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Variance gamma approximation to sums of triplewise independent random variables

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Abstract

In this article, we first discuss how triplewise independent random variables (rvs) are connected to a complete bipartite graph. Using the connection, we construct a sequence of triplewise independent rvs. We next consider a variance gamma (VG) approximation of sums of such triplewise independent rvs. Using Stein's method and the generalized zero-bias transformation, we obtain our bounds. Related limit theorems are also discussed.

Keywords: Variance gamma distribution, Stein's method, Triplewise independence, Counter examples to the central limit theorem.

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1 Introduction

For a sequence of mutually independent and identically distributed (i.i.d.) random variables (rvs) $X_1, X_2, \ldots, X_n$ with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$, for $1 \leq i \leq n$, where $0 < \sigma < \infty$, it is known that the standardized partial sums

$$S_n := \frac{1}{\sigma\sqrt{n}} \left(\sum_{k=1}^{n} X_k - n\mu \right) \xrightarrow{d} \mathcal{N}(0,1), \text{ as } n \to \infty, \tag{1.1}$$

where $\xrightarrow{d}$ denotes the convergence in law. This result is known as the Lindeberg-Lévy central limit theorem (CLT); see [1] and [2]. It is also known that mutual independence in general cannot be relaxed to the weaker notion of pairwise independence; see [3]. Moreover, it cannot even be relaxed to triplewise independence; see [4]. This article mainly focuses on approximating the partial sums of triplewise independent rvs.

In general, the K-tuplewise independence is defined as follows (see, Definition 1 of [5]).

Definition 1 Let $K \in \{2, 3, 4, \ldots\}$. An indexed family of rvs X_i, $i \in I$, is K-tuplewise independent if the rvs $X_{i_1}, X_{i_2}, \ldots, X_{i_K}$ are mutually independent for any K-tuple of distinct indices $i_1, i_2, \ldots, i_K$.

Counterexamples to the CLT can be traced back to [6], who constructed a sequence of pairwise i.i.d. rvs taking the values 1 and -1 with equal probabilities. By Theorem 1 of [6], the absolute values of the partial sums of that sequence can be bounded by a fixed random variable (rv), so that their standardized counterparts S_n converge in distribution to zero. Pruss [7] constructs a counterexample to the CLT which is a sequence of K-tuplewise i.i.d. rvs, where K can be arbitrary, and the marginal distribution can be any symmetric distribution with finite variance. Bradley and Pruss [8] construct a sequence of K-tuplewise i.i.d rvs, which is strictly stationary. Recently, Avanzi *et al.* [3] provide a survey of further constructions and construct a broad family of counterexamples for pairwise independence. Later, Beaulieu *et al.* [4] modify the construction of Avanzi *et al.* [3] to one which is based on a suitable sequence of graphs, each graph giving a family of K-tuplewise i.i.d. rvs. The rvs obtained from all graphs can be arranged into an array, each graph giving one row. They provide an increasing sequence of graphs giving triplewise independent rows and standardized row sums converging in law to a variance-gamma (VG) distribution, which is not normal, see Subsection 4.1 of Beaulieu *et al.* [4]. From that array, a sequence can be extracted such that its standardized partial sums do not converge to a normal distribution because it has a subsequence that converges to the VG distribution.

In this article, we consider the VG approximation to the partial sums of triplewise independent rvs. The present article is, to the best of our knowledge, the first work to derive bounds with respect to a probability metric for triplewise independent rvs. Using Stein's method and the generalized zero-bias transformation, we obtain error bounds for the approximation problem considered.

The organization of the article is as follows. In Section 2, we discuss some preliminary results, which will be useful later. In Section 3, we discuss the VG distribution and its related results. In Section 4, we discuss our approximation results. We derive error bounds for the VG approximation to the partial sums of triplewise independent rvs.

2 Notations and Preliminary Results

In this section, we introduce some notations and discuss important preliminary results. Let us first recall the definition of a complete bipartite graph (see, for instance, [9] for more details).

Definition 2 A complete bipartite graph is a special type of bipartite graph where the set of vertices is partitioned into two disjoint sets and each vertex of the first set gives an edge to each vertex of the second set.

Next, we see how the triplewise independent rvs are connected to a complete bipartite graph (see, [4] for more details). Consider a complete bipartite graph G_m with m vertices in each set and m^2 edges. Let $\{v_1, v_2, \ldots, v_m\}$ be the vertices in first set and $\{w_1, w_2, \ldots, w_m\}$ be the vertices in second set. Let $n = m^2$. Let $M_1, M_2, ..., M_{2m}$ be the sequence of i.i.d discrete uniform rvs on the set $\{1, 2, \ldots, l\}$. More precisely, $\mathbb{P}(M_j = i) = \frac{1}{l}$, for all $i \in \{1, 2, \ldots, l\}$. We now assign the rvs $M_1, M_2, ..., M_m$ to each vertex of first set of vertices of G_m. We also assign the rvs $M_{m+1}, \ldots, M_{2m}$ to the second set of vertices of G_m. For every pair $1 \leq i \leq m$ and $m+1 \leq j \leq 2m$, let us define

