

Supplementary Information: Critical non-thermal fixed point and the dynamical condensation phase transition

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I. NUMERICAL RESOLUTION OF THE KINETIC EQUATION

In this section, we provide details about the numerical procedure used to solve the kinetic equation (1) of the main text. After a few particle-scattering events, the solution of the kinetic equation becomes isotropic, i.e., $n_{\mathbf{p}}(t) = n_p(t)$. Under this assumption, the angular integrations in Eq. (1) can be carried out analytically, yielding

$$\frac{\partial n_p}{\partial t} = \frac{4\pi}{\mathcal{N}^2} \int_{p_1^2 + p_2^2 - p^2 > 0} dp_1 dp_2 p_1 p_2 W_{\text{eff}}(p, p_1, p_2, p_3) [n_{p_1} n_{p_2} (1 + n_p + n_{p_3}) - n_p n_{p_3} (1 + n_{p_1} + n_{p_2})], \quad (\text{S1})$$

where $p_3 \equiv \sqrt{p_1^2 + p_2^2 - p^2}$ and $\mathcal{N} \equiv 4\pi^2 \rho \xi^3 \gg 1$. Here and throughout this Supplemental Material, we use dimensionless variables, with momenta measured in units of $1/\xi$ and times in units of $1/g\rho$. Introducing the variable $u \equiv \cos \theta$, with θ the angle between $\mathbf{p}$ and $\mathbf{p}_2$, the dimensionless interaction kernel reads

$$W_{\text{eff}}(p, p_1, p_2, p_3) = \frac{p_2}{2} \int_{-1}^1 \frac{du}{y_{p,p_2}(u)} g_{\text{eff}}^2(p, p_2, u) [\Theta(y_{p,p_2}(u) - |p_3 - p_1|) - \Theta(y_{p,p_2}(u) - (p_3 + p_1))], \quad (\text{S2})$$

where $y_{p,p_2}(u) \equiv |\mathbf{p} - \mathbf{p}_2| = \sqrt{p^2 + p_2^2 - 2pp_2u}$ and Θ denotes the Heaviside step function. The effective coupling, $g_{\text{eff}}(p, p_2, u) \equiv g_{\text{eff}}(p^2 - p_2^2, |\mathbf{p} - \mathbf{p}_2|)$, is given by Eq. (2), with

$$g\Pi^R(p^2 - p_2^2, |\mathbf{p} - \mathbf{p}_2|) = \frac{1}{2\mathcal{N}} \int \frac{kdk n_k}{y_{p,p_2}(u)} \times \left[\ln \left(\frac{y_{p,p_2}^2(u) + |p^2 - p_2^2| + 2ky_{p,p_2}(u) + i0^+}{y_{p,p_2}^2(u) + |p^2 - p_2^2| - 2ky_{p,p_2}(u) + i0^+} \right) - \ln \left(\frac{-y_{p,p_2}^2(u) + |p^2 - p_2^2| + 2ky_{p,p_2}(u) + i0^+}{-y_{p,p_2}^2(u) + |p^2 - p_2^2| - 2ky_{p,p_2}(u) + i0^+} \right) \right]. \quad (\text{S3})$$

For the numerical implementation, it is convenient to rewrite the interaction kernel as

$$W_{\text{eff}}(p, p_1, p_2, p_3) = \frac{p_2}{2} \int_{y_{p,p_2}(u) \in I} du \frac{g_{\text{eff}}^2(p, p_2, u)}{y_{p,p_2}(u)} \quad (\text{S4})$$

where $I \equiv [\max(|p_3 - p_1|, |p - p_2|), \min(p_3 + p_1, p + p_2)]$. To evaluate the various integrals, momenta are discretized on a logarithmic grid (see below for details), extending from $p_{\min}$ to $p_{\max}$. These bounds are chosen to ensure that the relative deviations of both the total energy and the total particle number remain below 1%. Throughout this work, we typically use $p_{\min} = 10^{-5}p_0$ to $10^{-3}p_0$ and $p_{\max} = 10^2p_0$. The different momentum integrals are evaluated using optimized quadrature schemes:

- Evaluation of $g\Pi^R$. The integral is split into the three intervals $k \in [p_{\min}, k^-] \cup [k^-, k^+] \cup [k^+, p_{\max}]$, where

$$k^{\pm} = \frac{1}{2} \left(y_{p,p_2}(u) \pm \frac{|p^2 - p_2^2|}{y_{p,p_2}(u)} \right), \quad (\text{S5})$$

corresponding to the logarithmic singularities in Eq. (S3). Each interval is integrated using fourth-order sinh-tanh Gaussian-type quadratures.

