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Teaching symmetry through mathematical thought experiments: A quasi-experimental study

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Abstract

This study examined the impact of educating axial and central symmetry, as key concepts in geometric transformations, through mathematical thought experiments on fifth-grade students' academic achievement and APOS-based learning levels. This study employed an explanatory sequential mixed-methods design. The quantitative phase used a quasi-experimental pretest–posttest design with experimental and control groups, while the qualitative phase involved thematic analysis of task-based interviews and classroom observations to explain and enrich the quantitative findings. The population consisted of all male fifth-grade students in Tehran, Iran, during the 2023–2024 academic year. A convenience sample of 34 students took part in the research. A researcher-made test grounded in the APOS theory (Cronbach's alpha = 0.72) assessed academic achievement and learning levels. Over twelve 45-minute sessions, the experimental group learned symmetry through mathematical thought experiments, whereas the control group received conventional instruction. Descriptive results included means and standard deviations for all variables. The Kolmogorov–Smirnov test verified that the data followed a normal distribution. Hypotheses were tested with ANCOVA and Quade's ANCOVA. Findings showed that instruction based on mathematical thought experiments produced significant gains in both learning levels and academic achievement. Thematic analysis further identified key patterns in students' reasoning that supported these outcomes. The approach supported the development of solid knowledge structures and deeper conceptual understanding of symmetry.

1. Introduction

In contemporary society, the ability to think clearly and solve problems effectively is essential in both professional and daily contexts. Mathematics equips individuals with tools to confront challenges and develop solutions (Amalina & Vidákovich, 2023; Rivera, 2011). Consequently, a primary objective of mathematics education is to cultivate thinking skills and strengthen problem-solving capabilities. Geometry occupies a distinctive position as a branch that interconnects various areas of mathematics. Geometric skills are therefore fundamental competencies that students must develop, making geometry education critically important (National Council of Teachers of Mathematics [NCTM], 2000). The NCTM has advocated for the integration of geometry instruction at all educational levels, from elementary through high school. Two central aims characterize geometry learning: developing stronger reasoning abilities and enhancing spatial visualization skills (Alex & Mammen, 2016). Spatial visualization emerges from how individuals perceive and mentally represent objects, meaning that much of geometric reasoning fundamentally depends on visualization (Finke, 1985, as cited in Smith & Thelen, 2003). Throughout the history of mathematics education, visualization has played a substantial role in teaching and learning, particularly when addressing geometric problems (NCTM, 2000). In mathematical contexts, visualization enables students to discover concepts and achieve deeper understanding by forming and manipulating mental or drawn images. Students can generate these representations using paper and pencil, digital tools, or purely mental imagery (Gilligan-Lee et al., 2022; Jones, 2001). However, teaching mathematical concepts through visualization often proves more challenging than traditional lecture methods. Identifying clear, concrete examples for abstract ideas can be difficult, and many teachers lack extensive experience with visualization-based pedagogical approaches (Wilkerson et al., 2025). Mathematical thought experiments offer an alternative instructional pathway. These activities rely heavily on visualization while simultaneously guiding systematic reasoning. Teachers and researchers employ them deliberately within cognitive strategies to address scientific questions and open new perspectives on mathematical concepts (Clement, 2006, 2008; Reiner, 2006; Heinzmann, 2022; Buzzoni, 2022; Buzzoni & Agazzi, 2023).

Research indicates that thought experiments activate students' imagination, creativity, and problem-solving abilities (Helm et al., 1985; Lattery, 2001; Matthews, 1994; Reiner, 1998; Reiner & Burko, 2003; Gilbert & Reiner, 2000; Stephens & Clement, 2006). For these reasons, thought experiments have gained recognition as valuable pedagogical tools that develop creativity, intuition, and problem-solving skills (Gilbert & Reiner, 2000). Gilbert and Reiner (2000) demonstrated that students frequently solve problems by constructing imaginary scenarios in their minds. Such thought experiments engage students in active interaction with the material and its context, leading to enhanced understanding. For genuine understanding-sometimes characterized as cognitive change-to occur, students must undertake intentional mental, verbal, or physical actions (Barsalou, 1995). Thought experiments provide substantial support when students encounter difficult scientific concepts. By working through these experiments, students acquire experiential knowledge that produces understanding, and this understanding is intrinsically rewarding (Helm et al., 1985; Matthews, 1994; Nersessian, 1992; Reiner & Gilbert, 1990; Sharifi et al., 2025).

Geometric transformations constitute a fundamental topic within geometry. They appear across numerous mathematical branches and establish connections among those domains (Knuchel, 2004; Giménez, Thaqi, & Vanegas, 2021). The NCTM (2000) specifies that students from pre-kindergarten through grade 12 should recognize geometric transformations and apply them when analyzing mathematical situations. Mastering these concepts enables students to decompose shapes, analyze them systematically, reconstruct them, solve problems, and reason spatially (Bayazit & Aksoy, 2009; Xie et al., 2025). Additionally, understanding geometric transformations allows students to make sense of phenomena in their environment and deepens their comprehension of the physical world (Knuchel, 2004). Symmetry represents one of the most prevalent applications of geometric transformations in both mathematics and everyday life. This fundamental concept helps students perceive geometric shapes and their properties more clearly. Over time, teachers can connect symmetry to other mathematical topics, including geometry, algorithms, and number patterns (Knuchel, 2004). When students learn symmetry concepts, their visual and spatial skills improve, enhancing their ability to identify patterns and symmetries in shapes and figures (Clements & Sarama, 2009, 2011). Research on learners' understanding of geometric transformations, including symmetry, reveals that achieving complete comprehension is challenging. Students require extended time and diverse experiences to construct a clear and coherent understanding of the topic (Yanik & Flores, 2009; Mukuka et al, 2024). Because symmetry is inherently abstract, and because thought experiments support abstract reasoning, combining these two approaches appears promising for improving student learning (Hinzmann, 2022; Sharifi et al., 2025). However, before thought experiments can be designed effectively, teachers and researchers must first identify the mental models and misconceptions students hold regarding symmetry. Cognitive theories such as APOS (Action, Process, Object, Schema) provide a useful framework for uncovering those models and errors.

