benchmark suite.

Problem	Indictors	PPS	C3M	ToP	MOEAD	MOCGDE	DTCMMORL	AALR-CMMOEA
CMMOP	IGD	2.71	3.79	6.29	4.43	6.00	2.50	2.29
	HV	2.93	3.43	5.71	4.79	5.79	2.71	2.64
	IGDp	2.86	3.57	6.50	4.64	5.86	2.43	2.14

Table 2. Mean IGD values of AALR-CMMOEA and six comparison algorithms on the CMMOP benchmark suite.

Problem	PPS	C3M	ToP	MOEAD	MOCGDE	DTCMMORL	AALR-CMMOEA
CMMOP1	1.9092e-2 (2.15e-3) -	1.6753e-2 (1.64e-3) =	8.6587e-2 (9.67e-2) -	1.0099e-1 (9.32e-2) -	2.7238e-1 (2.93e-1) =	1.6094e-2 (3.43e-4) +	1.6274e-2 (1.58e-3)
CMMOP2	2.7857e-2 (4.80e-2) -	2.6060e-2 (6.20e-3) -	2.1282e-1 (1.59e-1) -	1.7247e-1 (1.32e-1) -	1.1503e-1 (1.21e-1) -	4.6764e-2 (7.36e-2) -	9.6888e-3 (2.66e-3)
CMMOP3	1.7830e-2 (7.57e-3) =	4.9633e-2 (6.05e-3) -	1.3506e-1 (1.56e-1) -	9.4397e-2 (5.80e-2) -	2.5828e-1 (2.99e-1) =	1.2304e-2 (3.68e-3) =	2.2093e-2 (2.23e-2)
CMMOP4	1.8445e-2 (2.93e-3) +	4.8361e-2 (6.69e-3) -	1.0551e-1 (1.61e-1) -	8.6885e-2 (9.62e-2) -	3.5216e-2 (1.09e-1) =	1.4817e-2 (2.36e-3) +	2.3002e-2 (5.42e-2)
CMMOP5	9.3554e-2 (2.78e-4) +	9.4852e-2 (2.11e-4) +	1.0468e-1 (2.24e-2) =	1.0389e-1 (3.11e-2) -	1.9571e-1 (1.58e-1) =	9.5898e-2 (2.54e-4) -	9.5399e-2 (2.20e-4)
CMMOP6	3.2550e-2 (4.51e-2) -	1.2982e-2 (3.13e-4) +	7.5832e-2 (1.06e-1) =	2.6670e-1 (2.52e-1) -	4.3162e-2 (1.47e-1) -	1.3022e-2 (1.68e-4) +	1.7085e-2 (2.39e-2)
CMMOP7	1.7375e-2 (4.13e-3) =	3.9028e-2 (8.24e-3) -	1.3775e-1 (1.04e-1) -	8.2996e-2 (3.47e-2) -	1.9310e-1 (2.07e-1) =	1.9626e-2 (4.05e-3) -	1.5846e-2 (9.61e-4)
CMMOP8	1.1803e-2 (3.36e-3) +	4.3489e-2 (4.55e-3) -	8.8598e-2 (1.32e-1) -	1.0779e-1 (9.53e-2) -	2.8831e-2 (5.26e-2) =	1.6804e-2 (6.79e-3) =	2.8929e-2 (2.56e-2)
CMMOP9	2.0164e-2 (6.48e-3) -	5.0022e-2 (4.82e-3) -	6.3812e-2 (4.69e-2) -	4.8080e-2 (2.95e-2) -	1.9671e-1 (2.44e-1) =	1.6677e-2 (2.57e-3) +	1.8744e-2 (1.84e-2)
CMMOP10	8.6771e-2 (9.84e-5) -	8.7359e-2 (3.58e-4) -	4.8479e-1 (1.73e-1) -	1.1607e-1 (5.76e-2) -	2.2222e-1 (1.32e-1) -	8.7052e-2 (3.13e-4) -	8.6535e-2 (6.53e-5)
CMMOP11	1.0206e-1 (4.24e-3) +	1.0239e-1 (2.29e-2) +	4.2584e-1 (2.20e-1) -	1.0143e-1 (3.18e-3) +	4.2565e-1 (3.15e-1) =	1.0850e-1 (8.05e-3) -	1.1175e-1 (7.14e-3)
CMMOP12	1.0966e-1 (1.46e-2) -	1.5559e-1 (1.62e-2) -	4.5063e-1 (1.71e-1) -	8.1422e-2 (2.29e-3) +	6.3473e-1 (3.28e-1) =	9.3797e-2 (3.87e-3) -	8.9264e-2 (3.43e-3)
CMMOP13	1.1632e-1 (1.13e-2) -	1.5097e-1 (9.65e-3) -	5.4230e-1 (1.92e-1) -	7.5888e-2 (2.17e-3) +	4.9299e-1 (3.22e-1) =	9.7033e-2 (7.00e-3) -	8.7334e-2 (3.55e-3)
CMMOP14	1.2838e-1 (8.42e-3) -	1.7265e-1 (5.94e-3) -	5.8941e-1 (1.70e-1) -	1.1653e-1 (3.38e-3) -	7.4359e-1 (2.25e-1) =	8.4280e-2 (3.48e-3) -	7.7522e-2 (2.92e-3)
+/-/=	4/8/2	3/10/1	0/12/2	3/11/0	0/3/11	4/8/2	

(“−” means AALR-CMMOEA better than compared algorithm, “+” means compared algorithm better than AALR-CMMOEA, “=” means not comparable).

Table 3. Mean HV values of AALR-CMMOEa and six comparison algorithms on the CMMOP benchmark suite.

Problem	PPS	C3M	ToP	MOEAD	MOCGDE	DTCMMORL	AALR-CMMOEa
CMMOP1	6.6207e-1 (2.96e-3) -	6.6461e-1 (2.45e-3) -	6.0987e-1 (7.18e-2) -	6.1093e-1 (5.81e-2) -	4.4318e-1 (2.29e-1) =	6.6661e-1 (4.98e-4) =	6.6626e-1 (1.98e-3)
CMMOP2	5.6698e-1 (5.48e-2) -	5.5672e-1 (9.65e-3) -	4.1424e-1 (1.14e-1) -	5.1841e-1 (6.78e-2) -	4.7868e-1 (1.03e-1) -	5.5288e-1 (5.96e-2) -	5.8459e-1 (2.01e-3)
CMMOP3	6.3198e-1 (9.55e-3) -	5.8248e-1 (8.54e-3) -	5.0699e-1 (1.25e-1) -	5.8394e-1 (3.42e-2) -	4.3949e-1 (2.37e-1) =	6.4044e-1 (4.36e-3) =	6.3418e-1 (1.77e-2)
CMMOP4	6.7839e-1 (2.67e-3) -	6.3015e-1 (9.13e-3) -	6.0085e-1 (1.05e-1) -	6.3124e-1 (3.88e-2) -	6.5755e-1 (1.09e-1) -	6.7918e-1 (4.21e-3) -	6.8113e-1 (2.80e-2)
CMMOP5	5.5751e-1 (7.41e-4) +	5.5661e-1 (4.09e-4) +	5.4406e-1 (2.45e-2) -	5.4832e-1 (2.81e-2) -	4.6753e-1 (1.26e-1) =	5.5404e-1 (4.70e-4) -	5.5463e-1 (3.16e-4)
CMMOP6	5.5953e-1 (1.22e-2) -	5.6473e-1 (2.53e-4) +	5.3892e-1 (4.90e-2) -	4.4482e-1 (1.31e-1) -	5.4522e-1 (8.81e-2) -	5.6522e-1 (1.05e-4) +	5.6438e-1 (6.23e-3)
CMMOP7	6.2419e-1 (6.02e-3) =	5.9026e-1 (1.16e-2) -	5.0790e-1 (8.92e-2) -	5.8006e-1 (2.20e-2) -	4.5660e-1 (1.84e-1) =	6.2074e-1 (5.87e-3) -	6.2644e-1 (1.51e-3)
CMMOP8	6.8333e-1 (4.92e-3) +	6.3559e-1 (6.84e-3) -	6.1798e-1 (8.13e-2) -	6.3196e-1 (3.85e-2) -	6.6649e-1 (3.66e-2) =	6.7220e-1 (9.98e-3) =	6.6775e-1 (2.00e-2)
CMMOP9	6.7013e-1 (5.86e-3) -	6.2425e-1 (6.74e-3) -	6.0972e-1 (5.15e-2) -	6.5086e-1 (1.66e-2) -	5.1424e-1 (1.94e-1) =	6.7560e-1 (2.98e-3) -	6.7751e-1 (1.18e-2)
CMMOP10	5.1231e-1 (7.20e-5) +	5.1051e-1 (5.84e-4) -	2.7880e-1 (9.91e-2) -	4.9382e-1 (1.90e-2) -	4.1117e-1 (7.92e-2) -	5.1152e-1 (3.47e-4) -	5.1224e-1 (9.51e-5)
CMMOP11	4.5645e-1 (4.73e-3) +	4.3286e-1 (3.79e-2) -	1.8408e-1 (1.01e-1) -	3.9569e-1 (4.78e-3) -	2.2282e-1 (1.95e-1) =	4.4323e-1 (8.52e-3) -	4.5338e-1 (7.72e-3)
CMMOP12	4.2636e-1 (2.55e-2) -	3.3723e-1 (2.35e-2) -	1.3221e-1 (7.24e-2) -	4.5284e-1 (4.33e-3) -	1.3546e-1 (1.69e-1) =	4.6734e-1 (5.09e-3) -	4.7159e-1 (6.16e-3)
CMMOP13	4.0935e-1 (1.68e-2) -	3.4369e-1 (1.55e-2) -	1.0126e-1 (6.81e-2) -	4.4629e-1 (4.09e-3) -	1.7755e-1 (1.84e-1) =	4.5961e-1 (9.45e-3) -	4.7990e-1 (6.26e-3)
CMMOP14	3.9381e-1 (1.61e-2) -	3.1548e-1 (9.59e-3) -	6.4681e-2 (5.36e-2) -	4.5566e-1 (3.66e-3) -	4.9722e-2 (8.48e-2) =	4.7687e-1 (5.95e-3) -	4.8780e-1 (5.38e-3)
+/-/=	1/10/3	2/12/0	0/12/2	0/14/0	0/3/11	1/10/3	

