

Effective Quantum Dynamics from Normalized Additive Amplitude Perturbations

Rajnish Kumar

rajnish.kumar@jkl.u.edu.in

JK Lakshmipat University

Research Article

Keywords: Additive amplitude noise, Depolarizing channel, Quantum noise, State fidelity, Stochastic perturbation

Posted Date: October 16th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-10546674/v1>

License:   This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: No competing interests reported.

Effective Quantum Dynamics from Normalized Additive Amplitude Perturbations

Rajnish Kumar^{1*}

^{1*}Department of Electronics and Communication Engineering, J K Lakshmipat University, , Jaipur, 302026, Rajasthan, India.

Corresponding author(s). E-mail(s): rajnish.kumar@jklu.edu.in;

Abstract

The characterization of quantum noise is fundamental to quantum information processing, quantum control and reliable operation of near-term quantum devices. Conventional descriptions of quantum noise are based on completely positive trace-preserving (CPTP) maps acting on density operators. These models describe effective quantum dynamics after the underlying physical interactions have been represented as a quantum channel. However, in practical cases, imperfections such as calibration errors, control-field fluctuations, finite-resolution waveform generation, and analog hardware noise can act upon the probability amplitudes of a quantum state earlier. The effect of such perturbation in quantum state after normalization has largely been unexplored. In this work, we develop a rigorous mathematical analysis of the normalized additive amplitude noise in a finite-dimensional quantum system. Assuming a zero-mean complex perturbation, we derive a second-order closed form expression for the ensemble-averaged density operator. The analysis leads to state-dependent evolution of the quantum state which cannot be represented by a CPTP map. We further prove that in case of isotropic noise, the ensemble-averaged density operator becomes an effective depolarizing channel. For anisotropic perturbations, we derive the orientation-dependent Bloch-vector contraction which is determined by the covariance geometry of the additive noise. Closed-form expressions are further obtained for measurement probabilities, state fidelity, and ensemble purity, showing that fidelity degradation depends only on the component of the perturbation covariance orthogonal to the input state. Monte Carlo simulations closely match the theoretical analysis, especially for small noise regime. The proposed model establishes a connection between classical additive-noise models and effective quantum dynamics generated by normalized state-space perturbations.

Keywords: Additive amplitude noise, Depolarizing channel, Quantum noise, State fidelity, Stochastic perturbation

1 Introduction

Quantum noise is one of the main obstacles to obtain a reliable quantum communication and quantum information processing [1–3]. Despite many advances to develop Noisy Intermediate-Scale Quantum (NISQ) hardware, quantum systems suffer from various noise sources such as decoherence, imperfect control, calibration errors, and stochastic fluctuations [4–6]. Such effects limit the computational accuracy and circuit depth [7, 8]. Therefore, accurate characterization of quantum noise is essential for quantum information science. This is required for quantum error correction, error mitigation, quantum control, quantum sensing, and fault-tolerant quantum computing [9–12].

The completely positive trace-preserving (CPTP) maps are standard mathematical framework to describe quantum noise [13, 14]. Some well-known examples include depolarizing, dephasing, amplitude-damping, and generalized Pauli channels [15]. Each of them represents a physically realizable interaction between a quantum system and its surrounding environment. These operator-based models provide a rigorous description of open quantum dynamics. They form the foundation of most theoretical analyses of noise, quantum error correction, and quantum error mitigation [14, 16].

Despite their mathematical success, CPTP models describe noise only after the underlying physical interaction has been abstracted into an operator acting on density matrices [17, 18]. In practical quantum hardware, however, many imperfections originate at a much earlier stage. Control electronics, microwave pulses, laser fields, analog circuitry, calibration inaccuracies, and finite-resolution waveform generators introduce small random perturbations [19, 20]. Such effects can directly alter the complex amplitudes governing the quantum-state preparation and manipulation. A useful mathematical abstraction of such imperfections is to model them as additive perturbations acting on the complex amplitudes prior to state normalization. Although additive noise constitutes one of the most fundamental models in classical signal processing, an analogous amplitude-domain framework has received comparatively little theoretical attention in quantum information [21, 22].

This observation motivates a fundamental question: *How does additive perturbation of quantum probability amplitudes manifest itself after normalization, and what effective quantum dynamics emerge upon averaging over the noise ensemble?* The answer is far from obvious because normalization introduces a nonlinear dependence between the signal and the perturbation. Therefore, the resulting stochastic evolution differs fundamentally from conventional linear quantum channels. It cannot be, in general, analyzed using standard operator techniques.

Recent studies have begun investigating additive amplitude perturbations and their influence on quantum-state fidelity, measurement probabilities, and gate robustness under small stochastic perturbations [23]. However, these studies focused primarily on observable performance measures and did not develop an operator-level description of the resulting ensemble dynamics or establish their relationship to conventional quantum-channel models. In that study, authors have demonstrated the practical impact of amplitude-domain fluctuations through analytical and numerical analysis, but did not address a more fundamental question: *what effective quantum dynamics emerge from normalized additive amplitude perturbations when viewed from the*

perspective of quantum channel theory? The present work answers this question by developing a rigorous operator-level framework for normalized additive amplitude noise. Specifically, we derive the ensemble-averaged state evolution, establishing its connection with depolarizing dynamics in the isotropic case, and further characterize the state-dependent behavior arising from anisotropic perturbations.

In this work, we develop a rigorous theoretical framework for normalized additive amplitude noise in finite-dimensional quantum systems. Without any assumption on the quantum channel a priori, we start with additive stochastic perturbations of the state vector. We then investigate the statistical properties of the resulting normalized quantum states. By performing a second-order perturbation analysis, we derive a closed-form expression for the ensemble-averaged density operator. We then study the fundamental role played by the covariance operator of the additive perturbation.

Our analysis shows that isotropic additive amplitude perturbations have a remarkable property that despite the nonlinear normalization process, the ensemble-averaged state coincides, to second order, to the depolarizing channel. This result establishes a direct theoretical connection between additive amplitude perturbations and one of the most important canonical quantum noise models. In contrast, for general anisotropic perturbations, the ensemble-averaged evolution becomes explicitly state dependent. This demonstrates that normalized additive noise does not generally reduce to a linear CPTP channel. This distinction shows an important limitation of interpretation of the arbitrary amplitude perturbations through conventional channel models. It further highlights the fundamentally nonlinear nature of normalized state-space perturbations.

We derive analytical expressions describing the influence of additive amplitude noise on several fundamental performance measures, including measurement probabilities, state fidelity, Bloch-vector contraction, and ensemble purity. These results show that normalization significantly alters the statistical behavior of additive perturbations. It leads to compact closed-form relationships between the covariance structure of the noise and observable quantities. Furthermore, we introduce an orientation-dependent sensitivity framework that quantifies how structured amplitude perturbations affect quantum states under different noise geometries.

Rather than replacing conventional quantum-channel descriptions, the proposed framework complements them by providing a state-space perspective on stochastic perturbations before normalization. The results establish a mathematical connection between amplitude-domain perturbations and effective quantum dynamics, thereby providing a theoretical basis for analyzing state-preparation errors, coherent-control imperfections, and calibration-induced fluctuations in near-term quantum hardware. To the best of our knowledge, this work provides the first general operator-level framework that derives the ensemble-averaged quantum dynamics resulting from normalized additive amplitude perturbations with arbitrary covariance, without assuming an underlying quantum channel.