According to APOS theory, students who possess appropriate mental structures for a mathematical concept learn that concept more readily. Without those structures, learning becomes considerably more difficult or even impossible (Dubinsky, 1991). APOS theory describes how learners construct and organize mathematical knowledge cognitively, illustrating how individuals gradually build a concept through a sequence of mental steps upon first encountering it. The theory presents a developmental cycle through which a mathematical idea takes shape, becomes stable, and grows more sophisticated. The primary goal of APOS theory is to explain the process of mathematical learning through four mental structures: Action, Process, Object, and Schema. Assessments based on APOS theory can help identify students' misconceptions (Dubinsky & McDonald, 2001; Arnon et al., 2014; Tuktamyshov & Gorskaya, 2024; Palobo, Sulaiman, & Rahaju, 2025).

Since symmetry is a foundational concept that prepares students for more advanced mathematics in later grades and also influences analytical thinking, cognitive abilities, and creativity, the instructional methods used to teach symmetry in elementary school are therefore of considerable importance. The present study investigates how teaching symmetry through thought experiments influences the academic achievement and cognitive learning levels of elementary students.

2. Thought Experiments and Their Role in Education

A thought experiment is a mental activity in which an individual imagines a situation or makes hypothetical assumptions to reach new understanding. Scientists use such experiments as a standard component of knowledge construction, and they constitute established practices within scientific inquiry (Reiner, 1998; Hadzidaki, 2008; Gilbert & Reiner, 2000; Galili, 2009). When various characterizations are synthesized, a thought experiment emerges as a three-step process: (1) an individual creates an imagined situation in the mind, (2) the individual examines that situation by forming a mental hypothesis, and (3) the individual draws a conclusion from analyzing the hypothesis (Gendler, 2014). Thought experiments typically begin as private mental activities, but teachers and students frequently employ them in group discussions to clarify concepts. This social application aligns well with learning principles because dialogue, interaction, and idea sharing help students adjust and refine their understanding (Reiner, 2012). In essence, thought experiments function both as personal cognitive tools and as prompts for collaborative discourse, which facilitates deeper comprehension of challenging ideas in mathematics and science. Vygotsky and subsequent research demonstrate that authentic understanding develops through social interaction (Vygotsky, 1978; Tharp & Gallimore, 1988; Callanan et al., 1995). When students engage in dialogue, pose questions, and negotiate meanings collectively, their concepts gradually align and deepen. Genuine learning occurs when individuals articulate their thoughts, test them against others' perspectives, and arrive at a shared, more refined understanding. In classroom settings, thought experiments invite students to imagine situations and then reason systematically toward conclusions (Reiner & Gilbert, 2000). These activities connect everyday experiences to new mathematical concepts and render abstract ideas more concrete (Helm et al., 1985). Simultaneously, a thought experiment remains fundamentally a private reasoning process within an individual's mind. However, when a student explains a mental representation to the group or uses it to justify an answer, the activity becomes collaborative. Different students may construct slightly different scenarios or notice distinct details. By listening to one another and synthesizing these varied images and insights, they solve problems together more effectively (Reiner, 1998).

3. Methods

This study employed an explanatory sequential mixed-methods design, combining quantitative and qualitative approaches to examine the effect of teaching symmetry through mathematical thought experiments on students' academic achievement, and to gain deeper insight into their reasoning about symmetry. In the quantitative phase, a quasi-experimental pretest–posttest design with an experimental group and a control group was used to assess the effect of instruction based on mathematical thought experiments. The study population consisted of all male fifth-grade students in Tehran during the 2023–2024 academic year. A total of 34 students were selected through convenience sampling and randomly assigned to either the experimental group (n = 17) or the control group (n = 17). Academic achievement was measured using a researcher-developed test constructed based on APOS theory (Action, Process, Object, and Schema). All items were grounded in the fifth-grade mathematics curriculum covering axial symmetry and central symmetry. Questions were developed using the official textbook specifications table, with additional ideas drawn from supplementary workbooks and items from TIMSS and Kangaroo mathematics

competitions. The test comprised 16 multiple-choice questions, each accompanied by a space where students could provide a written explanation of their reasoning. Experienced mathematics professors and teachers with at least ten years of classroom practice reviewed the items and confirmed content and face validity against the textbook specifications table. Cronbach's alpha was 0.72, indicating acceptable reliability. According to APOS levels, the items were distributed as follows: 3 Action items, 4 Process items, 4 Object items, and 5 Schema items.

Previous research on symmetry instruction has consistently identified common misconceptions and erroneous concepts among students (Moravcová, Robová, Hromadová, & Halas, 2021). These errors, documented across multiple studies, frequently obstruct correct and deep learning. To detect such difficulties, several test items were deliberately designed to include some of these misconceptions. This approach enabled a more precise evaluation of students' actual understanding of symmetry concepts. Details of these items are given in Table 1.