(“−” means AALR-CMMOEa better than compared algorithm, “+” means compared algorithm better than AALR-CMMOEa, “=” means not comparable).

Table 4. Mean IGDp values of AALR-CMMOEa and six comparison algorithms on the CMMOP benchmark suite.

Problem	PPS	C3M	ToP	MOEAD	MOCGDE	DTCMMORL	AALR-CMMOEa
CMMOP1	1.5505e-2 (9.29e-4) -	1.5425e-2 (8.23e-4) -	5.8507e-2 (5.94e-2) -	6.0955e-2 (5.19e-2) -	2.1431e-1 (2.36e-1) =	1.4955e-2 (3.00e-4) -	1.4822e-2 (6.41e-4)
CMMOP2	2.6987e-2 (4.77e-2) -	2.5728e-2 (6.22e-3) -	1.8139e-1 (1.25e-1) -	1.0648e-1 (9.47e-2) -	1.0514e-1 (1.06e-1) -	4.2701e-2 (6.28e-2) -	9.3239e-3 (2.66e-3)
CMMOP3	1.6504e-2 (6.43e-3) -	4.9189e-2 (6.13e-3) -	1.1188e-1 (1.10e-1) -	5.5326e-2 (3.00e-2) -	1.9289e-1 (2.29e-1) =	1.1256e-2 (3.01e-3) =	1.5923e-2 (1.21e-2)
CMMOP4	1.7458e-2 (2.58e-3) -	4.8094e-2 (6.76e-3) -	8.5922e-2 (1.14e-1) -	5.4433e-2 (5.42e-2) -	2.9988e-2 (8.92e-2) =	1.4106e-2 (2.34e-3) +	1.6646e-2 (3.26e-2)
CMMOP5	9.3476e-2 (2.72e-4) +	9.4800e-2 (2.04e-4) +	1.0401e-1 (2.00e-2) =	1.0194e-1 (2.53e-2) -	1.7111e-1 (1.12e-1) =	9.5813e-2 (2.42e-4) -	9.5328e-2 (2.20e-4)
CMMOP6	2.2804e-2 (2.35e-2) -	1.1901e-2 (1.41e-4) +	4.8665e-2 (6.41e-2) =	1.8272e-1 (1.87e-1) -	3.6158e-2 (1.15e-1) -	1.1900e-2 (1.22e-4) +	1.3881e-2 (1.21e-2)
CMMOP7	1.5975e-2 (3.91e-3) =	3.8280e-2 (8.28e-3) -	1.0217e-1 (7.16e-2) -	5.0006e-2 (1.85e-2) -	1.5090e-1 (1.61e-1) =	1.7922e-2 (4.14e-3) -	1.3904e-2 (6.88e-4)
CMMOP8	1.1156e-2 (3.25e-3) =	4.3253e-2 (4.58e-3) -	6.9173e-2 (8.87e-2) -	6.1887e-2 (5.35e-2) -	2.1346e-2 (3.23e-2) =	1.4585e-2 (4.78e-3) =	1.8695e-2 (1.27e-2)
CMMOP9	1.8538e-2 (3.98e-3) -	4.9768e-2 (4.88e-3) -	6.0285e-2 (3.65e-2) -	3.2178e-2 (1.42e-2) -	1.5262e-1 (1.79e-1) =	1.5516e-2 (2.08e-3) -	1.3837e-2 (8.93e-3)
CMMOP10	8.6611e-2 (8.78e-5) -	8.7301e-2 (3.59e-4) -	3.7686e-1 (1.09e-1) -	1.0698e-1 (4.11e-2) -	1.9640e-1 (9.83e-2) -	8.6902e-2 (3.02e-4) -	8.6391e-2 (6.11e-5)
CMMOP11	5.8628e-2 (2.15e-3) =	7.8991e-2 (2.45e-2) -	2.9384e-1 (1.18e-1) -	8.2572e-2 (2.13e-3) -	3.1783e-1 (2.37e-1) =	6.6632e-2 (3.32e-3) -	5.9576e-2 (3.17e-3)
CMMOP12	7.8521e-2 (1.38e-2) -	1.4637e-1 (1.71e-2) -	3.3779e-1 (9.37e-2) -	5.7763e-2 (2.16e-3) -	4.8018e-1 (2.52e-1) =	5.8318e-2 (3.14e-3) -	5.1542e-2 (2.05e-3)
CMMOP13	8.9700e-2 (8.75e-3) -	1.4268e-1 (1.10e-2) -	4.0821e-1 (1.25e-1) -	5.1790e-2 (2.93e-3) -	3.8313e-1 (2.37e-1) =	6.5067e-2 (5.29e-3) -	4.9737e-2 (2.50e-3)
CMMOP14	9.7964e-2 (9.02e-3) -	1.6511e-1 (7.58e-3) -	4.2713e-1 (8.22e-2) -	5.9881e-2 (1.71e-3) -	5.7003e-1 (1.66e-1) =	5.8115e-2 (3.97e-3) -	4.6985e-2 (2.74e-3)
+/-/=	4/9/1	2/12/0	0/14/0	0/14/0	0/4/10	1/10/3	

(“−” means AALR-CMMOEa better than compared algorithm, “+” means compared algorithm better than AALR-CMMOEa, “=” means not comparable).

