Supplementary Information for

A state-resolved Hawking channel for an area-law quantum memory

1 Exact contour factorization

The fermions of the matrix model carry a two-component spinor index, a matrix row index on which the bosonic matrices act, and a terminal flavour index [1, 2]. Set

$$A_N = \mathbb{C}^2 \otimes \mathbb{C}^N, \qquad F_N = \mathbb{C}^N, \qquad \mathfrak{h}_N = A_N \otimes F_N. \tag{1}$$

For every bosonic and bifundamental-link history Φ on a real-, Euclidean-, or closed-time contour, the one-particle operator acts as

$$h[\Phi] = h_A[\Phi] \otimes \mathbf{1}_{F_N}. \tag{2}$$

The same terminal index is left unchanged by the node–link Yukawa term of the two-block theory. Therefore the single-particle contour propagator has the form

$$U[\Phi] = U_A[\Phi] \otimes \mathbf{1}_{F_N}. \tag{3}$$

Theorem 1 (Contour noiseless-subsystem theorem)**.** *On the half-filled sector*

$$\mathcal{C}_N = \bigwedge^{N^2}(A_N \otimes F_N), \tag{4}$$

skew–Howe duality gives

$$\mathcal{C}_N \cong \bigoplus_{\substack{\lambda \vdash N^2 \\ \ell(\lambda) \le 2N,\ \lambda_1 \le N}} S_\lambda A_N \otimes S_{\lambda^\intercal} F_N. \tag{5}$$

For every contour history,

$$\bigwedge^{N^2}(U_A[\Phi] \otimes \mathbf{1}_{F_N}) = \bigoplus_\lambda S_\lambda(U_A[\Phi]) \otimes \mathbf{1}_{S_{\lambda^\intercal} F_N}. \tag{6}$$

Consequently, integrating over any collection of bosonic forward and backward histories preserves the identity superoperator on every factor $S_{\lambda^\intercal} F_N$.

Proof. Equation (5) is the dual Cauchy decomposition of the exterior power. Functoriality of Schur functors gives Eq. (6). A Schwinger–Keldysh integrand is a linear combination of forward and backward operators of this form, each with a scalar bosonic measure. Linearity therefore preserves the identity factor. □

The theorem is stronger than a typicality statement: no ensemble average, random Hamiltonian or large-N limit is used. It applies sector by sector throughout the full half-filled Fock space. What can remain sector dependent is the active geometric response and therefore the relative probability assigned to different sectors; within each protected multiplicity factor the branch acts identically on all logical states.

2 An explicit area-law protected factor

Define the odd-staircase highest weight

$$\mu^{(N)} = (2N-1, 2N-3, \dots, 3, 1), \qquad \lambda^{(N)} = (\mu^{(N)})^{\mathsf{T}}. \tag{7}$$

It contains N^2 boxes and fits inside the $N \times 2N$ rectangle required by Eq. (5).

Theorem 2 (Odd-staircase dimension)**.** *The protected flavour factor*

$$\mathcal{Q}_N = S_{\mu^{(N)}}\mathbb{C}^N \tag{8}$$

has exact dimension

$$\boxed{\dim \mathcal{Q}_N = 3^{N(N-1)/2}.} \tag{9}$$

Proof. The Weyl dimension formula gives

$$\dim S_\mu \mathbb{C}^N = \prod_{1\le i<j\le N} \frac{\mu_i - \mu_j + j - i}{j-i}. \tag{10}$$

For $\mu_i = 2(N-i)+1$, one has $\mu_i - \mu_j + j - i = 3(j-i)$. Every factor is therefore exactly three, proving Eq. (9). □

The protected entropy is

$$S_{\mathcal{Q}} = \frac{N(N-1)}{2}\ln 3. \tag{11}$$

Against the half-filled state count $S_{\text{model}} = \ln\binom{2N^2}{N^2} = 2N^2\ln 2 + O(\ln N)$,

$$\frac{S_{\mathcal{Q}}}{S_{\text{model}}} \longrightarrow \frac{\ln 3}{4\ln 2} = 0.396240625\dots. \tag{12}$$

Thus the exact noiseless sector is exponentially large in horizon area rather than a polynomial corner of the state space. It is also distinguished at half filling by self-complementarity: complementing occupied and empty cells in the $N \times 2N$ rectangle sends μ_i to $2N - \mu_{N+1-i}$ and leaves the odd staircase invariant.