Table 1
Common Misconceptions in Symmetry

APOS Cognitive Level	Challenges
Action	Some students fail to distinguish between central symmetry and axial symmetry and incorrectly classify shapes with central symmetry as axially symmetric (for example, they treat a parallelogram as axially symmetric) (Moravcová et al., 2019, 2021; Kmetová et al., 2024).
Process, Object, Schema	Students often confuse axial symmetry with central symmetry and treat the two concepts as interchangeable (Son, 2006; Jagoda, 2008; Moravcová et al., 2019, 2021).
Schema	Students struggle to identify or draw shapes that have oblique (diagonal) axes of symmetry (Moravcová et al., 2020; Jagoda, 2008).
Action	Many students treated any line that divided a shape into two parts as a line of symmetry, even when the two parts were not mirror images.
Process, Object	several students accepted only vertical lines as possible axes of symmetry and overlooked horizontal or diagonal axes in figures that clearly possessed them.

In the qualitative phase, data were collected throughout the intervention to gain a deeper understanding of students' reasoning about symmetry. These data included classroom observations, student interviews, participation in thought experiments, and written explanations provided in the post-test. These data were used to explore the reasoning processes students employed when solving symmetry tasks and to complement the quantitative findings. Data analysis was conducted using thematic analysis following the procedures proposed by Braun and Clarke (2006). The analysis involved familiarization with the data, generation of initial codes, identification of recurring patterns, development of themes and subthemes, and refinement of thematic categories. This process resulted in the identification of five major themes and multiple subthemes representing students' reasoning about symmetry.

3. 1. Instructional Procedure

Prior to the intervention, lesson plans and instructional materials on the topic of symmetry were first prepared for the control group using a conventional teaching approach. Following the preparation of these conventional lesson plans, lesson plans and materials for the experimental group were then developed based on the mathematical

thought experiment approach. All lesson plans and instructional materials were subsequently reviewed and critiqued by three teachers (each with more than ten years of teaching experience) and three mathematics education faculty members. After the suggested revisions were made, the materials were approved.

Both groups completed the same researcher-developed achievement test as a pretest before instruction commenced and as a posttest following the intervention. Teaching occurred in person during twelve 45-minute sessions. The control group followed conventional teaching approaches, while the experimental group learned axial and central symmetry through mathematical thought experiments. In the experimental group, each central concept of the symmetry unit was presented through a thought experiment structured as a step-by-step activity. The instructional sequence consisted of: (1) introducing the activity, (2) presenting a conceptual challenge, (3) constructing a mental or imaginary situation, (4) testing ideas through reasoning, (5) discussing in small groups, and (6) reaching a whole-class conclusion. When students encountered difficulties, the teacher facilitated their progress through targeted questioning. Students initially worked independently, then collaborated in groups of three, and finally shared findings with the entire class. Appendix 1 lists all activities and their overall structure. Appendix 2 presents one complete thought experiment. After data collection, descriptive statistics were computed, the Kolmogorov–Smirnov test was applied to verify normality, and inferential tests were conducted using ANCOVA and Quade's ANCOVA.

4. Findings

Data from the research instrument were entered into SPSS for statistical analysis. Initially, we organized the scores and calculated descriptive statistics for each variable. For inferential testing, we assessed normality using the Kolmogorov–Smirnov and Shapiro–Wilk tests. The Kolmogorov–Smirnov results indicated normal distribution. We then tested the hypotheses using parametric ANCOVA for the first hypothesis and nonparametric Quade's ANCOVA for the seconhypothesis, executed in R. The significance level for all tests was set at $\alpha = 0.05$.

4. 1. Hypothesis Testing

Here, consider two following hypothesis testing:

Hypothesis 1

Teaching symmetry concepts through mathematical thought experiments has a positive and significant effect on the academic achievement of fifth-grade elementary students.

This hypothesis was tested using univariate ANCOVA, with the pretest score as the covariate and the posttest score as the dependent variable. All required assumptions were satisfied. The Kolmogorov–Smirnov test confirmed normal distribution of posttest scores in both groups ($p = 0.20$ for control, $p = 0.20$ for experimental). Levene's test verified equal variances ($F(1, 32) = 1.133$, $p = 0.295$). The test for homogeneity of regression slopes was also satisfied ($F(1, 30) = 1.744$, $p = 0.197$). The ANCOVA results revealed a significant main effect for group ($F(1, 31) = 20.656$, $p < 0.001$, $\eta^2 = 0.400$), indicating that the experimental group significantly outperformed the control group on adjusted posttest academic achievement scores (see Table 2). The pretest score also significantly predicted the posttest score ($F(1, 31) = 7.441$, $p = 0.010$). The partial eta-squared value of 0.40 indicates a large effect size for the thought-experiment teaching method according to Cohen's guidelines.

Table 2

Overall Significance Results for the Model of Academic Achievement in Symmetry Course Using Analysis of Variance

Source	Sum of Squares	*df*	Mean Square	*F*	*p*-value	η^2
Corrected Model	182. 500	2	91.250	15.987	< .001*	0.508
Intercept	141.065	1	141.065	24.715	< .001*	0.444
Group	117.900	1	117.900	20.656	< .001*	0.400
Pretest-Posttest Academic Achievement	42.471	1	42.471	7.441	.010*	0.194
Error	176.941	31	5.708			
Total	4235	34				
Corrected Total	359.441	33				

Note. R^2 = 0.508 (Adjusted R^2 = 0.476), p < .05.