$$D_{i,j} := \begin{cases} 1, & \text{if } M_i = M_j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $D_{i,j}$ gives the edges that connect M_i and M_j. Let n be the total number of edges. We relabel the rvs $\{D_{i,j}\}$, $1 \leq i \leq j \leq n$, as $D_1, D_2, \ldots, D_n$. By Remark 2.1 of [4] and Definition 1, we observe that the sequence $\{D_1, D_2, \ldots, D_n\}$ is triplewise independent. Under the above setup, let us define

$$W := \frac{\sum_{k=1}^{n} D_k - nl^{-1}}{\sqrt{nl^{-1}(1 - l^{-1})}}. \tag{2.1}$$

The following theorem provides the asymptotic distribution of W (see, Section 4.1 of [4] for more detail).

Theorem 1 *Let $\{G_m\}_{m \geq 1}$ be the sequence of bipartite graphs, where i.i.d. discrete uniform rvs $M_1, \ldots, M_{2m}$ are assigned to the vertices of $\{G_m\}$. That is, $M_1, \ldots, M_m$ are assigned to the m vertices of first set, and $M_{m+1}, \ldots, M_{2m}$ to the m vertices of second set. Then, $W \xrightarrow{d} \frac{Z}{\sqrt{l-1}}$, as $m \to \infty$ (or, equivalently, as $n \to \infty$), where $Z \sim VG(l-1, 0, 1, 0)$ and VG denotes the variance-gamma distribution (see Definition 3).*

Next, we define Wasserstein and bounded Wasserstein distances (see [10]). Consider the function space

$$\mathcal{H}_W = \{h : \mathbb{R} \to \mathbb{R} \mid h \text{ is 1-Lipschitz and } |h(x) - h(y)| \leq |x - y|\}, \tag{2.2}$$

where $\|h\| = \sup_{x \in \mathbb{R}} |h(x)|$. Then, for any two random variables Y and Z,

$$\sup_{h \in \mathcal{H}_W} |\mathbb{E}[h(Y)] - \mathbb{E}[h(Z)]| := d_W(Y, Z) \text{ (say)} \tag{2.3}$$

is called the Wasserstein distance. Also, when we choose the function space

$$\mathcal{H}_{bW} = \{h : \mathbb{R} \to \mathbb{R} \mid h \text{ is 1-Lipschitz, } \|h\| \leq 1 \text{ and } \|h'\| \leq 1\},$$

so that $d_{\mathcal{H}_{bW}} := d_{bW}$ (say) is called the bounded Wasserstein distance.

Next, we discuss components of Stein's method. In general, the method is based on the fact that, any real-valued rv Z has a distribution F_Z if and only if there exists an operator A (also called the Stein operator) such that $\mathbb{E}(Af(Z)) = 0$, where $f \in \mathcal{F}$ (a suitable function space). This characterization leads us to the Stein equation

$$Af(x) = h(x) - \mathbb{E}h(Z), \tag{2.4}$$

where h is a real-valued test function. Replacing x with a rv Y and taking expectations on both sides of (2.4) gives

$$\mathbb{E}h(Y) - \mathbb{E}h(Z) = \mathbb{E}(Af(Y)). \tag{2.5}$$

The equality (2.5) plays a crucial role in Stein's method. For a real valued test function h, the problem of bounding the quantity $|\mathbb{E}h(Y) - \mathbb{E}h(Z)|$ relies on the bounds for the solution of (2.4) and behavior of Y. For more details on Stein's method, we refer to the reader [11–14] and the references therein.

3 Variance gamma distribution and related results

In this section, we discuss some important results related to the VG distribution. Let us first recall the definition of a VG distribution (see [15, 16]).

Definition 3 The VG distribution with parameters $r > 0$, $\theta \in \mathbb{R}$, $\sigma > 0$, $\mu \in \mathbb{R}$ has probability density function

$$p(x) = \frac{1}{\sigma\sqrt{\pi}\,\Gamma\left(\frac{r}{2}\right)} e^{\frac{\theta}{\sigma^2}(x-\mu)} \left(\frac{|x-\mu|}{2\sqrt{\theta^2+\sigma^2}}\right)^{\frac{r-1}{2}} K_{\frac{r-1}{2}}\left(\frac{\sqrt{\theta^2+\sigma^2}}{\sigma^2}|x-\mu|\right), \tag{3.1}$$

with support $\mathbb{R}$. Here $K_\nu(x)$ is a modified Bessel function of the second kind, defined by $K_\nu(x) = \int_0^\infty e^{-x\cosh(t)}\cosh(\nu t)\,dt$. For a rv Z with density (3.1), we write $Z \sim \mathrm{VG}(r, \theta, \sigma, \mu)$.