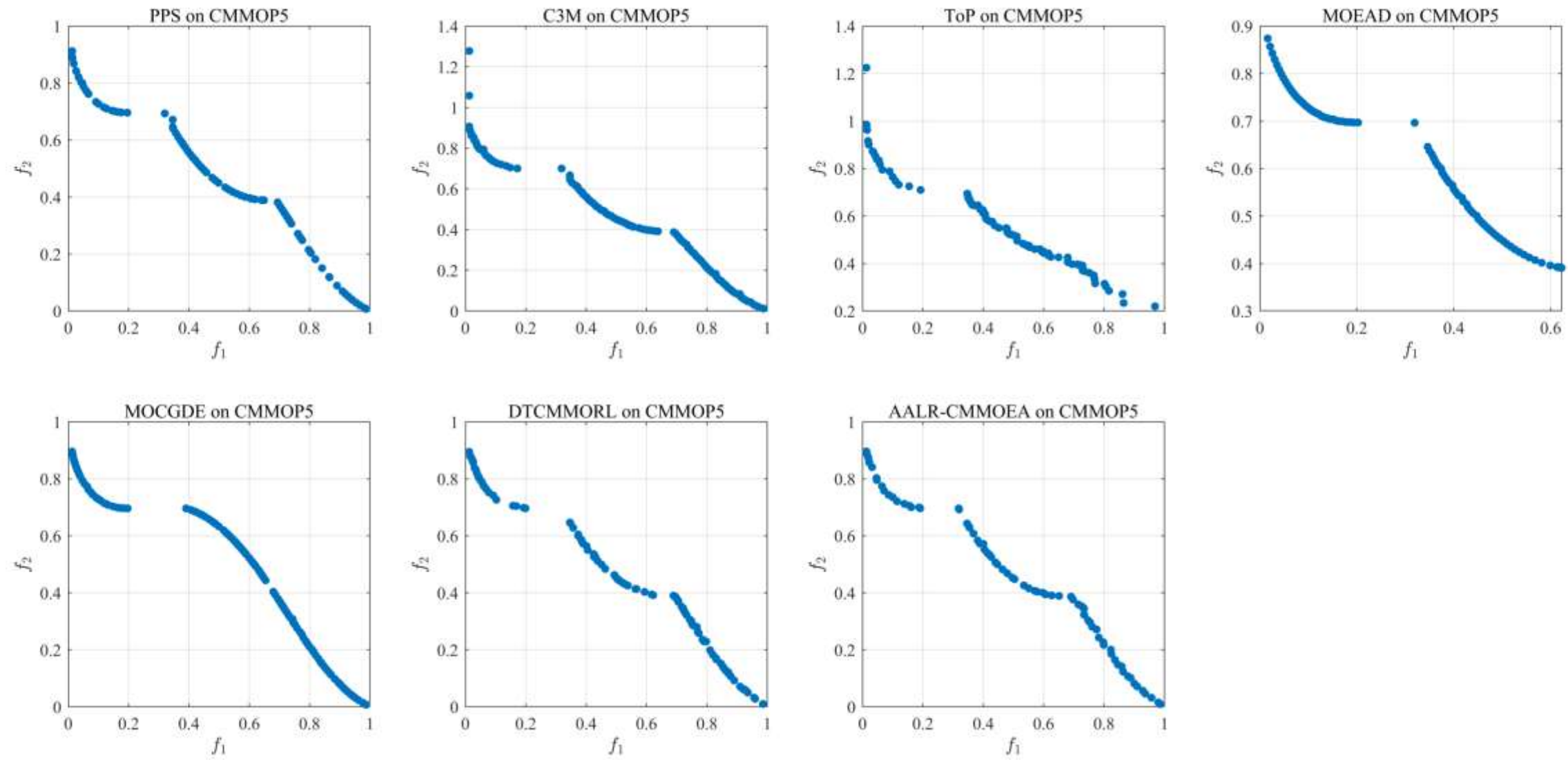


Fig. 3. PF visualization results of different algorithms on the CMMOP5 problem.

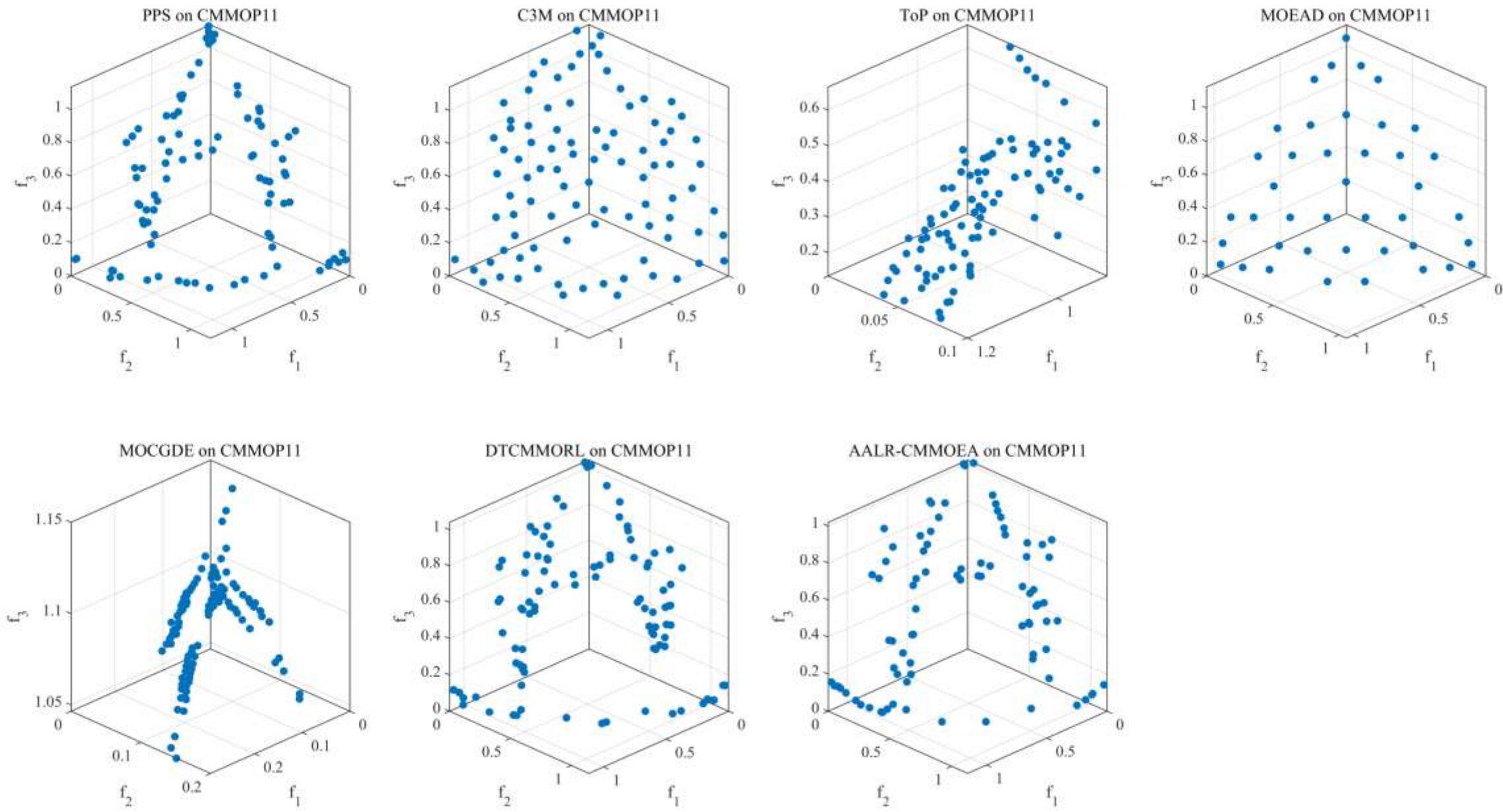


Fig. 4. PF visualization results of different algorithms on the CMMOP11 problem.

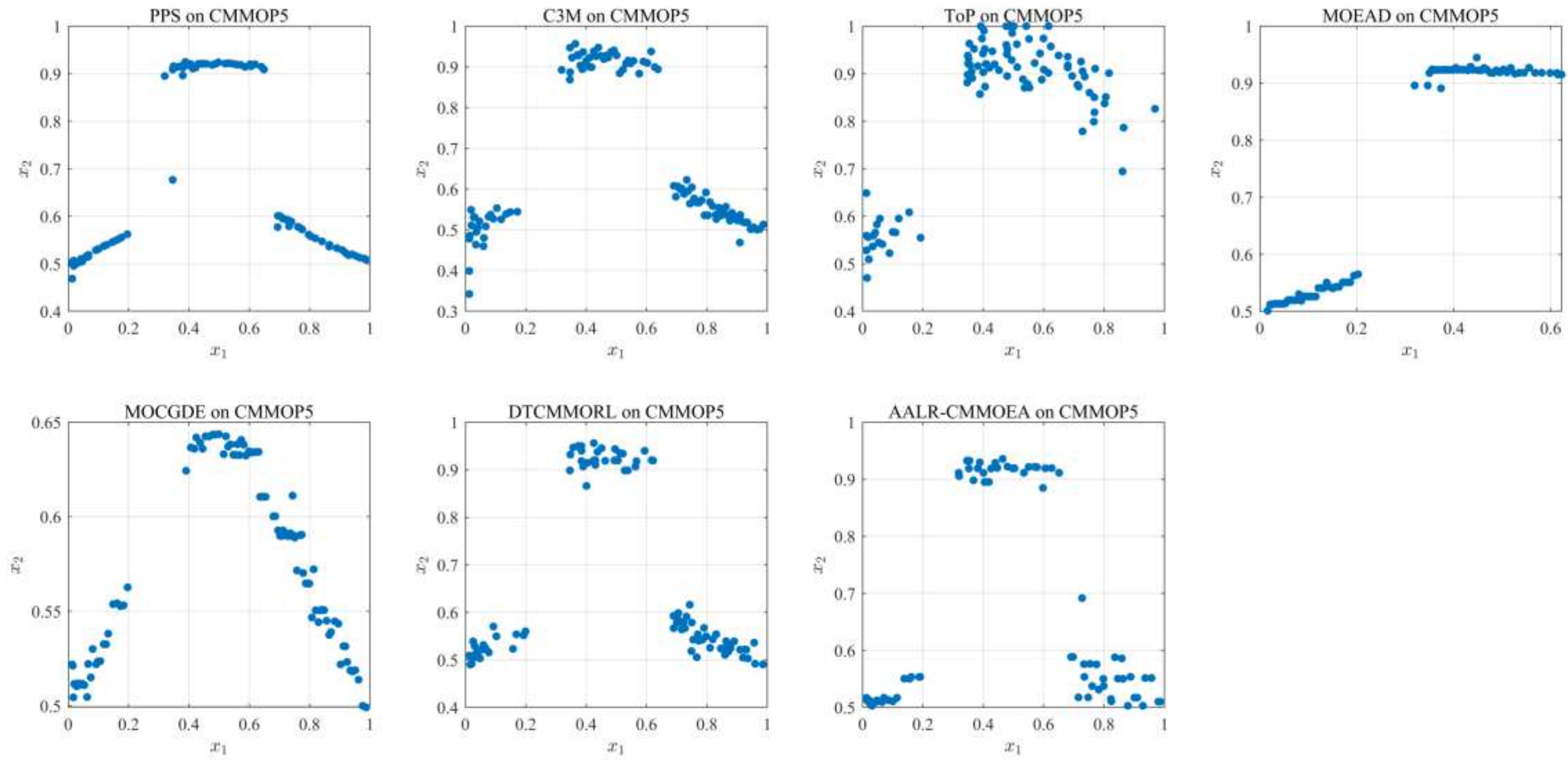


Fig. 5. PS visualization results of different algorithms on the CMMOP5 problem.