The coefficient of determination (R^2) for the model was 0.508, indicating that group membership and pretest scores together accounted for approximately 50.8% of the variance in posttest scores (Table 2). ANCOVA results demonstrated a significant difference in academic achievement between the experimental and control groups (F(1, 31) = 20.656, p < 0.001). After adjusting for pretest scores, the experimental group scored 3.75 points higher on average than the control group. The pretest score also significantly affected posttest performance (F(1, 31) = 7.441, p = 0.010); each additional point on the pretest was associated with a 0.731-point increase on the posttest (Table 3).

Table 3

ANCOVA Results for the Effect of Educational Intervention on Academic Achievement

Parameter	Beta	Standard Error	*t*-statistic	*p*-value	Eta Coefficient
Intercept	5.12	1.41	3.61	0.001*	0.297
Academic Achievement - Pretest	0.73	0.26	2.72	0.010*	0.194
Group (Control)	3.75	0.82	4.54	< 0.001*	0.400
Group (Educational Intervention)	–	–	–	–	–

Note. p < .05.

Hypothesis 2

Teaching symmetry through mathematical thought experiments has a positive and significant effect on students' cognitive levels as defined by APOS theory.

To test this hypothesis, responses to the researcher-developed test items were first scored separately for each of the four APOS levels (Action, Process, Object, and Schema). Whether the instructional intervention affected posttest performance at each cognitive level was then assessed after adjusting for pretest scores, which served as covariates. The Shapiro–Wilk test indicated that posttest scores at each level did not follow a normal distribution (p < .05). Because the normality assumption was violated, Quade's nonparametric ANCOVA was employed instead of

standard ANCOVA. This test compares posttest scores between groups while controlling for pretest scores without requiring normality. Separate analyses were conducted for the Action, Process, Object, and Schema levels. The means and standard deviations for pretest and posttest scores at each of the four cognitive levels are reported in Table 4.

Table 4
Descriptive Values for Different Cognitive Levels of Student Learning (Action, Process, Object, and Schema)

Variable	Measurement Stage	Experimental Group; Mean ± SD	Control Group; Mean ± SD
Action	Pretest	1.59 ± 0.71	1.76 ± 0.66
	Posttest	2.88 ± 0.33	1.76 ± 0.83
Process	Pretest	1.65 ± 1.17	1.71 ± 0.77
	Posttest	3.41 ± 0.71	2.65 ± 1.11
Object	Pretest	1.12 ± 0.70	0.82 ± 0.73
	Posttest	3.41 ± 0.71	2.41 ± 1.33
Schema	Pretest	0.88 ± 0.60	0.53 ± 0.62
	Posttest	3.00 ± 1.17	1.82 ± 1.29
Note. Values presented as mean ± standard deviation.			

As noted previously, nonparametric Quade's ANCOVA was applied to assess the effect of the teaching intervention on students' cognitive learning levels. The results are presented in Table 5.

Results demonstrated that the teaching method based on thought experiments produced significant gains at every APOS level. At the Action level, the experimental group improved by 1.29 points on average, while the control group showed no change ($p < 0.001$). Pretest scores at this level strongly predicted posttest scores ($p < 0.001$; see Table 5). At the Process level, the experimental group gained 1.76 points compared with 0.94 points in the control group ($p = 0.011$; see Table 4). Again, pretest scores exerted a strong influence on posttest scores ($p < 0.001$; see Table 5). At the Object level, the experimental group increased its mean score by 2.29 points, compared with 1.6 points in the control group ($p = 0.02$; see Table 4). However, pretest scores did not significantly predict posttest scores at this level (see Table 5). At the Schema level, the experimental group improved by 2.12 points, while the control group gained 1.29 points ($p = 0.021$; see Table 4). As with the Object level, pretest scores showed no significant effect on posttest scores (see Table 5).

Table 5

Results of Quade Nonparametric Analysis of Covariance Test: Effect of Educational Intervention on Cognitive Levels of Fifth-Grade Students (Action, Process, Object, and Schema)

Variable	Parameter	Sum of Squares	Degrees of Freedom	*F* Statistic	*p*-value
Action	Treatment	1231.85	1	42.6	< 0.001*
	Action Pretest	462.98	1	16.03	< 0.001*
	Residuals	895.4	31	–	–
Process	Treatment	399.98	1	7.3	0.011*
	Process Pretest	892.2	1	16.3	< 0.001*
	Residuals	1694	31	–	–
Object	Treatment	475.1	1	5.99	0.02*
	Object Pretest	165.1	1	2.08	0.16
	Residuals	2455.7	31	–	–
Schema	Treatment	486.26	1	5.85	0.021*
	Schema Pretest	163.28	1	1.96	0.17
	Residuals	2575.8	31	–	–

Note. $p < .05$.

4. 2. Mastery Levels Across APOS Stages

This study employed the Mastery Level index to measure students' conceptual understanding of symmetry. We selected this index because it aligns well with the APOS framework, which categorizes mathematical learning into four stages: Action, Process, Object, and Schema. The index provides a numerical score for performance at each stage, facilitating comparison of pretest and posttest results. The Mastery Level index also illustrates how understanding progresses sequentially across the APOS stages. Our objective was not merely to count correct answers but to track the depth of learning and conceptual development. This index served as a combined qualitative–quantitative tool that corresponded closely with the theoretical framework. Students achieved mastery at a given level only if they answered at least 80% of the items correctly for that level. The test included 3 items for Action, 4 for Process, 4 for Object, and 5 for Schema. Using this 80% threshold, we calculated the percentage of students in each group who achieved mastery at each level before and after the teaching intervention. These results are presented in Table 6.

