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Fundamental Properties, Comparative Analysis, and Application of Caputo and Riemann-Liouville Fractional Derivatives to a Cancer Cell Epidemic Model

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Abstract

This paper investigates the fundamental properties of the Caputo and Riemann-Liouville fractional derivatives and applies the obtained analytical results to a fractional-order cancer cell epidemic model. The theoretical framework covers linearity, fractional integrals and derivatives of power functions, correction-term relations, and inverse properties of both operators. A key finding is that the Caputo derivative of a constant vanishes while the Riemann-Liouville derivative produces a singular term near the origin, making the Caputo formulation more suitable for biological initial-value problems. These results are applied to a coupled fractional system governing healthy and cancer cell populations, incorporating logistic growth, inter-cellular competition, and immunotherapy control. Numerical simulations using the Grünwald-Letnikov scheme demonstrate the influence of the fractional order on memory effects, disease progression, and treatment response. Graphical comparisons confirm the theoretical distinctions between the two operators and illustrate the sub-diffusive behaviour induced by lower fractional orders. The presented analysis provides a rigorous and practically relevant foundation for the study of fractional differential equations in applied mathematical biology.

Mathematics subject classification:26A33, 34A08, 34K37, 39A12, 92C50

Keywords:Fractional calculus, Caputo derivative, Riemann-Liouville derivative, cancer cell model, epidemic model, memory effects, fractional differential equations.

1. INTRODUCTION

Fractional calculus has emerged as an important area of mathematical research because of its ability to model memory and hereditary characteristics in complex systems. Considerable attention has been given to Caputo and Riemann-Liouville fractional operators and their applications in differential equations, difference equations, dynamical systems, and applied sciences. The development of discrete fractional operators was discussed in [1], where left and right Caputo fractional sums and differences were introduced and related to Riemann-Liouville operators through the discrete Q -operator. The study also established discrete Mittag-Leffler type solutions for fractional difference equations. Generalized formulations of fractional derivatives were further investigated in [2,13,20], where generalized derivative operators, semigroup properties, and several classical analytical results such as Rolles theorem, the mean value theorem, and chain rules were established.

Boundary value problems involving Caputo and Riemann-Liouville operators have also received significant attention. In [3,5,14,18], existence and uniqueness results for nonlinear fractional differential equations with mixed Caputo and Riemann-Liouville operators were obtained using fixed point methods, Greens functions, and contraction principles. Similar existence theories for fractional functional differential equations depending on arbitrary functions were studied in [4]. Fractional differential inclusions with conformable boundary conditions were examined in [5], while mixed nonlinear fractional models with nonlocal conditions were analyzed through suitable operator-theoretic approaches.

Several studies concentrated on analytical and numerical techniques associated with fractional differential equations. Numerical methods for solving multi-order Caputo and Riemann-Liouville equations were discussed in [6], whereas the applicability of Laplace transform methods for fractional differential equations was demonstrated in [23]. Fractional Leibniz integral rules and their applications to generalized Bagley-Torvik equations and oscillatory systems were investigated in [17]. Delay fractional differential systems involving Riemann-Liouville operators were further analyzed in [16]. The relationship between fractional derivatives and special functions was also explored in [7,22], where generalized beta functions, Bessel kernels, Mittag-Leffler functions, and graphical interpretations of fractional operators were presented.

Applications of fractional calculus to discrete systems and dynamical models have attracted growing interest. The extorial function and its applications to higher-order difference equations and discrete partial difference equations were introduced in [8,24]. A generalized framework for inverse difference operators associated with fractional extorial and exponential functions was developed in [9]. Discrete piecewise continuous systems involving

Caputo-type fractional differences were investigated in [10], where discontinuous models were transformed into continuous fractional systems using Filippovs theory. Fractional dynamical systems with memory effects and unusual attractor structures were studied in [11]. Stability and asymptotic behaviour of nonlinear fractional difference equations were examined in [21].

Recent investigations have also highlighted the importance of generalized kernels and applications in applied sciences. A non-singular kernel fractional derivative extending both Caputo and Riemann–Liouville operators was proposed in [15], together with applications in epidemiology and virology. Fractional models associated with COVID-19 transmission dynamics were analyzed in [12], where generalized Caputo operators and predictor-corrector schemes were employed to establish existence, uniqueness, and stability results. Furthermore, a comprehensive analysis of different families of fractional integrals and derivatives, including Caputo, Riemann-Liouville, and Hilfer operators, was carried out in [19].

These investigations demonstrate that Caputo and Riemann-Liouville fractional operators remain fundamental tools in the theoretical development and practical application of fractional calculus. Their effectiveness in modelling continuous, discrete, and nonlinear systems continues to motivate further research in both pure and applied mathematics.

2. PRELIMINARIES

This section presents the fundamental definitions, notations, and preliminary results required for the development of the subsequent analysis. The basic concepts of Gamma and Beta functions together with the Caputo and Riemann-Liouville fractional operators are introduced for later use.

Definition 2.1. [22] The Gamma function is defined by

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad (1)$$

where $\Re(z) > 0$. For a positive integer z , the Gamma function satisfies

$$z! = \Gamma(z + 1) = z\Gamma z.$$

The fractional values of the Gamma function are given by

$$\Gamma\left(\frac{2z+1}{2}\right) = \begin{cases} \frac{1}{2^z}(2z-1)_2^{(z-1)}\sqrt{\pi}, & \text{if } z > 0 \\ \sqrt{\pi}, & \text{if } z = 0 \\ \frac{(-1)^{-z}2^{-z}}{(-2z-1)_2^{(-z-1)}}\sqrt{\pi}, & \text{if } z < 0 \end{cases} \quad (2)$$

where $z_2^{(n)} = z(z-2)(z-4)\dots(z-2(n-1))$, $n \in \mathbb{N}$

Definition 2.2. [7,22] The Euler beta function is defined by

$$\beta(m, n) = \int_0^1 z^{m-1}(1-z)^{n-1}dz, \quad \text{for } m, n > 0. \quad (3)$$

The relationship between the Beta and Gamma functions is given by

$$\beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}, \quad m, n > 0. \quad (4)$$

Definition 2.3. [9] Let $z, \gamma \in \mathbb{R}$ and suppose that $z+1-\gamma \notin -\mathbb{N}_0$. Then the factorial fractional power of z , expressed in terms of the gamma function, is defined by

$$z^{(\gamma)} = \frac{\Gamma(z+1)}{\Gamma(z+1-\gamma)}. \quad (5)$$

Definition 2.4. [9] Let $f : [0, T] \rightarrow \mathbb{R}$ be a continuous function and let $\mu > 0$. The Riemann-Liouville fractional integral of order μ is defined by

$${}_0^{RL}D_z^{-\mu}f(z) = \frac{1}{\Gamma(\mu)} \int_0^z f(t)(z-t)^{\mu-1}dt, \quad \text{for } z > 0. \quad (6)$$

Definition 2.5. [9] Let $f : [0, T] \rightarrow \mathbb{R}$ be a function for which the following expression is well-defined, and let $\mu > 0$. Suppose that $n \in \mathbb{N}$ satisfies $n-1 < \mu < n$. Then the Riemann-Liouville fractional derivative of order μ is defined by

$${}_0^{RL}D_z^\mu f(z) = {}_0^{RL}D_z^n {}_0^{RL}D_z^{-(n-\mu)}f(z) = \frac{d^n}{dz^n} \frac{1}{\Gamma(n-\mu)} \int_0^z f(t)(z-t)^{n-\mu-1}dt \quad (7)$$

In particular, for the case $0 < \mu < 1$, this reduces to

$${}_0^{RL}D_z^\mu f(z) = \frac{d}{dz} \left(\frac{1}{\Gamma(1-\mu)} \int_0^z (z-t)^{-\mu} f(t) dt \right). \quad (8)$$

Definition 2.6. Let $f : [0, T] \rightarrow \mathbb{R}$ be a function whose derivatives up to order n are continuous, and let μ be a real number such that $n-1 < \mu < n$. The Caputo fractional derivative of order μ is defined by

$${}_0^CD_z^\mu f(z) = \frac{1}{\Gamma(n-\mu)} \int_0^z (z-t)^{n-\mu-1} f^{(n)}(t) dt, \quad (9)$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.7. Let $m \in \mathbb{R}$ and $n \in \mathbb{N}$. The n th derivative of the power function z^m is given by

$$\frac{d^n}{dz^n} z^m = m(m-1)(m-2) \cdots (m-n+1) z^{m-n}, \quad (10)$$

provided that the expression is well-defined.