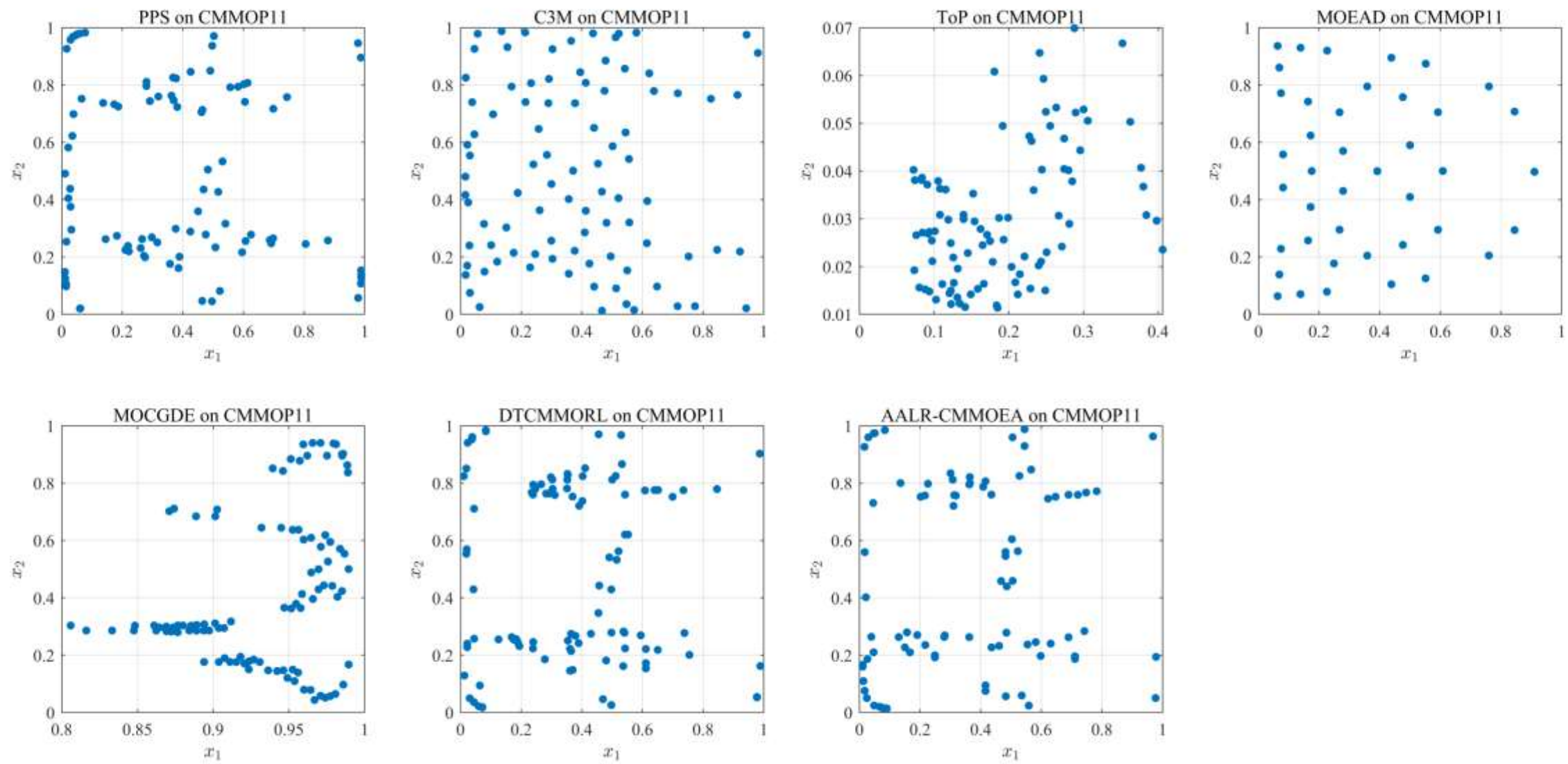


Fig. 6. PS visualization results of different algorithms on the CMMOP11 problem.

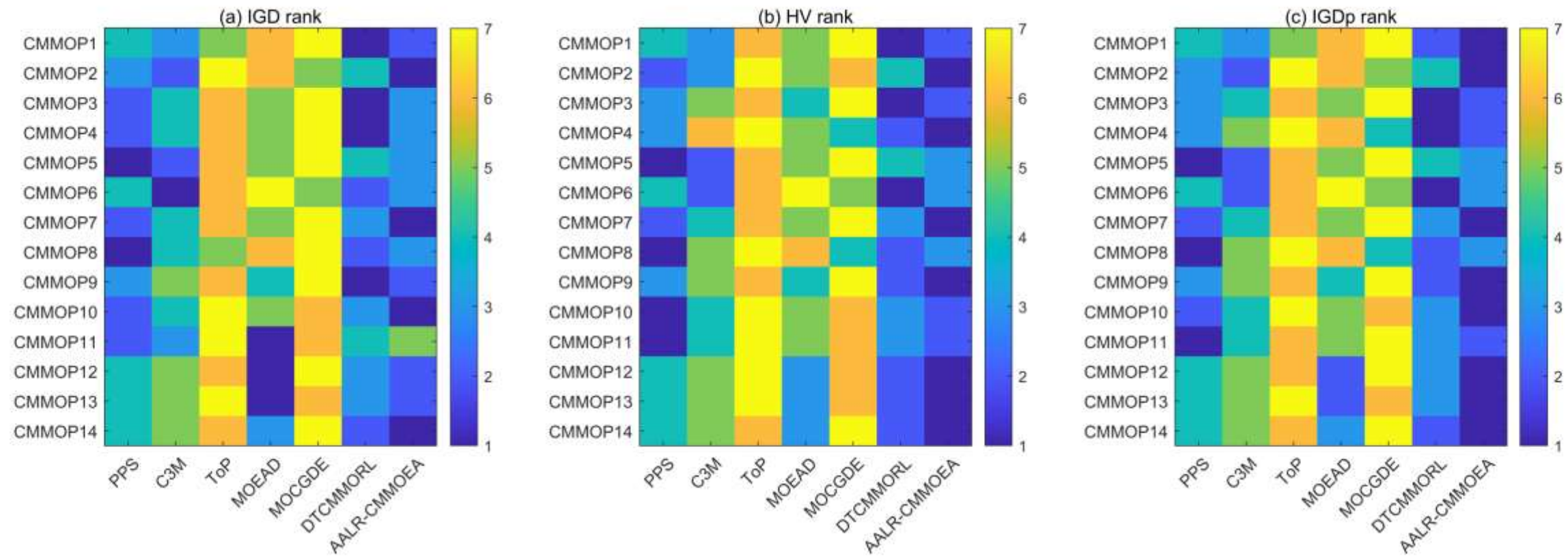


Fig. 7. Heatmap of ranking results of different algorithms on the CMMOP benchmark suite.

4.3. Ablation Study

In order to examine how much each primary constituent of the developed AALR-CMMOEa influences the algorithmic outcome, we carry out ablation studies across the CMMOP benchmark suite. To do this, we isolate each component by holding all other parameters and mechanisms constant. We formulate three distinct algorithmic modifications, with each variant systematically omitting a single core mechanism:

AALR-CMMOEa_NoLR: Here, the Lagrangian relaxation model is excluded. In its place, a standard constraint handling approach is utilized during the environmental selection phase, which serves to assess the specific impact of the Lagrangian relaxation technique within highly constrained landscapes.

AALR-CMMOEa_NoEps: the dynamic ϵ -constraint relaxation strategy is removed, and a strict feasibility-based selection criterion is used for population updating, in order to investigate the influence of adaptive constraint relaxation on the search process.

AALR-CMMOEa_NoAtt: the adaptive attention mechanism is removed, and a fixed search strategy is used during evolution, to assess the contribution of the attention-guided search mechanism to population exploration capability.

According to the statistics in Table 5, the fully integrated AALR-CMMOEa yields outstanding IGD results across a vast majority of the test instances. This confirms that incorporating the three core components successfully boosts the optimization performance of our proposed framework.

More precisely, comparing the full model against its NoAtt version reveals that omitting the self adaptive attention component triggers a substantial drop in efficiency across various test cases. In fact, on three specific benchmark problems, NoAtt falls far short of the performance level achieved by the integrated model. This outcome proves that our attention module effectively redistributes computational efforts in real time based on the current distribution of the population. Consequently, the algorithm consistently concentrates on highly promising zones and sparsely populated modal regions. This active redirection not only facilitates a closer approximation to the true constrained Pareto front but also enhances the overall quality of convergence while maintaining diverse equivalent solutions.

