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Finite Local Information Capacity in Continuum Dynamics

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Abstract

Finite-dimensional local Hilbert spaces are widely used as microscopic regulators and have also been proposed as a fundamental information-capacity property of space-time. A finite local dimension, however, is not equivalent to a spatial lattice spacing, and neither is equivalent to a coarse-graining scale. We formulate a regulator-resolved framework that keeps these three notions separate and defines finite-information deviations operationally as the failure of microscopic and effective time evolution to commute with a specified coarse-graining channel. The resulting deviation is a state- and observable-dependent quantity in a common effective state space, rather than a formal subtraction of generators acting on different domains. We derive a decomposition into capacity, spatial-discretization, and coarse-graining contributions; a finite-time capacity bound controlled by discarded state weight and leakage through the cutoff boundary; observable error bounds; and a sharp continuity bound for subsystem entropies. We further show that block entropy and averaged one-cell entropy differ by a relative-entropy measure of total correlation and therefore cannot be interchanged. An information-geometric expansion of neighboring reduced states yields a nonlinear sigma-model effective action on the manifold of coarse-grained states. A single entropy field follows only under an additional one-dimensional closure condition. Finally, we test the framework in an exactly evolved periodic Klein–Gordon chain with independently variable lattice spacing and local number-state cutoff. Spectral peaks are extracted from dynamical matrix elements rather than imposed. Ground-state, dispersion, reduced-state, and entropy errors converge rapidly with local capacity, while the remaining lattice–continuum discrepancy persists at fixed spacing. The results provide a controlled mathematical and numerical foundation

for finite-information models without claiming a unique microscopic theory of quantum gravity.

Keywords: finite local Hilbert space, continuum limit, quantum lattice systems, coarse-graining, trace distance, relative entropy, quantum information geometry, effective field theory, bosonic truncation, finite-information space-time

1 Introduction

Continuum quantum field theory describes nature with extraordinary precision, but its mathematical formulation contains infinitely many local degrees of freedom and admits arbitrarily short wavelengths. In practical and nonperturbative calculations, the continuum is approached through regulated theories in which a spatial lattice, a finite volume, a local basis truncation, or a restricted energy sector is introduced and then controlled through a scaling limit. The conceptual and mathematical distinction among these regulators is central to Wilsonian renormalization and lattice field theory [1–4]. It is equally central to quantum simulation, where continuous bosonic variables must be represented with finite hardware resources [5–9]. A separate line of research asks whether finite information capacity is not merely a numerical regulator but a microscopic property of nature. Black-hole entropy and holographic bounds motivate finite state counts for bounded gravitating regions [10, 11]; entanglement-based approaches connect geometric organization to quantum correlations [12–14]; and thermodynamic arguments relate gravitational dynamics to local entropy balance [15]. The Quantum Memory Matrix (QMM) program develops one realization of this broader idea by representing local regions as finite-dimensional information-bearing subsystems whose collective dynamics are intended to recover continuum space-time [16, 17]. A difficulty common to finite-information proposals is that several logically distinct statements are easily conflated. A finite local Hilbert-space dimension d limits the number of locally distinguishable states. A lattice spacing a limits spatial resolution and modifies the momentum spectrum. A coarse-graining scale ℓ specifies which microscopic distinctions are discarded in constructing macroscopic observables. None of these quantities can generally be replaced by another. A chain of qubits has $d = 2$ yet supports alternating configurations down to its lattice spacing; increasing d at fixed a does not remove lattice dispersion; and reducing a does not guarantee that a chosen finite local basis faithfully represents the state sector of interest. A genuine continuum limit is therefore a joint scaling statement, not simply the limit $d \rightarrow \infty$. A second difficulty is operational. Expressions such as “microscopic generator minus continuum generator” are not meaningful unless the two operators act on a common domain and codomain. The microscopic state may live in a tensor product of finite local spaces, whereas the effective state is described by fields, hydrodynamic variables, or reduced density matrices. A comparison requires explicit matching and coarse-graining maps. Once those maps are supplied, the relevant object is not a formal difference of Hamiltonians but the failure of the diagram formed by microscopic evolution, coarse-graining, and effective evolution to commute. A third difficulty concerns entropy. The von Neumann

entropy of a block, the average of one-site entropies, mutual information, discarded weight, and the entropy production of a reduced channel are different quantities. In an interacting quantum system, their differences contain precisely the correlations that an information-based description seeks to retain. Replacing a block entropy by an average of local entropies can therefore erase the relevant physics rather than merely alter a smoothing convention. This work resolves these issues and replaces a phenomenological finite-information ansatz with a regulator-resolved, operational framework. Its principal contributions are as follows.

First, we define a finite-information deviation as a trace-distance or channel-norm defect between two states in the same effective state space. The defect compares "evolve microscopically, then coarse-grain" with "coarse-grain, then evolve effectively." Second, we separate local-capacity, spatial-discretization, and coarse-graining errors and derive finite-time bounds for each comparison. The capacity term is controlled by the probability weight excluded by the local cutoff and by Hamiltonian leakage across the retained subspace. No universal inverse- d expansion is assumed. Third, we establish the precise relation between block entropy and averaged cell entropy and use sharp continuity inequalities to convert reduced-state errors into entropy errors. Fourth, we derive the leading local effective action from the information geometry of neighboring reduced states. The generic result is a nonlinear sigma model on a manifold of collective state coordinates; a single entropy scalar is a special closure, not an automatic consequence of finite capacity. Fifth, we implement the complete construction in an exactly evolved finite-capacity quantum lattice model. The numerical calculation uses genuine reduced density matrices and von Neumann entropies, extracts excitation frequencies from spectral weights, and keeps the local cutoff and lattice spacing independent. The purpose of the explicit model is not to claim that a four-site scalar chain is a model of microscopic space-time. It is to demonstrate, in a system for which all relevant quantities can be computed without uncontrolled approximations, what a valid finite-information analysis must contain. The implications for space-time frameworks are then stated as consistency requirements: a microscopic proposal must identify its local physical Hilbert space and constraints, its continuum matching map, its scaling trajectory, and the collective variables that close its low-energy dynamics.

The article is organized as follows. Section 2 defines finite-capacity lattice dynamics and the three independent regulators. Section 3 introduces operational deviations and derives capacity, observable, and entropy bounds. Section 4 distinguishes entropy from correlation measures and develops the information-geometric effective theory. Section 5 presents the exact quantum-lattice realization. Section 6 identifies the consequences for finite-information space-time models. Sections 7 and 8 summarize the limitations and conclusions. Appendix A proves the regulator decomposition, Appendix B proves the finite-time capacity bound, Appendix C supplies the information-geometric variation and conservation details, and Appendix D documents the numerical implementation and validation.

2 Finite-capacity microscopic dynamics and regulator hierarchy

A finite-information construction must be specified at three logically distinct levels before any continuum claim can be evaluated: the microscopic state space and dynamics, the regulators used to represent those dynamics, and the map that identifies the observables retained by the effective theory. This section fixes those ingredients in that order. We first define a general local quantum system with finite on-site capacity, then separate spatial resolution from local basis size and observational coarse-graining, and finally introduce the channels that place microscopic and effective evolution in a common state space. The separation is not merely terminological. It determines which limiting procedure is being tested and prevents a decrease in one error source from being misidentified as convergence of the full theory.

2.1 Local quantum systems and microscopic evolution

Let Λ_a denote a finite graph embedded, when needed, in a spatial discretization with characteristic spacing a . The graph is not assumed to be a regular lattice at the level of the general formalism. Each vertex $x \in \Lambda_a$ carries a local physical Hilbert space

$$\mathcal{H}_x^{(d_x)} \cong \mathbb{C}^{d_x}, \quad 2 \leq d_x < \infty, \quad (1)$$

and the microscopic space is

$$\mathcal{H}_{a,d} = \bigotimes_{x \in \Lambda_a} \mathcal{H}_x^{(d_x)}. \quad (2)$$

Equations (1) and (2) distinguish the finite local state space from the full microscopic tensor-product space. For notational economy we write $d_x = d$ when the local dimension is uniform. Gauge or gravitational models may contain a larger kinematic tensor product followed by local constraints; in that case $\mathcal{H}_{a,d}$ denotes the resulting physical subspace, not the unconstrained product. Finite-dimensional gauge representations and quantum-link constructions provide established examples in which the local Hilbert space and Gauss constraints must be considered together [18–20]. We assume a self-adjoint local Hamiltonian

$$H_{a,d} = \sum_{X \subset \Lambda_a} h_X^{(a,d)}, \quad (3)$$

where $h_X^{(a,d)}$ has finite support and the interaction norm decays sufficiently rapidly with the diameter of X . The associated unitary channel is

$$\Phi_{a,d}^t(\rho) = U_{a,d}(t)\rho U_{a,d}^\dagger(t), \quad U_{a,d}(t) = e^{-iH_{a,d}t}. \quad (4)$$

Equations (3) and (4) specify the microscopic interaction structure and its induced unitary channel. Locality in the microscopic graph is compatible with a finite effective propagation velocity through Lieb–Robinson bounds [21, 22]. Such a bound supplies a causal cone for lattice observables, but it does not by itself prove Lorentz invariance or identify the graph with physical space-time. The finite dimension in Eq. (1) has a

Table 1 Independent ingredients in a finite-capacity continuum construction.