The remainder of this paper is organized as follows. Section II introduces the normalized additive amplitude noise model and formulates the problem. Section III presents the theoretical analysis, including the general perturbation expansion, the isotropic and anisotropic noise models, and the analytical characterization of measurement statistics, fidelity, and ensemble purity. Section IV validates the theoretical

results through numerical simulations. Finally, Section V concludes the paper and outlines future research directions.

2 Normalized Additive Amplitude Noise Model

In this section, we introduce the normalized additive amplitude noise model. Unlike conventional quantum noise models which describe system evolution through completely positive trace-preserving (CPTP) maps, the proposed method models stochastic perturbations directly in the state-vector representation. Physical quantum states are then recovered through normalization. This results in a nonlinear stochastic transformation whose statistical properties are analyzed in the following section.

Consider a finite-dimensional quantum system defined on the Hilbert space $\mathcal{H} = \mathbb{C}^d$. A pure quantum state is represented by a normalized vector $|\psi\rangle \in \mathcal{H}$, with $\langle\psi|\psi\rangle = 1$ and its corresponding density operator given by $\rho = |\psi\rangle\langle\psi|$.

2.1 Additive Amplitude Perturbation

We model the imperfection as additive perturbations acting directly on the complex probability amplitudes. The perturbed state is written as

$$|\psi_n\rangle = |\psi\rangle + |\eta\rangle, \quad (1)$$

where $|\eta\rangle \in \mathcal{H}$ is a random complex vector satisfying $\mathbb{E}[|\eta\rangle] = 0$ and $\mathbb{E}[|\eta\rangle\langle\eta|] = \Sigma$. Here, $\Sigma \succeq 0$ denotes the covariance operator of the additive perturbation. Throughout this work, the perturbation is assumed to have finite second-order moments, while higher-order moments are neglected under the small-noise assumption. Unlike conventional operator-based quantum noise models, the perturbation acts before state normalization and therefore does not directly define a quantum channel.

2.2 State Normalization

Since the perturbed vector generally does not satisfy the normalization constraint, the physical quantum state is obtained as

$$|\tilde{\psi}\rangle = \frac{|\psi\rangle + |\eta\rangle}{\| |\psi\rangle + |\eta\rangle \|} \quad (2)$$

The corresponding density operator is $\rho_\eta = |\tilde{\psi}\rangle\langle\tilde{\psi}|$. Each realization therefore remains a valid pure quantum state satisfying $\text{Tr}(\rho_\eta) = 1$ and $\rho_\eta^2 = \rho_\eta$. However, the normalization introduces a nonlinear dependence on the perturbation, making the statistical analysis substantially different from that of conventional additive-noise models.

Rather than analyzing individual noisy realizations, we study the ensemble-averaged density operator $\bar{\rho} = \mathbb{E}[\rho_\eta]$, which characterizes the average effect of additive amplitude perturbations. Because normalization is nonlinear, the mapping $\rho \rightarrow \bar{\rho}$ is generally nonlinear and depends on both the input state and the covariance structure

of the perturbation. Therefore, it does not, in general, correspond to a linear CPTP quantum channel.

2.3 Problem Statement

The objective of this work is to characterize the statistical evolution of the normalized quantum state under additive amplitude perturbations. We will do the following

1. Derive a second-order approximation of the ensemble-averaged density operator in terms of the covariance operator Σ
2. Identify the conditions under which the resulting ensemble dynamics reduce to a standard depolarizing channel
3. Investigate how anisotropic covariance structures modify the ensemble evolution
4. Quantify the influence of additive amplitude noise on measurement statistics, state fidelity, Bloch-vector contraction, and ensemble purity

These objectives establish the foundation for the theoretical analysis presented in the next section.

3 Theoretical Analysis

This section develops the theoretical characterization of normalized additive amplitude noise. We first derive a general second-order expansion of the ensemble-averaged density operator. We then examine isotropic and anisotropic perturbations and derive analytical expressions for measurement statistics, fidelity, Bloch-vector evolution, and ensemble purity. Throughout the analysis, the perturbation covariance is assumed to satisfy $\Sigma = \mathcal{O}(\sigma^2)$, where σ denotes the characteristic standard deviation of the additive amplitude perturbation. The covariance operator scales quadratically with the perturbation amplitude, and the neglected terms in the subsequent expansion are of order $\mathcal{O}(\sigma^4)$. We further assume that the perturbation is a proper complex random vector satisfying $\mathbb{E}[|\eta\rangle\langle\eta|^T] = 0$, which is satisfied by circularly symmetric complex Gaussian noise. This assumption eliminates pseudo-covariance terms and simplifies the second-order perturbation analysis.

3.1 General Second-Order Expansion

The normalized noisy density operator can be written as

$$\rho_\eta = \frac{(|\psi\rangle + |\eta\rangle)(\langle\psi| + \langle\eta|)}{(\langle\psi| + \langle\eta|)(|\psi\rangle + |\eta\rangle)} \quad (3)$$

Let us define $s = \langle\psi|\eta\rangle$ and $q = \langle\eta|\eta\rangle$, such that the normalization denominator becomes $1 + s + s^* + q$. The following theorem gives the ensemble-averaged state in the small-noise regime.

Theorem 1 *Let $|\psi\rangle$ be a normalized pure state with density operator $\rho = |\psi\rangle\langle\psi|$, and let $|\eta\rangle$ be a zero-mean proper complex perturbation satisfying $\mathbb{E}[|\eta\rangle\langle\eta|] = \Sigma$. Then the ensemble-averaged*

normalized state satisfies

$$\begin{aligned}\bar{\rho} = \mathbb{E}[\rho_\eta] &= \rho + \Sigma - \text{Tr}(\Sigma)\rho \\ &+ 2 \text{Tr}(\rho\Sigma)\rho - \rho\Sigma - \Sigma\rho + \mathcal{O}(\sigma^4).\end{aligned}\tag{4}$$

Proof The numerator of ρ_η is expanded as

$$(|\psi\rangle + |\eta\rangle)(\langle\psi| + \langle\eta|) = \rho + L + Q\tag{5}$$

where $L = |\psi\rangle\langle\eta| + |\eta\rangle\langle\psi|$ and $Q = |\eta\rangle\langle\eta|$.