Furthermore, evaluating the comparative data between the NoEps variant and the full AALR-CMMOEa shows that the dynamic epsilon constraint relaxation scheme plays a vital role in determining the algorithm's overall search efficacy. Specifically, the NoEps version is defeated by the complete framework on five separate test instances, while managing to surpass it on only a single benchmark problem. These observations indicate that the dynamic epsilon relaxation scheme successfully arbitrates the conflict between satisfying constraints and exploring the decision landscape.

Specifically, in the early evolutionary stage, appropriately relaxing the constraint

conditions enables the population to traverse complex infeasible regions and discover more promising solutions. As the evolution proceeds, gradually tightening the constraint conditions ensures convergence toward the true constrained Pareto front. Therefore, this strategy can effectively alleviate the search stagnation caused by overly strict constraints at early stages, thereby improving the overall optimization performance.

For the NoLR variant, no significant performance differences are observed on most test problems compared with the complete AALR-CMMOEa; however, performance degradation can still be observed on several benchmark instances. This indicates that the Lagrangian relaxation mechanism does not directly determine the upper bound of the algorithm's performance, but it plays an important role in enhancing robustness under complex constrained environments.

By integrating penalty and constraint details directly into the Lagrangian formulation and actively modulating constraint pressure, our algorithm succeeds in sustaining a thorough exploration adjacent to the feasible boundary. This strategy successfully deters the population from stagnating in sub optimal feasible zones, thereby enhancing the algorithm's capability to capture intricate constrained Pareto optimal sets.

To visually inspect these dynamics, **Fig. 8** and **Fig. 9** depict the CPF and CPS approximations generated by the fully fledged AALR-CMMOEa alongside its three ablation variants on the CMMOP8 benchmark. Specifically, the plotted CPFs reveal how well the models converge and distribute their solutions in the objective environment. Meanwhile, the corresponding CPS plots reflect their capacity to identify several equivalent optimal subsets scattered across the variable landscape.

The visualizations clearly demonstrate that AALR-CMMOEa approximates the true CPF more closely and secures a much wider, more thorough distribution of CPSs throughout the decision landscape. On the contrary, omitting any single feature, such as the Lagrangian model, the adaptive epsilon strategy, or the attention module, leads to noticeable performance losses. These deficits manifest as poorer convergence in the objective space or reduced diversity across the variable space. These graphical outcomes strongly reinforce the crucial contribution of each designed algorithmic component.

Synthesizing the complete set of empirical findings, we can conclude that the self adaptive attention scheme, the dynamic epsilon constraint relaxation, and the Lagrangian relaxation mechanism collectively drive the optimization success of our algorithm. More precisely, the attention module and the dynamic epsilon relaxation are primarily responsible for the substantial gains observed under the IGD metric. On the other hand, the Lagrangian relaxation model is instrumental in fortifying algorithmic robustness and stability when dealing with highly challenging constrained landscapes.

The synergy of these three components enables the proposed AALR-CMMOEa to simultaneously achieve feasibility, convergence, and diversity in constrained multimodal multi-objective optimization problems, thereby obtaining higher-quality constrained Pareto solution sets. From the overall statistical results, the complete

algorithm consistently achieves superior or non-inferior performance compared with all variant algorithms on most test problems, further validating the rationality and effectiveness of the proposed components.

Table 5. Mean IGD values of AALR-CMMOEa and its three variant algorithms on the CMMOP benchmark suite.

Problem	NoLR	NoEps	NoAtt	AALR-CMMOEa
CMMOP1	2.0249e-2 (1.25e-2) =	3.3812e-2 (4.48e-2) =	1.7879e-2 (2.66e-3) -	1.6274e-2 (1.58e-3)
CMMOP2	1.1235e-2 (3.42e-3) =	2.4909e-1 (2.27e-1) -	1.9025e-2 (3.13e-2) -	9.6888e-3 (2.66e-3)
CMMOP3	1.7256e-2 (1.56e-2) =	2.2037e-2 (1.82e-2) =	2.7989e-2 (2.65e-2) =	2.2093e-2 (2.23e-2)
CMMOP4	1.1252e-2 (1.74e-3) =	6.3238e-2 (1.09e-1) -	2.1922e-2 (5.30e-2) =	2.3002e-2 (5.42e-2)
CMMOP5	9.5453e-2 (2.19e-4) =	9.6603e-2 (6.28e-3) =	9.5571e-2 (4.11e-4) =	9.5399e-2 (2.20e-4)
CMMOP6	1.2776e-2 (1.25e-4) =	8.9700e-2 (1.04e-1) -	1.7062e-2 (2.39e-2) =	1.7085e-2 (2.39e-2)
CMMOP7	1.8603e-2 (1.48e-2) =	4.4617e-2 (3.33e-2) -	2.0326e-2 (1.50e-2) -	1.5846e-2 (9.61e-4)
CMMOP8	2.0455e-2 (2.13e-2) +	3.8867e-2 (5.47e-2) =	2.6937e-2 (2.18e-2) =	2.8929e-2 (2.56e-2)
CMMOP9	1.2342e-2 (1.91e-3) =	1.4743e-2 (7.76e-3) =	1.8408e-2 (2.10e-2) =	1.8744e-2 (1.84e-2)
CMMOP10	8.6552e-2 (7.51e-5) =	2.7432e-1 (1.61e-1) -	1.0986e-1 (6.04e-2) =	8.6535e-2 (6.53e-5)
CMMOP11	1.1109e-1 (7.98e-3) =	1.2436e-1 (7.68e-2) =	1.1092e-1 (7.13e-3) =	1.0850e-1 (8.05e-3)
CMMOP12	9.0015e-2 (2.82e-3) =	8.7552e-2 (3.18e-3) +	8.8499e-2 (3.94e-3) =	8.9264e-2 (3.43e-3)
CMMOP13	8.7374e-2 (3.84e-3) =	8.8894e-2 (2.82e-3) =	8.8958e-2 (4.03e-3) =	8.7334e-2 (3.55e-3)
CMMOP14	7.7428e-2 (2.86e-3) =	7.9046e-2 (2.43e-3) =	7.7564e-2 (1.99e-3) =	7.7522e-2 (2.92e-3)
+/-/=	1/0/13	1/5/8	0/3/11	

(“-” means AALR-CMMOEa better than compared algorithm, “+” means compared algorithm better than AALR-CMMOEa, “=” means not comparable).

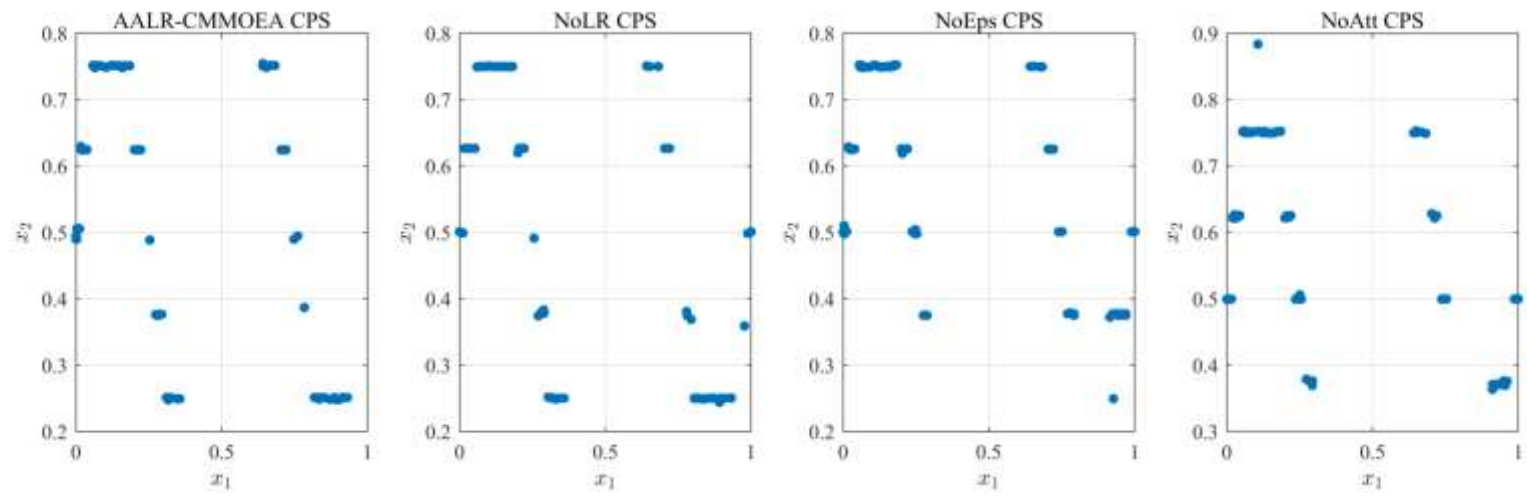


Fig. 8. PS visualization results of AALR-CMMOEA and its three variant algorithms on the CMMOP8 problem.