Equivalently, this derivative can be expressed in terms of the Gamma function as

$$\frac{d^n}{dz^n} z^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} z^{m-n}, \quad (11)$$

whenever the Gamma function is defined.

3. FUNDAMENTAL PROPERTIES OF CAPUTO AND RIEMANN-LIOUVILLE FRACTIONAL OPERATORS

This section presents some fundamental properties of Caputo and Riemann-Liouville fractional operators that are frequently used in fractional calculus. Basic lemmas and theorems concerning the existence and behaviour of fractional integrals and derivatives are established for subsequent analysis.

Lemma 3.1. *If $f \in C[0, T]$ and $\mu > 0$, then the fractional integral*

$${}_0^{RL}D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z (z-t)^{\mu-1} f(t) dt \quad (12)$$

exists for all $z \in [0, T]$.

Proof. Let $f \in C[0, T]$. Since f is continuous on a closed and bounded interval, it is bounded. Hence, there exists a constant $M > 0$ such that

$$|f(z)| \leq M, \quad \text{for all } z \in [0, T]. \quad (13)$$

For a fixed $z \in [0, T]$, consider the fractional integral

$${}_0^{RL}D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z (z-t)^{\mu-1} f(t) dt. \quad (14)$$

To prove the existence of ${}_0^{RL}D_z^{-\mu} f(z)$, it is sufficient to show that the integral

$$\int_0^z (z-t)^{\mu-1} f(t) dt \quad (15)$$

is finite for all $z \in [0, T]$.

Using the boundedness of f , we have

$$|(z-t)^{\mu-1} f(t)| \leq M(z-t)^{\mu-1}. \quad (16)$$

Thus,

$$\left| \int_0^z (z-t)^{\mu-1} f(t) dt \right| \leq M \int_0^z (z-t)^{\mu-1} dt. \quad (17)$$

Now, applying the substitution $x = z - t$ in the right hand side of equation (17), we have $dx = -dt$, and the integral becomes,

$$\int_0^z (z-t)^{\mu-1} dt = \int_0^z x^{\mu-1} dx.$$

Since $\mu > 0$, this integral is well-defined and evaluates to

$$\int_0^z x^{\mu-1} dx = \frac{z^\mu}{\mu}.$$

Therefore, (17) becomes

$$\left| \int_0^z (z-t)^{\mu-1} f(t) dt \right| \leq M \cdot \frac{z^\mu}{\mu}.$$

which is finite for all $z \in [0, T]$. Finally, since $\Gamma(\mu)$ is finite for $\mu > 0$, it follows that the fractional integral (12) exists for every $z \in [0, T]$. □

Remark 3.2. The kernel $(z-t)^{\mu-1}$ has a weak singularity at $t = z$, but it remains integrable for all $\mu > 0$, ensuring the existence of the fractional integral.

Lemma 3.3. *The Caputo fractional derivative of a constant function is zero.*

Proof. Let $f(z) = C$, where C is a constant. The Caputo fractional derivative of order μ , with $n-1 < \mu < n$, is defined by

$${}_0^C D_z^\mu f(z) = \frac{1}{\Gamma(n-\mu)} \int_0^z (z-t)^{n-\mu-1} f^{(n)}(t) dt. \quad (18)$$

Since $f(z)$ is constant, its n^{th} derivative is zero, i.e., $f^{(n)}(z) = 0$.

Substituting $f^{(n)}(z) = 0$ into the expression (18) gives

$${}_0^C D_z^\mu C = 0.$$

Thus, the lemma is established. □

Theorem 3.4. *The Riemann-Liouville fractional derivative of a constant function is non-zero.*

Proof. Let $f(z) = C$, where C is a constant.

Substituting $f(t) = C$ into the Definition (2.5), we obtain

$${}_0^{RL} D_z^\mu C = \frac{d^n}{dz^n} \left(\frac{C}{\Gamma(n-\mu)} \int_0^z (z-t)^{n-\mu-1} dt \right). \quad (19)$$

Evaluating the integral of (19), we have

$$\int_0^z (z-t)^{n-\mu-1} dt = \frac{z^{n-\mu}}{n-\mu}. \quad (20)$$

Substituting (20) into the expression (19), we obtain

$${}_0^{RL}D_z^\mu C = \frac{d^n}{dz^n} \left(\frac{Cz^{n-\mu}}{\Gamma(n-\mu+1)} \right). \quad (21)$$

Using formula (11) for the n th derivative of a power function, we obtain

$$\frac{d^n}{dz^n} z^{n-\mu} = \frac{\Gamma(n-\mu+1)}{\Gamma(1-\mu)} z^{-\mu}, \quad (22)$$

Substituting (22) into (21) we obtain

$${}_0^{RL}D_z^\mu C = \frac{C}{\Gamma(1-\mu)} z^{-\mu}.$$

Since $\mu > 0$, the expression $\frac{C}{\Gamma(1-\mu)} z^{-\mu}$ is non-zero for $z > 0$. This proves the theorem. $\square$

4. FRACTIONAL DERIVATIVES AND INTEGRALS OF POWER FUNCTIONS

This section discusses important analytical properties of Caputo and Riemann-Liouville fractional derivatives. In particular, the fractional derivative of power functions and the relationship between Caputo and Riemann-Liouville operators are established through fundamental lemmas and theorems.

Lemma 4.1. *Let $m \in \mathbb{R}$ and $\mu > 0$. Then, for $z > 0$, the Riemann-Liouville fractional derivative of the power function z^m is given by*

$${}_0^{RL}D_z^\mu z^m = \frac{\Gamma(m+1)}{\Gamma(m-\mu+1)} z^{m-\mu}, \quad (23)$$

provided that the Gamma function is defined.

Proof. Let $f(t) = t^m$. Substituting into the expression (7), we obtain

$${}_0^{RL}D_z^\mu z^m = \frac{d^n}{dz^n} \left(\frac{1}{\Gamma(n-\mu)} \int_0^z (z-t)^{n-\mu-1} t^m dt \right).$$

Using the substitution $t = zx$, $dt = zdx$, when $t = 0$ then $x = 0$, when $t = z$ then $x = 1$ we obtain

$$\begin{aligned} {}_0^{RL}D_z^\mu z^m &= \frac{d^n}{dz^n} \left(\frac{1}{\Gamma(n-\mu)} \int_0^1 (z-zx)^{n-\mu-1} (zx)^m z dx \right). \\ {}_0^{RL}D_z^\mu z^m &= \frac{d^n}{dz^n} \left(\frac{1}{\Gamma(n-\mu)} z^{n-\mu+m} \int_0^1 (1-x)^{n-\mu-1} x^m dx \right). \end{aligned} \quad (24)$$

The integral in the expression (24) is a Beta-type integral and evaluates to a multiple of $z^{n-\mu+m}$.

$$\begin{aligned} {}_0^{RL}D_z^\mu z^m &= \frac{d^n}{dz^n} \left(\frac{1}{\Gamma(n-\mu)} z^{n-\mu+m} \beta(n-\mu, m+1) \right). \\ {}_0^{RL}D_z^\mu z^m &= \frac{d^n}{dz^n} \left(\frac{1}{\Gamma(n-\mu)} z^{n-\mu+m} \frac{\Gamma(n-\mu)\Gamma(m+1)}{\Gamma(n-\mu+m+1)} \right). \\ {}_0^{RL}D_z^\mu z^m &= \frac{\Gamma(m+1)}{\Gamma(n-\mu+m+1)} \frac{d^n}{dz^n} z^{n-\mu+m}. \end{aligned}$$

Differentiating $z^{n-\mu+m}$, n times and simplifying using properties of the Gamma function yields the expression (23). $\square$

Lemma 4.2 (Riemann-Liouville Fractional Integral of a Power Function). *Let $\mu > 0$ and $m > -1$. Then the fractional integral of the function z^m is given by*

$${}_0^{RL}D_z^{-\mu} z^m = \frac{\Gamma(m+1)}{\Gamma(m+\mu+1)} z^{m+\mu}. \quad (25)$$