Table 6

Percentage Frequency of Individuals with Mastery Based on 80% Criterion in Experimental and Control Groups at Pretest and Posttest Stages Across Four APOS Levels

APOS Levels	Experimental Group				Control Group			
	n **(Pretest)**	**% (Pretest)**	***n*** **(Posttest)**	**% (Posttest)**	***n*** **(Pretest)**	**% (Pretest)**	***n*** **(Posttest)**	**% (Posttest)**
Action	0	0.0%	15	88.2%	1	5.9%	3	17.6%
Process	0	0.0%	9	52.9%	1	5.9%	5	29.4%
Object	0	0.0%	9	52.9%	0	0.0%	4	23.5%
Schema	0	0.0%	5	29.4%	0	0.0%	2	11.8%

Note. *n* = number of students achieving mastery; % = percentage of group (*n* = 17 per group; *N* = 34).

Frequency analysis of students' mastery levels (using an 80% threshold) revealed that the mathematical thought experiments substantially enhanced conceptual learning in the experimental group. In that group, all APOS levels improved from pretest to posttest, with no student achieving mastery before the intervention. The largest gain appeared at the Action level, where 88.2% of students achieved mastery on the posttest. Gains at the other levels were 52.9% for Process, 52.9% for Object, and 29.4% for Schema. These results confirm that thought experiments helped numerous students advance toward the higher and more complex stages of the APOS framework. The intervention both increased the number of students who reached mastery and produced clear advances in the depth of their understanding. In the control group, which followed traditional teaching methods, changes remained modest. The largest increase occurred at the Process level (29.4%), while gains at the Action, Object, and Schema levels were considerably lower. Overall, the data demonstrate that teaching symmetry through mathematical thought experiments produced stronger and more meaningful progress in students' grasp of the topic than standard instruction. Students transitioned from lower cognitive stages to higher ones more frequently in experimental conditions. The Mastery Level measure, which integrates quantitative and qualitative aspects, clearly highlighted these differences.

4. 3. Progression Across APOS Levels

To gain a more comprehensive picture of learning patterns, the study also tracked individual student progress across APOS levels and examined transitions from lower to higher stages in both groups (Table 7).

Table 7

Students' Progression to Higher APOS Levels Based on 80% Mastery Threshold in Experimental and Control Groups

Progression Path	Number at Base Level (Experimental)	Number Progressed (Experimental)	Number at Base Level (Control)	Number Progressed (Control)	Progression Percentage (Experimental)	Progression Percentage (Control)
Action to Process	15	8	3	2	47.06%	11.76%
Process to Object	8	8	2	1	47.06%	5.88%
Object to Schema	8	4	1	1	23.53%	5.88%
Action to Object (two levels)	15	8	3	1	47.06%	5.88%
Action to Schema (three levels)	15	4	3	1	23.53%	5.88%

Analysis of student transitions across the four APOS levels demonstrated that mathematical thought experiments most strongly affected progression from lower to middle stages. The experimental group outperformed the control group on every transition path, including Action to Process, Process to Object, Object to Schema, and more advanced combinations. For the Action to Process and Process to Object paths, the experimental group recorded a 47.06% progression rate, far exceeding the modest changes observed in the control group. In the control group, progress across all paths remained low, ranging from 5.9% to 11.8%. Paired-sample *t*-tests and McNemar tests, based on an 80% mastery threshold, compared pretest and posttest scores within each group. In the experimental group, both tests detected substantial gains at all four levels (Action, Process, Object, and Schema), with $p < 0.001$. The McNemar results confirmed that many students transitioned from non-mastery to mastery, particularly at the Action, Process, and Object levels. At the Schema level, the change approached significance ($p = 0.063$), indicating a somewhat smaller effect at this highest stage.

The control group showed no significant shifts on the McNemar test at any level; all *p*-values exceeded 0.05, indicating that mastery rates before and after instruction remained essentially unchanged. Overall, the results indicate substantial differences between the experimental and control groups in progression across APOS levels.

4. 4. Students' Reasoning in Symmetry Tasks

Qualitative data obtained from interviews, classroom observations, students' participation in thought experiments, and post-test explanations were analyzed using thematic analysis to examine how students reasoned about symmetry tasks. Five major themes emerged from the analysis: Geometric Transformational Reasoning, Spatial and Functional Reasoning, Geometric Property Reasoning, Embodied Spatial Reasoning, and Surface-Level Reasoning. Together, these themes illustrate the diverse cognitive, spatial, and embodied strategies students employed while solving axial and central symmetry tasks. Table 8 presents the themes, subthemes, representative codes, and frequencies identified through the analysis.

Table 8

Themes, Subthemes, and Representative Codes Identified in the Thematic Analysis of Students' Reasoning in Symmetry Tasks.