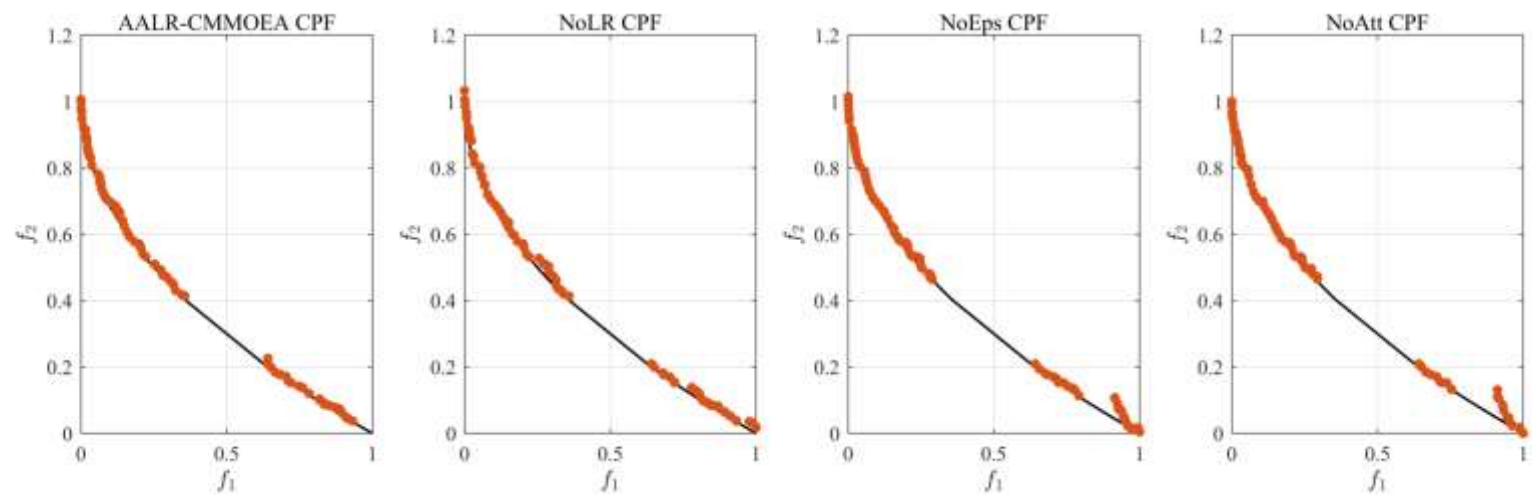


Fig. 9. PF visualization results of AALR-CMMOE and its three variant algorithms on the CMMOP8 problem.

4.4. Parameter Sensitivity Analysis

(1) epsPow

Within the AALR-CMMOEA framework, the parameter epsPow serves to regulate how quickly the relaxation threshold decays during the dynamic constraint relaxation process. Specifically, its magnitude dictates the transition speed when moving from a relaxed constraint phase to a strict constraint phase. A higher epsPow value triggers a more rapid contraction of the relaxation threshold. This strengthens the selection pressure driving the population toward feasible regions during the final phases, thereby speeding up convergence to the constrained Pareto front. Nevertheless, restricting the search space too quickly can limit early stage exploration, which impairs the algorithm's capacity to thoroughly investigate latent multimodal regions.

On the contrary, a lower epsPow value extends the duration of the relaxed constraint phase, boosting the population's capacity to navigate infeasible areas and locate several equivalent constrained Pareto optimal sets. At the same time, this slower transition can retard the rate of convergence during the final optimization steps. Consequently, selecting an optimal setting for epsPow is highly important to maintain a proper trade off between preserving diversity during exploration and driving convergence through feasibility during exploitation.

To further investigate the impact of the parameter epsPow on algorithm performance, the proposed AALR-CMMOEA is evaluated on all CMMOP benchmark problems with different parameter settings, i.e., $\text{epsPow} \in \{0.5, 1, 1.5, 2, 2.5, 3\}$. The Friedman test is then employed to statistically analyze the average ranking results in terms of IGD and HV metrics, as shown in **Fig. 10**.

As observed from the statistics, the Friedman mean ranks vary only slightly across different epsPow configurations. This suggests that the developed algorithm is highly insensitive to variations in this parameter. Specifically, setting epsPow to 2 yields the most favorable average rank under the IGD metric, which demonstrates its capacity to approach the constrained Pareto front while successfully satisfying feasibility requirements. Regarding the HV indicator, epsPow values of 0.5 and 3 deliver similar outcomes, though they do not present any substantial overall advantage.

Furthermore, **Fig. 11** and **Fig. 12** illustrate the PF and PS profiles produced under different epsPow configurations on the CMMOP8 test instance. These plots reveal that our method successfully converges to the true CPF while preserving adequate CPS coverage across a broad spectrum of parameter settings. This behavior confirms the outstanding stability of our dynamic constraint relaxation approach.

Particularly, when $\text{epsPow} = 2$, the optimization process establishes a more harmonious trade off between early stage global search and final phase local convergence. Taking into account both the convergence quality and the coverage of optimal solution sets, we designate $\text{epsPow} = 2$ as the default parameter configuration in our framework. These empirical assessments further verify that the dynamic constraint relaxation scheme remains highly stable over a wide range of parameter

values. This confirms that our algorithm is not overly susceptible to parameter tuning, highlighting its high practical utility.

(2) $\lambda_{\max}$

In AALR-CMMOEA, the parameter $\lambda_{\max}$ represents the upper bound of the Lagrangian multipliers, which directly affects the adjustment range of constraint pressure. To investigate its influence on algorithm performance, different values of $\lambda_{\max} = \{1, 5, 10, 20\}$ are tested, and the IGD and HV results are statistically analyzed on selected CMMOP benchmark problems. As shown in **Fig. 13**, the overall performance variation under different $\lambda_{\max}$ settings is relatively small, indicating that the proposed algorithm is not highly sensitive to this parameter and exhibits good robustness.

Although a slightly better performance can be observed with larger $\lambda_{\max}$ values on some problems, the overall improvement is not significant. As illustrated in **Fig. 14** and **Fig. 15**, the PF and PS visualizations under different $\lambda_{\max}$ settings further confirm that the proposed method maintains good convergence and diversity in the decision space across most configurations, demonstrating that the upper bound of the Lagrangian multipliers has a limited impact on the overall performance.

Considering both convergence and distribution quality, $\lambda_{\max} = 10$ is selected as the default setting, as it achieves a relatively balanced trade-off between constraint pressure regulation and population diversity preservation.

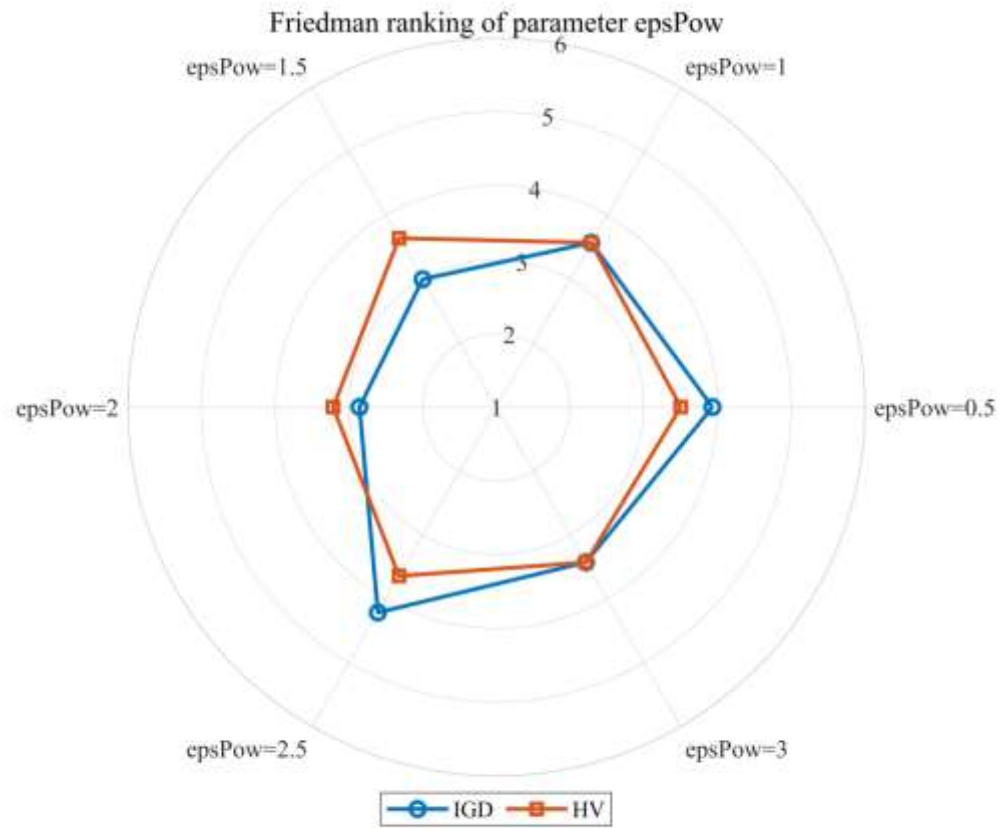


Fig. 10. Friedman ranking results of AALR-CMMOEA with different epsPow settings on the CMMOP benchmark suite.