Many probability distributions such as Laplace, product-normal, gamma, normal, among many others, belong to the VG family. For more details, we refer the reader to [15–17], and the references therein.

Next, we discuss Stein's method for the VG distribution. The following proposition gives a Stein equation for the VG distribution (see [17]).

Proposition 2 *Let $Z \sim \mathrm{VG}(r, \theta, \sigma, \mu)$ with density given in (3.1). Then a Stein equation for Z is given by*

$$\sigma^2(x-\mu)f''(x) + \left(\sigma^2 r + 2\theta(x-\mu)\right) f'(x) + (r\theta - (x-\mu))\, f(x) = h(x) - \mathbb{E}(h(Z)), \tag{3.2}$$

where $h \in \mathcal{H}$ is a real-valued test function.

The following corollary gives the Stein equation for VG$(1,0,1,0)$ distribution. Note that the VG$(1,0,1,0)$ distribution corresponds to the distribution of the product of two independent standard normal $\mathcal{N}(0,1)$ rvs, whereas the VG$(n,0,1,0)$ distribution arises as a sum of n independent copies of such rvs (see Theorem 1 of [18]).

Corollary 3 Let $Z \sim$ VG$(1,0,1,0)$ with density given in (3.1). Then a Stein equation for Z is given by

$$xf''(x) + f'(x) - xf(x) = h(x) - \mathbb{E}(h(Z)), \tag{3.3}$$

where $h \in \mathcal{H}$ is a real-valued test function.

The next result gives the solution to the Stein equation (3.3), which essentially follows from Lemma 2.3 of [19]. Let $C_b^k(\mathbb{R})$ be the space of k-times continuously differentiable functions on $\mathbb{R}$ that, along with all their derivatives up to order k, are bounded.

Proposition 4 *Let $h \in C_b^1(\mathbb{R})$. Then the unique bounded solution $f_h : \mathbb{R} \to \mathbb{R}$ to the Stein equation (3.3) is given by*

$$f_h(x) = -K_0(|x|)\int_0^x I_0(y)\tilde{h}(y)dy - I_0(x)\int_x^\infty K_0(|y|)\tilde{h}(y)dy, \tag{3.4}$$

where $\tilde{h}(y) = h(y) - \mathbb{E}h(Z)$, $I_0(x)$ and $K_0(x)$ are modified Bessel functions, defined, for all $x \in \mathbb{R}, \nu \in \mathbb{R}$, by

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu+k+1)k!}\left(\frac{x}{2}\right)^{\nu+2k} \text{ and } K_\nu(x) = \frac{\pi}{2sin(\nu\pi)}(I_{-\nu}(x) - I_\nu(x)).$$

Next, we discuss the properties of the solution to the Stein equation (3.4). The derivation of the following properties follows from Theorem 2.1 and Lemma 2.4 of [19].

Lemma 5 Suppose that $h \in C_b^3(\mathbb{R})$. Let f_h be defined in (3.4). Then,

$$\|f\| = 3\|\tilde{h}\|,\ \|f'\| = \frac{3}{2}\|\tilde{h}\|,\ \|f''\| = 2\|h'\| + 5\|\tilde{h}\|,$$

$$\|f^{(3)}\| = 4\|h''\| + 5\|h'\| + 4.89\|\tilde{h}\|,\ \|f^{(4)}\| = 8\|h^{(3)}\| + 9\|h''\| + 6.81\|h'\| + 15.75\|\tilde{h}\|,$$

and we also have

$$\|xf(x)\| \leq 2\|\tilde{h}\|,\ \|xf'(x)\| \leq \frac{3}{2}\|\tilde{h}\|,\ \|xf''(x)\| \leq \frac{9}{2}\|\tilde{h}\|,$$

where $\|f\| = \sup_{x\in\mathbb{R}}|f(x)|$ and $\tilde{h}(x) = h(x) - \mathbb{E}(h(Z))$, and $Z \sim VG(1,0,1,0)$.

Lemma 6 Let f be the solution of Z of the Stein equation (3.3). Also, let $A_2 f(x) := xf''(x) + f'(x)$, $x \in \mathbb{R}$. Then

$$\|(A_2 f)'\| \leq \|h'\| + \frac{9}{2}\left\|\tilde{h}(x)\right\| \text{ and } \|(A_2 f)''\| \leq \|h''\| + \frac{15}{2}\left\|\tilde{h}(x)\right\|. \tag{3.5}$$

Remark 1 Note that the quantity $\|(A_2 f)'\|$ in (3.5) can alternatively be bounded using the inequality (3.16) of [20] as

$$\|(A_2 f)'\| \leq \frac{99}{8}\|h'\|. \tag{3.6}$$

The advantages of this bound are discussed in Theorems 11 and 12. Indeed, (3.6) leads us to obtain the error bounds for the Wasserstein distance for the VG approximation to sums of triplewise independent rvs.