Let us assume that $x = s + s^*$. Since $x = \mathcal{O}(\sigma)$ and $q = \mathcal{O}(\sigma^2)$, the inverse normalization factor will have the expansion

$$\frac{1}{1+x+q} = 1 - x - q + x^2 + \mathcal{O}(\sigma^3).\tag{6}$$

Multiplying the numerator and the expanded normalization factor gives

$$\rho_\eta = \rho + L + Q - \rho x - Lx - \rho q + \rho x^2 + \mathcal{O}(\sigma^3)\tag{7}$$

where, terms such as Lq , Lx^2 , and Qx are omitted because they are of third or higher order in the perturbation amplitude. Taking expectation and using $\mathbb{E}[L] = 0$ and $\mathbb{E}[x] = 0$, we obtain

$$\mathbb{E}[\rho_\eta] = \rho + \mathbb{E}[Q] - \mathbb{E}[Lx] - \rho\mathbb{E}[q] + \rho\mathbb{E}[x^2] + \mathcal{O}(\sigma^4)\tag{8}$$

The quadratic terms are evaluated separately. We note that $\mathbb{E}[Q] = \mathbb{E}[|\eta\rangle\langle\eta|] = \Sigma$ and $\mathbb{E}[q] = \mathbb{E}[\langle\eta|\eta\rangle] = \text{Tr}(\Sigma)$. For the term $\mathbb{E}[Lx]$, the covariance relation and the properness condition of the zero-mean complex perturbation gives $\mathbb{E}[Lx] = \rho\Sigma + \Sigma\rho$ (Proof in Appendix).

Finally, using $x = s + s^*$ with $s = \langle\psi|\eta\rangle$, we can write that $\mathbb{E}[x^2] = \mathbb{E}[s^2] + \mathbb{E}[(s^*)^2] + 2\mathbb{E}[|s|^2]$. The properness condition $\mathbb{E}[|\eta\rangle\langle\eta|^T] = 0$ implies $\mathbb{E}[s^2] = \langle\psi|\mathbb{E}[|\eta\rangle\langle\eta|^T]|\psi^*\rangle = 0$ and similarly $\mathbb{E}[(s^*)^2] = 0$. Therefore, $\mathbb{E}[x^2] = 2\mathbb{E}[|s|^2] = 2\langle\psi|\mathbb{E}[|\eta\rangle\langle\eta|]|\psi\rangle = 2\langle\psi|\Sigma|\psi\rangle = 2\text{Tr}(\rho\Sigma)$. Substituting these expressions yields

$$\begin{aligned}\mathbb{E}[\rho_\eta] &= \rho + \Sigma - \text{Tr}(\Sigma)\rho \\ &+ 2 \text{Tr}(\rho\Sigma)\rho - \rho\Sigma - \Sigma\rho + \mathcal{O}(\sigma^4),\end{aligned}\tag{9}$$

which completes the proof. $\square$

Theorem 1 shows that normalization generates terms that cannot be obtained by simply dividing the expected unnormalized density operator by the expected normalization factor. In particular,

$$\mathbb{E}\left[\frac{\rho + L + Q}{1 + x + q}\right] \neq \frac{\mathbb{E}[\rho + L + Q]}{\mathbb{E}[1 + x + q]}.\tag{10}$$

The state-dependent term $2\text{Tr}(\rho\Sigma)\rho$ is especially important because it demonstrates that general anisotropic perturbations do not necessarily generate a linear quantum channel.

3.2 Isotropic Noise and Emergent Depolarization

We first consider isotropic perturbations described by $\Sigma = \sigma^2 I_d$, where I_d is the identity operator on a d -dimensional Hilbert space.

Theorem 2 *Let the perturbation covariance be isotropic $\Sigma = \sigma^2 I_d$. Then, in the small-noise regime, the ensemble-averaged density operator satisfies*

$$\bar{\rho} = (1 - d\sigma^2)\rho + d\sigma^2 \frac{I_d}{d} + \mathcal{O}(\sigma^4) \quad (11)$$

Therefore, to second order in the perturbation amplitude, the ensemble-averaged normalized state coincides with the output of the depolarizing transformation $\mathcal{D}_p(\rho) = (1-p)\rho + p\frac{I_d}{d}$ with the depolarization probability $p = d\sigma^2 + \mathcal{O}(\sigma^4)$.

Proof For $\Sigma = \sigma^2 I_d$, the quantities appearing in Theorem 1 become

$$\begin{aligned} \text{Tr}(\Sigma) &= \text{Tr}(\sigma^2 I_d) = \sigma^2 \text{Tr}(I_d) = d\sigma^2 \\ \text{Tr}(\rho\Sigma) &= \text{Tr}(\rho\sigma^2 I_d) = \sigma^2 \text{Tr}(\rho) = \sigma^2 \\ \rho\Sigma + \Sigma\rho &= \rho(\sigma^2 I_d) + (\sigma^2 I_d)\rho = 2\sigma^2 \rho \end{aligned} \quad (12)$$

Substituting into Theorem 1 gives

$$\begin{aligned} \bar{\rho} &= \rho + \sigma^2 I_d - d\sigma^2 \rho + 2\sigma^2 \rho - 2\sigma^2 \rho + \mathcal{O}(\sigma^4) \\ &= (1 - d\sigma^2)\rho + \sigma^2 I_d + \mathcal{O}(\sigma^4). \end{aligned} \quad (13)$$

Since $\sigma^2 I_d = d\sigma^2 \frac{I_d}{d}$, the result takes the standard depolarizing form

$$\bar{\rho} = (1 - p)\rho + p\frac{I_d}{d} + \mathcal{O}(\sigma^4) \quad (14)$$

with $p = d\sigma^2$. This completes the proof. $\square$

For a single qubit, $d = 2$, and hence

$$\bar{\rho} = (1 - 2\sigma^2)\rho + 2\sigma^2 \frac{I}{2} + \mathcal{O}(\sigma^4) \quad (15)$$

Corollary 1 Let the input qubit state be represented as

$$\rho = \frac{1}{2} (I + \mathbf{r} \cdot \boldsymbol{\sigma}) \quad (16)$$

Under isotropic additive amplitude noise, the Bloch vector transforms as

$$\mathbf{r}' = (1 - 2\sigma^2)\mathbf{r} + \mathcal{O}(\sigma^4) \quad (17)$$

Proof Substituting the Bloch representation into the qubit result gives

$$\begin{aligned} \bar{\rho} &= (1 - 2\sigma^2) \frac{1}{2} (I + \mathbf{r} \cdot \boldsymbol{\sigma}) + \sigma^2 I + \mathcal{O}(\sigma^4) \\ &= \frac{1}{2} [I + (1 - 2\sigma^2)\mathbf{r} \cdot \boldsymbol{\sigma}] + \mathcal{O}(\sigma^4) \end{aligned} \quad (18)$$

Comparison with $\bar{\rho} = \frac{1}{2} (I + \mathbf{r}' \cdot \boldsymbol{\sigma})$ gives the desired result. $\square$

3.3 Anisotropic Noise and State-Dependent Evolution

For a qubit, an arbitrary Hermitian covariance operator can be expressed as

$$\Sigma = aI + \mathbf{b} \cdot \boldsymbol{\sigma} \quad (19)$$

where $a \geq \|\mathbf{b}\|$ ensures that Σ is positive semidefinite.

The following result characterizes the effect of such structured noise on the Bloch vector.