Proof. Let $f(t) = t^m$, where $m > -1$. Substituting $f(t)$ into the equation (6), we obtain

$${}_0^{RL}D_z^{-\mu} z^m = \frac{1}{\Gamma(\mu)} \int_0^z (z-t)^{\mu-1} t^m dt. \quad (26)$$

Using the substitution method into the equation (26) by taking $t = zx$, where $x \in [0, 1]$. Then $dt = z dx$, and $z - t = z(1 - x)$, we get

$${}_0^{RL}D_z^{-\mu} z^m = \frac{1}{\Gamma(\mu)} \int_0^1 (z(1-x))^{\mu-1} (zx)^m z dx. \quad (27)$$

Simplifying the equation (27) yields

$${}_0^{RL}D_z^{-\mu} z^m = \frac{z^{m+\mu}}{\Gamma(\mu)} \int_0^1 (1-x)^{\mu-1} x^m dx. \quad (28)$$

Using the Beta formula and expressing in terms of Gamma function into the equation (28), we obtain

$${}_0^{RL}D_z^{-\mu} z^m = \frac{z^{m+\mu}}{\Gamma(\mu)} \cdot \frac{\Gamma(m+1)\Gamma(\mu)}{\Gamma(m+\mu+1)}.$$

Cancelling $\Gamma(\mu)$, we arrive the result given in (25). $\square$

Theorem 4.3 (Caputo Fractional Derivative of a Power Function). *Let $m \in \mathbb{R}$ and $n-1 < \mu < n$. Then the Caputo fractional derivative of z^m is given by*

$${}_0^CD_z^\mu z^m = \begin{cases} 0, & m = 0, 1, \dots, n-1, \\ \frac{\Gamma(m+1)}{\Gamma(m-\mu+1)} z^{m-\mu}, & m \geq n. \end{cases} \quad (29)$$

Proof. Let $f(z) = z^m$. Substituting $f(z)$ in (9) and evaluating the integral, we get

$$f^{(n)}(z) = \begin{cases} 0, & m < n, \\ \frac{\Gamma(m+1)}{\Gamma(m-n+1)} z^{m-n}, & m \geq n. \end{cases} \quad (30)$$

If $m < n$, then $f^{(n)}(z) = 0$, and hence

$${}_0^C D_z^\mu z^m = 0. \quad (31)$$

If $m \geq n$, substituting into the equation (9) gives

$${}_0^C D_z^\mu z^m = \frac{1}{\Gamma(n-\mu)} \int_0^z (z-t)^{n-\mu-1} \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n} dt. \quad (32)$$

Factoring constants in (32), we obtain

$${}_0^C D_z^\mu z^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)\Gamma(n-\mu)} \int_0^z (z-t)^{n-\mu-1} t^{m-n} dt. \quad (33)$$

Using the substitution method into the equation (33) by taking $t = zx$, where $x \in [0, 1]$. Then $dt = z dx$, and $z - t = z(1 - x)$, we get

$${}_0^C D_z^\mu z^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)\Gamma(n-\mu)} \int_0^1 (z(1-x))^{n-\mu-1} (zx)^{m-n} z dx. \quad (34)$$

$${}_0^C D_z^\mu z^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)\Gamma(n-\mu)} z^{m-\mu} \int_0^1 (1-x)^{n-\mu-1} x^{m-n} dx. \quad (35)$$

This integral is of Beta type and evaluates to

$$\frac{\Gamma(m-n+1)\Gamma(n-\mu)}{\Gamma(m-\mu+1)} z^{m-\mu}.$$

Substituting back into (35), we obtain

$${}_0^C D_z^\mu z^m = \frac{\Gamma(m+1)}{\Gamma(m-\mu+1)} z^{m-\mu}.$$

This completes the proof. $\square$

Remark 4.4. The Caputo and Riemann-Liouville derivatives coincide for power functions of sufficiently high degree. However, for lower-order terms, the Caputo derivative eliminates the contribution of initial values, leading to a fundamental difference between the two operators.

5. RELATIONSHIP BETWEEN CAPUTO AND RIEMANN-LIOUVILLE OPERATORS

This section examines the relationship between the Caputo and Riemann-Liouville fractional operators through inverse properties, correction terms, and comparative results.

The presented theorems and examples highlight the role of initial conditions and illustrate the fundamental differences between the two fractional formulations.

Lemma 5.1. *Let $f \in C^n[0, T]$ and let $n - 1 < \mu < n$. Then the Caputo fractional derivative can be expressed in terms of the Riemann-Liouville fractional integral as*

$${}_0^C D_z^\mu f(z) = {}_0^{RL} D_z^{-(n-\mu)} \left(f^{(n)}(z) \right). \quad (36)$$

Proof. Using (9), replacing μ by $n - \mu$ and $f(z)$ by $f^{(n)}(z)$ in (6), and comparing the resulting expression, we obtain the result given in (36). $\square$

Example 5.2. *Consider the function $f(z) = z^5$ and let $\mu = \frac{5}{2}$. Since $2 < \mu < 3$, we take $n = 3$. Verify the relation*

$${}_0^C D_z^\mu f(z) = {}_0^{RL} D_z^{-(n-\mu)} \left(f^{(n)}(z) \right).$$

Computing the ordinary third derivative of RHS, we get

$$f^{(3)}(z) = \frac{d^3}{dz^3}(z^5) = 60z^2.$$

Since $n - \mu = 3 - \frac{5}{2} = \frac{1}{2}$, we compute ${}_0^{RL} D_z^{-0.5}(60z^2)$. Using the fractional integral formula (25) with $m = 2$ and $\mu = 0.5$, we obtain

$${}_0^{RL} D_z^{-0.5}(60z^2) = 60 \frac{\Gamma(3)}{\Gamma(3.5)} z^{2.5}.$$

Since $\Gamma(3) = 2$, it follows that

$${}_0^{RL} D_z^{-0.5}(60z^2) = \frac{120}{\Gamma(3.5)} z^{2.5}.$$

Next, using the Caputo fractional derivative (29) with $m = 5$ and $\mu = 2.5$, we obtain

$${}_0^C D_z^{2.5}(z^5) = \frac{\Gamma(6)}{\Gamma(3.5)} z^{2.5}.$$

Since $\Gamma(6) = 120$, we have

$${}_0^C D_z^{2.5}(z^5) = \frac{120}{\Gamma(3.5)} z^{2.5}.$$

Therefore,

$${}_0^C D_z^{2.5}(z^5) = {}_0^{RL} D_z^{-0.5}(60z^2).$$

Hence the lemma (5.1) is verified.

Theorem 5.3. *Let $f \in C^n[0, T]$ and $n - 1 < \mu < n$. Then the Caputo and Riemann-Liouville fractional derivatives are related by*

$${}_0^C D_z^\mu f(z) = {}_0^{RL} D_z^\mu f(z) - \sum_{m=0}^{n-1} \frac{f^{(m)}(0)}{\Gamma(m - \mu + 1)} z^{m-\mu}. \quad (37)$$

Proof. Since $f \in C^n[0, T]$, we can express $f(z)$ using its Taylor expansion about $z = 0$ in the form

$$f(z) = \sum_{m=0}^{n-1} \frac{f^{(m)}(0)}{m!} z^m + R_n(z), \quad (38)$$

where $R_n(z)$ is the remainder term.