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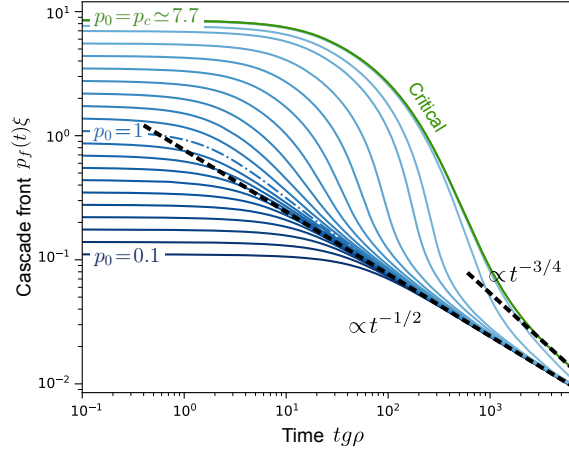


FIG. S1. Inverse-cascade front momentum $p_f(t)$ as a function of time for increasing values of the post-quench momentum p_0 , obtained by numerically solving Eq. (S1) and using definition in Eq. (S7). For subcritical quenches ($p_0 < p_c$), the front momentum follows $p_f(t) \propto t^{-1/2}$ at long times. For weak quenches ($1/\xi \leq p_0 < p_c$), the system first passes through a transient weak-turbulence regime in which $p_f(t) \propto |t - t^*|^{-\beta}$ with $\beta \simeq -1.1$, before eventually crossing over to the long-time coarsening fixed point where $\beta = 1/2$. For very deep quenches $p_0 \leq 1/\xi$ driving the Bose gas close to its ground state, the weak-turbulence regime disappears entirely. The dashed-dotted curve shows the front momentum evolution for $p_0 = 1/\xi$, for which the coarsening fixed point is reached most rapidly. At the critical quench ($p_0 = p_c$, green curve), the asymptotic scaling changes to $p_f(t) \propto t^{-3/4}$. In the figure, p_0 is expressed in units of $1/\xi = \sqrt{2mg\rho}$.

- Evaluation of W_{eff} . The angular integral over $u \in [-1, 1]$ is computed using a 150-point Gauss–Legendre quadrature. A mask is then applied to retain only the values satisfying $y_{p,p_2}(u) \in [\max(|p_3 - p_1|, |p - p_2|), \min(p_3 + p_1, p + p_2)]$ for all u .
- Evaluation of the collision integral. The (p_1, p_2) integration domain is divided into the four regions $[p_{\min}, p] \times [p_{\min}, p] \cup [p, p_{\max}] \times [p, p_{\max}] \cup [p_{\min}, p] \times [p, p_{\max}] \cup [p, p_{\max}] \times [p_{\min}, p]$, which are integrated using exponential Gaussian-type quadratures with 300 points (numerical density parameter set to -2) on $[p_{\min}, p]$ and 100 points (numerical density parameter set to 5) on $[p, p_{\max}]$. These choices are motivated by the structure of the integrand in the (p_1, p_2) plane.

The kinetic equation is solved for all $p \in [p_{\min}, p_{\max}]$ on a grid designed to maximize the accuracy of the interpolation of n_p while keeping the number of grid points as small as possible. We use a monotonic piecewise cubic Steffen interpolation for n_p , and the grid consists of the union of a logarithmically spaced grid of 500 points over $[p_{\min}, p_{\max}]$ and an additional set of $3 \times 200 = 600$ points distributed over the intervals $[p_{\min}, 0.01p_0] \cup [0.01p_0, 10p_0] \cup [10p_0, p_{\max}]$, using exponential Gaussian-type quadratures to increase the point density in both the low-momentum range (where the inverse cascade develops) and the large-momentum range (tail of the distribution).

The numerical solution of the kinetic equation is implemented in Julia 1.12, while data analysis is performed in Python 3.11. Time integration is carried out using the Tsit5 solver from Julia’s DifferentialEquations package, with an initial time step $\delta t = 10^{-6}$, a relative tolerance of 10^{-3} , and an absolute tolerance of 10^{-7} . For the perturbative equation ($g_{\text{eff}} = g$), the blow-up time t^* is estimated by fitting the long-time behavior of $n_0(t)$ to the scaling law (??) using the `optimize.curve_fit` routine from Python’s SciPy library.