3 Exact census of all protected sectors

Every partition μ of N^2 inside the $N \times 2N$ rectangle defines a protected factor $S_\mu\mathbb{C}^N$ and an active factor $S_{\mu^{\mathsf{T}}}\mathbb{C}^{2N}$. We enumerated every such partition through $N = 10$. The sector weight in the full half-filled code is

$$d_{\text{sec}}(\mu) = \dim S_\mu\mathbb{C}^N \ \dim S_{\mu^{\mathsf{T}}}\mathbb{C}^{2N}. \tag{13}$$

The exact closure test

$$\sum_\mu d_{\text{sec}}(\mu) = \binom{2N^2}{N^2} \tag{14}$$

was satisfied with maximum logarithmic error 5.7×10^{-14}.

At $N = 10$ the state-count-weighted median of $\ln\dim S_\mu\mathbb{C}^N / S_{\text{model}}$ is 0.3121, the 10–90% interval is 0.2840–0.3376, and sectors above 0.30 carry 71.5% of the full state count. The self-complementary odd staircase has ratio 0.3642 at this rank and lies above 81.7% of sectors by protected dimension. The closed-form code is therefore an upper-tail representative of a broad protected-sector distribution rather than an isolated choice.

4 The daughter–radiation operator is a branching isometry

For a rank split $N = K + n$, the terminal flavour space splits as

$$F_N = F_K \oplus F_n. \tag{15}$$

The restriction of a Schur functor to a direct sum is

$$S_\mu(F_K \oplus F_n) \cong \bigoplus_{\alpha,\beta} \mathbb{C}^{c^\mu_{\alpha\beta}} \otimes S_\alpha F_K \otimes S_\beta F_n, \tag{16}$$

where $c^\mu_{\alpha\beta}$ are Littlewood–Richardson coefficients. Choosing orthonormal bases in the summands makes Eq. (16) a unitary change of basis,

$$W_{N\to K,n} : \mathcal{Q}_N \longrightarrow \bigoplus_{\alpha,\beta} \mathbb{C}^{c^\mu_{\alpha\beta}} \otimes S_\alpha F_K \otimes S_\beta F_n, \qquad W^\dagger W = \mathbf{1}_{\mathcal{Q}_N}. \tag{17}$$

For $n = 1$, the branching is multiplicity free and is labelled by Gelfand–Tsetlin interlacing rows.

Let $B_{n,\lambda}$ denote the complete matrix, link, gauge-dressing and scalar saddle operator on the geometric factor. Equations (6) and (17) give the branch normal form

$$\boxed{K_{n,\lambda} = B_{n,\lambda} \otimes W_{N\to K,n}.} \tag{18}$$

For a fixed initial geometric state $|g\rangle$ and a complete final geometric branch,

$$K^\dagger_{n,\lambda} K_{n,\lambda}\Big|_{\mathcal{Q}_N} = \langle g|B^\dagger_{n,\lambda} B_{n,\lambda}|g\rangle\, \mathbf{1}_{\mathcal{Q}_N}. \tag{19}$$

Hence the branch probability is exactly independent of the logical microstate. The parent monopole saddle derives

$$P(\omega) = e^{-\omega/T}, \qquad T = \frac{\Omega}{2\pi} = \frac{M_P}{3N}, \tag{20}$$

and compares it with $T_H = bM_P/(8N)$, giving coefficient agreement for the parent-model choice $b = 8/3$ [3]. This scalar law and parameter match are inherited. The new result is its operator-valued completion, $K = \sqrt{p}\, W$, with W the exact branching isometry and $K^\dagger K = p\mathbf{1}$ on the logical factor.

5 Gauss-edge completion

Suppose the horizon code decomposes under a gauge group G as

$$\mathcal{C}_N = \bigoplus_\rho V_\rho \otimes M_\rho, \tag{21}$$

and the edge system contains conjugate representations

$$\mathcal{E} = \bigoplus_\rho V^*_\rho \otimes E_\rho. \tag{22}$$

Schur orthogonality gives

$$\mathrm{Inv}_G(\mathcal{C}_N \otimes \mathcal{E}) \cong \bigoplus_\rho M_\rho \otimes E_\rho. \tag{23}$$

To retain the original $\dim V_\rho \dim M_\rho$ states in each sector, one needs and suffices to have

$$\boxed{\dim E_\rho \geq \dim V_\rho.} \tag{24}$$

The minimal sector-preserving reservoir is the code-truncated regular representation

$$\mathcal{E}_{\min} = \bigoplus_\rho V^*_\rho \otimes \mathbb{C}^{\dim V_\rho}, \qquad \dim \mathcal{E}_{\min} = \sum_\rho (\dim V_\rho)^2. \tag{25}$$

The exact finite-rank enumeration shown in Fig. 4 of the main text verifies saturation to logarithmic error below 4.3×10^{-14} through rank nine.