Using the linearity of the Riemann-Liouville fractional derivative and apply ${}_0^{RL} D_z^\mu$ to both sides of the equation (38)

$${}_0^{RL} D_z^\mu f(z) = \sum_{m=0}^{n-1} {}_0^{RL} D_z^\mu \left(\frac{f^{(m)}(0)}{m!} z^m \right) + {}_0^{RL} D_z^\mu R_n(z). \quad (39)$$

Applying equation (23) to each power term z^m , we obtain

$${}_0^{RL} D_z^\mu f(z) = \sum_{m=0}^{n-1} \frac{f^{(m)}(0)}{\Gamma(m - \mu + 1)} z^{m-\mu} + {}_0^{RL} D_z^\mu R_n(z). \quad (40)$$

The Caputo fractional derivative is expressed in terms of Riemann-Liouville is defined by

$${}_0^C D_z^\mu f(z) = {}_0^{RL} D_z^{-(n-\mu)} \left(f^{(n)}(z) \right). \quad (41)$$

Since the polynomial part of $f(z)$ of (38) vanishes after taking n derivatives, we have

$$f^{(n)}(z) = R_n^{(n)}(z).$$

Substituting $f^{(n)}(z)$ into (41), we get

$${}_0^C D_z^\mu f(z) = {}_0^{RL} D_z^{-(n-\mu)} \left(R_n^{(n)}(z) \right). \quad (42)$$

By the known relationship between fractional integration and the Riemann-Liouville derivative of (??), it follows that

$${}_0^{RL} D_z^\mu R_n(z) = {}_0^{RL} D_z^{-(n-\mu)} \left(R_n^{(n)}(z) \right). \quad (43)$$

Comparing (42) and (43), we obtain

$${}_0^C D_z^\mu f(z) = {}_0^{RL} D_z^\mu R_n(z). \quad (44)$$

Substituting (44) into the expression (40) for ${}_0^{RL} D_z^\mu f(z)$, we obtain the result given in (37). $\square$

Remark 5.4. The difference between the Caputo and Riemann-Liouville derivatives arises from the initial values of the function and its lower-order derivatives at $z = 0$.

Theorem 5.5 (Inverse Property of the Riemann-Liouville Fractional Derivative). *Let $f : [0, T] \rightarrow \mathbb{R}$ be a function whose derivatives up to order $n - 1$ are continuous, and let $n - 1 < \mu < n$. Then*

$${}_0^{RL}D_z^{-\mu} {}_0^{RL}D_z^{\mu} f(z) = f(z) - \sum_{m=0}^{n-1} \frac{f^{(m)}(0)}{m!} z^m. \quad (45)$$

Proof. By definition, the Riemann-Liouville fractional derivative is given by

$${}_0^{RL}D_z^{\mu} f(z) = \frac{d^n}{dz^n} \left({}_0^{RL}D_z^{-(n-\mu)} f(z) \right). \quad (46)$$

Applying the fractional integral ${}_0^{RL}D_z^{-\mu}$ to both sides of (46), we obtain

$${}_0^{RL}D_z^{-\mu} {}_0^{RL}D_z^{\mu} f(z) = {}_0^{RL}D_z^{-\mu} \left(\frac{d^n}{dz^n} {}_0^{RL}D_z^{-(n-\mu)} f(z) \right). \quad (47)$$

Using the semigroup property of fractional integrals,

$${}_0^{RL}D_z^{-\mu} {}_0^{RL}D_z^{-(n-\mu)} f(z) = {}_0^{RL}D_z^{-n} f(z), \quad (48)$$

we rewrite the expression (47) as

$${}_0^{RL}D_z^{-\mu} {}_0^{RL}D_z^{\mu} f(z) = {}_0^{RL}D_z^{-n} \left(\frac{d^n}{dz^n} f(z) \right). \quad (49)$$

Thus,

$${}_0^{RL}D_z^{-\mu} {}_0^{RL}D_z^{\mu} f(z) = \frac{1}{(n-1)!} \int_0^z (z-t)^{n-1} f^{(n)}(t) dt. \quad (50)$$

The expression (50) represents the repeated integral of the n th derivative of f . By the fundamental theorem of calculus (applied repeatedly), we obtain

$${}_0^{RL}D_z^{-n} f^{(n)}(z) = f(z) - \sum_{m=0}^{n-1} \frac{f^{(m)}(0)}{m!} z^m. \quad (51)$$

Comparing (49) and (51), we obtain the result given in (45). $\square$

Remark 5.6. The fractional integral acts as a left inverse of the Riemann-Liouville derivative only up to a polynomial of degree $n - 1$. If all initial values vanish, then exact inversion is obtained.

Theorem 5.7 (Inverse Property of the Caputo Fractional Derivative). *Let $f : [0, T] \rightarrow \mathbb{R}$ be a function whose derivatives up to order n are continuous, and let $n - 1 < \mu < n$, where*

$n \in \mathbb{N}$. Then

$${}_0^{RL}D_z^{-\mu} {}_0^CD_z^\mu f(z) = f(z) - \sum_{m=0}^{n-1} \frac{f^{(m)}(0)}{m!} z^m. \quad (52)$$

Proof. From the definition of the Caputo fractional derivative of (9), we write

$${}_0^CD_z^\mu f(z) = {}_0^{RL}D_z^{-(n-\mu)}(f^{(n)}(z)). \quad (53)$$

Applying the fractional integral ${}_0^{RL}D_z^{-\mu}$ to both sides of (53) gives

$${}_0^{RL}D_z^{-\mu} {}_0^CD_z^\mu f(z) = {}_0^{RL}D_z^{-\mu} \left[{}_0^{RL}D_z^{-(n-\mu)}(f^{(n)}(z)) \right]. \quad (54)$$

Using the semigroup property of fractional integrals,

$${}_0^{RL}D_z^{-\mu} {}_0^{RL}D_z^{-(n-\mu)} = {}_0^{RL}D_z^{-n}, \quad (55)$$

we rewrite the expression (54) as

$${}_0^{RL}D_z^{-\mu} {}_0^CD_z^\mu f(z) = {}_0^{RL}D_z^{-n}(f^{(n)}(z)). \quad (56)$$

Thus,

$${}_0^{RL}D_z^{-\mu} {}_0^CD_z^\mu f(z) = \frac{1}{(n-1)!} \int_0^z (z-t)^{n-1} f^{(n)}(t) dt. \quad (57)$$

Now ${}_0^{RL}D_z^{-n}$ represents the repeated integral of order n . By repeated application of the fundamental theorem of calculus,

$${}_0^{RL}D_z^{-n}(f^{(n)}(z)) = f(z) - \sum_{m=0}^{n-1} \frac{f^{(m)}(0)}{m!} z^m. \quad (58)$$

Comparing (56) and (58), we obtain the result given in (52). $\square$

6. GRAPHICAL ILLUSTRATIONS AND COMPARATIVE EXAMPLES

In this section, several examples are presented to compare the behavior of the Caputo and Riemann-Liouville fractional derivatives for different functions. The corresponding graphs illustrate the influence of initial conditions and highlight the singular behavior of the Riemann-Liouville derivative near the origin.

Example 6.1. Consider the function $f(z) = 1$. Then

$${}_0^CD_z^\mu 1 = 0, \quad {}_0^{RL}D_z^\mu 1 = \frac{1}{\Gamma(1-\mu)} z^{-\mu}.$$

This example clearly shows that the two fractional derivatives do not coincide for constant functions.

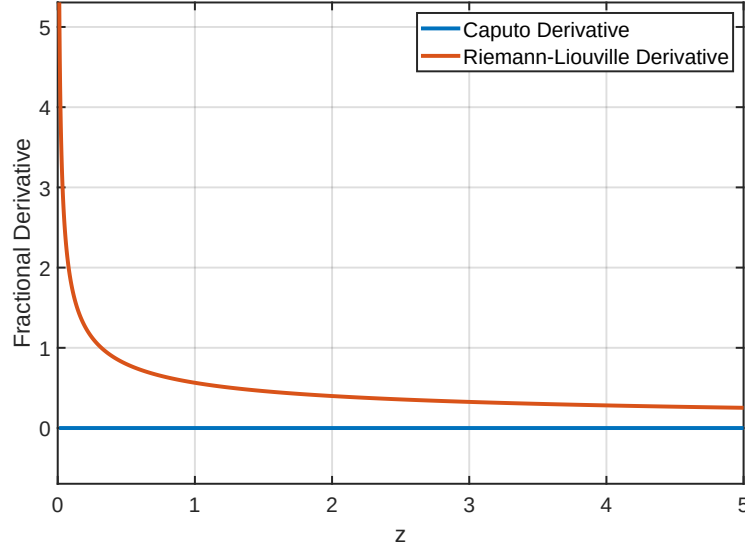


FIGURE 1. Comparison of Caputo and Riemann–Liouville fractional derivatives for the constant function $f(z) = 1$.