Theorem 3 *Let*

$$\rho = \frac{1}{2} (I + \mathbf{r} \cdot \boldsymbol{\sigma}), \quad \|\mathbf{r}\| = 1, \quad (20)$$

and let the covariance operator for a single qubit be $\Sigma = aI + \mathbf{b} \cdot \boldsymbol{\sigma}$. Then the ensemble-averaged Bloch vector for the qubit satisfies

$$\mathbf{r}' = [1 + 2(\mathbf{b} \cdot \mathbf{r} - a)] \mathbf{r} + \mathcal{O}(\sigma^4) \quad (21)$$

Proof For the above representations, $\text{Tr}(\Sigma) = 2a$ and $\text{Tr}(\rho\Sigma) = a + \mathbf{b} \cdot \mathbf{r}$. Using the Pauli identity $(\mathbf{u} \cdot \boldsymbol{\sigma})(\mathbf{v} \cdot \boldsymbol{\sigma}) = (\mathbf{u} \cdot \mathbf{v})I + i(\mathbf{u} \times \mathbf{v}) \cdot \boldsymbol{\sigma}$, the anticommutator becomes

$$\rho\Sigma + \Sigma\rho = (a + \mathbf{b} \cdot \mathbf{r})I + (a\mathbf{r} + \mathbf{b}) \cdot \boldsymbol{\sigma} \quad (22)$$

Substituting these expressions into Theorem 1 gives

$$\bar{\rho} = \frac{1}{2}I + \left[\frac{1}{2} + \mathbf{b} \cdot \mathbf{r} - a \right] \mathbf{r} \cdot \boldsymbol{\sigma} + \mathcal{O}(\sigma^4). \quad (23)$$

Writing the result in Bloch form,

$$\bar{\rho} = \frac{1}{2} (I + \mathbf{r}' \cdot \boldsymbol{\sigma}), \quad (24)$$

yields

$$\mathbf{r}' = [1 + 2(\mathbf{b} \cdot \mathbf{r} - a)] \mathbf{r} + \mathcal{O}(\sigma^4). \quad (25)$$

This completes the proof. $\square$

Theorem 3 shows that the contraction factor depends on the orientation of the input state relative to the anisotropy vector $\mathbf{b}$. Therefore, anisotropic additive amplitude noise does not generally produce a state-independent linear transformation of the Bloch sphere.

For diagonal covariance,

$$\Sigma = \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix} \quad (26)$$

we have

$$a = \frac{\sigma_0^2 + \sigma_1^2}{2}, \quad \mathbf{b} = \left(0, 0, \frac{\sigma_0^2 - \sigma_1^2}{2} \right) \quad (27)$$

Substituting the diagonal covariance parameters into Theorem 3 yields the state-dependent contraction factor as

$$\kappa(r_z) = 1 - (\sigma_0^2 + \sigma_1^2) + (\sigma_0^2 - \sigma_1^2)r_z \quad (28)$$

For diagonal anisotropic covariance, the Bloch vector transforms as

$$\mathbf{r}' = [1 - (\sigma_0^2 + \sigma_1^2) + (\sigma_0^2 - \sigma_1^2)r_z] \mathbf{r} + \mathcal{O}(\sigma^4) \quad (29)$$

Hence, states with different values of r_z experience different contraction factors under the same covariance matrix.

3.4 Measurement Statistics

Let (M) denote a positive operator-valued measure (POVM) effect associated with a measurement outcome, satisfying $0 \preceq M \preceq I$. The ideal and ensemble-averaged probabilities of this outcome are, respectively, $P_M = \text{Tr}(M\rho)$ and $\bar{P}_M = \text{Tr}(M\bar{\rho})$.

Proposition 1 *The ensemble-averaged measurement probability satisfies*

$$\begin{aligned} \bar{P}_M &= P_M + \text{Tr}(M\Sigma) - \text{Tr}(\Sigma)P_M \\ &\quad + 2\text{Tr}(\rho\Sigma)P_M - \text{Tr}[M(\rho\Sigma + \Sigma\rho)] + \mathcal{O}(\sigma^4) \end{aligned} \quad (30)$$

Proof The result follows by multiplying the expansion in Theorem 1 by M and taking the trace. Specifically,

$$\bar{P}_M = \text{Tr}(M\mathbb{E}[\rho_\eta]). \quad (31)$$

Applying linearity of the trace to each term in Theorem 1 gives the stated expression.

For isotropic qubit noise and the computational-basis projector $M_0 = |0\rangle\langle 0|$, we obtain

$$\mathbb{E}[P_0] = (1 - 2\sigma^2)|\alpha|^2 + \sigma^2 + \mathcal{O}(\sigma^4) \quad (32)$$

where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Equivalently,

$$\mathbb{E}[P_0] = |\alpha|^2 + \sigma^2(1 - 2|\alpha|^2) + \mathcal{O}(\sigma^4) \quad (33)$$

Thus, isotropic perturbations drive the measurement probability toward 1/2, rather than introducing a uniformly positive bias. $\square$

3.5 State Fidelity

For a pure reference state $\rho = |\psi\rangle\langle\psi|$, the fidelity with the ensemble-averaged noisy state is

$$F(\rho, \bar{\rho}) = \langle\psi|\bar{\rho}|\psi\rangle \quad (34)$$

Proposition 2 *The fidelity satisfies*

$$F(\rho, \bar{\rho}) = 1 - \text{Tr}(\Sigma) + \text{Tr}(\rho\Sigma) + \mathcal{O}(\sigma^4) \quad (35)$$

Equivalently, the fidelity loss is

$$1 - F(\rho, \bar{\rho}) = \text{Tr}[(I - \rho)\Sigma] + \mathcal{O}(\sigma^4) \quad (36)$$

Proof Using Theorem 1,

$$F = \text{Tr}(\rho\bar{\rho}) \quad (37)$$

$$= \text{Tr}(\rho^2) + \text{Tr}(\rho\Sigma) - \text{Tr}(\Sigma) \text{Tr}(\rho^2) \quad (38)$$

$$+ 2 \text{Tr}(\rho\Sigma) \text{Tr}(\rho^2) - \text{Tr}(\rho^2\Sigma) - \text{Tr}(\rho\Sigma\rho) + \mathcal{O}(\sigma^4) \quad (39)$$

Since ρ is pure,

$$\rho^2 = \rho, \quad \text{Tr}(\rho^2) = 1. \quad (40)$$

Furthermore we use the property that $\text{Tr}(\rho^2\Sigma) = \text{Tr}(\rho\Sigma)$ and $\text{Tr}(\rho\Sigma\rho) = \text{Tr}(\rho\Sigma)$. Therefore,

$$F = 1 - \text{Tr}(\Sigma) + \text{Tr}(\rho\Sigma) + \mathcal{O}(\sigma^4), \quad (41)$$

which proves the result. $\square$

For isotropic noise in dimension d ,

$$F = 1 - (d-1)\sigma^2 + \mathcal{O}(\sigma^4) \quad (42)$$

For a single qubit,

$$F = 1 - \sigma^2 + \mathcal{O}(\sigma^4) \quad (43)$$

The fidelity expression has a direct geometric interpretation. Noise parallel to the original state alters the state norm and is removed by normalization, whereas covariance in the orthogonal subspace produces leading-order fidelity loss.

3.6 Ensemble Purity

Each realization ρ_η is a pure state and therefore satisfies

$$\text{Tr}(\rho_\eta^2) = 1 \quad (44)$$

However, the ensemble-averaged state $\bar{\rho}$ is generally mixed.