Quantity	What it controls	What it does not control by itself
a	spatial resolution, allowed momenta, lattice dispersion	local amplitude or occupation accuracy
d	number of locally representable states in a chosen encoding	shortest spatial wavelength or continuum symmetry restoration
ℓ	distinctions discarded in the effective variables	microscopic state capacity or lattice ultraviolet structure
ξ	physical correlation length of the microscopic state	numerical basis accuracy
E_*	energy sector in which matching is demanded	global high-energy behavior outside that sector

precise informational meaning: a single site can support at most $\log d_x$ nats of von Neumann entropy. It does not impose a unique energy cutoff unless a Hamiltonian and a mapping from basis states to physical excitations are specified. A d -level spin, a truncated oscillator, and a quantum link can have the same local dimension while representing different physical sectors. Consequently, every statement about "finite information" must be indexed not only by d but also by the encoding and the state class under consideration. It is therefore useful to regard the microscopic specification as the tuple $(\Lambda_a, \{\mathcal{H}_x^{(d_x)}\}, H_{a,d}, \mathfrak{S})$, where $\mathfrak{S}$ is the family of states on which the approximation is required to be accurate. Two models can share the same graph and the same local dimension while having very different finite-capacity behavior because their Hamiltonians populate the cutoff boundary differently. Likewise, the entropy ceiling $\log d_x$ constrains the spectrum of a reduced density matrix, but it does not by itself determine the density of spatial modes, the maximum physically relevant energy, or the accuracy of a continuum observable. Those properties follow only after the encoding, dynamics, and target sector have been fixed. This state-sector dependence will reappear explicitly in the discarded-weight and leakage terms derived in Section 3.

2.2 Regulator hierarchy, matching maps, and coarse-graining channels

We distinguish three scales throughout:

- (i) the microscopic spacing a , which controls spatial discretization and the Brillouin zone;
- (ii) the local dimension d , which controls the number of retained local states in a specified encoding;
- (iii) the coarse-graining length ℓ , which controls the spatial or informational resolution of the effective description.

The roles of these quantities are summarized in Table 1.

A continuum limit is a scaling trajectory. In a conventional critical lattice system one requires, schematically,

$$a \rightarrow 0, \quad \frac{\xi}{a} \rightarrow \infty, \quad (5)$$

while bare couplings are tuned so that selected renormalized observables remain fixed [1, 2, 4]. A finite local representation is adequate along this trajectory only if its truncation error also vanishes uniformly in the target state and energy sector:

$$\sup_{\rho \in \mathfrak{S}(E_*)} \varepsilon_d(\rho) \rightarrow 0, \quad (6)$$

where ε_d is a representation-dependent discarded weight defined below. Equation (5) controls the spatial scaling trajectory, whereas Eq. (6) imposes uniform accuracy of the local representation on the target sector. The limits $a \rightarrow 0$ and $d \rightarrow \infty$ need not commute, and some continuum theories can be approached with fixed finite d when critical collective degrees of freedom emerge from a spin system. Conversely, $d \rightarrow \infty$ at fixed a leaves all lattice-spacing corrections intact. The effective scale ℓ need not be taken to zero. Hydrodynamics and low-energy effective field theory are useful precisely because ℓ is kept larger than microscopic scales while smaller than the scale of variation of the retained observables. The correct condition is a separation of scales, for example

$$a \ll \ell \ll L_{\text{obs}}, \quad (7)$$

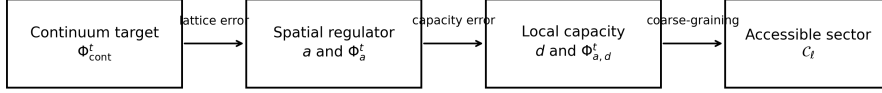
Equation (7), rather than the identification $a = \ell$, is the scale hierarchy used below. These scale conditions become operational only after states and observables at different resolutions have been matched. A statement that two descriptions agree is otherwise ambiguous: agreement may refer to the full microscopic state, to a selected subalgebra, to a collection of correlation functions, or only to a few hydrodynamic variables. The coarse-graining prescription therefore belongs to the physical definition of the effective theory. It identifies which distinctions are intentionally discarded and, consequently, which discrepancies should count as capacity, lattice, or closure errors. Let $\mathcal{H}_{\text{eff}}$ denote the Hilbert space, algebraic state space, or finite set of collective degrees of freedom used by the effective theory. A microscopic-to-effective comparison requires a completely positive trace-preserving (CPTP) map

$$\mathcal{C}_{a,d,\ell} : \mathcal{D}(\mathcal{H}_{a,d}) \rightarrow \mathcal{D}(\mathcal{H}_{\text{eff}}), \quad (8)$$

where $\mathcal{D}(\mathcal{H})$ is the set of density operators on $\mathcal{H}$ [23, 24]. Depending on the problem, $\mathcal{C}_{a,d,\ell}$ can include partial traces, block isometries, a quantum channel that stores selected expectation values, or a map to a manifold of maximum-entropy states. A reconstruction map

$$\mathcal{R}_{a,d,\ell} : \mathcal{D}(\mathcal{H}_{\text{eff}}) \rightarrow \mathcal{D}(\mathcal{H}_{a,d}) \quad (9)$$

may be introduced when the information lost by coarse-graining is to be quantified. Equations (8) and (9) therefore define the forward and reverse comparison maps, respectively. Neither map is unique. Their specification is part of the physical model, just as a renormalization prescription is part of an effective-field-theory calculation. Figure 1



$$\delta_{a,d,\ell}(t) \leq \delta_{\text{cap}}(d, t) + \delta_{\text{lat}}(a, t) + \delta_{\text{cg}}(\ell, t)$$

The three regulators are independent: increasing d does not remove finite- a or finite- ℓ effects.

Fig. 1 Regulator-resolved finite-information framework. The microscopic theory is specified by a spacing a , local dimension d , and Hamiltonian $H_{a,d}$. A CPTP coarse-graining map $\mathcal{C}_{a,d,\ell}$ produces an effective state at resolution ℓ . Finite-information deviation is the noncommutativity of microscopic and effective evolution in the common effective state space. Capacity, lattice, and coarse-graining errors are separated rather than absorbed into a single parameter.

shows the resulting hierarchy. The upper route evolves a microscopic state and then restricts it to the accessible description. The lower route first constructs the effective state and then evolves it with an effective dynamical channel. The difference between the two routes is the operational finite-information deviation defined in Section 3.

3 Operational finite-information deviations

With the microscopic and effective descriptions now connected by explicit channels, their disagreement can be defined without subtracting operators that act on different spaces. This section introduces the channel-level commuting-diagram defect, resolves the total discrepancy into capacity, lattice, and coarse-graining contributions, and then derives a finite-time bound for a projected local basis. The same trace-distance framework is finally translated into bounds on observables and subsystem entropies. Keeping these steps together makes clear that a finite-information deviation is not a universal function of d ; it is a relational quantity determined by a state, an evolution time, a matching prescription, and the accessible observable sector.

3.1 Operational defect and regulator decomposition

Let Φ_{eff}^t be the effective dynamical channel on $\mathcal{D}(\mathcal{H}_{\text{eff}})$. It may be unitary, dissipative, or explicitly time dependent. The microscopic and effective generators need not be subtracted. Instead, both routes are compared after mapping to the same codomain.

Definition 1 (Finite-information deviation superoperator). For fixed (a, d, ℓ) and time t , define

$$\mathfrak{D}_{a,d,\ell}^t = \mathcal{C}_{a,d,\ell} \circ \Phi_{a,d}^t - \Phi_{\text{eff}}^t \circ \mathcal{C}_{a,d,\ell}. \quad (10)$$

For a microscopic state ρ , the state-dependent deviation is

$$\delta_{a,d,\ell}(t; \rho) = \frac{1}{2} \|\mathfrak{D}_{a,d,\ell}^t(\rho)\|_1. \quad (11)$$

The worst-case channel deviation is

$$\Delta_{a,d,\ell}(t) = \frac{1}{2} \|\mathfrak{D}_{a,d,\ell}^t\|_\diamond. \quad (12)$$

Equation (11) is the state-restricted form of the defect, while Eq. (12) gives its reference-system-stable channel norm. The trace distance $T(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_1$ has an operational interpretation as optimal state distinguishability [23, 25]. The diamond norm extends that comparison to states entangled with an arbitrary reference system [24]. For a foundational microscopic proposal, the diamond norm is the cleanest channel-level criterion, but state-restricted trace distances are often physically more informative because no finite-dimensional approximation is expected to be uniformly accurate on an unrestricted, unbounded-energy bosonic state space. The defect in Eq. (10) also distinguishes a poor effective closure from a genuinely finite-capacity effect. If d is increased while a, ℓ , the matching map, and the effective dynamics are held fixed, the change in δ isolates capacity dependence. If a is changed at sufficiently converged d , the change isolates lattice dependence. If ℓ or the set of retained observables is changed, the resulting variation diagnoses coarse-graining dependence. This comparison must be performed with a fixed matching convention. Retuning the effective channel independently at every value of d can absorb a truncation error into fitted parameters and thereby conceal rather than measure convergence. A useful regulator study therefore specifies which renormalized quantities are held fixed, how initial states are paired, and which observables define the effective sector. Under those conditions the total discrepancy can be decomposed into a sequence of physically interpretable comparisons.