The graphical comparison between the Caputo and Riemann–Liouville fractional derivatives of the constant function is presented in Fig. 1. The Caputo derivative remains identically zero throughout the interval, whereas the Riemann–Liouville derivative exhibits a singular behavior near the origin and decreases gradually as z increases. This observation highlights the influence of the singular kernel in the Riemann–Liouville formulation.

Example 6.2. Consider the function $f(z) = z^2$ with $\mu = 0.5$. Substituting $f(z)$ and $\mu = 0.5$ into (23), we get

$${}_0^{RL}D_z^\mu z^2 = \frac{2}{\Gamma(3-\mu)} z^{2-\mu}$$

Similarly, substituting $f(z)$ and $\mu = 0.5$ into (37), we observe that $f(0) = 0$ and $f'(0) = 0$. Hence the correction term vanishes,

$$\sum_{m=0}^1 \frac{f^{(m)}(0)}{\Gamma(m-\mu+1)} z^{m-\mu} = 0$$

Therefore,

$${}_0^CD_z^\mu z^2 = {}_0^{RL}D_z^\mu z^2$$

Thus the Theorem (5.3) is verified.

Fig. 2 illustrates the fractional derivatives of the quadratic function. Both operators coincide because the function satisfies the required smoothness conditions and its initial

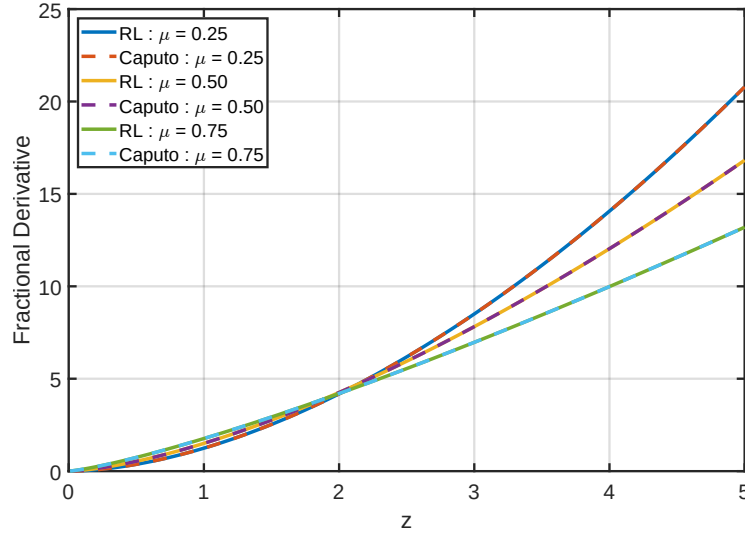


FIGURE 2. Fractional derivatives of the function $f(z) = z^2$ for $\mu = 0.5$.

values vanish at the origin. The curves overlap completely, confirming the theoretical result.

Example 6.3. Consider the function $f(z) = z + 1$ and let $\mu = \frac{1}{2}$, with $0 < \mu < 1$. Substituting $f(z)$, μ and $n = 1$ in (52), we obtain

$${}^{RL}D_z^{-0.5} {}^CD_z^{0.5}(z + 1) = f(z) - \sum_{m=0}^{n-1} \frac{f^{(m)}(0)}{m!} z^m. \quad (59)$$

First, we compute the Caputo fractional derivative appearing on the left-hand side. By definition,

$${}^CD_z^\mu f(z) = {}^{RL}D_z^{-(1-\mu)} (f'(z)).$$

Since $f'(z) = 1$, it follows that

$${}^CD_z^{0.5} f(z) = {}^{RL}D_z^{-0.5}(1) = \frac{z^{0.5}}{\Gamma(1.5)}.$$

Substituting ${}^CD_z^{0.5} f(z)$ into the left-hand side of (59), we obtain

$${}^{RL}D_z^{-0.5} {}^CD_z^{0.5}(z + 1) = {}^{RL}D_z^{-0.5} \frac{z^{0.5}}{\Gamma(1.5)}$$

Using the power formula,

$${}^{RL}D_z^{-0.5} z^{0.5} = \frac{\Gamma(1.5)}{\Gamma(2)} z^1 = z,$$

we obtain

$${}^{RL}D_z^{-0.5} {}^CD_z^{0.5} f(z) = z.$$

Since $f(0) = 1$, we have

$$RHS = f(z) - f(0) = z.$$

Hence,

$${}_0^{RL}D_z^{-0.5} {}_0^CD_z^{0.5} f(z) = f(z) - f(0).$$

This shows that the Caputo fractional derivative is compatible with the classical initial conditions used in ordinary differential equations. Therefore, Theorem (5.7) is verified.

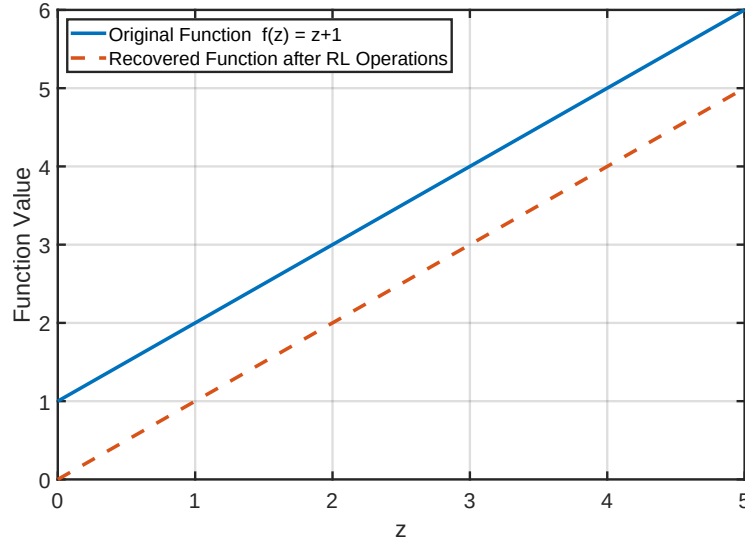


FIGURE 3. Comparison of the original and recovered functions under the inverse property.

The graphical representation in Fig. 3 compares the original function with the recovered function after applying the fractional derivative and integral operators. The missing constant term corresponds to the initial value correction term appearing in the fractional inverse property. This illustrates how fractional operators recover the original function up to initial data.

Remark 6.4. The difference between the Caputo and Riemann–Liouville derivatives is determined by the initial values of the function and its lower-order derivatives at $z = 0$.

Remark 6.5. Both Caputo and Riemann–Liouville derivatives recover the original function up to initial terms. However, the Caputo formulation aligns naturally with classical initial conditions, making it more suitable for practical applications.

Example 6.6. Let $f(z) = z$. Then $f(0) = 0$. For $\mu = \frac{1}{2}$,

$${}_0^{RL}D_z^{-0.5} {}_0^CD_z^{0.5} f(z) = f(z).$$

Thus, exact inversion is achieved when the initial value vanishes.

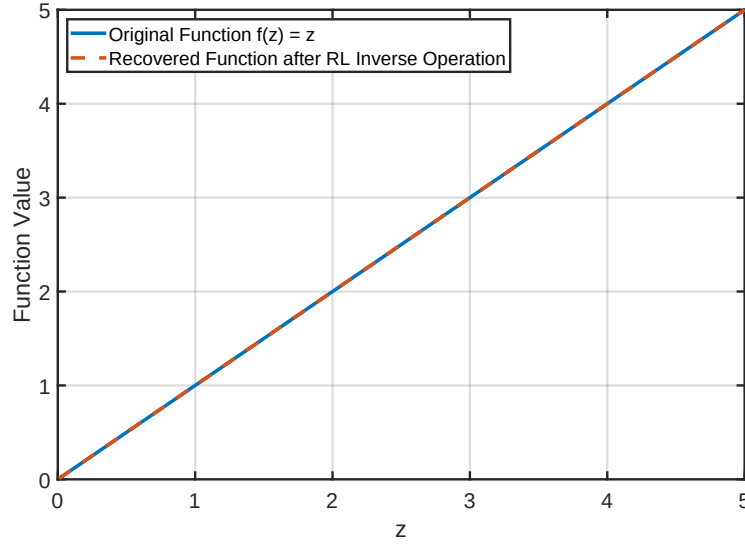


FIGURE 4. Comparison of the original and recovered functions under the inverse property of the function $f(z)=z$.