Proposition 3 *For isotropic additive amplitude noise acting on a pure state in dimension d , the purity of the ensemble-averaged state satisfies*

$$\text{Tr}(\bar{\rho}^2) = 1 - 2(d-1)\sigma^2 + \mathcal{O}(\sigma^4) \quad (45)$$

For a single qubit,

$$\text{Tr}(\bar{\rho}^2) = 1 - 2\sigma^2 + \mathcal{O}(\sigma^4) \quad (46)$$

Proof. From Theorem 2,

$$\bar{\rho} = \rho + \sigma^2(I_d - d\rho) + \mathcal{O}(\sigma^4). \quad (47)$$

Consequently,

$$\bar{\rho}^2 = \rho^2 + \sigma^2[\rho(I_d - d\rho) + (I_d - d\rho)\rho] + \mathcal{O}(\sigma^4) \quad (48)$$

Since $\rho^2 = \rho$, we can write that

$$\bar{\rho}^2 = \rho + 2\sigma^2(1-d)\rho + \mathcal{O}(\sigma^4) \quad (49)$$

Taking the trace gives

$$\text{Tr}(\bar{\rho}^2) = 1 - 2(d-1)\sigma^2 + \mathcal{O}(\sigma^4) \quad (50)$$

which completes the proof.

It is important to distinguish the purity of individual noisy realizations from the purity of their ensemble average. The former remains unity, while the latter decreases because statistical averaging produces an effective mixed state.

4 Numerical Validation

The analytical results have been validated through Monte Carlo simulations (50000 samples) of a single-qubit system subject to normalized additive amplitude perturbations. The noiseless input state is represented as $|\psi\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{j\phi}\sin(\frac{\theta}{2})|1\rangle$, where $\theta = 1.10$ rad and $\phi = 0.65$ rad unless otherwise specified. For each noise realization, a complex perturbation vector $|\eta\rangle = \eta_0|0\rangle + \eta_1|1\rangle$ is generated, where $\eta_i \sim \mathcal{CN}(0, \sigma_i^2)$. The resulting physical state is obtained by normalization. The ensemble-averaged density operator is estimated according to

$$\bar{\rho}_{\text{MC}} = \frac{1}{N_{\text{MC}}} \sum_{m=1}^{N_{\text{MC}}} |\tilde{\psi}_{\eta}^{(m)}\rangle \langle \tilde{\psi}_{\eta}^{(m)}|, \quad (51)$$

where N_{MC} denotes the number of independent Monte Carlo realizations.

For the fidelity, ensemble-purity, and measurement-probability results, the noise standard deviation is varied over $0 \leq \sigma \leq 0.30$, using 21 uniformly spaced values. The isotropic case is obtained by setting $\sigma_0^2 = \sigma_1^2 = \sigma^2$.

To illustrate Bloch-sphere contraction, randomly distributed pure qubit states are generated using normalized complex Gaussian vectors. In the isotropic example, the noise standard deviation is fixed at $\sigma = 0.22$. The anisotropic case is modeled using the diagonal covariance matrix. For the comparison between isotropic and anisotropic perturbations, the isotropic component variance is set to $\sigma_0^2 = \sigma_1^2 = 0.035$, whereas the anisotropic component variances are selected as $\sigma_0^2 = 0.010$ and $\sigma_1^2 = 0.070$.

For the orientation-dependent contraction study, the component variances are set to $\sigma_0^2 = 0.010$ and $\sigma_1^2 = 0.080$. The input Bloch-vector component r_z is varied uniformly over $[-1, 1]$. The simulated radial contraction is compared with the second-order analytical expression [28](#).

The measurement-probability analysis considers the input states $|0\rangle$, $|+\rangle$, and $|\psi_b\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$. These states respectively correspond to ideal computational-basis probabilities of $P_0 = 1$, $P_0 = \frac{1}{2}$, and $P_0 = \frac{1}{4}$ respectively. Therefore, they illustrate convergence toward the maximally mixed-state value $P_0 = 1/2$ under isotropic perturbations.

All simulations are implemented in Python using NumPy, while Matplotlib is used for visualization. A fixed random seed of 42 is employed to ensure reproducibility. The numerical density matrices are additionally verified to be Hermitian, positive semidefinite, and unit trace.

[Fig. 1](#) compares the analytical expression for the state fidelity with the corresponding Monte Carlo estimates as the noise standard deviation increases. We observe an excellent agreement for the small-noise regime that clearly confirms the accuracy of the second-order approximation derived in [Proposition 2](#). As predicted theoretically,

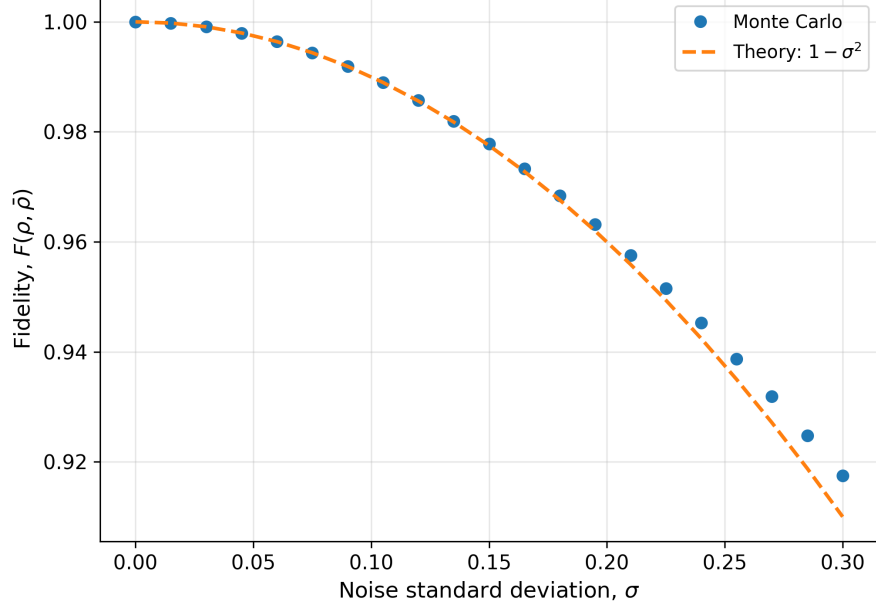


Fig. 1 State fidelity as a function of the noise standard deviation under isotropic additive amplitude perturbations. Monte Carlo simulations are compared with the second-order analytical approximation. The close agreement validates the theoretical fidelity expression in the small-noise regime, while slight deviations at larger noise levels arise from neglected higher-order terms.

the fidelity decreases approximately linearly with σ^2 , which indicates that the leading-order degradation originates from the component of the perturbation orthogonal to the input state. Slight deviations between theory and simulation become noticeable at larger noise levels, where the higher-order terms neglected in the perturbation analysis start to contribute.

Fig. 2 shows the purity of the ensemble-averaged density operator as a function of the noise standard deviation. Although every normalized realization remains a pure quantum state, statistical averaging over multiple realizations produces an effective mixed state whose purity decreases with increasing perturbation strength. The simulated results closely follow the analytical expression derived in Proposition 3, validating the predicted quadratic dependence of the ensemble purity on the noise variance.