II. INVERSE-CASCADE FRONT

While the dynamical exponent α is determined numerically from the time evolution of the zero-momentum occupation $n_0(t)$, the exponent β characterizes the evolution of the inverse-cascade front momentum $p_f(t)$. In particular, the long-time scaling form (4) of the main text predicts

$$p_f(t) \sim t^{-\beta}. \quad (\text{S6})$$

To extract β numerically, we adopt the following systematic definition of $p_f(t)$. We fix a positive constant $\lambda \lesssim 1$ and define $p_f(t)$ as the lowest momentum satisfying

$$|\ln n_0(t) - \ln n_{p_f}(t)| \geq \lambda. \quad (\text{S7})$$

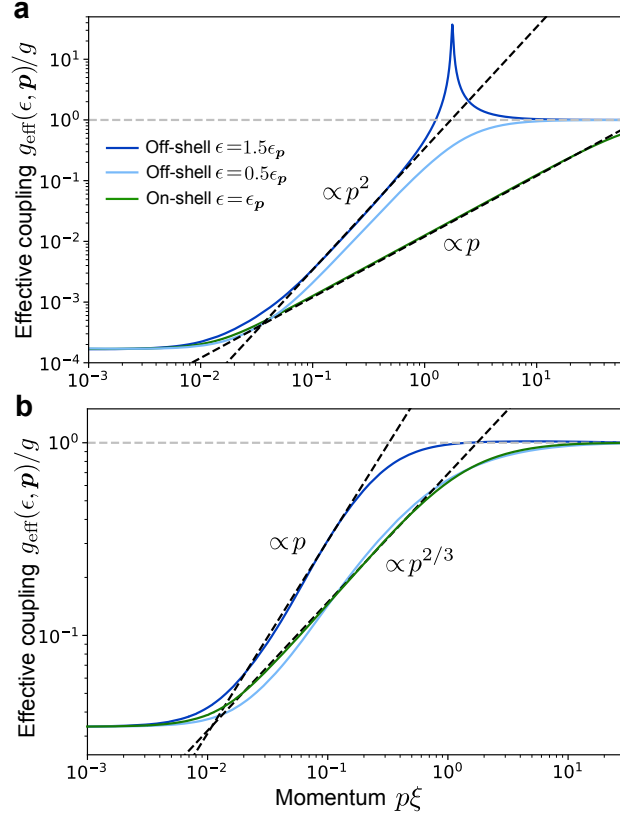


FIG. S2. Effective coupling strength $g_{\text{eff}}(\epsilon, \mathbf{p})/g$, evaluated with the exact momentum distribution $n_{\mathbf{p}}(t)$ computed numerically at long times by solving the kinetic equation (S1). Here time is fixed to $t = 7000/g\rho$. Panel **a** shows $g_{\text{eff}}(\epsilon, \mathbf{p})$ for $p_0/\xi = 0.63$, corresponding to a deep quench below the critical point, and panel **b** shows $g_{\text{eff}}(\epsilon, \mathbf{p})$ for $p_0 = p_c$, corresponding to a quench to the critical point. In each panel, the dark blue curve corresponds to $\epsilon = 1.5\epsilon_{\mathbf{p}}$, the light blue one to $\epsilon = 0.5\epsilon_{\mathbf{p}}$ (off-shell cases) and the green one to $\epsilon = \epsilon_{\mathbf{p}}$ (on-shell case).

This criterion allows us to determine the front momentum at each time step. In our work, we take $\lambda = 0.6$. The resulting evolution of $p_f(t)$ is shown in Fig. S1 for different values of the quench parameter p_0 . The dynamics of $p_f(t)$ displays qualitatively distinct behaviors depending on whether the quench is subcritical ($p_0 < p_c$) or critical ($p_0 = p_c$). In both cases, the front momentum decays algebraically at long times according to $p_f(t) \propto t^{-\beta}$, but with $\beta \simeq 1/2$ for subcritical quenches and $\beta \simeq 3/4$ at the condensation threshold. For weak quenches below the threshold, $1/\xi \leq p_0 < p_c$, the system first passes through a transient weak-turbulence regime in which $p_f(t) \propto |t - t^*|^{-\beta}$ with $\beta \simeq -1.1$, before eventually crossing over to the asymptotic coarsening fixed point characterized by $\beta = 1/2$.

III. EFFECTIVE COUPLING AT LONG TIMES

In addition to the momentum distribution, it is instructive to examine the structure of the effective coupling $g_{\text{eff}}(\epsilon, \mathbf{p})$ during the evolution, particularly at long times, where it is expected to differ substantially from the Boltzmann approximation, $g_{\text{eff}} = g$. Fig. S2 shows the ratio $g_{\text{eff}}(\epsilon, \mathbf{p})/g$ at the fixed long time $t = 7000/g\rho$ for two values of the post-quench momentum. Panel a corresponds to a deep quench below the critical point, $p_0 = 0.63/\xi$, while panel b corresponds to a quench precisely to the critical point, $p_0 = p_c$. In each panel, we plot $g_{\text{eff}}(\epsilon, \mathbf{p})$ for three representative energies: the on-shell value, $\epsilon = \epsilon_{\mathbf{p}}$, and two off-shell values, $\epsilon = 1.5\epsilon_{\mathbf{p}}$ and $\epsilon = 0.5\epsilon_{\mathbf{p}}$. We emphasize that both on-shell and off-shell processes contribute, in principle, to the collision integral in Eq. (1).