Conceptual Category (Theme)	Subtheme	Representative Codes	Open Codes	Frequency
Geometric Transformational Reasoning	Mental Folding	• Imagined Folding	• If the paper is folded, the shapes will overlap (coincide). • I folded the paper in my mind • Without folding the paper, I realized that the shapes would overlap.	14
	Rotational Reasoning	• Mental Rotation • Half-Turn Reasoning	• I rotated the shape. • If the shape is rotated by a half-turn (180°), it coincides with itself.	8
	Mental Transformation	• Imaginary Manipulation • Hypothetical Reasoning	• I moved the shape around in my mind. • First, I thought about what would happen if the shape was flipped over.	4
Spatial and Functional Reasoning	Point Correspondence	• Point Mapping • Point Reflection	• If this point is here, the other one should be on the other side of the line of symmetry. • This point should be opposite the center of symmetry. • In my mind, I drew a line to the center of symmetry and continued it the same distance.	15
	Spatial Projection	• Point-to-Axis Extension	• I measured the distance from each point to the line of symmetry and extended the same distance on the other side of the line.	7
	Spatial Structuring	• Grid Counting Strategy • Partitioning Strategy	• I counted the squares on the grid. • I divided the shape in half.	14
	Distance-Based Spatial Analysis	• Distance Mapping	• I counted the grid squares from the point to the line of symmetry.	13
	Relational Spatial Analysis	• Spatial Comparison	• I compared the distances between the shapes.	12

Conceptual Category (Theme)	Subtheme	Representative Codes	Open Codes	Frequency
	Physical Interaction	• Trace-Based Spatial Correspondence	• I can use the trace of the shape on the back of the paper to find its reflection.	3
Geometric Property Reasoning	Distance-Based Reasoning	• Equidistance Reasoning	• Each point should be the same distance from the line of symmetry on the opposite side. • Each point should be the same distance from the center of symmetry on the opposite side.	16
		• Distance Measurement Strategy	• I counted from the shape to the center of symmetry and continued the same distance. • I measured the distances with a ruler.	
	Structural Property Reasoning	• Structural Comparison • Linear Alignment Strategy	• The distances on both sides are equal. • I used a ruler and saw that the points were on the same straight line. • Using a ruler, I drew a line from each point to the center of symmetry and extended it the same distance beyond the center.	12
Embodied Spatial Reasoning	Gesture-Based Reasoning	• Gesture-Based Projection	• Finger movement from the point to the line of symmetry and extension beyond it. • Hand gesture indicating measurement of the distance from the shape to the center of symmetry and extension beyond it.	8
		• Gesture-Based Transformation	• Hand gesture indicating folding. • Hand movement indicating rotation.	12
		• Gesture-Based Spatial Tracking	• Tracing the line of symmetry with a finger. • Pointing to the line of symmetry with a hand. • I drew a line to the center of symmetry and continued it the same distance.	13

Conceptual Category (Theme)	Subtheme	Representative Codes	Open Codes	Frequency
		• Gesture-Based Correspondence	• Pointing to two corresponding points with a finger.	10
Surface-Level Reasoning	Appearance-Based Judgments	• Visual Comparison	• The shapes look the same on both sides of the line of symmetry.	4
		• Appearance-Based Judgment	• I considered it symmetrical simply because the two sides looked identical.	

Geometric Transformational Reasoning was one of the most prevalent themes identified in students' reasoning about symmetry. In this form of reasoning, students used mental geometric transformations to interpret and justify symmetry relationships. They examined symmetry by mentally folding, rotating, or otherwise manipulating shapes to determine how corresponding parts related to one another. The most frequently observed strategy within this theme was mental folding (14 students). Students imagined folding a figure along its line of symmetry and evaluated whether the two parts would overlap perfectly. In addition, some students used mental rotation to interpret central symmetry, mentally rotating a figure by a half-turn (180°) to determine whether it would coincide with itself (8 students). A smaller group of students engaged in more general forms of mental manipulation and hypothetical transformation of shapes (4 students). These findings suggest that, for many students, understanding symmetry was grounded in the visualization and mental execution of geometric transformations rather than merely in the observation of a figure's visible features. Spatial and Functional Reasoning emerged as a prominent theme in the data, indicating that many students understood symmetry through locating points, identifying corresponding points, and organizing spatial relationships among the elements of a figure rather than through mental transformations or formal geometric properties. The most frequently observed subtheme was Point Correspondence (15 students). Students evaluated their solutions by identifying the locations of corresponding points relative to a line or center of symmetry. Their responses suggest that symmetry was primarily interpreted through the spatial relationships established between corresponding points.

Spatial Structuring was also evident in a substantial number of responses (14 students). Students used strategies such as counting grid squares and partitioning figures into equal parts to organize spatial information and determine the positions of figure elements. Distance-Based Spatial Analysis (13 students) and Relational Spatial Analysis (12 students) were also common. In these cases, students examined symmetry by comparing distances and relative positions. For example, some counted the grid squares between a point and the line of symmetry, whereas others compared the distances between shapes or figure components.

The subtheme of Spatial Projection was identified in several responses (7 students). Students determined the location of a corresponding point by extending the distance from a point to the line of symmetry by an equal amount on the opposite side. A smaller number of students demonstrated Physical Interaction (3 students), using direct interaction with the figure to support their reasoning. For example, some relied on the trace of a shape visible through the paper to identify its reflected position. This finding suggests that these students drew upon concrete and observable spatial cues when reasoning about symmetry. Overall, the findings indicate that many students understood symmetry through the organization and interpretation of spatial relationships. Within this form of reasoning, the identification of symmetry was primarily based on locating elements, establishing correspondences,

extending spatial relationships, and comparing distances rather than on the application of formal geometric properties.