Fig. 3 shows the action of isotropic additive amplitude noise on the Bloch sphere. The initially uniform distribution of pure states contracts radially toward the origin while preserving spherical symmetry. This behavior is similar to as expected from Corollary 1, which predicts uniform scaling of the Bloch vector by the factor $(1 - 2\sigma^2)$. The figure therefore provides a geometric interpretation of the emergent depolarizing behavior proposed in Theorem 2.

Fig. 4 compares the analytical and Monte Carlo measurement probabilities for several representative input states. States with measurement probabilities larger than one-half gradually move downward toward the maximally mixed-state value. The states with probabilities below one-half increase toward the same limit. The balanced

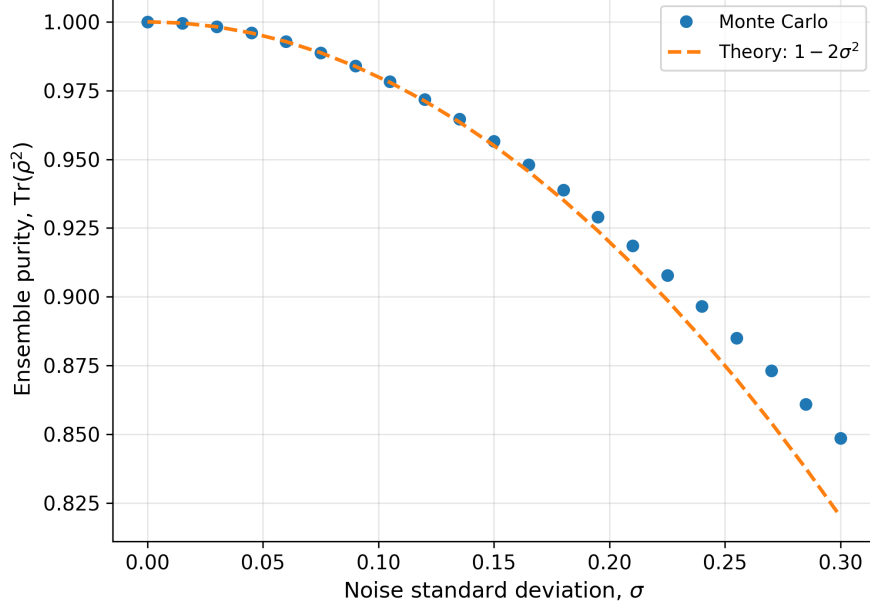


Fig. 2 Purity of the ensemble-averaged density operator versus the noise standard deviation for isotropic additive amplitude noise. The analytical prediction is validated by Monte Carlo simulations, demonstrating that statistical averaging transforms an ensemble of pure states into an effective mixed state.

superposition state remains approximately unchanged because its ideal measurement probability already equals one-half. These observations agree with Proposition 1 and demonstrate that isotropic additive perturbations drive measurement statistics toward those of the maximally mixed state.

Fig. 5 compares the ensemble evolution produced by isotropic and anisotropic covariance operators. Under isotropic perturbations, all Bloch vectors undergo nearly uniform radial contraction, preserving the spherical symmetry of the state space. In contrast, anisotropic perturbations produce orientation-dependent contraction, with different regions of the Bloch sphere exhibiting different shrinkage factors. This qualitative change confirms that anisotropic additive amplitude noise cannot generally be represented by a single state-independent depolarizing transformation, in agreement with Theorem 3.

Fig. 6 validates the analytical contraction factor derived for diagonal anisotropic covariance matrices. The Monte Carlo estimates closely follow the theoretical prediction over the entire range of input-state orientations. This confirms that the contraction depends on the projection of the input state onto the principal direction of the covariance operator. States aligned with the stronger perturbation direction experience larger contraction than those aligned with the weaker direction. This result shows the fundamental difference between isotropic and anisotropic additive amplitude noise and demonstrates that the statistical evolution depends on both the covariance structure and the input-state orientation.

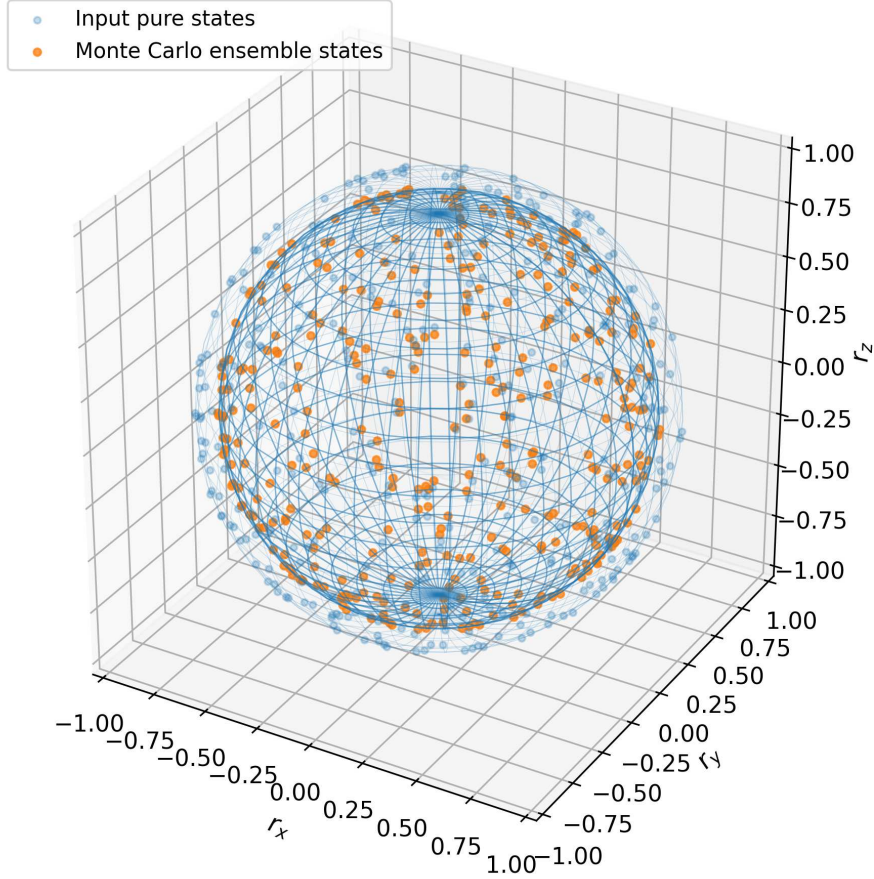


Fig. 3 Bloch-sphere representation of isotropic additive amplitude noise. The original distribution of pure states (outer sphere) contracts uniformly toward the origin after ensemble averaging, showing the Bloch-vector scaling predicted by the theoretical analysis.