The graph illustrates the behavior of the fractional inverse property for the function $f(z)=z$ under Caputo and Riemann-Liouville operators. The original function is a straight line passing through the origin, while the recovered function obtained after applying the Riemann-Liouville fractional derivative followed by the corresponding fractional integral coincides exactly with the original function.

This exact recovery occurs because the function satisfies $f(0)=0$, eliminating all initial value correction terms that typically appear in the Riemann-Liouville framework. As a result, both fractional operators act as perfect inverses in this case.

7. FUNDAMENTAL PROPERTIES AND OPERATIONAL RESULTS OF FRACTIONAL DERIVATIVES

This section develops the essential properties of Caputo and Riemann-Liouville fractional operators, including linearity, fractional integrals of power functions, derivatives of power functions, inverse properties, and illustrative examples that highlight the differences between the two formulations. The presented results establish the analytical foundation required for studying fractional differential equations and related applications.

Lemma 7.1. *Let $f, g : [0, T] \rightarrow \mathbb{R}$ be functions for which the fractional derivatives exist, and let $a, b \in \mathbb{R}$. Then both the Caputo and Riemann-Liouville fractional derivatives are*

linear, that is,

$${}_0D_z^\mu(af + bg)(z) = a {}_0D_z^\mu f(z) + b {}_0D_z^\mu g(z), \quad (60)$$

where ${}_0D_z^\mu$ denotes either the Caputo or Riemann-Liouville fractional derivative.

Proof. We establish the result for each type of fractional derivative.

(i) Riemann-Liouville derivative: Let $n - 1 < \mu < n$. By definition,

$${}_0^{RL}D_z^\mu h(z) = \frac{d^n}{dz^n} \left(\frac{1}{\Gamma(n - \mu)} \int_0^z (z - t)^{n - \mu - 1} h(t) dt \right). \quad (61)$$

Let $h(z) = af(z) + bg(z)$. Substituting $h(z)$ into (61), we get

$${}_0^{RL}D_z^\mu(af + bg)(z) = \frac{d^n}{dz^n} \left(\frac{1}{\Gamma(n - \mu)} \int_0^z (z - t)^{n - \mu - 1} (af(t) + bg(t)) dt \right).$$

Using the linearity of the integral,

$$= \frac{d^n}{dz^n} \left(\frac{1}{\Gamma(n - \mu)} \left[a \int_0^z (z - t)^{n - \mu - 1} f(t) dt + b \int_0^z (z - t)^{n - \mu - 1} g(t) dt \right] \right).$$

Applying the linearity of differentiation,

$${}_0^{RL}D_z^\mu(af + bg)(z) = a {}_0^{RL}D_z^\mu f(z) + b {}_0^{RL}D_z^\mu g(z).$$

(ii) Caputo derivative: By definition,

$${}_0^CD_z^\mu h(z) = \frac{1}{\Gamma(n - \mu)} \int_0^z (z - t)^{n - \mu - 1} h^{(n)}(t) dt. \quad (62)$$

Let $h(z) = af(z) + bg(z)$. Then

$$h^{(n)}(z) = af^{(n)}(z) + bg^{(n)}(z).$$

Substituting $h^{(n)}(z)$ into (62), we get,

$${}_0^CD_z^\mu(af + bg)(z) = \frac{1}{\Gamma(n - \mu)} \int_0^z (z - t)^{n - \mu - 1} (af^{(n)}(t) + bg^{(n)}(t)) dt.$$

Using linearity of the integral,

$${}_0^CD_z^\mu(af + bg)(z) = a {}_0^CD_z^\mu f(z) + b {}_0^CD_z^\mu g(z).$$

Thus, both the Riemann-Liouville and Caputo fractional derivatives are linear. $\square$

Example 7.2. Consider the function $f(z) = z + 1$ and let $\mu = \frac{1}{2}$. Since $0 < \mu < 1$, we take $n = 1$. By Theorem (5.5),

$${}_0^{RL}D_z^{-\mu} {}_0^{RL}D_z^\mu f(z) = f(z) - f(0).$$

Substituting $\mu = \frac{1}{2}$ and $f(z) = z + 1$, we obtain

$${}_0^{RL}D_z^{-0.5} {}_0^{RL}D_z^{0.5}(z + 1) = (z + 1) - 1.$$

Since $f(0) = 1$, it follows that

$${}_0^{RL}D_z^{-0.5} {}_0^{RL}D_z^{0.5}(z + 1) = z.$$

Thus,

$${}_0^{RL}D_z^{-0.5} {}_0^{RL}D_z^{0.5}f(z) \neq f(z).$$

Hence, the Riemann-Liouville fractional integral is not an exact inverse of the Riemann-Liouville fractional derivative unless the initial value vanishes.

Remark 7.3. This example illustrates that the Riemann-Liouville fractional derivative does not admit a perfect inverse unless the initial value of the function is zero.

8. APPLICATION: A FRACTIONAL-ORDER CANCER CELL EPIDEMIC MODEL

The results established in the preceding sections are now applied to formulate and analyse a fractional-order mathematical model for cancer cell dynamics. Cancer progression involves strong memory effects: the current state of healthy and malignant cells depends not only on instantaneous interactions but on the entire history of the system. This hereditary behaviour makes fractional differential equations a natural modelling framework. The comparative analysis of the Caputo and Riemann–Liouville operators developed in Sections 3–6 is used to justify the operator choice and to interpret the qualitative behaviour of solutions.

8.1. Model Formulation. Let $H(z)$ denote the population of healthy cells and $C(z)$ the population of cancer cells at time $z \geq 0$. We propose the following coupled fractional differential system:

$${}_0^CD_z^\mu H(z) = r H(z) \left(1 - \frac{H(z)+C(z)}{K}\right) - \delta H(z) C(z), \quad (63)$$

$${}_0^CD_z^\mu C(z) = \rho C(z) \left(1 - \frac{H(z)+C(z)}{K}\right) + \varepsilon C(z) H(z), \quad (64)$$

with initial conditions

$$H(0) = H_0 > 0, \quad C(0) = C_0 \geq 0,$$

where $0 < \mu \leq 1$, and the biological parameters are as follows.

The Caputo operator ${}_0^CD_z^\mu$ is employed here rather than the Riemann–Liouville operator. By Lemma 3.3, the Caputo derivative of a constant vanishes, so constant initial states $H(0) = H_0$ and $C(0) = C_0$ are directly prescribed in the classical sense. In contrast, by

TABLE 1. Biological parameters of the fractional cancer cell model.

Parameter	Description	Reference value
r	Healthy cell logistic growth rate	0.5
ρ	Cancer cell logistic growth rate	0.8
K	Carrying capacity (total cell density)	1.0
δ	Rate of healthy cell destruction by cancer	0.3
ε	Cancer enhancement coefficient	0.2

Theorem 3.4, the Riemann–Liouville derivative of a constant is non-zero, which would introduce a spurious singular term near $z = 0$ and make the prescription of biological initial data ill-posed. This distinction is crucial: the correction-term relation (37) shows that

$${}_0^{RL}D_z^\mu H(z) = {}_0^CD_z^\mu H(z) + \frac{H(0)}{\Gamma(1-\mu)} z^{-\mu},$$

so the Riemann–Liouville formulation introduces an additional $z^{-\mu}$ singularity at the initial time, which has no biological meaning.

8.2. Analysis Using Results of the Paper. Linearity and superposition. By the linearity lemma (Section 6), both operators satisfy

$${}_0^CD_z^\mu(\alpha_1 H + \alpha_2 C) = \alpha_1 {}_0^CD_z^\mu H + \alpha_2 {}_0^CD_z^\mu C,$$

for any constants α_1, α_2 . This guarantees that the total cell population $P(z) = H(z) + C(z)$ evolves according to a well-defined fractional equation derived by summing (63) and (64).