To separate the three regulators, choose a matched family of initial states and define four effective states at time t :

$$\sigma_{a,d,\ell}(t) = \mathcal{C}_{a,d,\ell} \Phi_{a,d}^t(\rho_{a,d}), \quad (13)$$

$$\sigma_{a,\infty,\ell}(t) = \mathcal{C}_{a,\infty,\ell} \Phi_{a,\infty}^t(\rho_{a,\infty}), \quad (14)$$

$$\sigma_{0,\infty,\ell}(t) = \mathcal{C}_{0,\infty,\ell} \Phi_{0,\infty}^t(\rho_{0,\infty}), \quad (15)$$

$$\sigma_{\text{eff}}(t) = \Phi_{\text{eff}}^t \mathcal{C}_{0,\infty,\ell}(\rho_{0,\infty}). \quad (16)$$

Equations (13), (14), (15), and (16) define, in order, the finite-capacity lattice state, the untruncated lattice state, the continuum state after coarse-graining, and the independently evolved effective state. Here ∞ labels the untruncated local representation at fixed a , and 0 labels the continuum theory. The notation is formal enough to cover different microscopic encodings but presumes explicit matching of the physical initial state and renormalized parameters.

Proposition 1 (Regulator-resolved deviation). *The total state-space discrepancy satisfies*

$$T(\sigma_{a,d,\ell}, \sigma_{\text{eff}}) \leq \delta_{\text{cap}} + \delta_{\text{lat}} + \delta_{\text{cg}}, \quad (17)$$

where

$$\delta_{\text{cap}} = T(\sigma_{a,d,\ell}, \sigma_{a,\infty,\ell}), \quad (18)$$

$$\delta_{\text{lat}} = T(\sigma_{a,\infty,\ell}, \sigma_{0,\infty,\ell}), \quad (19)$$

$$\delta_{\text{cg}} = T(\sigma_{0,\infty,\ell}, \sigma_{\text{eff}}). \quad (20)$$

The three terms in Eqs. (18), (19), and (20) isolate capacity, lattice, and closure errors in the bound (17). The proof is the triangle inequality and is given in Appendix A. Its importance is conceptual rather than algebraic: an observed plateau with increasing d does not contradict convergence. It indicates that δ_{cap} has fallen below the remaining $\delta_{\text{lat}} + \delta_{\text{cg}}$. Conversely, a small lattice error cannot compensate for a local truncation that excludes appreciable state weight.

3.2 Finite-time capacity bounds and observable consequences

We next quantify the capacity contribution for a cutoff defined by an orthogonal projector P_d onto a retained subspace $\mathcal{H}_d \subset \mathcal{H}$, with $Q_d = \mathbb{I} - P_d$. Let H be a self-adjoint Hamiltonian on $\mathcal{H}$, and let

$$H_d = P_d H P_d \quad (21)$$

act on $\mathcal{H}_d$. For a full state ρ , define its retained probability and discarded weight by

$$p_d(\rho) = \text{Tr}(P_d \rho), \quad \varepsilon_d(\rho) = 1 - p_d(\rho), \quad (22)$$

and its normalized projection by

$$\tilde{\rho}_d = \frac{P_d \rho P_d}{p_d(\rho)}, \quad p_d(\rho) > 0. \quad (23)$$

Equations (21), (22), and (23) fix the retained dynamics, excluded probability, and normalized finite-capacity initial state used in the theorem.

Theorem 2 (Finite-time capacity error). *Assume that $Q_d H P_d$ is bounded on the state sector considered. With $U(t) = e^{-iHt}$ and $U_d(t) = e^{-iH_d t}$, where U_d is embedded in $\mathcal{H}$, one has*

$$T\left(U(t)\rho U^\dagger(t), U_d(t)\tilde{\rho}_d U_d^\dagger(t)\right) \leq \sqrt{\varepsilon_d} + \frac{\varepsilon_d}{2} + |t| \|Q_d H P_d\|. \quad (24)$$

Any CPTP coarse-graining channel can only decrease the left-hand side.

The first two terms quantify mismatch already present in the initial state; they follow from the gentle-measurement inequality and normalization of the projected state. The last term is dynamical leakage across the truncation boundary. The proof uses

Duhamel's formula and contractivity of trace distance and is given in Appendix B. Equation (24) demonstrates why there is no universal series in $1/d$. The scaling is inherited from both $\varepsilon_d(\rho)$ and $\|Q_d H P_d\|$, which depend on the encoding, Hamiltonian, state sector, and time. For a bosonic number cutoff P_d retaining local occupations $0, \dots, d-1$, a union bound and Markov's inequality give the conservative estimate

$$\varepsilon_d(\rho) \leq \sum_{x \in \Lambda_a} \Pr_{\rho}(n_x \geq d) \leq \frac{\langle N_{\text{tot}} \rangle_{\rho}}{d}, \quad N_{\text{tot}} = \sum_x n_x. \quad (25)$$

Equation (25) makes the state-sector dependence explicit even at the level of a conservative occupation-moment estimate. States with exponentially decaying occupation tails converge much faster than this worst-case bound, while highly occupied or strongly squeezed states may require much larger d . For the single-site coherent state used in Section 5, the initial discarded weight is known exactly because the occupation statistics are Poissonian:

$$\varepsilon_d^{\text{coh}}(\alpha) = 1 - e^{-|\alpha|^2} \sum_{n=0}^{d-1} \frac{|\alpha|^{2n}}{n!}. \quad (26)$$

This quantity isolates initial representational loss. It does not include later population transfer toward the cutoff boundary or the operator-construction term in Eq. (27); both can make the real-time error appreciably larger. More refined rigorous bounds for bosonic and gauge-theory simulations likewise depend on energy moments, locality, and evolution time rather than on d alone [9]. The theorem applies directly when the finite model is the projected Hamiltonian $P_d H P_d$. Other digitizations, including truncating canonical matrices before forming polynomial Hamiltonian terms, add an operator-construction error

$$\delta H_d = H_d^{\text{enc}} - P_d H P_d, \quad (27)$$

Equation (27) separates this digitization error from excluded-state and leakage errors; its finite-time contribution can be bounded by $|t| \|\delta H_d\|$ on the relevant sector. We retain this distinction in the numerical model below. The state-space estimate becomes physically useful only when it is converted into statements about measurements performed in the retained theory. Trace distance is particularly well suited to this task because it is monotone under coarse-graining and directly controls expectation values of every bounded observable in the common effective space. The same reduced-state distance also controls the continuity of finite-dimensional entropy. Thus the theorem above supplies a single certified error budget for dynamical states, accessible measurements, and information-theoretic quantities rather than three unrelated convergence tests.

For any bounded effective observable O , trace distance immediately implies

$$|\text{Tr}[O(\rho - \sigma)]| \leq 2 \|O\|_{\infty} T(\rho, \sigma). \quad (28)$$

Equation (28) shows that an operational deviation is not merely a norm diagnostic: it is a uniform upper bound on all normalized accessible observables. When the observable is unbounded, as for canonical fields, the comparison must be restricted to an energy-constrained state set or replaced by a bounded function of the observable. This is another reason that unrestricted worst-case channel errors are inappropriate for bosonic continuum limits. Let B be a finite effective subsystem with dimension D_B , and let

$$T_B = T(\rho_B, \sigma_B). \quad (29)$$

The subsystem distance in Eq. (29) is the input to the entropy-continuity estimate. Continuity of finite-dimensional entropy was established by Fannes and sharpened to the optimal dimension-dependent form by Audenaert [26, 27]. For $0 \leq T_B \leq 1 - 1/D_B$, the sharp bound is

$$|S(\rho_B) - S(\sigma_B)| \leq T_B \log(D_B - 1) + h_2(T_B), \quad (30)$$

where

$$h_2(T) = -T \log T - (1 - T) \log(1 - T) \quad (31)$$

is the binary entropy defined in Eq. (31). For larger T_B , the trivial bound $|\Delta S_B| \leq \log D_B$ applies. Equation (30) converts a reduced-state convergence test into a certified entropy convergence test and will be evaluated explicitly in Section 5.

4 Entropy, correlations, and information-geometric effective theory

The operational bounds of Section 3 do not by themselves determine which collective variables should enter an effective action. That choice requires attention to correlations that are invisible to one-site entropy averages and to the local geometry of the reduced-state manifold. This section first separates block entropy, local entropy, total correlation, and reconstruction loss. It then uses relative entropy to derive the metric governing slowly varying families of coarse-grained states. The resulting continuum theory is generically a multi-field nonlinear sigma model; an entropy-only scalar sector emerges only after an additional closure condition has reduced the accessible state manifold to one dimension.