Geometric Property Reasoning was observed when students justified their judgments about symmetry by referring to formal geometric properties and relationships. Unlike the previous themes, which were primarily grounded in spatial organization or mental transformations, this form of reasoning relied on stable geometric principles to support students' responses. The most frequently observed subtheme was Distance-Based Reasoning (16 students). Students considered equal distance from a line or center of symmetry to be the primary criterion for identifying symmetry. They examined symmetry relationships by counting distances, making direct measurements, or comparing the positions of points. These findings suggest that many students interpreted symmetry through the geometric principle of equidistance rather than through visual similarity alone. Another prominent subtheme was Structural Property Reasoning (12 students). In these cases, students evaluated symmetry through structural comparisons, linear alignment, and the analysis of geometric relationships among figure elements. They examined whether points were located on the same straight line, compared structural relationships between different parts of a figure, and used rulers to verify geometric relationships. Their attention to line extensions, point alignment, and the structural equivalence of figure components suggests that they understood symmetry as a relationship grounded in the geometric organization of a figure.

Overall, the findings indicate that a number of students based their understanding of symmetry on formal geometric properties. Within this form of reasoning, the identification of symmetry was grounded not in perceptual similarity or spatial organization, but in principles such as equidistance, linear alignment, and structural relationships among figure elements. Embodied Spatial Reasoning was observed when students used bodily movements and gestures to support their spatial thinking and explain symmetry relationships. In this form of reasoning, gestures served not merely as a means of communication but also as a tool for organizing, visualizing, and exploring spatial relationships. The most frequently observed subtheme was Gesture-Based Spatial Tracking (13 students). Students traced lines of symmetry with their fingers, pointed to symmetry axes, or used hand movements to follow spatial relationships within figures. These gestures enabled them to represent and track symmetry relationships through bodily actions. Gesture-Based Transformation was also commonly observed (12 students). Students used hand movements to represent geometric transformations such as folding and rotation. Through these gestures, they represented transformation processes associated with symmetry and used bodily actions to support their reasoning. Gesture-Based Correspondence (10 students) was another prominent subtheme. In these cases, students pointed to corresponding points with their fingers to identify and justify symmetry relationships between figure elements. The subtheme of Gesture-Based Projection was identified in several responses (8 students). Students used finger or hand movements to indicate the extension of distances from a point to a line or center of symmetry and beyond it. These gestures were used to represent the locations of corresponding elements and to support judgments about symmetry. Overall, the findings suggest that, for some students, reasoning about symmetry extended beyond mental and verbal processes and was supported by bodily interaction with spatial representations. Gestures played an important role in helping students visualize, organize, and interpret symmetry relationships. Surface-Level Reasoning was the least frequently observed theme in the data (4 students). In this form of reasoning, students judged symmetry primarily on the basis of the visual similarity of figure components without attending to geometric relationships, element correspondences, or other structural properties. This finding suggests that their judgments were primarily grounded in the perceptual appearance of figures rather than in the analysis of spatial and geometric relationships.

4. 5. Students' Misconceptions in Symmetry Tasks

Students' responses to the symmetry test showed that most students could correctly identify symmetry in basic shapes such as triangles, circles, squares, and rectangles. However, they frequently experienced difficulties when the same shapes were slightly modified, for example by adding or removing a small component or by slightly rotating the figure. Some students experienced conceptual confusion when identifying lines of symmetry in composite figures formed by combining several different geometric shapes. Many of them considered any line that divided a figure into two regions to be a line of symmetry. This misconception indicates that students had not yet developed a clear understanding of the defining properties of a line of symmetry, particularly the distinction between a straight line that produces two mirror-image halves and other lines that merely divide a figure. For example, when presented with a crescent-shaped figure, several students reported symmetry after drawing a line that appeared to divide the shape into two equal parts, even though the resulting regions were not geometrically congruent and could not be perfectly superimposed. These responses suggest that students' judgments were based primarily on visual appearance rather than on formal geometric properties. In addition, some students identified angle bisectors and diagonals as lines of symmetry, reflecting weaknesses in their understanding of related geometric concepts and an insufficient distinction among diagonals, angle bisectors, and lines of symmetry. A further misconception was identified in relation to central symmetry. Several students appeared to confuse central symmetry with axial symmetry, positioning corresponding elements directly opposite one another as though reflecting them across a line, rather than rotating them 180 degrees about the center of symmetry. This suggests that students had not yet distinguished between a reflective transformation and a rotational transformation, treating central symmetry as a form of mirror-image correspondence rather than point-based rotation.

5. Discussion and Conclusion

Symmetry enables individuals to recognize and interpret patterns in the world around them, from natural phenomena to everyday objects (Knuchel, 2004). Nevertheless, many students experience persistent difficulties in understanding both axial and central symmetry (Moravcová et al., 2021). These difficulties have long been a concern for mathematics teachers and researchers because insufficient conceptual understanding of symmetry can hinder students’ subsequent learning of geometric transformations and related mathematical ideas. Consequently, selecting instructional approaches that foster deep conceptual understanding remains a central challenge in mathematics education. Addressing this challenge requires not only careful attention to the mathematical structure of symmetry but also an understanding of the misconceptions and reasoning processes that shape students’ learning (Yanik & Flores, 2009).

One instructional approach that addresses these challenges is the use of mathematical thought experiments. They provide students with opportunities to explore hypothetical situations, manipulate geometric objects mentally, and reason through their conclusions. When implemented in classroom settings, they also promote collaborative discussion, where students share their thinking and gradually refine their ideas. Such learning environments create meaningful contexts for developing abstract mathematical concepts (Dönertaş, 2011). Instead of relying on what they can directly observe, students build understanding by imagining situations and drawing on prior knowledge (Reiner, 1998). Beyond their instructional role, thought experiments also serve as cognitive tools that support conceptual change, spatial reasoning, and problem solving (Reiner, 2013).