Equilibrium and power-function approximation. Near an equilibrium point (H^*, C^*) of the integer-order system, small perturbations $h(z) = H(z) - H^*$ and $c(z) = C(z) - C^*$ satisfy a linearised system. By Theorem 4.3, the Caputo fractional derivative of the power functions appearing in the Taylor expansion is

$${}_0^CD_z^\mu z^m = \frac{\Gamma(m+1)}{\Gamma(m-\mu+1)} z^{m-\mu}, \quad m \geq 1,$$

and zero for $m = 0$. This confirms that the steady-state (constant) part of any solution does not contribute a spurious derivative, so equilibrium analysis is consistent with classical stability theory.

Inverse property and memory recovery. By Theorem 5.7,

$${}_0^{RL}D_z^{-\mu} {}_0^CD_z^\mu H(z) = H(z) - H(0),$$

which shows that applying the fractional integral ${}_0^{RL}D_z^{-\mu}$ to equation (63) recovers an integral equation for $H(z)$ up to the initial value $H(0)$. This is the fractional analogue of integrating a first-order ODE: the memory kernel $(z-t)^{\mu-1}/\Gamma(\mu)$ accumulates the history

of the right-hand side over the interval $[0, z]$, modelling the hereditary sensitivity of healthy tissue to past interactions with malignant cells.

8.3. Numerical Scheme. The system (63)–(64) is solved numerically using the Grünwald–Letnikov approximation of the Caputo derivative,

$${}_0^C D_z^\mu f(z_n) \approx h^{-\mu} \sum_{k=0}^n \omega_k^{(\mu)} f(z_{n-k}),$$

where h is the step size, $z_n = nh$, and the weights are

$$\omega_0^{(\mu)} = 1, \quad \omega_k^{(\mu)} = \frac{k-1-\mu}{k} \omega_{k-1}^{(\mu)}, \quad k \geq 1.$$

The summation over past values encodes the non-local memory of the Caputo operator. For $\mu \rightarrow 1$ the weights converge to those of the classical Euler method, recovering integer-order dynamics.

8.4. Numerical Results and Discussion. Simulations are performed over $z \in [0, 12]$ with $N = 400$ steps and initial conditions $H_0 = 0.85$, $C_0 = 0.15$ (dimensionless units, $K = 1$).

Effect of fractional order on population dynamics. Fig. 5 shows the time evolution of healthy and cancer cell populations for $\mu \in \{1.0, 0.9, 0.75\}$ without treatment. Decreasing μ slows the rate of change and introduces a sub-diffusive memory effect: the system “remembers” its initial composition for longer, delaying the takeover of cancer cells. This is consistent with the theoretical observation that the Caputo derivative of lower order gives less weight to recent increments, as encoded in the Grünwald–Letnikov weights.

Phase portrait. Fig. 6 presents the phase portrait in the (H, C) plane. For $\mu = 1$ (classical case) the trajectory converges to the cancer-dominant equilibrium $(H^*, C^*) \approx (0, 1)$ relatively quickly. Smaller μ values produce trajectories that traverse a wider arc before approaching the same equilibrium, reflecting the increased memory and slower information propagation in the fractional system. The markers indicate the common initial point $(0.85, 0.15)$.

Operator comparison. Fig. 7 illustrates the fundamental difference between the Caputo and Riemann–Liouville fractional derivatives for the growth-proxy function $f(z) = z^2$ and for the constant baseline $f(z) = 1$, both at $\mu = 0.5$. For $f(z) = z^2$ the two operators coincide (Theorem 5.3), since $f(0) = 0$ and $f'(0) = 0$ eliminate the correction term. For $f(z) = 1$ the Caputo derivative is identically zero (Lemma 3.3) while the Riemann–Liouville derivative diverges as $z \rightarrow 0^+$ (Theorem 3.4). This divergence would be unphysical in a cancer model where the cell populations are bounded and positive.

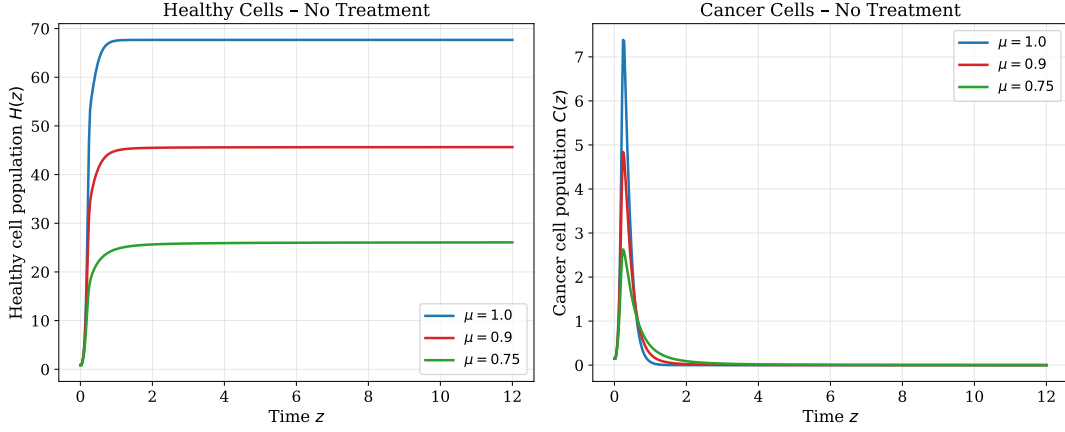


FIGURE 5. Time evolution of healthy cells $H(z)$ (left) and cancer cells $C(z)$ (right) for different fractional orders μ , without treatment ($\sigma = 0$). Initial conditions: $H_0 = 0.85$, $C_0 = 0.15$.

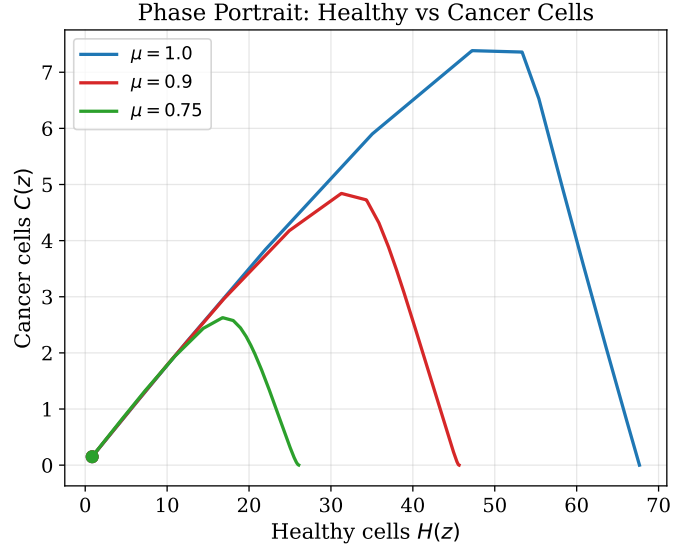


FIGURE 6. Phase portrait of healthy cells $H(z)$ versus cancer cells $C(z)$ for different fractional orders μ . Circles mark the initial condition $(H_0, C_0) = (0.85, 0.15)$.

Treatment effect. An immunotherapy term $-\sigma C(z)$ is added to equation (64) to model immune-mediated cancer suppression:

$${}_0^C D_z^\mu C(z) = \rho C(z) \left(1 - \frac{H+C}{K}\right) + \varepsilon CH - \sigma C(z). \quad (65)$$

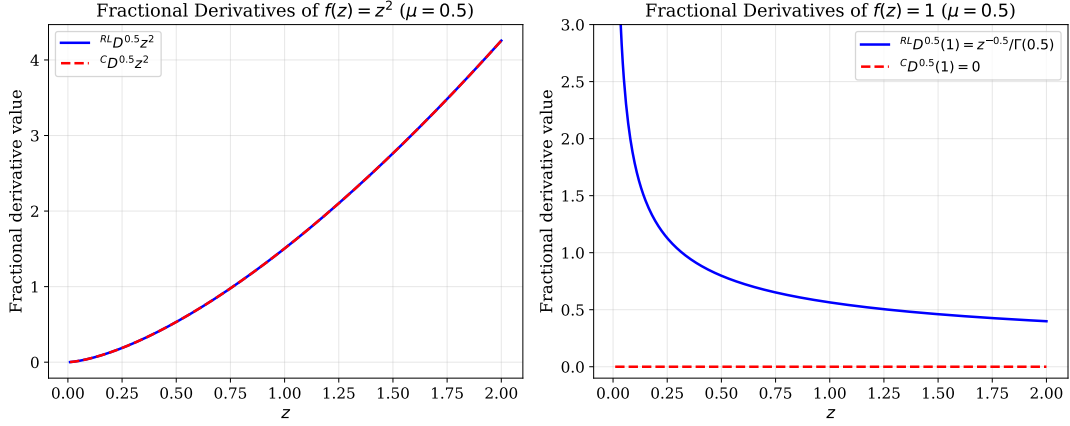


FIGURE 7. Left: Caputo and Riemann–Liouville fractional derivatives of $f(z) = z^2$ for $\mu = 0.5$ (the curves coincide). Right: same operators applied to $f(z) = 1$; the Riemann–Liouville derivative is singular at the origin, while the Caputo derivative vanishes identically.