4.1 Correlation-sensitive entropy and information geometry

For a block B of microscopic sites, define

$$\rho_B = \text{Tr}_{\bar{B}} \rho, \quad S_B = -\text{Tr}(\rho_B \log \rho_B). \quad (32)$$

The average of one-site entropies is

$$\bar{S}_B = \frac{1}{|B|} \sum_{x \in B} S(\rho_x). \quad (33)$$

Equations (32) and (33) define inequivalent block-level and site-averaged entropy observables. These quantities coincide only in special states. Their unnormalized difference is the total correlation in the block:

Proposition 3 (Entropy–correlation identity). *For any multipartite block state ρ_B ,*

$$\sum_{x \in B} S(\rho_x) - S(\rho_B) = D\left(\rho_B \left\| \bigotimes_{x \in B} \rho_x\right.\right) \geq 0. \quad (34)$$

Equation (34) identifies their difference with a nonnegative relative-entropy correlation measure. The equality follows from the definition of quantum relative entropy [28]. The right-hand side vanishes exactly when the block factorizes. For a two-site block it is the mutual information

$$I(x : y) = S(\rho_x) + S(\rho_y) - S(\rho_{xy}). \quad (35)$$

For two sites, Eq. (35) is the corresponding specialization. Thus replacing S_B by an average of local entropies removes a correlation term rather than merely changing normalization. In a finite-information theory that attributes physical significance to stored correlations, this replacement is generally inadmissible. The entropy of a reduced state also should not be confused with information lost by a chosen effective representation. The latter can be quantified by the reconstruction relative entropy

$$\mathcal{I}_{\text{loss}}(\rho) = D(\rho \| \mathcal{R} \circ \mathcal{C}(\rho)), \quad (36)$$

when the support condition for relative entropy is satisfied. Equation (36) is explicitly map dependent and therefore measures information lost by the chosen representation rather than entropy intrinsic to the microscopic state. This quantity depends on the coarse-graining and reconstruction prescription and vanishes exactly for states perfectly recoverable by that prescription. It is therefore closer to an operational “finite-information loss” than raw subsystem entropy. The preceding identities also show why entropy alone is generally too coarse to parameterize local physics. Many inequivalent density matrices share the same spectrum, and therefore the same von Neumann entropy, while differing in charges, coherences, correlation structure, or response to perturbations. A systematic effective theory should instead begin with a family of reduced states described by enough collective coordinates to distinguish the physically accessible configurations. Relative entropy then provides a canonical local measure of distinguishability on that family. Suppose a family of faithful coarse-grained block states is parameterized by collective coordinates $\theta^A(x)$,

$$\rho_x = \rho(\theta^1(x), \dots, \theta^n(x)). \quad (37)$$

Equation (37) defines the collective-state manifold on which the effective information geometry is constructed. The coordinates may be expectation values, conserved densities, order parameters, modular parameters, or combinations thereof. The choice is

determined by the microscopic closure, not by covariance alone. For neighboring states, we define the symmetrized relative entropy by

$$D_s(\rho, \sigma) = \frac{1}{2} [D(\rho\|\sigma) + D(\sigma\|\rho)]. \quad (38)$$

The quantity in Eq. (38) has the local expansion

$$D_s(\rho(\theta), \rho(\theta + d\theta)) = \frac{1}{2} G_{AB}^{\text{BKM}}(\theta) d\theta^A d\theta^B + \mathcal{O}(\|d\theta\|^3), \quad (39)$$

where G_{AB}^{BKM} is the Bogoliubov–Kubo–Mori monotone metric associated with relative entropy [29–31]. One convenient expression is

$$G_{AB}^{\text{BKM}} = \int_0^1 ds \operatorname{Tr} [\rho^s (\partial_A \log \rho) \rho^{1-s} (\partial_B \log \rho)]. \quad (40)$$

Equation (40) gives the explicit BKM metric used in the expansion. It is positive semidefinite and becomes positive definite after quotienting redundant coordinates. Consider a microscopic distinguishability cost that penalizes neighboring blocks,

$$\mathcal{A}_{\text{disc}} = \sum_{\langle x, y \rangle} w_{xy} D_s(\rho_x, \rho_y), \quad (41)$$

with nonnegative weights w_{xy} . Equation (41) is a microscopic distinguishability functional, not a postulated scalar-field action. If the state variables vary slowly across the graph, Eq. (39) gives

$$\mathcal{A}_{\text{disc}} \longrightarrow \frac{M_I^2}{2} \int d^D x \sqrt{|g|} G_{AB}(\theta) g^{\mu\nu} \partial_\mu \theta^A \partial_\nu \theta^B + \text{higher derivatives}, \quad (42)$$

Equation (42) is the resulting Euclidean sigma-model limit. Its second-order relative-entropy derivation and the associated variational identities are detailed in Appendix C. Here M_I carries the dimensions required by the continuum measure and depends on the microscopic weights and scaling trajectory. The leading gradient term is therefore not selected by a Taylor expansion of a scalar entropy value alone. It follows from the second-order distinguishability of neighboring quantum states. The target-space metric G_{AB} retains information about all collective coordinates needed to distinguish those states.

4.2 Lorentzian effective action and closure conditions

After analytic continuation or an independent real-time matching calculation, the most general local, two-derivative Lorentzian action for the collective coordinates and a

metric $g_{\mu\nu}$ has the scalar–tensor form

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \left[\frac{M_{\text{P}}^2}{2} F(\theta) R - \frac{M_I^2}{2} G_{AB}(\theta) \nabla_\mu \theta^A \nabla^\mu \theta^B - V(\theta) + \sum_i \frac{c_i}{\Lambda^{\Delta_i-4}} \mathcal{O}_i \right]. \quad (43)$$

We use metric signature $(-, +, +, +)$. The operators $\mathcal{O}_i$ contain higher derivatives or additional fields and are ordered by their scaling dimension. Equation (43) makes explicit several structures that cannot be excluded merely by ”low-derivative dominance”: a potential V , a nonminimal curvature coupling F , and a multidimensional information metric. Symmetries or a microscopic calculation are required to eliminate them. The ordering in Eq. (43) should be understood as an effective-field-theory expansion around a specified background and state sector. It does not imply that all collective fields remain light, that the target-space metric is constant, or that integrating out the discarded coordinates is local in time. If a supposedly irrelevant coordinate relaxes slowly, its elimination can produce memory kernels and colored noise rather than a local potential. Conversely, when a spectral gap separates it from the retained sector, its effects can be organized in higher-dimension operators with coefficients fixed by matching. This distinction is essential when an information-theoretic variable is promoted from a diagnostic of reduced states to a dynamical source of geometry. In the minimally coupled sector $F = 1$, variation with respect to the metric gives

$$T_{\mu\nu}^{(\text{info})} = M_I^2 G_{AB} \left(\nabla_\mu \theta^A \nabla_\nu \theta^B - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \theta^A \nabla^\alpha \theta^B \right) - g_{\mu\nu} V. \quad (44)$$

The stress tensor in Eq. (44) retains the full target-space metric and potential. The collective equations of motion are

$$\square \theta^A + \Gamma_{BC}^A(\theta) \nabla_\mu \theta^B \nabla^\mu \theta^C - \frac{1}{M_I^2} G^{AB} \partial_B V = 0, \quad (45)$$

where Γ_{BC}^A is the Levi–Civita connection of G_{AB} . On solutions of Eq. (45), diffeomorphism invariance implies

$$\nabla^\mu T_{\mu\nu}^{(\text{info})} = 0. \quad (46)$$

Equation (46) follows on shell from Eq. (45); Appendix C gives the direct calculation. An entropy field defined kinematically from reduced density matrices cannot simply be varied as an independent scalar unless a closure relation establishes that its dynamics are described by Eq. (45) or an appropriate open-system generalization. Projection-operator and open-system methods provide the natural framework when the closure includes dissipation, noise, or memory kernels [32]. A single scalar $S(x) = S[\rho_x]$ is sufficient only under a restrictive condition. Suppose the accessible state manifold is one-dimensional, $\rho = \rho(\vartheta)$, and $dS/d\vartheta \neq 0$ in the region of interest. Then ϑ can locally

be replaced by S , yielding

$$-\frac{M_I^2}{2}K(S)\nabla_\mu S\nabla^\mu S, \quad K(S) = G_{\vartheta\vartheta}(\vartheta(S))\left(\frac{d\vartheta}{dS}\right)^2. \quad (47)$$

If distinct states have the same entropy, as they generically do, entropy does not provide a coordinate on the state manifold and no entropy-only action can reproduce all local distinguishability. The scalar reduction is therefore an additional constitutive assumption, not a universal consequence of finite local Hilbert dimension. This derivation also fixes dimensions. Von Neumann entropy is dimensionless in natural units, so a canonical term $(\nabla S)^2$ has mass dimension two. Its coefficient must carry mass dimension two, represented by $M_I^2 K(S)$ in Eq. (47). A dimensionless coefficient multiplying $(\nabla S)^2$ in four space-time dimensions would be inconsistent unless the field itself had first been rescaled to mass dimension one.

5 Exact quantum-lattice realization

The formal construction is now evaluated in a model chosen specifically to separate local-capacity error from spatial-discretization error. A periodic Klein–Gordon chain is useful because the untruncated lattice spectrum is known analytically, while a finite number-state basis provides a controlled and independently variable local dimension. We use this structure to test both static and real-time quantities: ground energies, dynamically extracted excitation frequencies, reduced density matrices, trace distances, subsystem entropies, and mutual information. The calculation is not presented as a microscopic model of space-time; it is an exact benchmark showing how a finite-information claim can be formulated so that every approximation is visible and independently checkable.