4 Main results

In this section, we discuss our main results and their relevance to the literature. Before stating our results, we need the following setup.

4.1 The setup

Let $\{G_m\}_{m\geq 1}$ be the sequence of bipartite graphs. Let $M_1, \ldots, M_{2m}$ be a sequence of i.i.d. discrete uniform rvs that are assigned to the vertices of $\{G_m\}$. That is, $M_1, \ldots, M_m$ are assigned to the m vertices of first set, and $M_{m+1}, \ldots, M_{2m}$ to the m vertices of second set. Observe next that the number of vertices $= 2m$ and the number of edges $n = m^2$. Define

$$N_i^{(1)} := N_i{}^{(1)}(m) = \text{the number of } M_j\text{'s equal to } i \text{ within the sample } \{M_j\}_{j=1}^{m}, \text{and}$$

$$N_i^{(2)} := N_i{}^{(2)}(m) = \text{the number of } M_j\text{'s equal to } i \text{ within the sample } \{M_j\}_{j=m+1}^{2m}.$$

Then $N^{(p)} = (N_1{}^{(p)}, \ldots, N_l{}^{(p)}) \sim \text{Multinomial}\Big(m, (\frac{1}{l}, \ldots, \frac{1}{l})\Big)$ for $p = 1, 2$. Moreover $N^{(1)}$ and $N^{(2)}$ are independent. Let

$$I_j := 1_{\{M_j = i\}} = \begin{cases} 1, & \text{if } M_j = i, \\ 0, & \text{otherwise.} \end{cases}$$

By definition, we have $N_i{}^{(1)} = \sum_{j=1}^{m} I_j$ and $N_i{}^{(2)} = \sum_{j=m+1}^{2m} I_j$, since $I_j \sim$ Bernoulli($\frac{1}{l}$).
Hence, we can write (2.1) as

$$W = \frac{\sum_{i=1}^{l} N_i{}^{(1)} N_i{}^{(2)} - nl^{-1}}{\sqrt{nl^{-1}(1 - l^{-1})}}. \tag{4.1}$$

Note that $\mathbb{E}[N_i{}^{(1)}] = \mathbb{E}[N_i{}^{(2)}] = \frac{m}{l}$, $\text{Var}(N_i{}^{(1)}) = \text{Var}(N_i{}^{(2)}) = \frac{m}{l}(1 - \frac{1}{l})$ and $\mathbb{E}[N_i{}^{(1)} N_i{}^{(2)}] = \frac{m^2}{l^2}$.

Define,

$$X_i := \frac{N_i{}^{(1)} - \frac{m}{l}}{\sqrt{\frac{m}{l}(1 - \frac{1}{l})}} \qquad \text{and} \qquad Y_i := \frac{N_i{}^{(2)} - \frac{m}{l}}{\sqrt{\frac{m}{l}(1 - \frac{1}{l})}}, \tag{4.2}$$

such that $\mathbb{E}[X_i] = \mathbb{E}[Y_i] = 0$ and $\mathrm{Var}(X_i) = \mathrm{Var}(Y_i) = 1$.

Let

$$X_i' := N_i^{(1)} - \frac{m}{l} \qquad \text{and} \qquad Y_i' := N_i^{(2)} - \frac{m}{l}. \tag{4.3}$$

Hence from (4.2) and (4.3) we get

$$N_i^{(1)} N_i^{(2)} = \left(X_i' + \frac{m}{l}\right)\left(Y_i' + \frac{m}{l}\right) = X_i'Y_i' + \frac{m}{l}(X_i' + Y_i') + \frac{m^2}{l^2}. \tag{4.4}$$

So, from (4.1) and (4.4) we can write

$$\sum_{i=1}^{l} N_i^{(1)} N_i^{(2)} = \sum_{i=1}^{l} X_i'Y_i' + \sum_{i=1}^{l} \frac{m}{l}(X_i' + Y_i') + \frac{m^2}{l}. \tag{4.5}$$

Note that

$$\sum_{i=1}^{l} X_i' = \sum_{i=1}^{l}(N_i^{(1)} - \frac{m}{l}) = \sum_{i=1}^{l} N_i^{(1)} - \sum_{i=1}^{l} \frac{m}{l} = m - m = 0 \text{ and } \sum_{i=1}^{l} Y_i' = 0. \tag{4.6}$$

From (4.2) and (4.3), we have

$$\sum_{i=1}^{l} X_i'Y_i' = \sum_{i=1}^{l} \frac{m}{l}(1 - \frac{1}{l}) X_iY_i = \frac{m}{l}(1 - \frac{1}{l}) \sum_{i=1}^{l} X_iY_i. \tag{4.7}$$

Using (4.7) and (4.6) in (4.5), we have

$$\sum_{i=1}^{l} N_i^{(1)} N_i^{(2)} = \frac{m}{l}(1 - \frac{1}{l}) \sum_{i=1}^{l} X_iY_i + \frac{m^2}{l}. \tag{4.8}$$

Using (4.8) in (4.1), noting that $n = m^2$, we get

$$W = \sqrt{\frac{1}{l}\left(1 - \frac{1}{l}\right)} \sum_{i=1}^{l} X_iY_i. \tag{4.9}$$

We now have the following lemmas, which will be used later.