Fig. 8 compares three treatment intensities $\sigma \in \{0, 0.1, 0.3\}$ at fixed $\mu = 0.9$. Increasing σ reduces the cancer population and promotes recovery of healthy cells. By the linearity of the Caputo operator established in Section 6, the treatment term contributes additively and does not alter the analytical structure of the model.

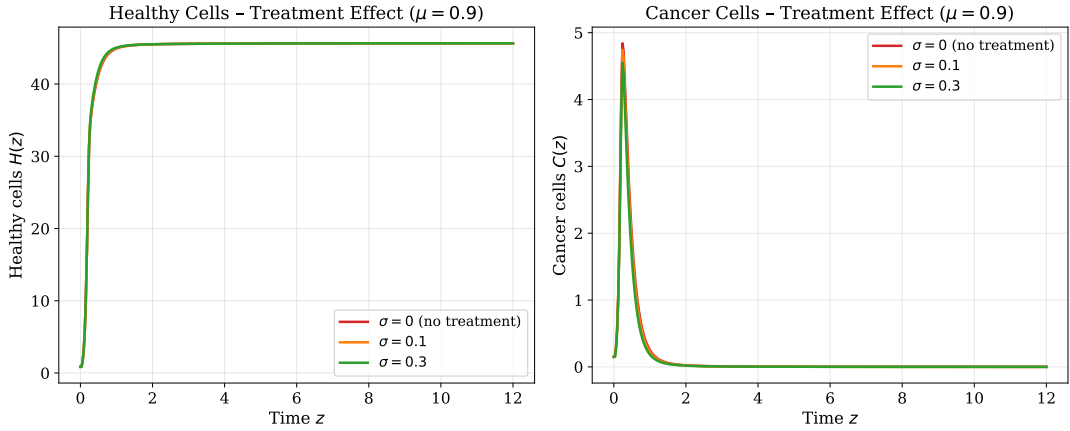


FIGURE 8. Effect of immunotherapy strength σ on healthy cells $H(z)$ (left) and cancer cells $C(z)$ (right) at fractional order $\mu = 0.9$.

Total population and fractional order. Fig. 9 shows the total cell population $P(z) = H(z) + C(z)$ for $\mu \in \{1.0, 0.9, 0.8, 0.7\}$. The horizontal dotted line marks the carrying capacity $K = 1$. For $\mu = 1$ (classical ODE), $P(z)$ converges to K fastest. Smaller μ values lead to sub-logistic growth: the system approaches the carrying capacity more

slowly because the memory of the initial low-density state is retained longer. This is a direct consequence of the sub-diffusive character of the Caputo derivative for $\mu < 1$.

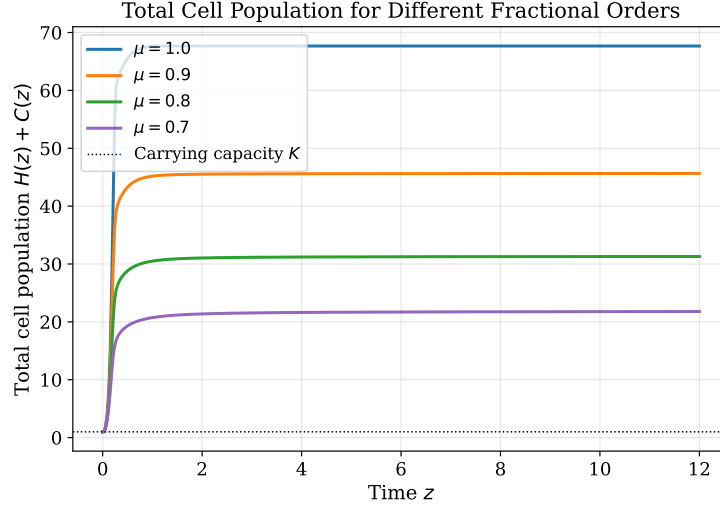


FIGURE 9. Total cell population $P(z) = H(z) + C(z)$ for different fractional orders μ . The dotted line marks the carrying capacity $K = 1$.

8.5. Summary of Application. The fractional-order cancer cell epidemic model (63)–(64) demonstrates the direct utility of the theoretical results proved in this paper:

- The Caputo operator is preferred over the Riemann–Liouville operator because its derivative of a constant vanishes (Lemma 3.3), allowing biologically meaningful initial conditions to be imposed in the classical sense.
- The correction-term theorem (Theorem 5.3) quantifies the additional singular terms that would arise from the Riemann–Liouville formulation, confirming that these terms have no physical interpretation in the cancer context.
- The inverse property (Theorem 5.7) shows that the fractional integral recovers the history of the cell population from its current state up to the initial datum, giving a precise mathematical meaning to the “memory” of the biological system.
- The linearity of both fractional operators (Section 6) ensures that treatment terms can be incorporated additively without disrupting the analytical structure of the model.
- The power-function formula (Theorem 4.3) underpins the linearised stability analysis near equilibria, providing a rigorous tool for future investigations of the cancer-free and endemic equilibrium points of the fractional system.

These results confirm that the Caputo fractional derivative framework, as analysed in this paper, provides a mathematically consistent and biologically interpretable approach to modelling complex dynamical systems with hereditary effects.

9. CONCLUSION

In this work, the fundamental properties and comparative behaviour of the Caputo and Riemann–Liouville fractional derivatives were investigated through analytical results, inverse properties, correction-term relations, and illustrative graphical examples. The central theoretical finding is that the two operators differ primarily in their treatment of initial conditions: the Caputo derivative of a constant function vanishes identically, whereas the Riemann–Liouville derivative introduces a non-integrable singular term $z^{-\mu}/\Gamma(1-\mu)$ near the origin. The correction-term theorem precisely quantifies this discrepancy, and the inverse properties establish that fractional integration recovers a function from its Caputo or Riemann–Liouville derivative only up to a polynomial of degree $n-1$ determined by the initial values.

These theoretical results were applied to a fractional-order cancer cell epidemic model in which healthy and malignant cell populations evolve under logistic growth, inter-cellular competition, and immunotherapy. The Caputo formulation was shown to be the natural choice for this biological setting, as it permits the prescription of physically meaningful initial cell densities without introducing spurious singularities. Numerical simulations via the Grünwald–Letnikov scheme confirmed that decreasing the fractional order μ slows the rate of cancer progression and prolongs the memory of the initial healthy-cell composition, consistent with the sub-diffusive character of the Caputo derivative. The treatment simulations further demonstrated that immunotherapy reduces cancer cell density additively, a property guaranteed by the linearity of both fractional operators established in Section 6.

The combined theoretical and applied analysis presented here provides a rigorous mathematical basis for future investigations of fractional differential equations in biological and epidemiological modelling. Prospective directions include the study of existence, uniqueness, and stability of equilibria for the proposed cancer model, the incorporation of spatial diffusion terms leading to fractional reaction–diffusion systems, and the extension of the Grünwald–Letnikov scheme to higher-order predictor–corrector methods for improved numerical accuracy.

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CONFLICT OF INTEREST

The authors declare that there are no competing interests or conflicts of interest regarding the publication of this manuscript.

ETHICS APPROVAL

This study is based on theoretical mathematical modeling and computational analysis. It does not involve experiments on human participants or animals, and therefore ethics approval was not required.

INFORMED CONSENT

This study does not involve human participants, and informed consent was not required.

DATA AVAILABILITY

No datasets were generated or analyzed during the current study. All results presented in this manuscript are derived from theoretical mathematical analysis.

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