6 Link-mode projection and reduced Euclidean cap

The published quadratic link Hessian decomposes into fixed total-magnetic-number sectors. In the helicity basis (w_0, w_+, w_-), we formed the full finite-rank matrices and projected onto the background-covariant transverse subspace

$$D^a W_a = 0. \tag{26}$$

For ranks $N = 5, 8, 12, 20, 32, 64$, the external tachyonic interval contains one negative projected eigenvalue. It is always the outward extremal mode and agrees with

$$k_-(\xi) = 2(\xi - J)^2 - 4(J + 1) \tag{27}$$

within 1.8×10^{-13}. The second projected eigenvalue remains positive. This is a finite-rank Gauss-compatible quadratic test; it does not replace solving the complete nonlinear, time-dependent Gauss constraint.

For the normalized mode $w = \sqrt{J + 1}\, y$ and $s = (\xi - J)/\sqrt{2(J + 1)}$, the reduced Euclidean equation is

$$y'' = (s^2 - 1)y + y^3. \tag{28}$$

We used

$$s(\tau) = 1.2 \left[1 - \operatorname{sech}^2(\tau/3)\right], \qquad \tau \in [-15, 15], \tag{29}$$

and imposed $y(\pm15) = 0$. The boundary-value solver used 1,576 adaptive nodes, had maximum residual 2.35×10^{-9}, and produced a nonzero cap with reduced action difference -1.05535 relative to $y = 0$. This benchmark tests the link envelope once the collective coordinate is supplied by the rank-changing background; it is not an independent derivation of that background trajectory.

7 Gelfand–Tsetlin trajectories and recovery

For $n = 1$, a basis vector in $\mathcal{Q}_N$ is a Gelfand–Tsetlin triangular pattern. Cutting the pattern at a daughter rank K yields a radiation prefix, a daughter suffix and their shared representation label. Random complex Gaussian coefficients in the complete pattern basis generate Haar-distributed states. For each shared row, the coefficient array is reshaped into a radiation-by-daughter matrix, and its singular values give the exact Schmidt spectrum.

The Petz map was computed for a four-dimensional random code inside the $N = 3$ protected sector. If A_j are the Kraus matrices of the radiation channel and

$$\sigma_R = \frac{1}{q}\sum_j A_j A_j^\dagger, \tag{30}$$

then the Petz Kraus matrices are

$$R_j = \frac{1}{\sqrt{q}} A_j^\dagger \sigma_R^{-1/2}. \tag{31}$$

The entanglement fidelity of the recovered channel is

$$F_e = \sum_{j,k} \left|\frac{1}{q}\operatorname{Tr}(R_j A_k)\right|^2. \tag{32}$$

Complete radiation-basis dephasing was applied by replacing every A_j with the row-resolved Kraus family $P_r A_j$. The coherent fidelity rises to one at complete emission, whereas the dephased channel saturates at $1/q$.

8 Numerical verification and source data

All plots in the main text and Extended Data were generated by the supplied Python scripts. The protected dimension was checked against the Weyl product through $N = 20$ with maximum log error 3.70×10^{-13}. The all-sector census through $N = 10$ contains 517,971 sectors at the largest rank and reproduces the total half-filled dimension with maximum logarithmic closure error 5.7×10^{-14}. One-step branching probabilities normalize through $N = 12$ with maximum error 2.13×10^{-14}. The thermal direct-sum channel was tested on 100 random logical states; the largest branch-probability deviation was 4.44×10^{-16}. Two orthogonal states with identical basis populations had exact output trace distance one and dephased trace distance zero.

The corresponding numerical controls and finite-rank scans are shown in Extended Data Figs. 1–5 of the main manuscript.

9 Claim boundaries

The exact commutant theorem applies to every skew–Howe sector, but branch rates can differ between active geometric factors and the explicit closed-form recovery simulations use the distinguished self-complementary code. The Hawking-temperature factor and its $b = 8/3$ coefficient match are inherited from the previously calculated monopole saddle. The background-transverse link scan is not the complete nonlinear Gauss solution. Finally, a complete AMPS analysis requires a reconstructed interior partner with the correct infalling two-point function; the present work supplies the state-resolved channel on which that calculation can be performed, but does not claim the smooth-infall result.

References

[1] Chong-Sun Chu. A matrix model proposal for quantum gravity and the quantum mechanics of black holes. *arXiv:2406.01466*, 2024. v3, 2025.

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[3] Chong-Sun Chu. Hawking radiation from tunneling in black hole quantum mechanics. *arXiv:2603.12199*, 2026.