Lemma 7 Let X_i and Y_i be two rvs as defined in (4.2). Then

$$\mathbb{E}[X_i^3] = \mathbb{E}[Y_i^3] = \frac{1 - \frac{2}{l}}{\sqrt{\frac{m}{l}(1 - \frac{1}{l})}}, \qquad l \geq 2. \tag{4.10}$$

Moreover,

$$\mathbb{E}[|X_i^3|] = \mathbb{E}[|Y_i^3|] \leq \sqrt{\frac{l}{m}}, \qquad l \geq 2. \tag{4.11}$$

Proof Note that,

$$\mathbb{E}[|X_i|^3] = \mathbb{E}\Big[\Big(\frac{N_i^{(1)} - \frac{m}{l}}{\sqrt{\frac{m}{l}(1-\frac{1}{l})}}\Big)^3\Big] = \frac{1}{(\frac{m}{l}(1-\frac{1}{l}))^{3/2}}\mathbb{E}\left[\Big(N_i^{(1)} - \frac{m}{l}\Big)^3\right], \tag{4.12}$$

and

$$N_i^{(1)} = I_1 + I_2 + \ldots + I_m. \tag{4.13}$$

Therefore,

$$N_i^{(1)} - \frac{m}{l} = \sum_{j=1}^{m}(I_j - \frac{1}{l}).$$

Hence,

$$\mathbb{E}\Big[\Big(N_i^{(1)} - \frac{m}{l}\Big)^3\Big] = \mathbb{E}[(N_i^{(1)})^3] - 3\frac{m}{l}\,\mathbb{E}[(N_i^{(1)})^2] + 3\frac{m^2}{l^2}\,\mathbb{E}[N_i^{(1)}] - \frac{m^3}{l^3}.$$

Using (4.13), we have

$$(N_i^{(1)})^2 = \Big(\sum_{j=1}^{m} I_j\Big)^2.$$

Hence,

$$(N_i^{(1)})^2 = \sum_{j=1}^{m} I_j^2 + \sum_{j\neq k} I_j I_k.$$

Since $I_j^2 = I_j$,

$$(N_i^{(1)})^2 = \sum_{j=1}^{m} I_j + \sum_{j\neq k} I_j I_k. \tag{4.14}$$

Taking the expectation on both sides of (4.14), we get

$$\mathbb{E}[(N_i^{(1)})^2] = \sum_{j=1}^{m}\mathbb{E}[I_j] + \sum_{j\neq k}\mathbb{E}[I_j I_k].$$

So,

$$\mathbb{E}[(N_i^{(1)})^2] = \frac{m}{l} + m(m-1)\frac{1}{l^2}.$$

Using (4.13), we write

$$(N_i^{(1)})^3 = \sum_{j=1}^{m} I_j^3 = \sum_{j=1}^{m} I_j^3 + 3\sum_{j\neq k} I_j^2 I_k + \sum_{\substack{j,k,l \\ \text{distinct}}} I_j I_k I_l.$$

Since $I_j^2 = I_j$ and $I_j^3 = I_j$,

$$(N_i^{(1)})^3 = \sum_{j=1}^{m} I_j + 3\sum_{j\neq k} I_j I_k + \sum_{\substack{j,k,l \\ \text{distinct}}} I_j I_k I_l. \tag{4.15}$$

Taking the expectation on both sides of (4.15), we get

$$\mathbb{E}[(N_i^{(1)})^3] = \sum_{j=1}^{m}\mathbb{E}[I_j] + 3\sum_{j\neq k}\mathbb{E}[I_j I_k] + \sum_{\substack{j,k,l \\ \text{distinct}}}\mathbb{E}[I_j I_k I_l].$$

Hence, we get

$$\mathbb{E}[(N_i^{(1)})^3] = \frac{m}{l} + 3m(m-1)\frac{1}{l^2} + m(m-1)(m-2)\frac{1}{l^3}.$$

Therefore,

$$\mathbb{E}\Big[\Big(N_i^{(1)} - \frac{m}{l}\Big)^3\Big] = \frac{m}{l}(1-\frac{1}{l})(1-\frac{2}{l}).$$

Moreover,

$$\mathbb{E}[|X_i|^3] = \frac{1}{\Big(\frac{m}{l}(1-\frac{1}{l})\Big)^{3/2}}\frac{m}{l}(1-\frac{1}{l})(1-\frac{2}{l}) = \frac{1-\frac{2}{l}}{\sqrt{\frac{m}{l}(1-\frac{1}{l})}}, \qquad l \geq 2. \tag{4.16}$$