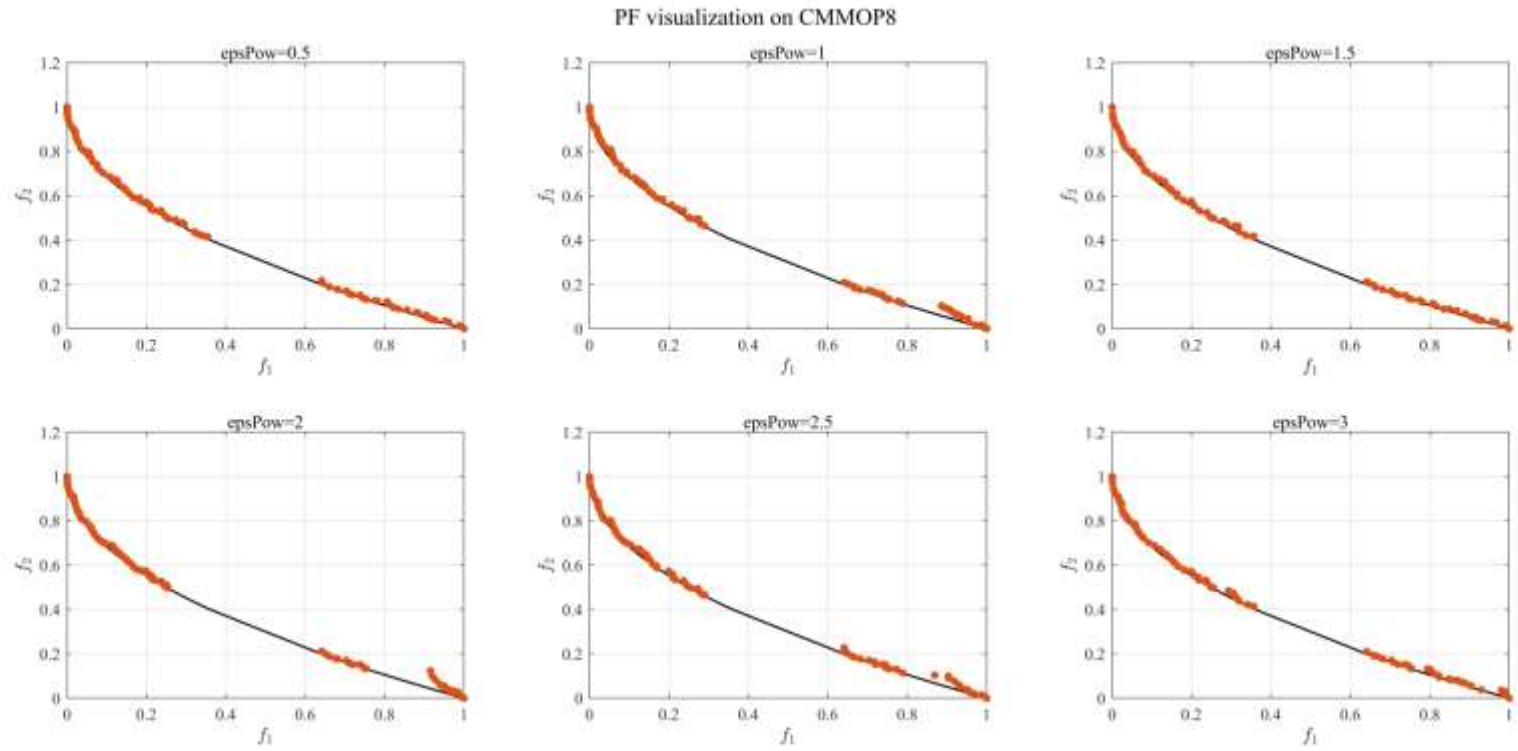


Fig. 11. PF visualization results of AALR-CMMOEA with different epsPow settings on the CMMOP8 problem.

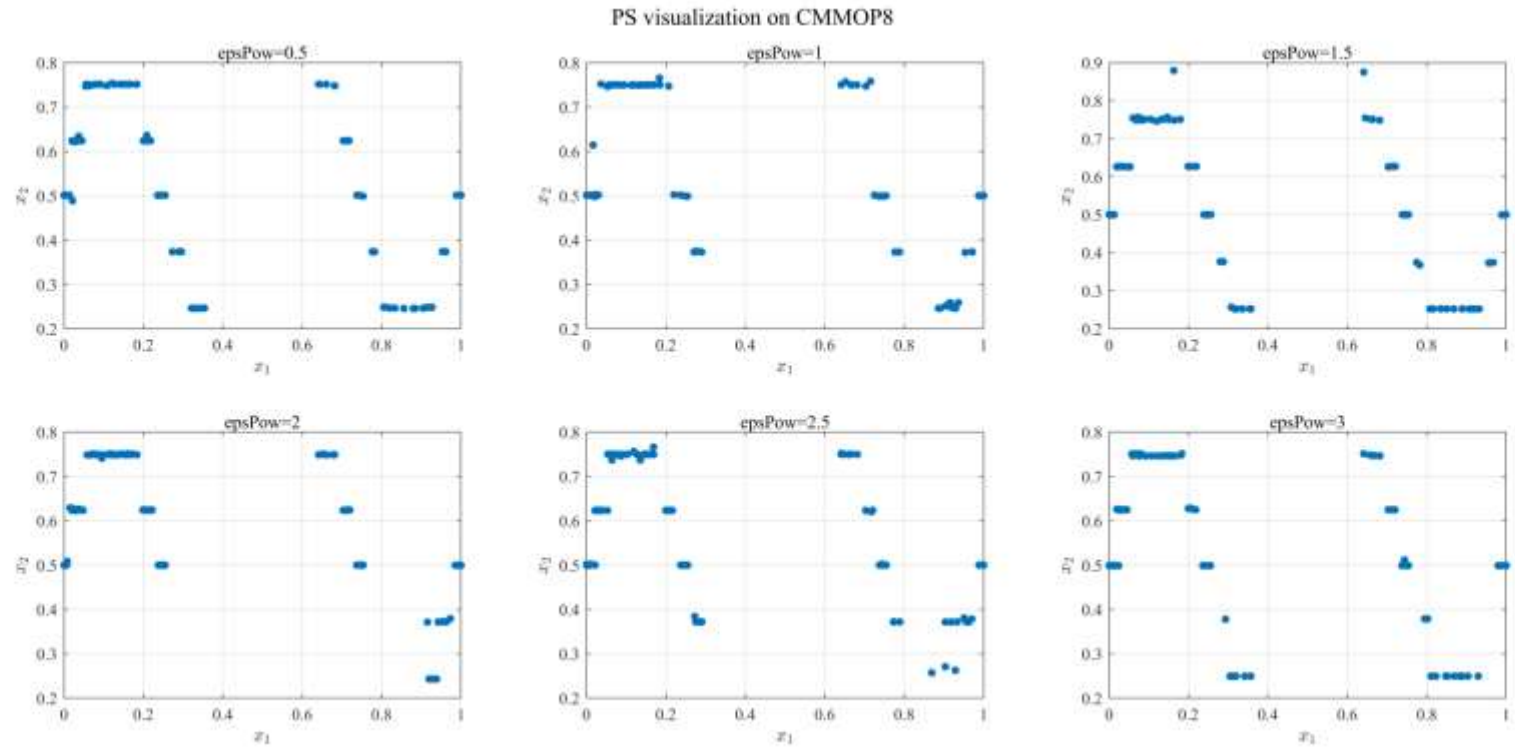


Fig. 12. PS visualization results of AALR-CMMOEa with different epsPow settings on the CMMOP8 problem.

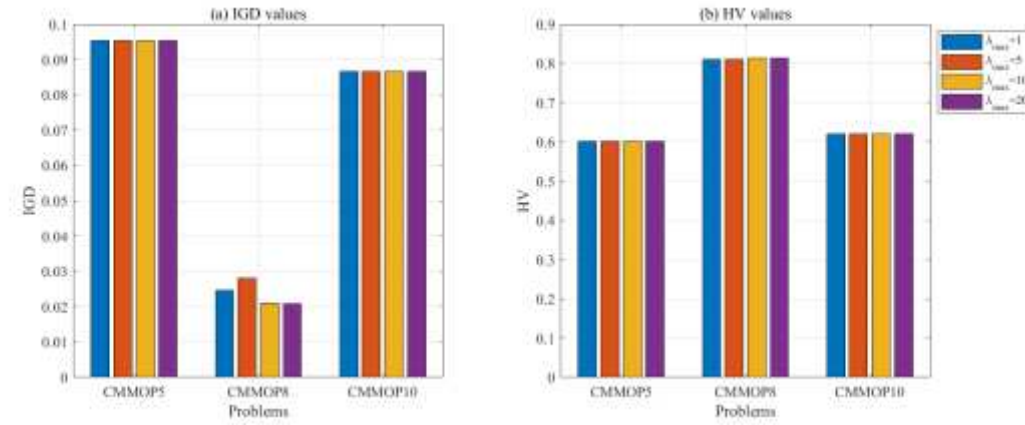


Fig. 13. Sensitivity analysis results of AALR-CMMOEA with respect to the parameter $\lambda_{\max}$ on the CMMOP benchmark suite.