5 Conclusion

This paper presented a mathematical framework for analyzing normalized additive amplitude noise in finite-dimensional quantum systems. Unlike conventional approaches that model noise through completely positive trace-preserving (CPTP) maps, the proposed technique considers stochastic perturbations acting directly on the probability amplitudes. A general second-order expansion of the ensemble-averaged density operator was derived, showing the role of the perturbation covariance in determining the resulting quantum-state evolution. The theoretical analysis showed that isotropic additive amplitude perturbations are a special case in which the ensemble-averaged dynamics reduce, to second order, to an effective depolarizing channel. In contrast, anisotropic perturbations were shown to produce orientation-dependent state evolution. This demonstrates that normalized additive amplitude noise does not, in

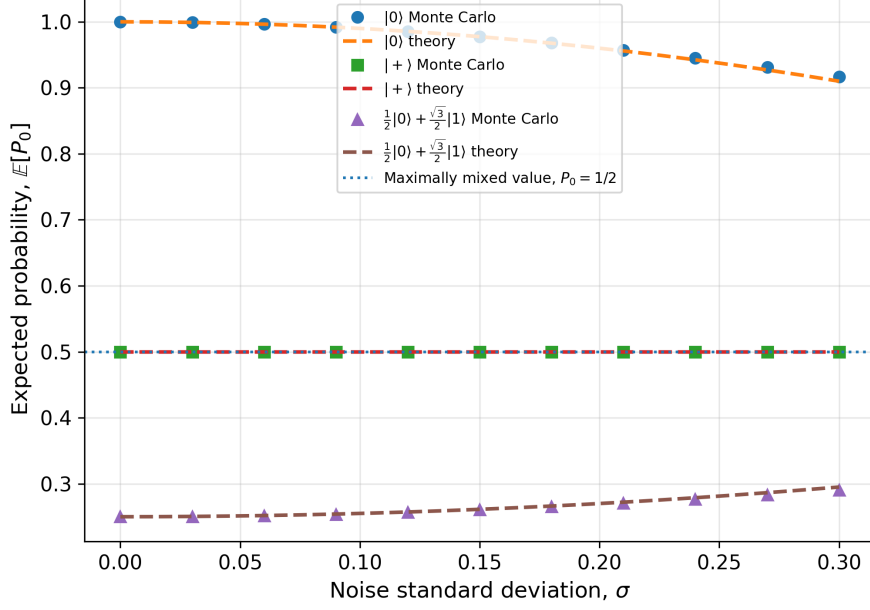


Fig. 4 Expected computational-basis measurement probability as a function of the noise standard deviation for representative input states. Both analytical and Monte Carlo results show that isotropic additive perturbations drive the measurement probabilities toward the maximally mixed-state value of $1/2$.

general, correspond to a linear quantum channel. Closed-form expressions were further obtained for Bloch-vector contraction, measurement probabilities, state fidelity, and ensemble purity. Numerical simulations were presented to validate each theoretical result. Excellent agreement between the analytical approximations and Monte Carlo simulations was observed in the small-noise regime. The results confirm the accuracy of the second-order analysis and illustrate the transition from isotropic depolarization to anisotropic state-dependent behavior. Future work will extend the analysis to multi-qubit systems, correlated and time-dependent perturbations, and investigate potential applications in quantum error mitigation, quantum control, and noise characterization for near-term quantum hardware.

Appendix A Dependence on ρ and Σ

For the term $\mathbb{E}[Lx]$, let us define $s = \langle \psi | \eta \rangle$ and $x = s + s^*$, where $s^* = \langle \eta | \psi \rangle$. Expanding the product gives

$$\begin{aligned} \mathbb{E}[Lx] &= \mathbb{E}[|\psi\rangle\langle\eta|s] + \mathbb{E}[|\psi\rangle\langle\eta|s^*] \\ &\quad + \mathbb{E}[|\eta\rangle\langle\psi|s] + \mathbb{E}[|\eta\rangle\langle\psi|s^*]. \end{aligned} \tag{A1}$$

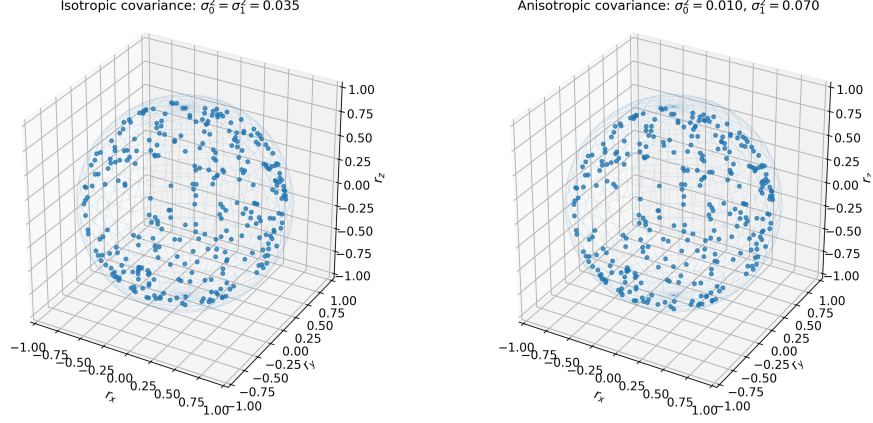


Fig. 5 Comparison of ensemble state evolution under isotropic and anisotropic additive amplitude perturbations. Isotropic noise produces uniform radial contraction of the Bloch sphere, whereas anisotropic perturbations generate orientation-dependent contraction governed by the covariance structure of the additive noise.

The first term is evaluated as

$$\begin{aligned}
 \mathbb{E}[|\psi\rangle\langle\eta|s] &= |\psi\rangle\mathbb{E}[\langle\eta|\langle\psi|\eta\rangle] \\
 &= |\psi\rangle\mathbb{E}[\langle\psi|(|\eta\rangle\langle\eta|)] \\
 &= |\psi\rangle\langle\psi|\mathbb{E}[|\eta\rangle\langle\eta|] \\
 &= |\psi\rangle\langle\psi|\Sigma = \rho\Sigma
 \end{aligned} \tag{A2}$$

Similarly, the fourth term becomes

$$\begin{aligned}
 \mathbb{E}[|\eta\rangle\langle\psi|s^*] &= \mathbb{E}[|\eta\rangle\langle\eta|] |\psi\rangle\langle\psi| \\
 &= \Sigma|\psi\rangle\langle\psi| \\
 &= \Sigma\rho
 \end{aligned} \tag{A3}$$

For the second term, we write

$$\mathbb{E}[|\psi\rangle\langle\eta|s^*] = |\psi\rangle\mathbb{E}[\langle\eta|\langle\eta|\psi\rangle] \tag{A4}$$

which depends on the conjugate pseudo-covariance $\mathbb{E}[|\eta^*\rangle\langle\eta|]$. Since the perturbation is assumed to be proper, $\mathbb{E}[|\eta\rangle\langle\eta|^T] = 0$, this pseudo-covariance vanishes, implying

$$\mathbb{E}[|\psi\rangle\langle\eta|s^*] = 0 \tag{A5}$$

By the same argument $\mathbb{E}[|\eta\rangle\langle\psi|s] = 0$. Combining the four terms yields