Similarly, we obtain

$$\mathbb{E}[|Y_i|^3] = \frac{1-\frac{2}{l}}{\sqrt{\frac{m}{l}(1-\frac{1}{l})}}, \qquad l \geq 2. \tag{4.17}$$

To find the upper bound of the third moment of X_i and Y_i, we write

$$\frac{1-\frac{2}{l}}{\sqrt{\frac{m}{l}(1-\frac{1}{l})}} = \frac{l-2}{\sqrt{m(l-1)}}. \tag{4.18}$$

Note that,

$$(l-2)^2 = l^2 - 4l + 4 \leq l^2 - l = l(l-1),\ l \geq 2.$$

Hence,

$$\mathbb{E}[|X_i|^3] = \mathbb{E}[|Y_i|^3] \leq \sqrt{\frac{l}{m}}, \qquad l \geq 2. \tag{4.19}$$

This proves the result. □

We now define the zero-biased distribution of order n (see, [19, Definition 1.1] for more details).

Definition 4 Let W be a mean-zero rv with finite, non-zero variance σ^2. We say that $W^{*(n)}$ has the W-zero biased distribution of order n if for all n times differentiable functions f for which $\mathbb{E}[Wf(W)]$ exists,

$$\mathbb{E}[Wf(W)] = \sigma^2\mathbb{E}\Big[A_n f\left(W^{*(n)}\right)\Big],$$

where $A_n f(x) = x^{-1}T^n f(x)$ and $Tf(x) = xf'(x)$. Here T^n denotes n-times compositions of T, that is, $T^n = T \circ T \circ \cdots \circ T$ (n times).

Next, we present an important property of a zero-biased distribution similar to [19, Lemma 1.1].

Lemma 8 Let $Z_1, \ldots, Z_l$ be independent mean zero rvs with $\mathbb{E}Z_i^2 = 1$. Set $W := \sum_{i=1}^{l} Z_i$ and $\mathbb{E}W^2 = l$. Let I be a random index independent of the rv Z_i such that $\mathbb{P}(I = i) = \frac{1}{l}$. Let $W_i := W - Z_i = \sum_{j\neq i} Z_j$. Then $W_I + Z_I^*$ has the W-zero biased distribution, where $Z_I^* = Z_I^{*(1)}$ has the Z_I-zero biased distribution of order 1.

Proof Since $\mathbb{E}Z_i = 0$ and $\mathbb{E}Z_i^2 = 1$ for all $i = 1, \ldots, l$, we have $\mathbb{E}W^2 = \sum_{i=1}^{l} \mathbb{E}Z_i^2 = l$. Hence, for all smooth functions f, we write $\mathbb{E}[Wf(W)] = l\mathbb{E}[f'(W^*)]$.
Now,

$$\mathbb{E}[Wf(W)] = \mathbb{E}\left[\left(\sum_{i=1}^{l} Z_i\right) f(W)\right] = \sum_{i=1}^{l} \mathbb{E}[Z_i f(W)].$$

Since $W = W_i + Z_i$ and W_i is independent of Z_i, applying the zero-bias identity we get

$$\mathbb{E}[Z_i f(W)] = \mathbb{E}[Z_i f(W_i + Z_i)] = \mathbb{E}Z_i^2 \mathbb{E}f'(W_i + Z_i^*).$$

Note that

$$\mathbb{E}[Z_i f(W)] = \mathbb{E}f'(W_i + Z_i^*),$$

since $\mathbb{E}Z_i^2 = 1$. Therefore,

$$l\mathbb{E}f'(W^*) = \sum_{i=1}^{l} \mathbb{E}f'(W_i + Z_i^*).$$

Now using the definition of random index and $P(I = i) = \frac{1}{l}$, we obtain

$$\sum_{i=1}^{l} \mathbb{E}f'(W_i + Z_i^*) = l \sum_{i=1}^{l} \frac{1}{l} \mathbb{E}f'(W_i + Z_i^*) = l\mathbb{E}f'(W_I + Z_I^*).$$

So,

$$\mathbb{E}f'(W^*) = \mathbb{E}f'(W_I + Z_I^*).$$

This completes the proof. □

Lemma 9 Let X_i and Y_i be defined in (4.2) where X_i follows the same distribution as X and Y_i follows the same distribution as Y for $i = 1, 2, \ldots, l$. Let $W_l = \frac{1}{\sqrt{l}} \sum_{i=1}^{l} Z_i$, where $Z_i = X_i Y_i$. Then

$$\mathbb{E}\left|W_l - {W_l}^{*(2)}\right| \leq \frac{5}{4} \frac{\sqrt{l}}{m}, \qquad l \geq 2. \tag{4.20}$$