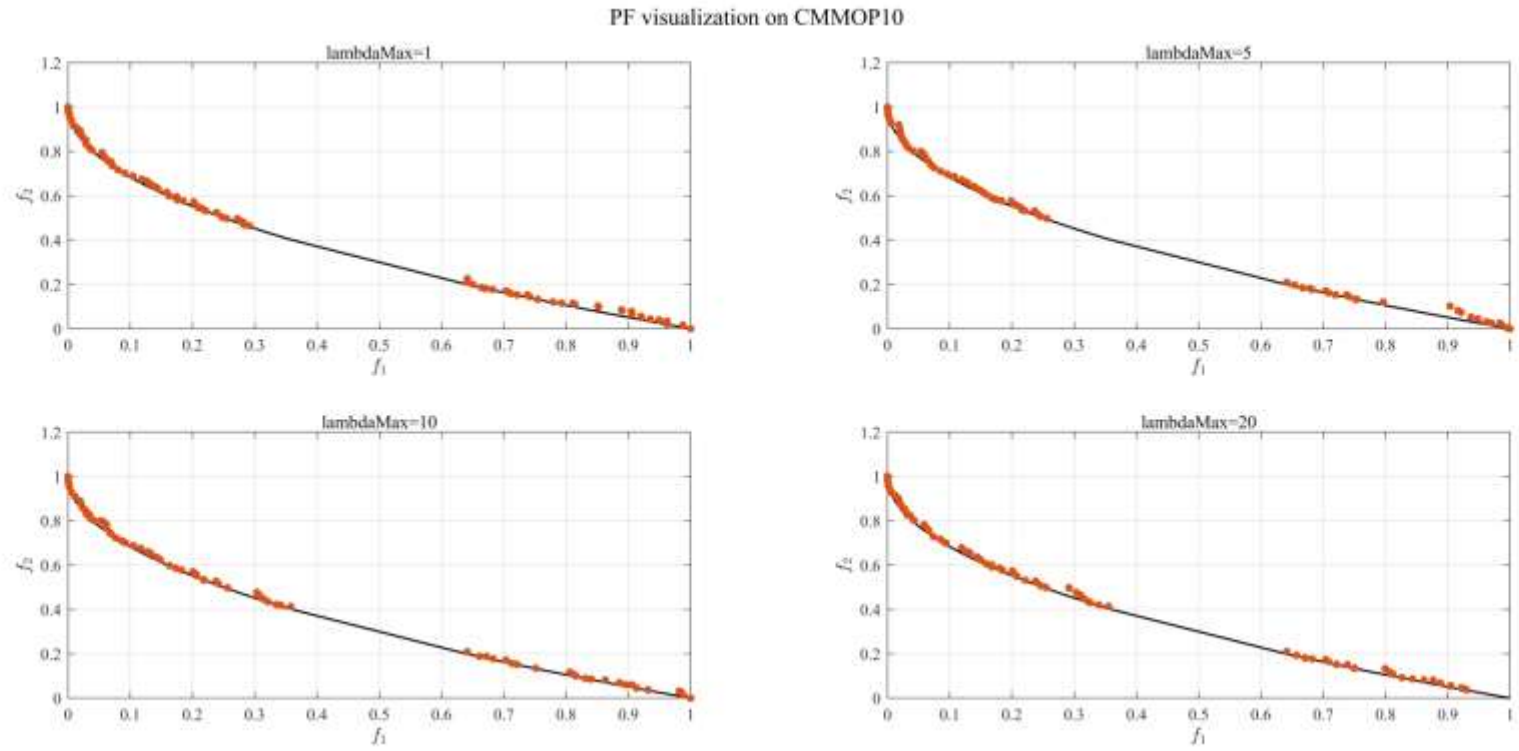


Fig. 14. PF visualization results of AALR-CMMOE with different $\lambda_{\max}$ settings on the CMMOP10 problem.

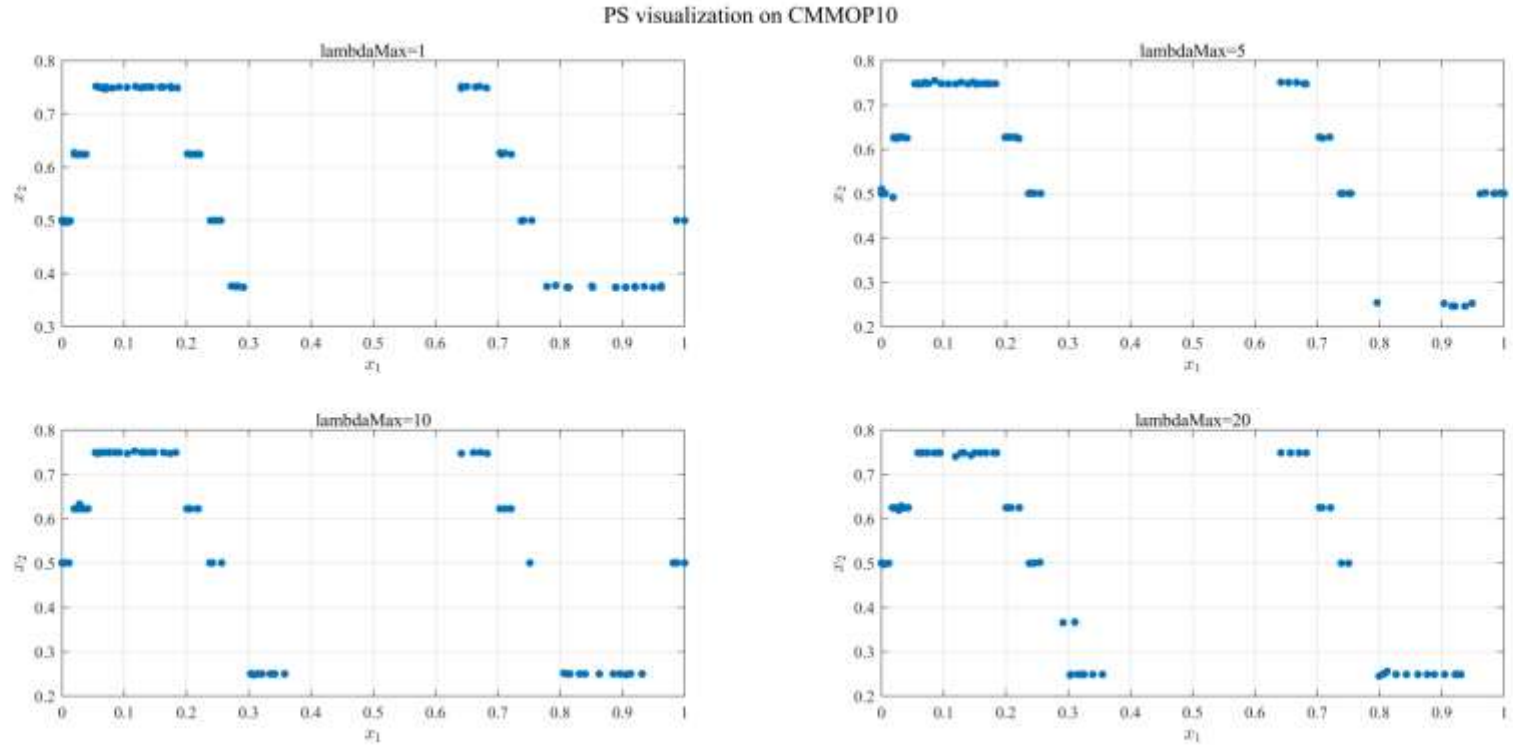


Fig. 15. PS visualization results of AALR-CMMOEa with different $\lambda_{\max}$ settings on the CMMOP10 problem.

5. Conclusion and Future Work

To tackle the mismatch between multimodal diversity, feasibility, and convergence within CMMOPs, this study proposes a novel evolutionary algorithm integrated with Lagrangian relaxation and adaptive attention, named AALR-CMMOEA. The algorithm utilizes a co-evolutionary scheme structured around two distinct populations. Within this setup, a dynamic ε -constraint relaxation technique is utilized to boost initial exploration, whereas a Lagrangian relaxation module dynamically calibrates constraint pressure in response to the population's feasibility proportion and violation extent. Concurrently, we incorporate an adaptive attention mechanism designed to direct search resources toward high-potential and sparse regions, which enhances the algorithm's performance in uncovering multiple constrained Pareto optimal sets.

Empirical assessments conducted on the CMMOP benchmark suite reveal that AALR-CMMOEA delivers highly competitive or even superior results across IGD, HV, and IGDp performance indicators relative to multiple state-of-the-art baseline algorithms. Moreover, ablation analyses validate the individual contributions and efficacy of each designed module.

Future research will focus on analyzing the scalability of the developed algorithm when applied to high-dimensional CMMOPs. Additionally, to optimize search efficiency, we intend to explore more advanced self-adaptive schemes, such as stage control and constraint regulation models driven by reinforcement learning. Ultimately, the developed framework will be applied to practical engineering design challenges to test both its robustness and real-world utility.

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