5.1 Model and numerical protocol

We test the formalism in a periodic one-dimensional Klein–Gordon chain with $N = 4$ sites,

$$H_a = \frac{1}{2} \sum_{n=0}^{N-1} \left[p_n^2 + m^2 q_n^2 + \frac{(q_{n+1} - q_n)^2}{a^2} \right], \quad q_N \equiv q_0, \quad (48)$$

Equation (48) defines the spatially regulated microscopic dynamics, with $[q_n, p_m] = i\delta_{nm}$. The exact lattice normal-mode frequencies are

$$\omega_a(k) = \sqrt{m^2 + \frac{4}{a^2} \sin^2\left(\frac{ka}{2}\right)}, \quad (49)$$

whereas the target continuum dispersion is

$$\omega_{\text{cont}}(k) = \sqrt{m^2 + k^2}. \quad (50)$$

The difference between Eqs. (49) and (50) is a pure spatial-discretization effect and persists even when the local oscillator representation is exact. Each site is represented

in the first d number states of an onsite oscillator with frequency

$$\omega_0 = \sqrt{m^2 + \frac{2}{a^2}}, \quad (51)$$

The basis frequency in Eq. (51) matches the local quadratic coefficient in Eq. (48). The finite matrices $q^{(d)}$ and $p^{(d)}$ are inserted into Eq. (48) before the polynomial terms are formed. This is a standard number-basis truncation, but, because $(P_d p P_d)^2$ differs at the cutoff boundary from $P_d p^2 P_d$, it is not identical to the exact projected Hamiltonian $P_d H_a P_d$. The distinction corresponds to the encoding error in Eq. (27) and vanishes rapidly in the low-energy sector studied here. The calculation uses

$$N = 4, \quad a = 1, \quad m = 0.7, \quad d \in \{4, 6, 8, 10, 12, 14\}. \quad (52)$$

Equation (52) fixes every dimensionless parameter and local cutoff used in the reported runs. The $d = 14$ calculation serves as the finite numerical reference for reduced-state dynamics; its spectral errors relative to the analytic lattice frequencies are below 1.5×10^{-10} for the modes reported below. Sparse Hermitian eigensolvers produce the lowest 24 eigenpairs. Real-time states are evolved with a Krylov action of the matrix exponential [33, 34]. All calculations use double-precision complex arithmetic. Excitation frequencies are not inserted into synthetic data. For each allowed momentum

$$k_r = \frac{2\pi r}{Na}, \quad r = 0, 1, 2, \quad (53)$$

listed in Eq. (53), we construct the momentum-resolved field operator

$$q_{k_r} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-ik_r n a} q_n. \quad (54)$$

The operator in Eq. (54) resolves the translation-invariant normal modes, and we identify the excitation through the largest spectral weight

$$W_j(k_r) = |\langle E_j | q_{k_r} | E_0 \rangle|^2. \quad (55)$$

Equation (55) supplies the extraction criterion. The extracted frequency is $\omega_d(k_r) = E_{j_*} - E_0$, where j_* maximizes W_j after excluding the ground state. This procedure can fail visibly at an inadequate cutoff, which is a feature rather than a hidden fit assumption. For real-time tests, site zero is initialized in a coherent state $|\alpha\rangle$ and the remaining sites in the onsite vacuum. We use $\alpha = 0.55$ and $\alpha = 1.00$ to test state dependence. At each time $t \in [0, 8]$, the one-site reduced density matrix $\rho_0^{(d)}(t)$ is computed by an exact partial trace and embedded into the $d = 14$ local basis. Capacity deviation is measured by

$$T_d(t) = \frac{1}{2} \left\| \rho_0^{(d)}(t) - \rho_0^{(14)}(t) \right\|_1. \quad (56)$$

Table 2 Convergence of the finite local basis at fixed lattice spacing $a = 1$. Frequencies are extracted from the dominant spectral weights in Eq. (55); they are not imposed. The reference exact-lattice frequencies are 0.7000000000, 1.5779733838, and 2.1189620100.

d	$ E_0^{(d)} - E_0^{(14)} /E_0^{(14)}$	$\omega_d(0)$	$\omega_d(\pi/2)$	$\omega_d(\pi)$
4	3.060×10^{-3}	0.5402417064	1.2742944778	2.1616221918
6	3.129×10^{-5}	0.6978824711	1.5764993408	2.1170902175
8	4.024×10^{-7}	0.6999647002	1.5779534375	2.1189430953
10	5.289×10^{-9}	0.6999994152	1.5779730765	2.1189617400
12	6.853×10^{-11}	0.6999999907	1.5779733791	2.1189620061
14	reference	0.6999999999	1.5779733837	2.1189620100

Equation (56) is the real-time local-capacity diagnostic. The entropy error uses the genuine von Neumann entropy of the same reduced matrices,

$$\Delta S_d(t) = \left| S(\rho_0^{(d)}(t)) - S(\rho_0^{(14)}(t)) \right|. \quad (57)$$

Equation (57) therefore compares genuine subsystem entropies; no classical point-wise surrogate such as $-p \log p$ is used. Full implementation details and independent numerical checks are provided in Appendix D.

5.2 Spectral convergence and separation of capacity from lattice errors

Table 2 gives the ground-state and mode-frequency results. The exact lattice ground-state energy is

$$E_{0,a} = \frac{1}{2} \sum_{r=0}^{N-1} \omega_a(k_r) = 2.987454388824\dots, \quad (58)$$

Equation (58) provides an analytic benchmark and agrees with the $d = 14$ value at the displayed precision. The relative ground-energy error decreases from 3.06×10^{-3} at $d = 4$ to 6.85×10^{-11} at $d = 12$. The $d = 4$ truncation strongly distorts the spectral identification, especially at $k = \pi/2$, while $d \geq 8$ resolves all three frequencies accurately.

Figure 2 displays the same convergence without assuming a power law in $1/d$. The decrease is much faster than the conservative Markov estimate in Eq. (25), as expected for the low-occupation Gaussian ground state. A single inverse- d expansion would both understate this convergence and obscure its dependence on the state sector.

Rapid convergence with d establishes that the selected low-energy sector is accurately represented, but it does not establish convergence to the spatial continuum. Once the finite-basis contribution has become negligible, the remaining discrepancy can be attributed to the lattice regulator and studied without contamination from local truncation. This second comparison is what distinguishes a genuine joint continuum analysis from a basis-convergence test alone. Figure 3 makes the regulator separation explicit. In the left panel, the converged finite- d points lie on the exact lattice dispersion

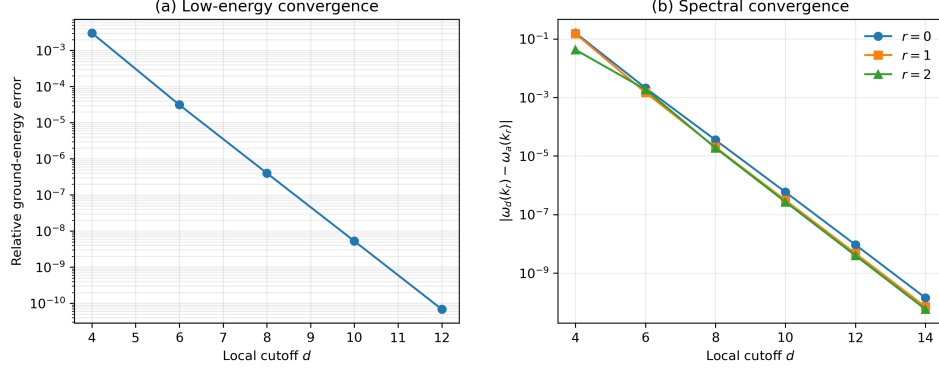


Fig. 2 Local-capacity convergence at fixed $a = 1$. Left: relative ground-state energy error against the $d = 14$ reference. Right: absolute errors of frequencies extracted from spectral weights for the three independent lattice momenta. Lines connect calculated values only and are not fits to an assumed $1/d$ law.

in Eq. (49), not on the continuum curve in Eq. (50). The local capacity error is negligible for $d = 12$, but the lattice-continuum difference remains large at $k = \pi$ because ka is not small. This is precisely the behavior that is lost when d is incorrectly treated as the spatial continuum parameter.

For $ka \ll 1$, expansion of Eq. (49) yields

$$\omega_a^2(k) = m^2 + k^2 - \frac{a^2 k^4}{12} + \mathcal{O}(a^4 k^6). \quad (59)$$

The k^4 correction is derived from the finite-difference Hamiltonian and has the dimensionally required coefficient a^2 . It is not generated by dividing k^4 by a Hilbert-space dimension. The right panel of Figure 3 evaluates the relative lattice-continuum error at fixed physical momentum while varying a and confirms its decrease as the spatial regulator is removed.

5.3 Real-time reduced-state convergence, state dependence, and entropy certification

Figure 4 shows $T_d(t)$ for the coherent pulse with $\alpha = 0.55$. The deviation is time dependent and nonmonotonic because a local cutoff changes both the initial representation and the subsequent transport of amplitude and correlations. Nevertheless, the envelope decreases systematically with d . The maximum one-site trace distances over $0 \leq t \leq 8$ are

d	4	6	8	10	12
$\max_t T_d(t)$	0.42475	0.08863	0.02513	0.00595	0.00143.

(60)

The numerical maxima in Eq. (60) quantify the envelope shown in Figure 4. Through Eq. (28), the final value implies that every unit-norm observable on site zero differs from the $d = 14$ reference by at most 2.86×10^{-3} at all sampled times for $d = 12$.