Proof Given $W_l = \frac{1}{\sqrt{l}} \sum_{i=1}^{l} Z_i$. By Lemma 8, we have

$${W_l}^{*(2)} = W_l - \frac{Z_I}{\sqrt{l}} + \frac{{Z_I}^{*(2)}}{\sqrt{l}}.$$

Now,

$${W_l}^{*(2)} = \frac{1}{\sqrt{l}} \sum_{i=1}^{l} Z_i - \frac{Z_I}{\sqrt{l}} + \frac{{Z_I}^{*(2)}}{\sqrt{l}}.$$

Therefore,

$$W_l - {W_l}^{*(2)} = \frac{Z_I}{\sqrt{l}} - \frac{{Z_I}^{*(2)}}{\sqrt{l}} = \frac{1}{\sqrt{l}}(Z_I - {Z_I}^{*(2)}).$$

Now,

$$\mathbb{E}\left|W_l - {W_l}^{*(2)}\right| = \frac{1}{\sqrt{l}} \mathbb{E}\left|(Z_I - {Z_I}^{*(2)})\right| \leq \frac{1}{\sqrt{l}} \left[\mathbb{E}|Z_I| + \mathbb{E}|{Z_I}^{*(2)}|\right].$$

Again we know,

$$\mathbb{E}|Z_I| = \mathbb{E}|X_I Y_I| \leq \sqrt{\mathbb{E}{X_I}^2} \sqrt{\mathbb{E}{Y_I}^2} = 1,$$

and

$$\mathbb{E}|Z_I^{*(2)}| = \mathbb{E}|X_I^* Y_I^*| = \mathbb{E}|X_I^*|\mathbb{E}|Y_I^*| = \frac{1}{4}\mathbb{E}|X|^3\mathbb{E}|Y|^3.$$

Therefore,

$$\mathbb{E}\left|W_l - {W_l}^{*(2)}\right| \leq \frac{1}{\sqrt{l}}\left[1 + \frac{1}{4}\mathbb{E}|X|^3\mathbb{E}|Y|^3\right], \qquad l \geq 2. \tag{4.21}$$

By Holder's inequality for any rv with finite third moment, we have $\left(\mathbb{E}|X|^3\right)^{1/3} \geq \left(\mathbb{E}|X|^2\right)^{1/2}$. Since $\mathbb{E}X^2 = 1$, we have $\left(\mathbb{E}|X|^3\right)^{1/3} \geq 1$, and therefore,

$$\mathbb{E}|X|^3 \geq 1. \tag{4.22}$$

Similarly,

$$\mathbb{E}|Y|^3 \geq 1. \tag{4.23}$$

Using (4.22) and (4.23) in (4.21), we have

$$\mathbb{E}\left|W_l - {W_l}^{*(2)}\right| \leq \frac{1}{\sqrt{l}}\mathbb{E}|X|^3\mathbb{E}|Y|^3\left[1 + \frac{1}{4}\right] = \frac{5}{4}\frac{1}{\sqrt{l}}\mathbb{E}|X|^3\mathbb{E}|Y|^3, \qquad l \geq 2. \tag{4.24}$$

Using (4.11) in (4.24), we get

$$\mathbb{E}\left|W_l - {W_l}^{*(2)}\right| \leq \frac{5}{4}\frac{\sqrt{l}}{m}, \qquad l \geq 2.$$

This proves the result. □

The following result is a special case of [19, Theorem 4.1], for the case $n = 2$.

Theorem 10 *Let W_l be defined as in Lemma 9 and $Z \sim \mathrm{VG}(1, 0, 1, 0)$. Let f be the solution of the $\mathrm{VG}(1, 0, 1, 0)$ Stein equation (3.3). Assume that $(W_l, W_l^{*(2)})$ is given on a joint probability space so that $W_l^{*(2)}$ has the W_l-zero biased distribution of order 2. Then for $h \in C_b^1(\mathbb{R})$, we have*

$$\left|\mathbb{E}[h(W_l)] - \mathbb{E}[h(Z)]\right| \leq \left\|(A_2 f)'\right\| \mathbb{E}\left|W_l - W_l^{*(2)}\right|. \tag{4.25}$$

Next, we obtain an error bound for VG approximation to partial sums of triplewise independent rvs. Recently, Beaulieu *et al.* [4, Subsection 4.1] established a limiting result, which shows that the partial sums of triplewise independent rvs converge to a VG distribution.

Theorem 11 *Let W_l be defined as in Lemma 9 and $Z \sim \mathrm{VG}(1, 0, 1, 0)$. Assume that $(W_l, W_l^{*(2)})$ is given on a joint probability space so that $W_l^{*(2)}$ has the W_l-zero biased distribution of order 2. Then*

$$d_{bW}(W_l, Z) \leq \frac{5}{4}\frac{\sqrt{l}}{m}\left[\left\|h'\right\| + \frac{9}{2}\left\|\tilde{h}(x)\right\|\right], \qquad l \geq 2. \tag{4.26}$$

Also, we have

$$d_W(W_l, Z) \leq \frac{495}{32}\frac{\sqrt{l}}{m}, \qquad l \geq 2. \tag{4.27}$$

Proof (i) Using (3.5) and Lemma 9 in (4.25), we get (4.26).