As is immediately apparent from Fig. S2, the effective coupling exhibits several characteristic momentum regimes in which it follows distinct algebraic scalings,

$$g_{\text{eff}}(\epsilon, \mathbf{p}) \propto \mathbf{p}^z, \quad z \in \{0; 2/3; 1; 2\}, \quad (\text{S8})$$

where the exponent $z = 2/3$ arises only for quenches to the critical point. For quenches across the transition, the three exponents $z = 0, 1$, and 2 , visible in Fig. S2a, were previously predicted in Ref. [1] using a semi-analytical approach

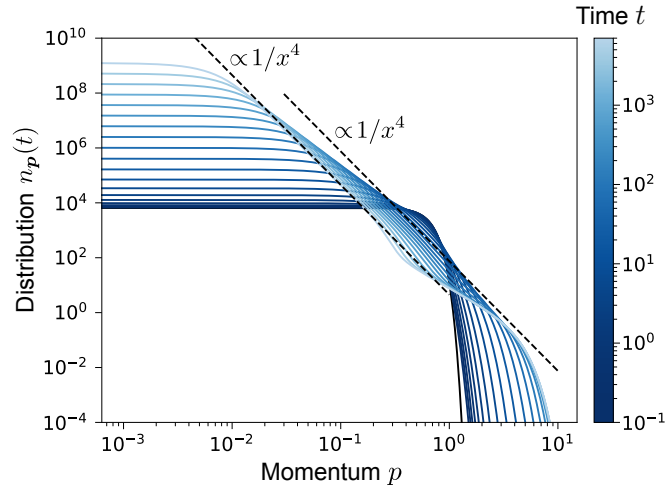


FIG. S3. Momentum distribution $n_{\mathbf{p}}(t)$ at different times for a post-quench momentum $p_0 = 0.63/\xi$, corresponding to a deep quench across the transition. At intermediate momenta, the distribution rapidly develops the characteristic algebraic scaling $n_{\mathbf{p}}(t) \sim p^{-4}$ associated with vortex coarsening, with no signature of an intermediate weak-turbulence regime, in agreement with Fig. 3 of the main text. At large momenta, the distribution also exhibits a pronounced direct cascade. Throughout the figure, momenta are expressed in units of $1/\xi$ and times in units of $1/g\rho$.

in which Eq. (2) was evaluated with a trial momentum distribution exhibiting algebraic decay.

We now examine which of these scaling regimes is relevant for describing the long-time nonthermal fixed points observed numerically for the two types of quenches. We first consider quenches across the transition. In this case, the long-time inverse cascade is governed by particle-number conservation, which imposes the relation $\alpha/\beta = d = 3$. Combined with the general exponent relation (5), this constraint uniquely selects $z = 2$ as the only exponent compatible with an inverse cascade, i.e., with $\alpha, \beta > 0$. This implies that, for quenches across the transition, the long-time dynamics is dominated by off-shell contributions to the effective coupling.

For a critical quench, by contrast, the self-similar scaling of the momentum distribution imposes the relation $\alpha/\beta = 2$ (see main text). Combined with the exponent relation (5), this uniquely selects $z = 2/3$ as the only exponent compatible with an inverse cascade. In marked contrast to quenches across the transition, the long-time dynamics at criticality is therefore dominated by on-shell contributions to the effective coupling.

IV. MOMENTUM DISTRIBUTION FOR STRONG QUENCHES

For deep quenches that drive the Bose gas close to its ground state, i.e., when the quench parameter satisfies $p_0 < 1/\xi$, the system again evolves at long times toward a coarsening fixed point. Fig. S3 shows the corresponding evolution of the momentum distribution. At long times, rescaling the distributions with the exponents $\alpha = 1.5$ and $\beta = 0.5$ yields the same data collapse as in Fig. (4) of the main text, demonstrating that the coarsening fixed point is qualitatively unchanged compared with the weak-quench regime.

The dynamics nevertheless differs from that of weak quenches in two important aspects. First, consistent with Fig. (3), no trace of a transient weak-turbulence regime is observed in the momentum distribution. Instead, the intermediate-momentum region exhibits the scaling $n_{\mathbf{p}}(t) \sim p^{-4}$ almost throughout the entire evolution. Second, a pronounced *direct* cascade develops at large momenta. This cascade was predicted theoretically and observed in *ab initio* simulations in Ref. [1], and was subsequently observed experimentally in Ref. [2]. It is also captured by the present quantum-kinetic framework and involves specific dynamical exponents. A dedicated study of direct cascades across the BEC transition will be investigated in detail in future work.

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