The result illustrates why a static reconstruction error is insufficient. A representation that accurately matches an initial field amplitude can still leak at the cutoff

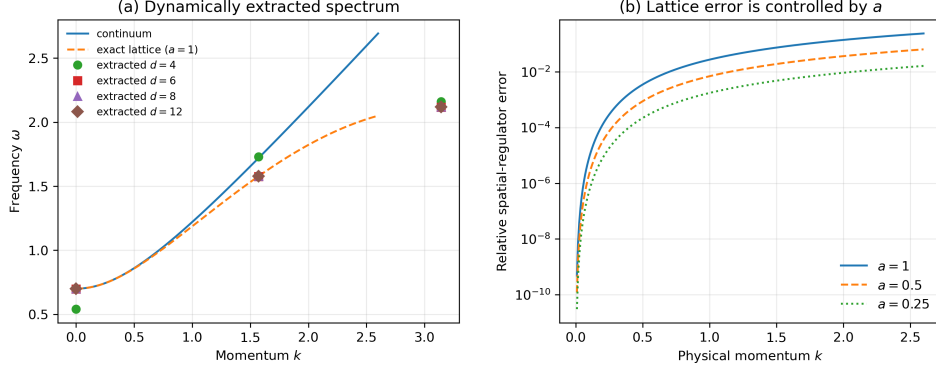


Fig. 3 Independent capacity and spatial-discretization effects. Left: continuum dispersion, exact lattice dispersion at $a = 1$, and frequencies extracted from finite- d spectral weights. Once d is converged, the points approach the lattice curve while retaining a finite lattice-continuum difference. Right: relative lattice-continuum error as a function of physical momentum for $a = 1, 0.5$, and 0.25 , evaluated analytically from Eq. (49).

boundary or alter entanglement generation under the Hamiltonian. Conversely, a moderate global state difference can become small after a physically motivated coarse-graining channel. The operational definition in Eq. (11) makes the accessible subsystem and comparison time explicit. A convergence statement must also specify the state family on which it is intended to hold. Raising the coherent amplitude changes the occupation tail, increases the probability of reaching the finite-basis boundary, and modifies the correlations generated during evolution. Comparing two amplitudes therefore probes whether the cutoff error behaves as predicted by the state-dependent theory rather than merely confirming convergence for one specially chosen initial condition. The same data permit a second independent test: whether trace-distance convergence certifies convergence of the actual von Neumann entropy through Eq. (30). The larger coherent amplitude $\alpha = 1.00$ places more weight at higher local occupations. Equation (26) makes the initial state dependence explicit: at $d = 4$, the discarded coherent-state weights are 2.74×10^{-4} for $\alpha = 0.55$ and 1.90×10^{-2} for $\alpha = 1.00$, while at $d = 12$ they are below 10^{-15} and 8.32×10^{-10} , respectively. The substantially larger real-time deviations therefore cannot be attributed to initial projection alone; they also measure cutoff-boundary dynamics and the finite-matrix encoding of the Hamiltonian. The maxima of $T_d(t)$ for $\alpha = 1.00$ are 0.64263, 0.14618, 0.04302, 0.01397, and 0.00423 for $d = 4, 6, 8, 10, 12$, respectively. This state dependence is not a numerical nuisance; it is an essential property of a finite local representation and confirms that d alone does not define a universal physical error. The left panel of Figure 5 compares the maximum trace-distance errors for both amplitudes. The right panel compares the actual entropy error with the Audenaert upper bound from Eq. (30). For $\alpha = 0.55$, the maximum entropy error falls from 0.58507 at $d = 4$ to 7.09×10^{-4} at $d = 12$. For $\alpha = 1.00$, it falls from 0.79173 to 2.27×10^{-3} . The continuity bound is conservative, as a worst-case theorem should be, but it certifies entropy convergence directly from reduced-state convergence.

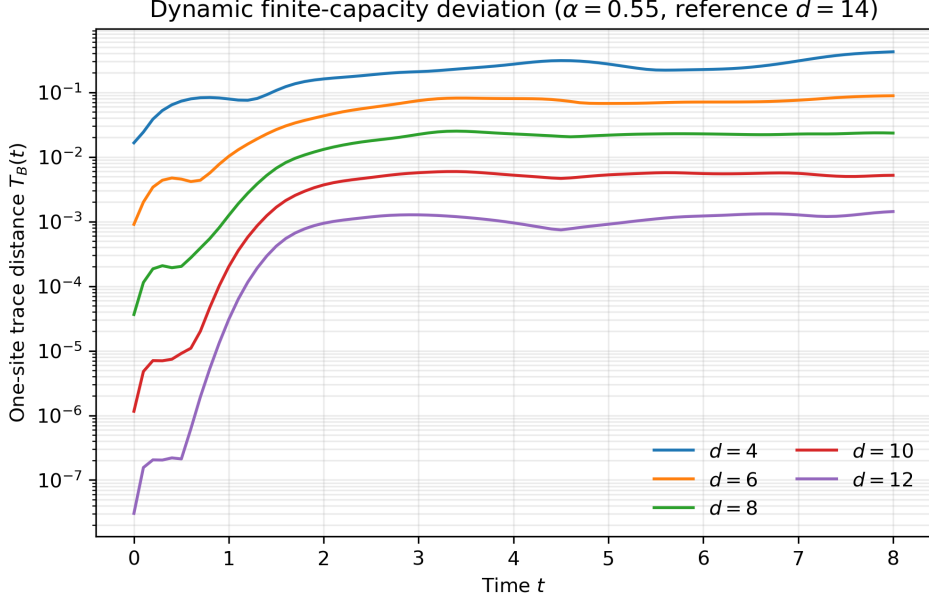


Fig. 4 Real-time local-capacity deviation for a coherent excitation with $\alpha = 0.55$. The exact global states are evolved under each finite- d Hamiltonian, reduced to site zero, embedded in the $d = 14$ reference basis, and compared by trace distance. The deviation is a state-space quantity and no fitted continuum curve is used.

Table 3 Block entropies and mutual information in the $d = 14$ ground state. All logarithms are natural. Translation invariance makes the two one-site entropies equal within numerical precision.

Sites	$S(\rho_i)$	$S(\rho_j)$	$S(\rho_{ij})$	$I(i : j)$
Adjacent (0, 1)	0.181858663	0.181858663	0.209398452	0.154318874
Opposite (0, 2)	0.181858663	0.181858663	0.281661286	0.082056040

Finally, Table 3 evaluates Proposition 3 in the converged $d = 14$ ground state. Every one-site entropy is 0.181858663, but the entropy of an adjacent pair is 0.209398452, not twice the one-site value. The difference is a mutual information of 0.154318874. Opposite sites are less strongly correlated but still have nonzero mutual information. An averaged one-site entropy field would therefore discard a quantitatively substantial part of the block information even in this elementary Gaussian model.

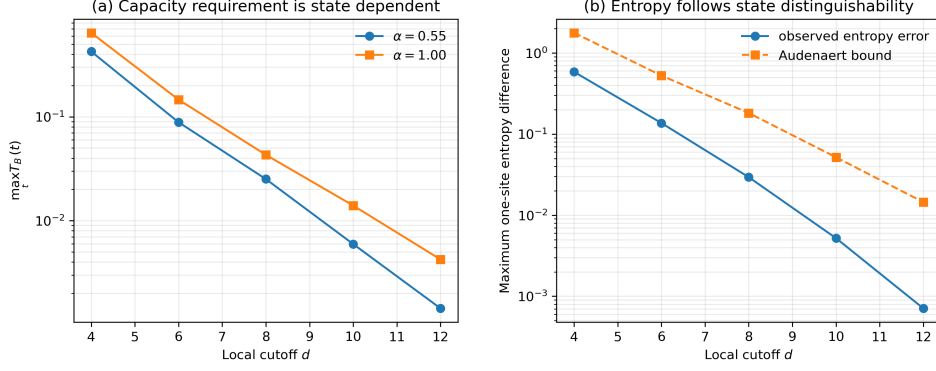


Fig. 5 State dependence and entropy certification. Left: maximum one-site trace distance over $0 \leq t \leq 8$ for coherent amplitudes $\alpha = 0.55$ and 1.00 . Right: maximum actual von Neumann entropy error and the corresponding sharp finite-dimensional continuity bound. Higher occupation increases the cutoff required for the same accuracy.

6 Consequences for finite-information space-time frameworks

The general framework and explicit model do not establish that space-time is a lattice of finite-dimensional cells. They do establish what must be supplied for that hypothesis to become a predictive microscopic theory. This section translates the regulator analysis into requirements for finite-information space-time proposals, including the QMM program. We distinguish the kinematic assertion of finite local state capacity from the dynamical task of recovering geometry and symmetries, explain when an information-gradient sector can consistently close, and specify how modified propagation or gravitational backreaction must be derived. The aim is to replace broad phenomenological association with a sequence of calculations that can succeed or fail independently.

6.1 From finite local kinematics to a closed information sector

Assigning a finite-dimensional space $\mathcal{H}_x$ to each microscopic region specifies a state-counting rule. It does not determine the Hamiltonian, the constraints, the relation between adjacency and geometry, or the physical meaning of the basis states. A space-time proposal must therefore provide at least:

- (1) a physical local algebra or constrained Hilbert space;
- (2) local interaction terms and a rule for their scaling with microscopic separation;
- (3) a definition of geometric observables or a map from correlations to an emergent metric;
- (4) a state sector in which the low-energy comparison is claimed;
- (5) a renormalization trajectory on which continuum observables are held fixed.