(ii) Using (3.6) and Lemma 9 in (4.25), we get (4.27). □

Next, we obtain an error bound in approximating a statistic that has an asymptotic VG$(l-1,0,1,0)$ distribution.

Theorem 12 *Let W_l be defined as in Lemma 9 and $Z \sim \mathrm{VG}(l-1,0,1,0)$. Then*

$$d_{bW}(W_l, Z) \leq \frac{5}{4}\frac{\sqrt{l}(l-1)}{m}\Big[\,\|h'\| + \frac{9}{2}\left\|\tilde{h}(x)\right\|\Big], \qquad l \geq 2. \tag{4.28}$$

Also, we have

$$d_{W}(W_l, Z) \leq \frac{495}{32}\frac{\sqrt{l}(l-1)}{m}, \qquad l \geq 2. \tag{4.29}$$

Proof (i) Let $W_{(i)} := X_i Y_i$, so that $W_l = \frac{1}{\sqrt{l}}\sum_{i=1}^{l} W_{(i)}$. Using the VG$(l-1,0,1,0)$ Stein equation (3.2), we have

$$\begin{aligned}
|\mathbb{E}h(W_l) - \mathbb{E}h(Z)| &= \mathbb{E}\Big[W_l f''(W_l) + (l-1)f'(W_l) - W_l f(W_l)\Big] \\
&= \sum_{i=1}^{l-1}\mathbb{E}\Big[W_{(i)} f''(W_l) + f'(W_l) - W_{(i)} f(W_l)\Big] \\
&= \sum_{i=1}^{l-1}\mathbb{E}\Big[\mathbb{E}[W_{(i)} f''(W_l) + f'(W_l) - W_{(i)} f(W_l)|(W_{(1)},\ldots, \\
&\qquad W_{(i-1)}, W_{(i+1)},\ldots, W_{(l)}]\Big].
\end{aligned} \tag{4.30}$$

Fix i, and define $V_i := \sum_{l\neq i} W_{(l)}$. Then $W_l = W_{(i)} + V_i$. Next let

$$\mathcal{F}_i := \sigma(W_{(1)},\ldots, W_{(i-1)}, W_{(i+1)},\ldots, W_{(l)}) \qquad (\sigma \text{ - algebra}).$$

Hence,

$$\begin{aligned}
&\mathbb{E}\Big[W_{(i)} f''(W_l) + f'(W_l) - W_{(i)} f(W_l)\,\Big|\,\mathcal{F}_k\Big] \\
&= \mathbb{E}\Big[W_{(i)} f''(W_{(i)} + V_i) + f'(W_{(i)} + V_i) - W_{(i)} f(W_{(i)} + V_i)\Big].
\end{aligned} \tag{4.31}$$

Define a shifted function $g_{V_i}(x) := f(x + V_i)$. Then $g_{V_i}^{(m)}(x) = f^{(m)}(x + V_i)$, and therefore,

$$\begin{aligned}
\|g_{V_i}^{(m)}\| &= \sup_{x\in\mathbb{R}}|f^{(m)}(x + V_i)| \\
&= \sup_{y\in\mathbb{R}}|f^{(m)}(y)| \qquad (4.32) \\
&= \|f^{(m)}\|. \qquad (4.33)
\end{aligned}$$

Hence, the derivative norms are invariant under translations. Using (4.31) with $g_{V_i}(x) := f(x + V_i)$ and then applying Theorem 11, we get

$$|\mathbb{E}[W_{(i)} g_{V_i}''(W_{(i)}) + g_{V_i}'(W_{(i)}) - W_{(i)} g_{V_i}(W_{(i)})]| \leq \frac{5}{4}\frac{\sqrt{l}}{m}\Big[\,\|h'\| + \frac{9}{2}\left\|\tilde{h}(x)\right\|\Big], \qquad l \geq 2. \tag{4.34}$$

Using (4.34) in (4.30), we get (4.28).

(ii) Following steps similar to case 1, noting that $\|(A_2 f)'\|$ is bounded as in (3.6), we get (4.29).

This completes the proof. □

Remark 2 Note that if $m \to \infty$, then from (4.28) and (4.29), we have $W_l \xrightarrow{d} Z$, where $Z \sim \text{VG}(l-1, 0, 1, 0)$, $l \geq 2$. Note that Beaulieu *et al.* [4] prove that the partial sums of triplewise independent rvs converge to a VG distribution. We obtain the order of convergence $O(m^{-1})$ (or, equivalently, $O(n^{-\frac{1}{2}})$) for this limiting result, which is novel in our opinion.

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Declarations

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