The findings of the present study indicate that teaching symmetry through mathematical thought experiments had a significant positive effect on students' academic achievement and APOS-based cognitive learning levels. Students in the experimental group outperformed those in the control group on posttest achievement after controlling for pretest scores. Significant gains were observed across all four APOS levels. These results suggest that the intervention supported not only performance outcomes but also deeper cognitive development in students' understanding of symmetry.

From an APOS perspective, students' learning progression reflects a gradual reconstruction of mental structures, moving from action-based procedures toward more advanced process, object, and schema constructions. Rather than focusing solely on procedural success, students developed more structured ways of reasoning about geometric relationships. This interpretation aligns with APOS theory, which conceptualizes mathematical learning as a dynamic and hierarchical construction of mental structures (Dubinsky & McDonald, 2001; Arnon et al., 2014; Tsafe, 2024).

The mastery and progression analyses provided further evidence of conceptual development. A larger proportion of students in the experimental group achieved mastery across APOS levels, particularly at the Action, Process, and Object levels. In addition, students exposed to mathematical thought experiments more frequently progressed to higher APOS levels compared to the control group. These findings indicate that the intervention supported students in moving beyond initial recognition of symmetry toward more structured mathematical understanding.

From a cognitive perspective, students are increasingly engaged in spatial and relational reasoning when solving symmetry tasks. Their strategies, including mental folding, mental rotation, point correspondence, and distance-based reasoning, reflect a shift from perceptual judgments toward structured mathematical reasoning. This pattern aligns with evidence that spatial visualization plays a central role in mathematical performance, particularly in geometry and transformation-based tasks (Lowrie & Logan, 2023).

Embodied spatial reasoning also played an important role in students' learning processes. Students used gestures such as tracing lines of symmetry, pointing to corresponding points, indicating equal distances, and simulating folding or rotation. These gestures functioned as cognitive tools that supported visualization and reasoning. This finding is consistent with embodied perspectives on mathematical learning, which emphasize that abstract understanding develops through physical actions and gestures (Boonstra et al., 2023; McCluskey et al., 2023).

Despite these gains, the analysis of students' misconceptions revealed persistent difficulties. Although many students correctly identified symmetry in familiar figures, they struggled when figures were modified, rotated, or presented in composite forms. Some students treated any dividing line as a line of symmetry or confused diagonals and angle bisectors with symmetry axes. These findings suggest that students' reasoning was still partly influenced by perceptual appearance rather than formal geometric definitions. This pattern aligns with previous research on persistent misconceptions in symmetry understanding (Moravcová et al., 2021). Educational perspectives further emphasize the importance of addressing such misconceptions through instruction that integrates manipulation, movement, and spatial reasoning activities (The Royal Society, 2024).

The findings of this study are consistent with previous research highlighting the value of mathematical thought experiments for supporting conceptual understanding, reasoning, creativity, and problem solving (Dönertaş, 2011; Karakuyu & Tortop, 2009; Tortop, 2015). However, this study extends prior work by demonstrating that such approaches are effective for typical fifth-grade students, not only gifted learners. Furthermore, integrating

quantitative APOS-based analysis with qualitative evidence from students' reasoning and gestures provides a more comprehensive explanation of how learning develops.

The findings also contribute to APOS-based research by extending its application to elementary mathematics. While prior studies have primarily focused on undergraduate contexts, this study shows that APOS is also useful for analyzing conceptual development in primary school geometry (Soku et al., 2025).

From a practical perspective, mathematical thought experiments offer teachers an effective instructional approach for teaching symmetry. Activities that encourage students to mentally manipulate figures, justify reasoning, and engage in collaborative discussion can promote deeper conceptual understanding. Incorporating gestures and spatial reasoning into lessons can help students connect concrete experiences to abstract ideas.

This study has several limitations. The sample size was relatively small, convenience sampling was used, and participants were drawn from a single educational context. Although the researcher-developed assessment demonstrated acceptable validity and reliability, future studies should replicate these findings with larger and more diverse samples. Further research is also needed across different grade levels and mathematical topics.

In conclusion, mathematical thought experiments represent a promising instructional approach for supporting students' understanding of symmetry. Beyond improving academic achievement, this approach fosters progression across APOS cognitive levels while strengthening spatial reasoning, embodied reasoning, and conceptual understanding. Although some misconceptions persisted, the findings suggest that structured mental exploration and collaborative reasoning can meaningfully support students' conceptual development in geometry.

Declarations

Author Contribution

The first author conceived and designed the study, developed the instructional materials, conducted the data collection and analysis, and drafted the manuscript. The second and third authors supervised the research, contributed to the theoretical framework, and revised the manuscript. All authors read and approved the final version of the manuscript.

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Data Availability. The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

Consent to Participate. Informed consent was obtained from the parents or legal guardians of all individual participants included in the study. (Additionally, assent was obtained from the participating students.)

Ethics Declaration. This study was conducted in accordance with the ethical standards of the institutional research committee of Shahid Rajaee Teacher Training University and with the 1964 Helsinki declaration and its later amendments. All procedures involving human participants were approved by the appropriate institutional ethics committee.

Competing Interests. The authors declare that they have no competing interests.

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