The existence of a Lieb–Robinson cone can support emergent causal propagation, but recovering an approximately Lorentz-invariant dispersion requires further tuning or symmetry arguments. Recovering diffeomorphism invariance is more demanding still and cannot be inferred from writing a covariant effective action after coarse-graining. In the QMM setting, local cells are intended to act both as microscopic geometric degrees of freedom and as quantum memory registers [16]. The present results refine that proposal in two ways. First, any claimed QMM continuum limit must specify independent a , d , and ℓ scalings, even if a fundamental construction later relates them dynamically. Second, the QMM imprint variables must be defined as collective coordinates on reduced states or operator algebras. Their entropy can be one useful invariant, but it is not generally a complete coordinate. The passage from this kinematic substrate to an information-sector field theory is a separate closure problem. It requires identifying a family of local states that remains approximately invariant under the microscopic dynamics, determining which coordinates on that family are slow, and showing that the discarded coordinates relax rapidly enough to be represented by local coefficients. Only after those conditions are met can an entropy gradient be interpreted as a dynamical term rather than as a descriptive statistic of an evolving reduced state. An entropy-gradient term can arise consistently when a one-dimensional family of locally equilibrated states closes the dynamics and the information metric is rewritten in terms of entropy, as in Eq. (47). To use such a sector in gravity, a microscopic derivation must determine:

$$K(S), \quad V(S), \quad F(S), \quad M_I, \quad (61)$$

Equation (61) lists the minimum matching data, and the derivation must establish the domain on which S is a valid coordinate. A bounded microscopic entropy also requires that the effective field range or potential encode the finite state manifold. Treating S as an unconstrained canonical scalar over all real values loses the microscopic capacity interpretation. If the accessible state manifold has dimension greater than one, the appropriate theory is the multi-field sigma model of Eq. (43). The additional coordinates may include energy density, charge, coherence, or correlation tensors. Integrating them out can generate nonlocal response, dissipation, or higher-derivative operators. These terms are not pathologies; they are the expected signatures of a reduced description. What must be avoided is attributing them universally to ∇S without a closure calculation.

6.2 Derived predictions and a falsifiable development path

Modified propagation does not follow from finite capacity in the abstract. It follows from specific microscopic dispersion or specific effective operators. In the scalar chain, the correction in Eq. (59) is derived from the spatial finite difference. In an effective space-time model, a correction to a matter field ϕ would require an operator such as

$$\frac{c_1}{\Lambda^2}(\Box\phi)^2 \quad \text{or} \quad \frac{c_2}{\Lambda^4}\nabla_\mu\theta^A\nabla_\nu\theta^B\nabla^\mu\phi\nabla^\nu\phi, \quad (62)$$

The two examples in Eq. (62) illustrate higher-derivative and information-background couplings whose coefficients must be fixed by matching. One then expands around a stated background and derives the mode equation. A phenomenological k^4/d term is neither dimensionally complete nor tied to a microscopic calculation. The same principle applies to gravitational or cosmological sources. An information-sector stress tensor is meaningful only after the collective equations of motion and matching coefficients are known. The present work therefore does not infer dark-matter-like sources, dark energy, black-hole spectra, or observational bounds from the generic existence of finite local Hilbert spaces. Those are separate model-dependent applications. This requirement is constructive rather than merely restrictive. Once an operator and its coefficient have been obtained by matching, the model produces a definite correction with a stated domain of validity. That correction can then be compared with laboratory, astrophysical, or cosmological data without ambiguity about whether it arose from local capacity, spatial discreteness, or the chosen coarse-graining closure. A useful development program should therefore preserve the chain from microscopic interaction to effective operator to observable response and attach a failure criterion to every link. A finite-information space-time theory can be developed in a sequence in which every step admits a failure criterion:

- (i) construct a microscopic constrained model with exact unitary dynamics;
- (ii) identify a family of low-energy states and observables;
- (iii) specify $\mathcal{C}_{a,d,\ell}$ and, where useful, $\mathcal{R}_{a,d,\ell}$;
- (iv) measure or bound δ_{cap} , δ_{lat} , and δ_{cg} separately;
- (v) derive the information metric and effective coefficients from reduced states;
- (vi) extract, rather than insert, a correction to spectra or response functions;
- (vii) compare the correction with a continuum theory along a controlled scaling trajectory.

Failure of convergence, absence of a symmetry-restoring trajectory, or inability to close the collective dynamics would falsify the corresponding microscopic realization without falsifying the broader idea of finite local capacity. This layered structure is scientifically preferable to attributing every mismatch to one undifferentiated "finite-information deviation."

7 Discussion and limitations

The operational framework resolves several ambiguities that otherwise obscure finite-information claims. The central quantity is a commuting-diagram defect in a common state space, not a subtraction of unrelated generators. The decomposition in Eq. (17) prevents local capacity from being confused with lattice spacing or coarse-graining. Theorem 2 identifies the physical quantities that control a truncation: excluded state weight and dynamical leakage. Proposition 3 shows that local entropy averages omit correlations. The information-geometric construction explains when a gradient action is justified and why the generic target is a state manifold rather than a single entropy

scalar. The numerical model was deliberately chosen to be exactly solvable in its untruncated lattice form. This permits three independent checks. The converged finite- d ground energy agrees with the analytic sum over normal modes. Frequencies extracted from spectral weights agree with the analytic lattice dispersion. The $d = 14$ reference can therefore be checked against exact results rather than trusted solely because it is the largest simulated basis. The real-time calculation then probes quantities for which no analytic finite- d formula was inserted: reduced density matrices, trace distances, and entropies. Several limitations are important. First, $N = 4$ is sufficient to isolate capacity convergence and regulator separation but not to study a thermodynamic or critical continuum limit numerically. The lattice-spacing comparison in Figure 3 therefore uses the exact analytic dispersion rather than larger exact Hilbert-space simulations. Extending the state-space calculations to larger N calls for tensor-network or Gaussian-state methods. Second, the $d = 14$ real-time state is a numerical reference, not the mathematical $d = \infty$ state. Its spectral agreement and occupation-tail convergence make it adequate for the reported precision, but a rigorous time-dependent *a posteriori* bound would strengthen the result. Third, the number-basis Hamiltonian is an encoded truncation rather than exactly $P_d H_a P_d$, as stated in Section 5.1. Fourth, the information-geometric action is a controlled local expansion of a family of reduced states, but Lorentzian continuation and coupling to dynamical geometry require a separate microscopic matching calculation. The formal bounds are also intentionally conservative. The operator norm $\|Q_d H P_d\|$ can be dominated by cutoff-boundary states that have negligible probability in the physical sector. Energy-constrained norms, state-dependent Duhamel estimates, or moment bounds can yield substantially sharper results [9]. Likewise, the Audenaert bound is optimal in the worst case for a given trace distance and dimension, but a particular Gaussian family occupies a much smaller subset of state space. The theory is compatible with finite-dimensional local systems that recover continuum physics at fixed d through criticality, as well as with bosonic encodings that require $d \rightarrow \infty$. It does not assert that one mechanism is universal. This is a strength: "finite information" becomes a testable property of a specified microscopic representation rather than a label attached to any discretized model.

8 Conclusion

We have developed a regulator-resolved framework for analyzing finite local information capacity and its relation to continuum effective dynamics. A central result is that local Hilbert-space dimension, spatial lattice spacing, and coarse-graining scale represent distinct regulators whose effects cannot be conflated and must therefore be studied independently. Within this framework, finite-information deviation is defined operationally through the noncommutativity of microscopic evolution, coarse-graining, and effective evolution. Finite-time capacity errors are governed by the discarded state weight and by leakage across the boundary of the retained subspace, rather than by any universal inverse- d expansion. Convergence in trace distance then provides uniform control over all accessible bounded observables and, through sharp continuity bounds, also constrains subsystem entropy errors. The analysis further shows that block entropy and the average of single-cell entropies differ by the total quantum

correlations contained within the block and therefore cannot, in general, be used interchangeably. At the effective-field-theory level, the leading local information-gradient description is generically a nonlinear sigma model defined on a manifold of reduced quantum states. A reduction to a single entropy field is possible only after imposing an additional one-dimensional closure on the accessible state manifold. In the explicit quantum-lattice realization developed here, finite-capacity errors decrease rapidly as the local Hilbert-space dimension is increased, while the difference between lattice and continuum dispersion persists at fixed lattice spacing. This provides a direct numerical confirmation that local capacity and spatial discretization control physically distinct sources of deviation. Taken together, these results provide a mathematically defined and numerically reproducible foundation for finite-information models. They also establish a clear standard for future applications to space-time and quantum gravity: the microscopic dynamics, coarse-graining and matching channels, relevant state sector, regulator-scaling trajectory, and effective coefficients must all be specified before finite local information capacity can yield a genuine physical prediction.

Declarations

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Data availability. All numerical data used to generate the tables and figures are included with the manuscript source package in machine-readable comma-separated-value files.

Materials availability. Not applicable.

Code availability. The complete Python program and an executed Jupyter notebook required to reproduce the numerical calculations and all PNG figures are included with the manuscript source package. They use NumPy, SciPy, and Matplotlib, record all model parameters, display every figure inline, and save the image files to the local `figures` directory.

Use of artificial intelligence. Generative artificial intelligence tools were used solely to assist with language editing and stylistic refinement of the manuscript. They were not used to generate the scientific content, theoretical arguments, mathematical derivations, numerical results, interpretations, or conclusions.

Author contribution. Conceptualization, E.M.; theoretical development and methodology, F.N.; formal analysis, F.N.; investigation, F.N. and E.M.; software and data curation, F.N.; writing—original draft, F.N.; writing—review and editing, F.N. and E.M.; visualization, F.N.; supervision and project administration, F.N. and E.M. Both authors reviewed and approved the final manuscript.