$$\mathbb{E}[Lx] = \rho\Sigma + \Sigma\rho \tag{A6}$$

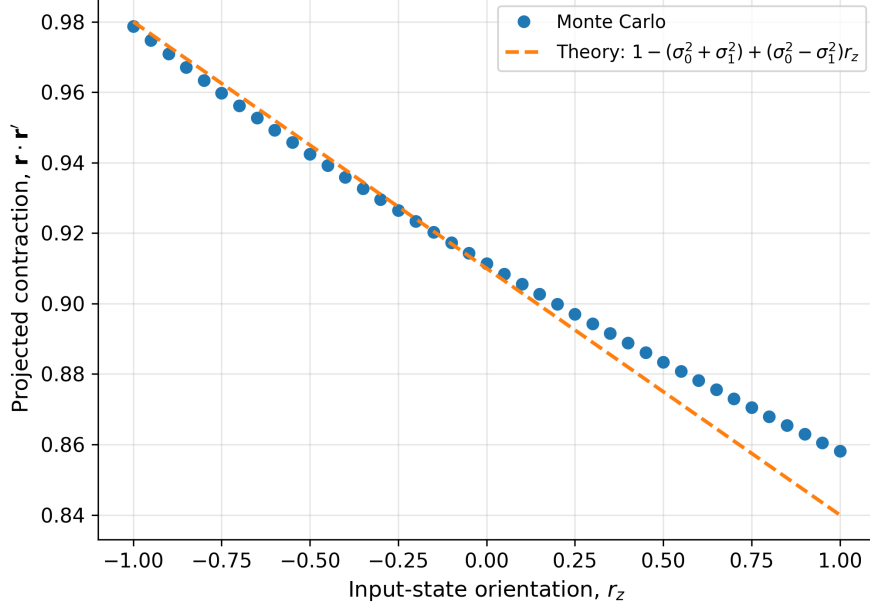


Fig. 6 Orientation-dependent Bloch-vector contraction under anisotropic additive amplitude noise. The Monte Carlo results closely follow the analytical contraction factor derived for diagonal covariance operators. This confirms that the contraction depends on both the input-state orientation and the principal directions of the perturbation covariance.

Declarations

- Funding: The author received no funding for this work.
- Conflict of interest/Competing interests: The author declare no conflict of interest.
- Ethics approval and consent to participate: Not applicable.
- Consent for publication: Not applicable
- Data availability: No datasets were generated or analyzed during the current study.
- Materials availability: not applicable
- Code availability: The code used in this study is available from the author upon reasonable request.
- Author contribution: Rajnish Kumar conceived the research idea, developed the theoretical framework, performed the mathematical analysis, conducted the simulations, interpreted the results, and wrote and revised the manuscript. The author has read and approved the final manuscript.

References

- [1] Mele, F.A., Lami, L., Giovannetti, V.: Quantum optical communication in the presence of strong attenuation noise. *Physical Review A* **106**(4), 042437 (2022)
- [2] Chakraborty, M., Mukherjee, A., Nag, A., Chandra, S.: Hybrid quantum noise

- model to compute gaussian quantum channel capacity. *IEEE Access* **12**, 14671–14689 (2024)
- [3] Cattaneo, M., Rossi, M.A., García-Pérez, G., Zambrini, R., Maniscalco, S.: Quantum simulation of dissipative collective effects on noisy quantum computers. *PRX quantum* **4**(1), 010324 (2023)
 - [4] Wallman, J.J., Emerson, J.: Noise tailoring for scalable quantum computation via randomized compiling. *Physical Review A* **94**(5), 052325 (2016)
 - [5] Preskill, J.: Quantum computing in the nisq era and beyond. *Quantum* **2**, 79 (2018)
 - [6] Bhalekar, O.: In: Rana, M., Pandey, B. (eds.) *Quantum Error Correction and Noise Mitigation*, pp. 81–95. Springer, Cham (2026). https://doi.org/10.1007/978-3-032-10775-6_5
 - [7] Azses, D., Dupont, M., Evert, B., Reagor, M.J., Dalla Torre, E.G.: Navigating the noise-depth tradeoff in adiabatic quantum circuits. *Physical Review B* **107**(12), 125127 (2023)
 - [8] AbuGhanem, M., Eleuch, H.: Full quantum tomography study of google’s sycamore gate on ibm’s quantum computers. *EPJ Quantum Technology* **11**(1), 36 (2024)
 - [9] Suzuki, Y., Endo, S., Fujii, K., Tokunaga, Y.: Quantum error mitigation as a universal error reduction technique: Applications from the nisq to the fault-tolerant quantum computing eras. *PRX quantum* **3**(1), 010345 (2022)
 - [10] Gottesman, D.: *An Introduction to Quantum Error Correction and Fault-Tolerant Quantum Computation* (2009). <https://arxiv.org/abs/0904.2557>
 - [11] Terhal, B.M.: Quantum error correction for quantum memories. *Rev. Mod. Phys.* **87**, 307–346 (2015) <https://doi.org/10.1103/RevModPhys.87.307>
 - [12] Bhat, H.A., Khanday, F.A., Kaushik, B.K., Bashir, F., Shah, K.A.: Quantum computing: Fundamentals, implementations and applications. *IEEE Open Journal of Nanotechnology* **3**, 61–77 (2022) <https://doi.org/10.1109/OJNANO.2022.3178545>
 - [13] Srivastav, A., Pandey, V., Mohan, B., Pati, A.K.: Family of exact and inexact quantum speed limits for completely positive and trace-preserving dynamics. *Physical Review A* **112**(5), 052204 (2025)
 - [14] Watrous, J.: *The Theory of Quantum Information*. Cambridge university press, Cambridge (2018)

- [15] Wei, Z.-D., Yan, W.-B., Man, Z.-X., Xia, Y.-J., Zhang, Y.-J., Cai, Q.-Y.: Mitigating noise effects in phase estimation via a quantum cyclic n-switch. *Physical Review A* **114**(1), 012604 (2026)
- [16] Caruso, F., Giovannetti, V., Lupo, C., Mancini, S.: Quantum channels and memory effects. *Rev. Mod. Phys.* **86**, 1203–1259 (2014) <https://doi.org/10.1103/RevModPhys.86.1203>
- [17] Nielsen, M.A., Chuang, I.L.: *Quantum Computation and Quantum Information*. Cambridge university press, Cambridge (2010)
- [18] Breuer, H.-P., Petruccione, F.: *The Theory of Open Quantum Systems*. Oxford University Press, Oxford (2007). <https://doi.org/10.1093/acprof:oso/9780199213900.001.0001>
- [19] Martinis, J.M.: Qubit metrology for building a fault-tolerant quantum computer. *npj Quantum Information* **1**(1), 1–3 (2015)
- [20] Arute, F., Arya, K., Babbush, R., Bacon, D., Bardin, J.C., Barends, R., Biswas, R., Boixo, S., Brandao, F.G., Buell, D.A., *et al.*: Quantum supremacy using a programmable superconducting processor. *nature* **574**(7779), 505–510 (2019)
- [21] Steven, M.K., *et al.*: *Fundamentals of statistical signal processing*. PTR Prentice-Hall, Englewood Cliffs, NJ **10**(151045), 148 (1993)
- [22] Zhang, X.-D.: *Modern Signal Processing: Spectral Estimation and Time-domain Analysis*. Walter de Gruyter GmbH & Co KG, Berlin/Beijing (2022)
- [23] Kumar, R., Arnon, S., Larsen, T.: Effect of additive noise on the performance of quantum computing: An engineering approach. *Electronics* **15**(13), 2975 (2026)