A Proof of the regulator decomposition

This appendix supplies the explicit trace-norm argument underlying the regulator-resolved decomposition in Eq. (17) and records how the construction generalizes when additional matching steps are present. The proof is elementary, but stating the intermediate states makes the physical meaning of each term unambiguous.

Let

$$\rho_0 = \sigma_{a,d,\ell}, \quad \rho_1 = \sigma_{a,\infty,\ell}, \quad \rho_2 = \sigma_{0,\infty,\ell}, \quad \rho_3 = \sigma_{\text{eff}}. \quad (63)$$

Equation (63) names the four states introduced in Eqs. (13)–(16). Because the trace norm obeys the triangle inequality,

$$\begin{aligned} 2T(\rho_0, \rho_3) &= \|\rho_0 - \rho_3\|_1 \\ &\leq \|\rho_0 - \rho_1\|_1 + \|\rho_1 - \rho_2\|_1 + \|\rho_2 - \rho_3\|_1. \end{aligned} \quad (64)$$

Equation (64) is the trace-norm triangle inequality for this chain of matched states. Dividing by two gives Eq. (17). The terms depend on the chosen matching family. This is unavoidable: capacity and lattice errors are relational quantities between regulated descriptions, not intrinsic scalars attached to d and a independently of a state and an observable sector.

If an additional model-reduction step is present, one may insert another intermediate state and obtain a fourth term. Likewise, if the initial-state matching is imperfect, the contractivity of CPTP maps permits its error to be added explicitly. The decomposition is therefore modular.

B Proof of the finite-time capacity bound

This appendix derives the finite-time estimate by separating initial projection disturbance from dynamical leakage through the cutoff boundary. The argument also shows precisely where contractivity under subsequent coarse-graining enters.

We prove Theorem 2. Let $p = \text{Tr}(P\rho) = 1 - \varepsilon$ and $\tilde{\rho} = P\rho P/p$. The gentle-measurement inequality for the projection P gives

$$\|\rho - P\rho P\|_1 \leq 2\sqrt{\varepsilon}. \quad (65)$$

The initial projection disturbance is bounded by Eq. (65). Moreover,

$$\left\| P\rho P - \frac{P\rho P}{p} \right\|_1 = \left| 1 - \frac{1}{p} \right| \text{Tr}(P\rho P) = \varepsilon. \quad (66)$$

Equation (66) isolates the additional normalization change. The triangle inequality therefore yields

$$T(\rho, \tilde{\rho}) \leq \sqrt{\varepsilon} + \frac{\varepsilon}{2}. \quad (67)$$

Equation (67) combines the two initial-state contributions. Unitary invariance of trace distance implies

$$T(U\rho U^\dagger, U\tilde{\rho}U^\dagger) = T(\rho, \tilde{\rho}). \quad (68)$$

It remains to compare evolution of the same retained state under H and $H_d = PHP$. Duhamel's identity gives, on the range of P ,

$$\begin{aligned} U(t)P - U_d(t)P &= -i \int_0^t U(t-s)(H - H_d)U_d(s)P \, ds \\ &= -i \int_0^t U(t-s)QHP U_d(s)P \, ds. \end{aligned} \quad (69)$$

Equation (69) expresses the evolution difference entirely through the cutoff-boundary coupling QHP . Hence

$$\|(U(t) - U_d(t))P\| \leq |t| \|QHP\|. \quad (70)$$

Equation (70) bounds the propagator difference on the retained subspace. For any density operator τ supported on P ,

$$\begin{aligned} T(U\tau U^\dagger, U_d\tau U_d^\dagger) &\leq \frac{1}{2} \left(\|(U - U_d)\tau U^\dagger\|_1 + \|U_d\tau(U^\dagger - U_d^\dagger)\|_1 \right) \\ &\leq \|(U - U_d)P\| \leq |t| \|QHP\|. \end{aligned} \quad (71)$$

Combining Eqs. (67), (68), and (71) by the triangle inequality proves Eq. (24). Finally, trace distance is contractive under every CPTP map $\mathcal{C}$:

$$T(\mathcal{C}(\rho), \mathcal{C}(\sigma)) \leq T(\rho, \sigma), \quad (72)$$

Equation (72) proves the final statement after coarse-graining.

For an encoded Hamiltonian $H_d^{\text{enc}} = H_d + \delta H_d$, a second Duhamel estimate gives

$$\|e^{-iH_d t} - e^{-iH_d^{\text{enc}} t}\| \leq |t| \|\delta H_d\|, \quad (73)$$

provided the relevant operators are bounded on the retained space. Equation (73) is the encoded-Hamiltonian analogue of Eq. (70); this term may be added to Eq. (24).

C Information-geometric expansion and conservation

This appendix provides the differential expansion of relative entropy used in the main text and verifies conservation of the collective stress tensor on solutions of the sigma-model equations. It also records the additional terms required by a nonminimal curvature coupling.

For faithful states, quantum relative entropy is

$$D(\rho\|\sigma) = \text{Tr} \, \rho(\log \rho - \log \sigma). \quad (74)$$

Equation (74) fixes the relative-entropy convention. Let $\rho(\theta)$ be twice differentiable. Expanding $\rho(\theta + d\theta)$ and using $\text{Tr}(\partial_A \rho) = 0$, the linear contribution to relative entropy

vanishes. The symmetric second-order part defines a positive bilinear form. For the relative-entropy Hessian this is the BKM metric, giving Eq. (39). Positivity follows from monotonicity of relative entropy under CPTP maps [29, 30].

For the sigma-model action in Eq. (43) with $F = 1$, variation with respect to θ^A gives

$$0 = M_I^2 G_{AB} \square \theta^B + M_I^2 \partial_C G_{AB} \nabla_\mu \theta^C \nabla^\mu \theta^B - \frac{M_I^2}{2} \partial_A G_{BC} \nabla_\mu \theta^B \nabla^\mu \theta^C - \partial_A V. \quad (75)$$

Equation (75) is the uncontracted Euler–Lagrange equation. Multiplication by G^{AD} and rearrangement yield Eq. (45). A direct covariant derivative of Eq. (44) gives

$$\nabla^\mu T_{\mu\nu}^{(\text{info})} = M_I^2 G_{AB} \mathcal{E}^A \nabla_\nu \theta^B, \quad (76)$$

Equation (76) makes conservation proportional to the collective equations of motion, where $\mathcal{E}^A$ is the left-hand side of Eq. (45). The stress tensor is therefore conserved on shell. If the collective variables are externally prescribed functions rather than dynamical fields, Eq. (46) does not follow and their backreaction cannot consistently be inserted into Einstein’s equation without accounting for the external sector that drives them.

When $F(\theta)$ is not constant, variation of FR produces the familiar scalar–tensor terms

$$M_P^2 [FG_{\mu\nu} + g_{\mu\nu} \square F - \nabla_\mu \nabla_\nu F], \quad (77)$$

Equation (77) gives the complete metric variation of the nonminimal term, and the collective equations acquire $(M_P^2/2)\partial_A F R$. These terms must be retained together; keeping only second derivatives in an effective stress tensor while omitting the nonminimal action that generates them is inconsistent.

D Numerical implementation and reproducibility

This appendix documents the complete numerical pipeline and the independent analytic checks used to validate the finite- d reference calculation. Every quantity shown in the tables and figures is generated by the accompanying code and written to a machine-readable data file.

The accompanying program `simulate.finite.capacity.py` performs the following steps.

- (1) It constructs sparse local annihilation matrices with elements $\langle n-1|b|n\rangle = \sqrt{n}$ and forms

$$q = \frac{b + b^\dagger}{\sqrt{2\omega_0}}, \quad p = -i\sqrt{\frac{\omega_0}{2}}(b - b^\dagger). \quad (78)$$

Equation (78) defines the finite canonical matrices.

- (2) It embeds these operators at each of four tensor-product sites and assembles Eq. (48) as a sparse Hermitian matrix.

- (3) It computes the lowest 24 eigenpairs with a residual tolerance of 10^{-11} and evaluates the spectral weights in Eq. (55).
- (4) It constructs truncated coherent states recursively, normalizes them, and applies the Hamiltonian exponential at 81 equally spaced times from 0 to 8 using a Krylov method.
- (5) It reshapes each pure state tensor and obtains one- and two-site density matrices by exact partial contraction.
- (6) It computes trace distance from the eigenvalues of the Hermitian state difference and entropy from the eigenvalues of each reduced density matrix, discarding only numerical eigenvalues below 10^{-14} .
- (7) It writes every plotted value to comma-separated files and generates the five 300-dpi PNG figures included in the manuscript.

The exact lattice frequencies and ground energy provide independent validation. For the reported $d = 14$ calculation,

$$\begin{aligned}
|\omega_{14}(0) - \omega_a(0)| &= 1.43 \times 10^{-10}, \\
|\omega_{14}(\pi/2) - \omega_a(\pi/2)| &= 7.06 \times 10^{-11}, \\
|\omega_{14}(\pi) - \omega_a(\pi)| &= 5.83 \times 10^{-11}.
\end{aligned} \tag{79}$$

The three frequency residuals in Eq. (79) independently validate the $d = 14$ reference. The complete raw outputs are `spectral_convergence.csv`, `dynamic_deviation_alpha_055.csv`, `dynamic_deviation_alpha_100.csv`, and `block_correlations.csv`. Running the program from the project root regenerates the data and figures without manual intervention.

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